On the definition of solid discharge in hydro-environment research and applications

FRANCESCO BALLIO (IAHR Member), Professor, *Department of Civil and Environmental Engineering, Politecnico di Milano, Italy*
Email: francesco.ballio@polimi.it (author for correspondence)

VLADIMIR NIKORA (IAHR Member), Professor, *School of Engineering, University of Aberdeen, UK*
Email: v.nikora@abdn.ac.uk

STEPHEN E. COLEMAN† (IAHR Member), Associate Professor, *Department of Civil and Environmental Engineering, University of Auckland, New Zealand*

†Professor Stephen E. Coleman passed away on 23 July 2012, when this manuscript was under preparation.

1 Introduction

A traditional approach to morphodynamics and sediment transport is based on an Eulerian framework where variables (solid fluxes, concentrations, velocities of solid phase, and other parameters) are defined and measured over finite (and relatively large) control volumes/areas, being often averaged over finite time periods. Within this conventional approach, the involved variables are typically considered as continuous and "well-behaved" with respect to time and space differentiation (i.e. as differentiable). However, high space–time resolution required for the analysis of morphodynamic processes at scales smaller than that of a river reach (e.g. local scour processes, bedform evolution, and gravel particle motion) often implies that the spatial and temporal extensions at which variables are defined and/or measured are not much larger than those of the particle motion, thus violating conventional continuum assumptions. Another example is a near-threshold sediment transport when the movement of particles is sparse in space and intermittent in time, and thus it may be difficult to assume the desired smoothness for the measured quantities (e.g. Furbish *et al.* 2012). Thus, it is important that the variables used in sediment transport studies are unambiguously defined (including the scale of consideration), and intrinsic limitations of these definitions are well understood and identified. Below we provide an example that helps to highlight this issue.

Let us consider a volumetric bedload transport rate per unit width (q_{bl}) as a variable of interest. According to the current practice, it can be expressed using two conventional forms (e.g. Garcia 2008, in *ASCE Manual on Practice* 110, p. 68):

$$q_{bl} = u_{bl}\, c_{bl}\, \delta_{bl} \tag{1}$$

and

$$q_{bl} = N_{bl}\, w_{bl}\, u_{bl} \tag{2}$$

where u_{bl} is the particle (or solid) velocity, c_{bl} is the sediment concentration, δ_{bl} is the thickness of bedload layer, N_{bl} is the number of moving particles per unit bed area, and w_{bl} is the particle volume. A number of conceptual issues arise from definitions (1) and (2) which are highlighted below.

Received 25 June 2013; accepted 20 November 2013/Currently open for discussion.

(i) Should the quantities in Eqs. (1) and (2) be considered as instantaneous or as time-averaged measures? In principle, expressions (1) and (2) cannot be valid for both frameworks simultaneously unless the quantities on the right sides of the equations are uncorrelated.

(ii) The variable q_{bl} is intrinsically defined as a flux *through a surface*. Self-consistent evaluation for q_{bl} should employ, therefore, only quantities defined over the same surface. This is, in principle, possible with Eq. (1), although in practice volume-averaged quantities are typically used (e.g. volumetric sediment concentration is used rather than an areal concentration). As for Eq. (2), it employs quantities that are intrinsically defined over a *volume*, i.e. some volume and/or time averaging is implicitly employed.

(iii) The bedload rate q_{bl} is typically used in differential equations (e.g. the Exner equation). Under which conditions (e.g. space and time resolution) can this variable be considered smooth enough for spatial and time differentiation?

The example given above suggests that even well-established and widely used concepts may require some special attention if accurate and unambiguous definitions are desired. This, for instance, may be needed if one tries to compare bedload definitions and relationships available in the literature, which often relate to different scales, from the particle scale to the river reach scale (see, for example, Böhm *et al.* 2004, Ancey 2010, Furbish *et al.* 2012). However, the scale issue is rarely explicitly considered making the comparison task ambiguous.

The aim of this paper is thus to discuss and clarify some basic definitions related to sediment transport, at least with respect to the Eulerian kinematic variables. Specifically, we will (1) propose univocal and self-consistent definitions for quantities involved in the integral continuity equation for sediments, (2) analyse their time/space (ir)regularity and their scale dependence, and (3) compare relations for different descriptions. For the sake of clarity, we will treat the problem in its simplest possible form (see next section for a discussion of basic assumptions), as extensions to more general frameworks are straightforward. In particular, we will not discuss quantities involved in the conservation equations other than the mass balance, although many of the considerations proposed here could be extended to other quantities (i.e. momentum, energy).

The paper is organized as follows. General definitions and the framework used for the analysis are given in the next section. Section 3 contains conceptual and mathematical body of the paper where alternative forms for instantaneous and time-averaged mass balances are derived (Sections 3.1 and 3.2) and discussed (Section 3.3). In Section 4, the concepts are further enlightened by applying them to an experimental example. Finally, the definitions given in Section 3 are compared in Section 5 with corresponding quantities proposed in other studies.

2 General framework and definitions

To underpin our considerations, we use a general definition sketch in Fig. 1a. The Eulerian form of the principle of mass

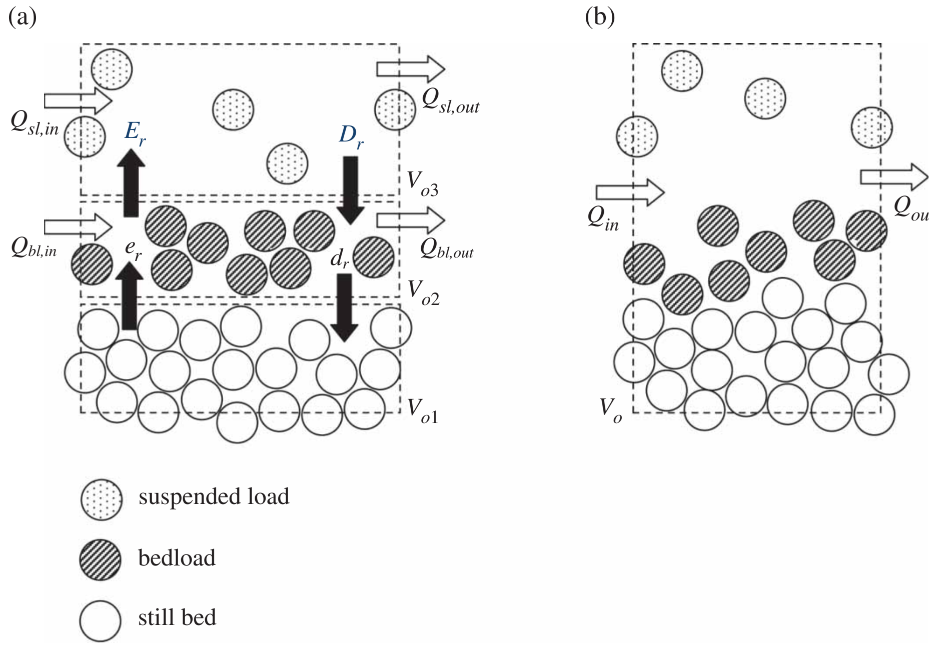

Figure 1 Definition sketch. Q_{sl} is suspended solid load; Q_{bl} is bedload; Q is total solid discharge; E_r is erosion rate; D_r is settling rate; e_r is entrainment (pickup) rate; d_r is deposition rate; and V_{oi} are control volumes. All quantities (Q_x, E_r, D_r, e_r, and d_r) are extensive variables calculated over their respective surfaces. (a) Observation domain is divided into volumes whose boundaries coincide with interfaces between sediment layers. (b) Domain is identified by single volume, without *a priori* subdivisions into layers or sub-domains

conservation (mass balance) of sediments with constant density, over a finite volume V_o and a finite time lag T_o, can be written as

$$V(t + T_o) - V(t) = V_{in}(t, T_o) - V_{out}(t, T_o) \quad (3)$$

where $V(t)$ is the volume of sediments within the control volume V_o at time t, $V_{in}(t,T_o)$ and $V_{out}(t,T_o)$ are volumes of sediments entering and exiting V_o through its boundary surface during the time period from t to $t + T_o$, respectively. By dividing all terms by T_o one obtains:

$$\frac{V(t + T_o) - V(t)}{T_o} = \frac{V_{in}}{T_o} - \frac{V_{out}}{T_o} \quad (4)$$

where terms $V_{in/out}/T_o = \overline{Q}^s_{in/out}$ can be interpreted as time-averaged sediment fluxes through the input/output surfaces (proper definitions for averaging will be given in Section 3.2), and the term on the left represents the average rate of change of sediment volume contained within V_o.

Equations (3) and (4) do not require any constraint for the choice of V_o and T_o, which can be "small" or "large" with respect to the sediment space and time scales. However, although the quantities involved in Eqs. (3) and (4) are defined they may not be well behaved depending on V_o and/or T_o. The explanatory concept is depicted in Fig. 2, which illustrates a possible behaviour of the time-averaged sediment flux $\overline{Q}^s$ as a function of the integration time period T_o. In this sketch, the process is governed by three time scales: the scale $T_p = d/u$ is the time period needed for a particle to cross the reference surface, where d and u are particle diameter and velocity, respectively; the scale T_i is the time period between arrivals of two subsequent particles at the reference surface; and the scale T_{bf} is time period of bedforms. The time scales in Fig. 2 are chosen to satisfy the condition $T_p < T_i \ll T_{bf}$, i.e. it is assumed that there is a small-scale separation between T_p and T_i and a large-scale separation between the particle scales T_p, T_i , on one hand, and bedform scale T_{bf}, on the other hand. At $T_o \ll T_p$, the reference surface is crossed by

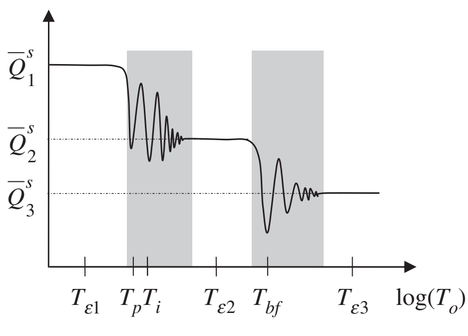

Figure 2 Qualitative representation of time-averaged sediment flux $\overline{Q}^s$ as a function of the integration time period T_o. T_p, T_i, and T_{bf} are characteristic time scales for individual particle crossing, particles inter-arrival, and bedforms, respectively. Grey shadings correspond to the intervals where function is not well behaved. Meaning of T_ε is explained in Section 2 of the article

particles without interruptions producing a "smooth" flux. However, with increasing T_o, i.e. at scales comparable to the particle scales T_p andT_i, the average flux becomes ill-defined (first grey area). At larger scales (i.e. $T_o \gg T_i$), the flux $\overline{Q}^s$ assumes smooth behaviour again preserving it until T_o reaches values comparable to T_{bf}. At $T_o \gg T_{bf}$, the smoothness of $\overline{Q}^s$ returns.

Thus, for the smallest values of the integration time period T_o, the flux $\overline{Q}^s$ is a smooth function representing the volume flux due to the uninterrupted particle movement through the surface ($\overline{Q}^s_1$ in Fig. 2). The flux $\overline{Q}^s$ progressively decreases with increasing T_o as the total cross-sectional area of the particles at the reference plane decreases in time when T_o approaches T_p. For $T_o > T_p$, the average flux $\overline{Q}^s$ oscillates due to alternation of periods when no particles are crossing the surface and periods when particles cross the surface. Oscillations eventually disappear for larger T_o that includes a large number of crossings, so that the time-averaged sediment flux $\overline{Q}^s$ becomes smooth again reaching $\overline{Q}^s_2$. When T_o is further increased, the fluctuations of the averaged sediment flux due to bedforms appear, eventually vanishing at $T_o \gg T_{bf}$,, with $\overline{Q}^s \to \overline{Q}^s_3$. In Fig. 2, we assumed that $\overline{Q}^s_2$ corresponds to the flux at the crests of bedforms, so that the time averaging over a series of bedforms gives lower values for the average flux, leading to $\overline{Q}^s_3 < \overline{Q}^s_2$.

The example in Fig. 2 highlights potential dependence of the "instantaneous" sediment flux on the time scale of consideration. From a conventional point of view, the "instantaneous" sediment flux should be defined as $Q = \lim_{T_o \to 0} \overline{Q}^s$ leading to the differential version of the mass balance:

$$\frac{dV}{dt} = Q_{in}(t) - Q_{out}(t) \quad (5)$$

where $Q = \lim_{T_o \to 0} \overline{Q}^s = \overline{Q}^s_1$. In morphological models, however, the scale for the analysis is typically much larger than particle-related scales. The concept of "vanishing" T_o in $Q = \lim_{T_o \to 0} \overline{Q}^s$ is therefore operationally relaxed to finite values of T_o. This relaxation can be interpreted, with the help of Fig. 2, as a generalization of the "instantaneous" sediment flux by incorporating scale dependence of Q, i.e. $Q = \lim_{T_o \to T_\varepsilon} \overline{Q}^s$ where T_ε is a small but finite value representing scale of consideration ($T_{\varepsilon 1}$, $T_{\varepsilon 2}$, and $T_{\varepsilon 3}$ in Fig. 2). Thus, the quantities $\overline{Q}^s_2$ and $\overline{Q}^s_3$ in Fig. 2 may be considered as "instantaneous", given that the analysis is performed within the corresponding "smooth" regions of T_o. In contrast, the variables within the shaded regions of Fig. 2 although defined are not well behaved.

In this paper we will focus on the "instantaneous" fluxes defined at $T_{\varepsilon 1}$, i.e. $Q = \lim_{T_o \to T_{\varepsilon 1} \ll T_p} \overline{Q}^s$ (Section 3.1). Although inconvenient for practical applications, this scale of consideration allows clarifying critical issues of scale effects on the definitions of sediment quantities (Furbish *et al.* 2012, Ancey and Heyman 2013), by upscaling to larger scales (e.g. from $T_{\varepsilon 1}$ to $T_{\varepsilon 2}$ to $T_{\varepsilon 3}$) through time integration (Section 3.2).

The discussion of the effects of the integration time T_o may be repeated with respect to spatial scales representing the integration volume V_o and the related surfaces in Eq. (4), eventually leading to the differential form of the mass balance in space. We omit this consideration here as conceptually it is similar to the discussion of the effects of T_o above.

As we are only interested in the sediment phase, it will be necessary to specify whether we are considering superficial or intrinsic-averaged quantities (definitions are given below). Further specifications may be necessary to differentiate between moving and non-moving sediments, or bedload versus suspended load. For such demarcations, approaches making use of a clipping function $\gamma(\boldsymbol{x}, t)$ can be adopted (e.g. Nikora *et al*. 2013); here and in the following $\boldsymbol{x}$ (bold) indicates the space coordinate vector. Specifically for the solid phase (e.g. Coleman and Nikora 2009), $\gamma(\boldsymbol{x}, t)$ is defined in space and time as

$$\gamma(\boldsymbol{x}, t) = 1 \text{ if a point } (\boldsymbol{x}, t) \text{ is occupied by solid,}$$
$$\gamma(\boldsymbol{x}, t) = 0 \text{ otherwise.} \tag{6}$$

Note that in different research areas, the function $\gamma(\boldsymbol{x}, t)$ is also known as "characteristic", "phase distribution", or "phase indicator" function (e.g. Lhuillier 1992, Zhang and Prosperetti 1994). Clipping functions are useful for defining (integral) quantities which assume some physical property within selected sub-domains of the integration domain (e.g. solids as above). Consequently, definitions for superficial $\langle\theta\rangle^s$ and intrinsic $\langle\theta\rangle$ volumetric averages of a variable θ are expressed as

$$\langle\theta\rangle^s = \frac{1}{V_o}\int_{V_o}\theta(\boldsymbol{x}, t)\gamma(\boldsymbol{x}, t)\mathrm{d}V = \frac{1}{V_o}\int_{V}\theta(\boldsymbol{x}, t)\mathrm{d}V \tag{7}$$

$$\langle\theta\rangle = \frac{1}{V}\int_{V_o}\theta(\boldsymbol{x}, t)\gamma(\boldsymbol{x}, t)\mathrm{d}V = \frac{1}{V}\int_{V}\theta(\boldsymbol{x}, t)\mathrm{d}V \tag{8}$$

where the solid volume V within the total volume V_o is given by

$$V = V(t) = \int_{V_o}\gamma(\boldsymbol{x}, t)\mathrm{d}V \tag{9}$$

Analogous expressions can also be written for superficial $\overline{\theta}^s$ and intrinsic $\overline{\theta}$ time averages. For our analysis, in addition to $\gamma(\boldsymbol{x}$,t) in Eq. (6) we also define:

$$\gamma_m(\boldsymbol{x}, t) = 1 \text{ if point } (\boldsymbol{x}, t) \text{ is occupied by } \textit{moving} \text{ solid,}$$
$$\gamma_m(\boldsymbol{x}, t) = 0 \text{ otherwise} \tag{10}$$
$$\gamma_b(\boldsymbol{x}, t) = 1 \text{ if point } (\boldsymbol{x}, t) \text{ is occupied by still (}\textit{bed}\text{) solid,}$$
$$\gamma_b(\boldsymbol{x}, t) = 0 \text{ otherwise} \tag{11}$$

so that $\gamma(\boldsymbol{x}, t) = \gamma_m(\boldsymbol{x}, t) + \gamma_b(\boldsymbol{x}, t)$.

Basic balance formulations (3) to (5) need to be expressed in terms of fundamental variables. One of them is the solid concentration within V_o that can be defined using $\gamma(\boldsymbol{x}, t)$ as

$$\phi_V = \frac{V(t)}{V_o} = \frac{\int_{V_o}\gamma(\boldsymbol{x}, t)\mathrm{d}V}{V_o} \tag{12}$$

As the validity of the conservation of mass has no limitation, we will try to maintain the analysis as general as possible. However, some (minor) restrictions will be introduced in order to limit ramifications of the discussion and complications of the nomenclature. Until now, only the condition of constant sediment density is assumed, so that mass balances for the solid phase can be reduced to volumetric balances. Further assumptions are explained below.

(i) No distinction between bedload and suspended load will be considered. This separation could be introduced by specifying additional clipping functions similar to those defined by Eqs. (6), (10), and (11). For the sake of simplicity, however, our discussion will relate to bedload only.
(ii) Although conceptually the analysis is valid for a fully three-dimensional space $[x, y, z]$, we will consider particle motion in the longitudinal direction x only, i.e. focusing on the longitudinal component of the velocity vector of the solid phase. In addition, the boundaries of control volumes employed are such that only yz planes are crossed by moving particles. These restrictions are posed solely to avoid using vectorial notations; generalization of results is conceptually straightforward.

Finally, we observe that there is no limitation for the choice of the size and position of the domain V_o. Size can span from a sub-particle domain to large volumes containing many particles (as in classical continuum approach). In this paper we will not use preferential volumes (V_{o1}, V_{o2}, V_{o3}) of the sketch in Fig. 1a, where their boundaries coincide with interfaces between sediment layers with different behaviour. Instead, volumes considered are fully generic (Fig. 1b) while phases are distinguished by means of appropriate clipping functions (i.e. liquid vs. solid, or still solid vs. moving solid). The main advantages of such an approach are: (i) it does not require identifying layers and interfaces *a priori* and (ii) it can be applied without special care at any scale, including a particle scale, where a clear definition of interfaces between layers may become cumbersome.

3 Solid discharge definitions: alternative forms

3.1 *Instantaneous fluxes*

Following definitions given in the previous section, the solid flux through a boundary can be expressed using surface integration as

$$Q = \int_{S_o} u(\boldsymbol{x}, t)\gamma(\boldsymbol{x}, t)\mathrm{d}S \tag{13}$$

where S_o is the area of the inflow/outflow surfaces and $u(\mathbf{x}, t)$ is the longitudinal velocity of the solid phase (i.e. within a particle). Variants of Eq. (13) are commonly used in continuum mechanics; see Furbish *et al.* (2012) for a specific discussion with respect to sediment transport. By substituting Eqs. (9) and (13) into Eq. (5) we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_o}\gamma(\mathbf{x},t)\mathrm{d}V = \int_{S_{o,in}} u(\mathbf{x},t)\gamma(\mathbf{x},t)\mathrm{d}S - \int_{S_{o,out}} u(\mathbf{x},t)\gamma(\mathbf{x},t)\mathrm{d}S \tag{14}$$

Considering solid fluxes it is also useful to employ (phase-) *areal* averages:

$$\{\theta\}^s = \frac{1}{S_o}\int_{S_o}\theta(\mathbf{x},t)\gamma(\mathbf{x},t)\mathrm{d}S = \frac{1}{S_o}\int_{S(t)}\theta(\mathbf{x},t)\mathrm{d}S \tag{15a}$$

$$\{\theta\} = \frac{1}{S(t)}\int_{S_o}\theta(\mathbf{x},t)\gamma(\mathbf{x},t)\mathrm{d}S = \frac{1}{S(t)}\int_{S(t)}\theta(\mathbf{x},t)\mathrm{d}S \tag{15b}$$

$$\{\theta\}^{sm} = \frac{1}{S_o}\int_{S_o}\theta(\mathbf{x},t)\gamma_m(\mathbf{x},t)\mathrm{d}S = \frac{1}{S_o}\int_{S_m(t)}\theta(\mathbf{x},t)\mathrm{d}S \tag{15c}$$

$$\{\theta\}^{m} = \frac{1}{S_m(t)}\int_{S_o}\theta(\mathbf{x},t)\gamma_m(\mathbf{x},t)\mathrm{d}S = \frac{1}{S_m(t)}\int_{S_m(t)}\theta(\mathbf{x},t)\mathrm{d}S \tag{15d}$$

where the brackets $\{\cdot\}$ denote areal averaging, $S(t)$ is the portion of S_o occupied by the solid phase at time t, and $S_m(t)$ is the portion of S_o occupied by the *moving* solid phase at time t (see Fig. 3). We also define the sediment areal concentrations ϕ_A and ϕ_{Am} as the areal counterparts of ϕ_V, i.e.:

$$\phi_A = \frac{1}{S_o}\int_{S_o}\gamma(\mathbf{x},t)\mathrm{d}S = \frac{S(t)}{S_o} \tag{16a}$$

$$\phi_{Am} = \frac{1}{S_o}\int_{S_o}\gamma_m(\mathbf{x},t)\mathrm{d}S = \frac{S_m(t)}{S_o} \tag{16b}$$

With Eqs. (15a)–(15d), we have $\{\theta\}^s = \phi_A\{\theta\}$ and $\{\theta\}^{sm} = \phi_{Am}\{\theta\}^m$.

It should be noted that $S(t)$ and $\phi_A(t)$ are continuous functions of time, while $S_m(t)$ and $\phi_{Am}(t)$ are not always so. Let us consider, as an example, a particle crossing surface S_o which is entrained at time t: in the portion of space occupied by the particle we have $\gamma(t) = \gamma(t + \mathrm{d}t) = 1$, so that $S(t)$ changes smoothly during entrainment, as a consequence of the infinitesimal displacement of the particle. Corresponding values for γ_m, on the contrary, change from $\gamma_m(t) = 0$ to $\gamma_m(t + \mathrm{d}t) = 1$ thus causing a finite increase for $S_m(t)$ within an infinitesimal time lag.

Using Eqs. (14)–(16) the sediment discharge over surface S_o can be expressed as

$$Q = \int_{S_o} u\gamma\ \mathrm{d}S = \int_{S(t)} u\ \mathrm{d}S = S_o\{u\}^s = S_o\phi_A\{u\} \tag{17a}$$

$$Q = \int_{S_o} u\gamma_m\ \mathrm{d}S = \int_{S_m(t)} u\ \mathrm{d}S = S_o\{u\}^{sm} = S_o\phi_{Am}\{u\}^m \tag{17b}$$

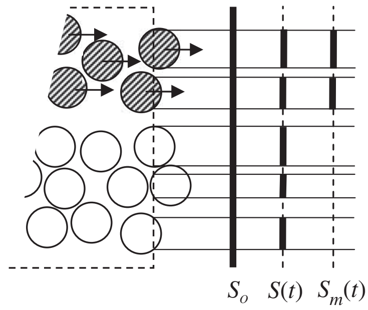

Figure 3 Definition of the different portions of the outflow surface for the volume V_o : S_o is complete surface; S is portion of S_o occupied by solid phase; S_m is portion of S_o occupied by moving solid phase

Equations (17) implicitly indicate that $\{u\}^s = \{u\}^{sm}$. In general, integrals calculated over $S_m(t)$ (moving particles) differ from those calculated over $S(t)$ (all particles), i.e. $\{\theta\}^s \neq \{\theta\}^{sm}$ and $\gamma \neq \gamma_m$. However, for the sediment velocity we have $\int_S u\ \mathrm{d}S = \int_{S_m} u\ \mathrm{d}S + \int_{S-S_m} u\ \mathrm{d}S = \int_{S_m} u\ \mathrm{d}S$ as $\int_{S-S_m} u\ \mathrm{d}S \equiv 0$ and thus $\{u\}^s = \{u\}^{sm}$.

Using Eqs. (12) and (17), the instantaneous integral mass balance (5) can be alternatively expressed as

$$\frac{V_o}{S_o}\frac{\mathrm{d}\phi_V}{\mathrm{d}t} = \{u\}^s_{in} - \{u\}^s_{out} = \{u\}^{sm}_{in} - \{u\}^{sm}_{out} \tag{18a}$$

$$\frac{V_o}{S_o}\frac{\mathrm{d}\phi_V}{\mathrm{d}t} = (\phi_A\{u\})_{in} - (\phi_A\{u\})_{out} \tag{18b}$$

$$\frac{V_o}{S_o}\frac{\mathrm{d}\phi_V}{\mathrm{d}t} = (\phi_{Am}\{u\}^m)_{in} - (\phi_{Am}\{u\}^m)_{out} \tag{18c}$$

where all terms of the volumetric mass balance have been divided by the boundary surface S_o. Equations (18a)–(18c) involve different forms for the solid fluxes. Equation (18a) is expressed in terms of superficial quantities, either over all particles or only moving ones; Eq. (18b) makes use of intrinsic quantities calculated over all particles; and Eq. (18c) is referred to intrinsic quantities over moving particles. A further variety of expressions may be obtained by dividing the mass balance terms by "intrinsic" surfaces S and S_m instead of S_o employed in Eqs. (18a)–(18c).

An interesting alternative to formulation (18c) can be derived by noticing that this expression uses a mixture of two variable types: (i) defined over all particles within the volume V_o (concentration ϕ_V), and (ii) defined over moving particles within the surface S_o (concentration ϕ_{Am} and velocity $\{u\}^m$). By separating moving and still particles within the volume V_o we can obtain:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_o}\gamma(\mathbf{x},t)\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_o}(\gamma_m(\mathbf{x},t) + \gamma_b(\mathbf{x},t))\mathrm{d}V = \frac{\mathrm{d}V_m}{\mathrm{d}t} + \frac{\mathrm{d}V_b}{\mathrm{d}t} \tag{19}$$

where V_m and V_b are the volumes of moving and still particles within the control volume V_o, respectively. Similar to the total concentration ϕ_V defined by Eq. (12), we can also consider concentrations of the moving (ϕ_{Vm}) and still (ϕ_{Vb}) particles. The time derivative of V_b can be expressed by noticing that it corresponds to the net exchange rate among the "still" and "moving" phases, i.e.:

$$\frac{\mathrm{d}V_b}{\mathrm{d}t} = d_r(t) - e_r(t) \tag{20}$$

where e_r and d_r are volumetric entrainment and deposition rates within the reference volume V_o. The integral mass balance (18c) then reads:

$$\frac{\mathrm{d}V_m}{\mathrm{d}t} = (Q_{in} - Q_{out}) - \frac{\mathrm{d}V_b}{\mathrm{d}t} = (Q_{in} - Q_{out}) + (e_r - d_r) \tag{21}$$

or

$$V_o \frac{\mathrm{d}\phi_{Vm}}{\mathrm{d}t} = S_o\left(\phi_{Am}\{u\}^m\right)_{in} - S_o\left(\phi_{Am}\{u\}^m\right)_{out} + (e_r - d_r) \tag{22}$$

Equation (22) has been derived from a mass balance over a generic volume containing moving and still particles (Fig. 1b). It is formally equivalent to a balance over a volume $V_o = V_{o1} + V_{o2} + V_{o3}$ in the sketch of Fig. 1a. However, the entrainment and deposition rates in Eqs. (20)–(22) do not represent physical fluxes across interfaces (as in Fig. 1a); rather they express rates of change of status (still or moving) within the reference volume.

3.2 *Time-averaged forms*

Expressions for the time-averaged balance or, equivalently, the balance over a finite time period T_f, may be similarly derived from Eq. (4) or (5). Note that the averaging is assumed over a time period larger than that used to define an "instantaneous" flux, i.e. $T_f > T_{\varepsilon 1}$. The volumetric term requires no further discussion in addition to the expressions already presented in the previous sections. A time-averaged flux terms can be expressed as

$$\overline{Q}^s_{in/out} = \frac{1}{T_f}\int_t^{t+T_f} Q_{in/out}(\tau)\mathrm{d}\tau = \frac{V_{in/out}}{T_f} \tag{23}$$

so that

$$\frac{V(t+T_f) - V(t)}{T_f} = \overline{Q}^s_{in} - \overline{Q}^s_{out} \tag{24}$$

where τ is an integration variable. Equations (23) and (24) make use of superficial time averaging; however, the intrinsic averaging performed only over time periods when particles are present at the *in*/*out* surfaces may be more beneficial. Similar to Nikora et al. (2013), we define the time porosity $\phi_{T\mu} = T_\mu / T_f$ where the quantity T_μ is the part of the total averaging period T_f when the surface S_o was crossed by solid particle(s). In turn, the quantity $(T_f - T_\mu)$ is the remaining part of T_f when the surface S_o was free from the particles. As in the section above, the time averages can be defined over all particles or over moving particles only. Consequently, the counterparts of Eq. (24), which make use of intrinsic time averages, are:

$$\frac{V(t+T_f) - V(t)}{T_f} = \left(\phi_{T\mu}\overline{Q}\right)_{in} - \left(\phi_{T\mu}\overline{Q}\right)_{out} \tag{25}$$

$$\frac{V(t+T_f) - V(t)}{T_f} = \left(\phi_{T\mu m}\overline{Q}^m\right)_{in} - \left(\phi_{T\mu m}\overline{Q}^m\right)_{out} \tag{26}$$

where

$$\overline{Q}_{in/out} = \frac{1}{T_\mu}\int_t^{t+T_f} Q_{in/out}(\tau)\mathrm{d}\tau = \frac{1}{T_\mu}V_{in/out} \tag{27}$$

$$\overline{Q}^m_{in/out} = \frac{1}{T_{\mu m}}\int_t^{t+T_f} Q_{in/out}(\tau)\mathrm{d}\tau = \frac{1}{T_{\mu m}}V_{in/out} \tag{28}$$

The quantity $T_{\mu m}$ and the corresponding time porosity $\phi_{T\mu m} = T_{\mu m}/T_f$ are defined for moving particles only while T_μ and $\phi_{T\mu} = T_\mu / T_f$ refer to any particles.

Our final expressions for the time-averaged integral mass balance can be derived by comparing Eqs. (24)–(26) with instantaneous balances given by Eq. (18). Combined space and time integration/averaging offers nine alternatives based on three averaging forms: (i) superficial, (ii) intrinsic over all particles, and (iii) intrinsic over moving particles (as $\{u\}^s = \{u\}^{sm}$ there is no need for considering superficial integrals over moving particles only). Thus, we can derive 3(space) × 3(time) = 9 alternative versions of the flux terms in the balance equation (where the volumetric term is kept the same):

$$\overline{Q}^s = \frac{1}{T_f}\int_t^{t+T_f} Q(\tau)\mathrm{d}\tau = S_o\overline{\{u\}^s}^s = S_o\overline{\phi_A\{u\}}^s = S_o\overline{\phi_{Am}\{u\}^m}^s \tag{29a}$$

$$\overline{Q} = \frac{1}{T_\mu}\int_t^{t+T_f} Q(\tau)\mathrm{d}\tau = S_o\overline{\{u\}^s} = S_o\overline{\phi_A\{u\}} = S_o\overline{\phi_{Am}\{u\}^m} \tag{29b}$$

$$\overline{Q}^m = \frac{1}{T_{\mu m}}\int_t^{t+T_f} Q(\tau)\mathrm{d}\tau = S_o\overline{\{u\}^s}^m = S_o\overline{\phi_A\{u\}}^m = S_o\overline{\phi_{Am}\{u\}^m}^m \tag{29c}$$

The interrelationships between different flux forms are given by

$$\overline{Q}^s = \phi_{T\mu}\overline{Q} = \phi_{T\mu m}\overline{Q}^m \tag{30}$$

In principle, there is no preference among the different expressions for the fluxes in Eq. (29) as they all contain the same information, although differently distributed between the velocity and space and/or time porosity (concentration) terms.

Further forms of the sediment flux stem from its decomposition into a sum of the mean and fluctuating parts, such as

$$\overline{Q}^s = S_o \overline{\phi_A \{u\}}^s = S_o \left(\overline{\phi_A}^s \overline{\{u\}}^s + \overline{\phi_A'^s \{u\}'^s}^s \right) \qquad (31)$$

where $\phi_A'^s$ and $\{u\}'^s$ are deviations of the instantaneous variables from mean values (i.e. prime denotes fluctuations). As different strategies for the time averages are possible, corresponding residual fluctuations require to be differentiated from each other (as they may have different meaning). Note that the decomposition into fluctuating and mean components is most meaningful for intrinsic variables as superficial averages my lead to a non-zero correlation term even if the physical variables are constant. Possible relevance of such decompositions will be discussed in Section 4.3.

3.3 *Discussion*

In the sections above we have shown that the integral balance of sediment mass (volume) can be written in different forms. The volumetric term is the same for all of them, being a product of the control volume and the concentration of sediments within it (with the exception of Eq. 22 where the volumetric concentration refers to moving sediments only). The sediment fluxes through the boundaries may be expressed as the product of *Area*, *Concentration*, and *Velocity*, definitions of which depend on the averaging strategy. Alternative forms of the sediment fluxes, such as in Eqs. (18) and (29), represent different ways of information partitioning between the concentration and velocity terms. The use of sediment velocities averaged over moving particles only is, probably, the most attractive from a phenomenological point of view, although it generates more complex expressions.

All variables which are based on clipping over all particles, as in Eq. (18b), are *defined* and *continuous* (although potentially *intermittent*) for *any* size and form of the control volume V_o, from the sub-particle scale to very large domains containing large numbers of particles. The term "*intermittent*" is used here to describe a quantity showing alternation of intervals where its value is zero with intervals when its value is non-zero and is changing in time (term "intermittency" here should not be confused with its use in the analyses of high-Reynolds-number turbulence and continuous records of bedload as in Singh *et al.* 2009). As already noted, variables based on clipping over moving sediments, as in Eqs. (18c) and (22), may have a more direct phenomenological relevance, but they typically suffer from being less well behaved. Indeed, the change of the "phase" of any particle contained in V_o from "still" to a "moving" state or vice versa implies sudden changes within dt of the quantities which are defined over either of these two phases. As a consequence, these variables can be not only intermittent but also discontinuous. A limiting case for such a behaviour is represented by the entrainment and deposition rates $e_r(t)$ and $d_r(t)$, which are likely to be discontinuous and infinite and best expressed as a sum of Dirac delta functions.

Within a classical continuum framework, irregular behaviours as those discussed above mean that the concerned variables are ill-defined, i.e. they are not sufficiently regularized by averaging over appropriately large spatial and/or time extents. As pointed out in Section 1 and in Section 2, such a regularization procedure is not always possible in sediment transport analysis, as the averaging extents should be large enough to contain many (moving) particles and at the same time sufficiently small with respect to the reference domain. The difficulties arise for the case of relatively large sediments and/or relatively low transport rates, for which a requirement of scale separation needed for the regularization is not met. On the other hand, the strength of the integral approach compared with equivalent differential balances is that it can be used even with non-well-behaved quantities.

The integral sediment balance involves both the volumetric concentration (in the time derivative) and the areal concentration and areal-averaged velocity (in sediment fluxes). Thus, our relations highlight approximate nature of the solid discharge expressions that employ volume-averaged velocities and concentrations (instead of areal quantities), as in Eq. (2) and in experimental assessments where sediment motion is optically measured from the top (for example, Fernandez Luque and Van Beek 1976, Radice and Ballio 2008). Such "volume"-based approximation of the area-based quantities should be considered with caution. For instance, it is legitimate if volume averaging is used to generate smoothed variables with the regularity required by the continuum approach, which is possible only with sufficient scale separation so that the regularization volume can be taken as representative of an infinitesimal point (where the variable is defined for a further mathematical treatment). Finally, differences between volumetric and areal integral quantities may disappear for time- or ensemble-averaged quantities, at least under proper conditions of spatial uniformity and/or time stationarity. For an extensive analysis of this issue, see Furbish *et al.* (2012).

Equations (29) show that expressions for mean values of sediment fluxes typically (though not necessarily) involve averaging of nonlinear terms resulting from the product of sediment concentration and velocity. As a consequence, if quantities are decomposed into the mean and fluctuating components, correlation terms arise, as in the example expressed by Eq. (31). This feature is acknowledged for suspended sediment transport (Hurther and Lemmin 2003) and has been recently highlighted in Radice and Ballio (2008) for bedload processes. Some studies suggest that solid discharge fluctuations for transport on a flat bed are primarily due to fluctuations of the concentration, with little contribution from velocity fluctuations (e.g. Ancey 2010). This suggestion implies that the concentration–velocity correlation term (as in Eq. 31) is negligible compared with the product of the corresponding mean values. On the other hand, it should not be excluded that significant correlations between the concentration and velocity may occur in more complex configurations (for example, migrating bedforms or scour due to an unsteady horseshoe vortex structure), as a consequence of large-scale temporal variations of the processes sustaining sediment transport.

Whatever is their quantitative impact, terms resulting from concentration and velocity fluctuations play an important conceptual role, as they generate diffusive terms in the averaged sediment transport equations (Furbish *et al*. 2012, Ancey and Heyman 2013).

Sensitivity of different forms of the sediment balance to change of scale is not the same. As an example, let us consider a spatially homogeneous field and sufficiently large time intervals. Referring to superficial quantities $\overline{Q}^s = S_o\overline{\{u\}^s}^s$, no scale effect is expected since although $\{u\}^s(t)$ depends on S_o its time-averaged value does not if T_f is large enough to represent ensemble averaging. Similarly, in the limit $S_o \to \infty$ we may also expect $\{u\}^s(t) \to \overline{\{u\}^s}^s$. On the other hand, if we refer to the intrinsic quantities based on mobile particles such as in $\overline{Q}^s = S_o\left(\overline{\phi_{Am}}^s\,\overline{\{u\}^m}^s + \overline{\phi'^s_{Am}\{u\}^{m's}}^s\right)$, we should expect all variables to be sensitive to S_o, which makes them much less attractive as descriptors of sediment transport. This scale sensitivity occurs because fluctuations around the mean are strongly dependent on S_o: instantaneous values are highly intermittent for $S_o \approx d^2$ but regularity increases with increase of S_o. Again, we should expect that fluctuations are no longer present in the time series if $S_o \to \infty$, and thus the correlation term disappears.

4 Example: weak bed load on a plane bed

In this section, an experimental data set related to uni-directional sediment transport on a plane bed is used to explore properties of the previously defined quantities. The full description of the experimental set-up and procedures is reported in Radice and Ballio (2008) and Campagnol *et al*. (2012), and therefore below only essential details of experimentation and data processing are given.

4.1 *Experimental procedure*

The experimental run was completed in a 5.8 m long, 0.4 m wide, and 0.16 m deep rectangular pressurized duct. Plastic PVC grains with density of 1.43 kg m^{-3} and characteristic size $d = 0.0036$ m were used as sediments. Water discharge was $Q_w = 0.0248$ m^3 s^{-1}, slightly larger than the threshold value for incipient motion of bed particles ($Q_{wc} = 0.0190$ m^3 s^{-1}), thus providing conditions of a weakly-mobile bed. The run duration was $T_f = 20$ s, along which the particle motion was filmed from above using a black-and-white CCD camera with a resolution set to 763 × 576 pixels and a frame rate of 50 fps. Particle movements were identified by means of the *Streams* package (Nokes 2007) which allowed to track the paths of individual particles and, consequently, to identify grain crossing events across a transverse reference line ($L = 0.165$ m long) at the centre of the channel, together with velocities and intercepted areas of the particles (Fig. 4). A total number of 566 particles crossed the line within the run duration; the consequently obtained average solid discharge was $\overline{Q}^s = 6.91 \times 10^{-7}$ m^{3}s^{-1}. Finally, trajectories around the reference line were interpolated in order to reconstruct time series at a sampling frequency as high as 750 Hz. The corresponding sampling time $\Delta t_s = 0.0013$ s is much less than the typical values for the particle time scale $T_p = d/u = 0.03 - 0.04$ s and thus the obtained time series can be considered as "instantaneous". The image-processing procedure identified only moving grains and, as a consequence, all measured quantities are intrinsically clipped on moving particles. Specifically, a grain was labelled as "moving" if it moved a distance larger than 0.1 mm from its rest position.

4.2 *Instantaneous quantities*

Figure 5 presents snapshots of the time series for the intrinsic concentration ϕ_{Am} and velocities $\{u\}^m$ and $\{u\}^{sm}$, which are computed over a reference area $S_o = L \times d$. These variables can be used to compute the sediment flux (per unit area) as $Q(t)/S_o = \{u\}^{sm}(t) = \phi_{Am}(t)\{u\}^m(t)$.

Plots in Fig. 5 clearly show the intermittency of the process, i.e. there are time periods when no particles are crossing the reference line, with the overall time porosity of $\phi_{T\mu m} = T_{\mu m}/T_f = 0.57$. The lower peaks in the concentration time series (Fig. 5a) correspond to crossings of single particles; such events are characterized by the peak concentration equal to $(\pi d^2/4)/S_o = 0.017$. Higher peak values relate to multiple crossings (up to 6 particles simultaneously crossing the line were observed). The intrinsic-averaged sediment velocities (Fig. 5b) vary within the

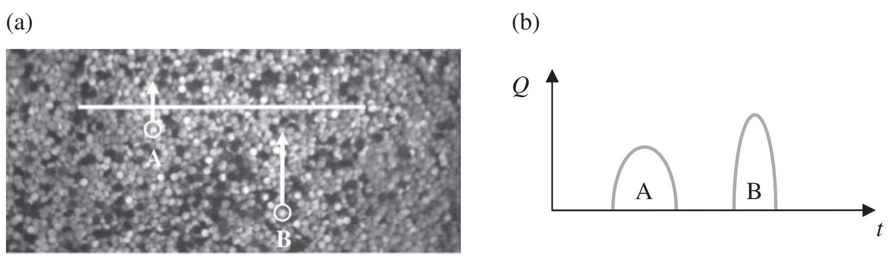

Figure 4 (a) Portion of a movie frame with indication of the reference line and two sample particles (A, B) upstream of the line moving with different velocities. (b) Qualitative sketch of resulting temporal evolution of solid discharge due to crossing of the two particles

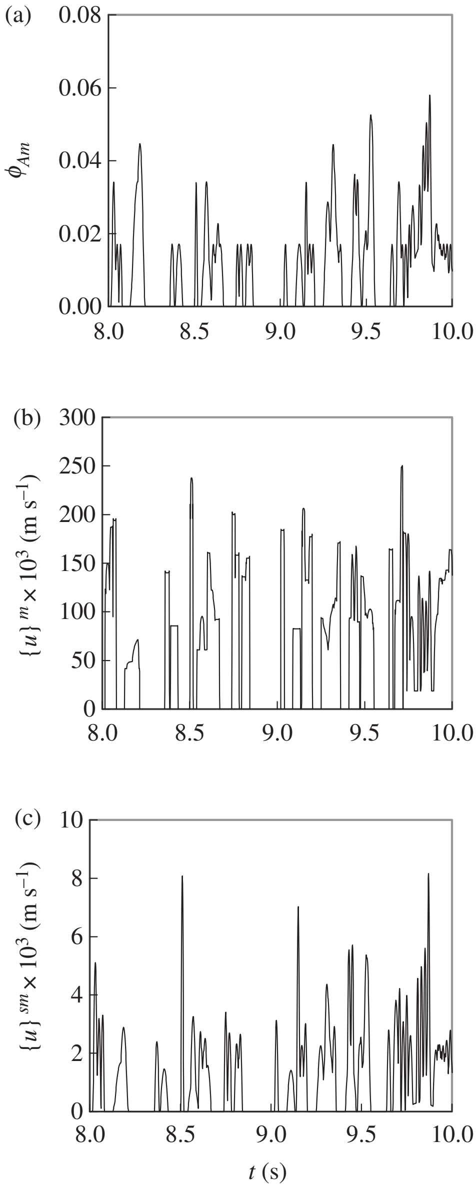

Figure 5 Time series of instantaneous (a) concentration ϕ_{Am}, (b) intrinsic velocity $\{u\}^m$, and (c) superficial velocity $\{u\}^{sm}$. Only a fraction of the total time series is shown

range $\{u\}^m = 0$ to $0.25\,\mathrm{m\,s^{-1}}$, with approximately 50% of the velocity values falling in the range from 0.07 to $0.14\,\mathrm{m\,s^{-1}}$.

Figure 5c shows the superficial-averaged velocity $\{u\}^{sm}$ that is equivalent to the sediment flux per unit area $Q(t)/S_o$. This velocity is fictitious as values of $\{u\}^{sm}(t)$ are not physical velocities of the moving particles (typical values of $\{u\}^{sm}(t)$ are within the range 10^{-3} to $10^{-2}\,\mathrm{m\,s^{-1}}$, i.e. much smaller than the physical velocities). Superficial and physical (intrinsic) velocities are connected as $\{u\}^{sm} = \phi_{Am}\{u\}^m$ (see Eq. 18).

4.3 *Time-averaged quantities*

Values of the relevant time-averaged quantities are listed in Table 1. Expressions (32)–(34) below provide flux estimates based on the definitions introduced in Section 3.2:

$$\frac{\overline{Q}^s}{S_o} = \overline{\phi_{Am}\,\{u\}^m}^s = 1.16 \times 10^{-3}\ \mathrm{m\,s^{-1}} \tag{32a}$$

$$\begin{aligned}\frac{\overline{Q}^s}{S_o} &= \overline{\phi_{Am}}^s\,\overline{\{u\}^m}^s + \overline{\phi'^s_{Am}\,\{u\}^{m's}}^s \\ &= (0.0105 \times 62.2 + 0.51) \times 10^{-3} \\ &= (0.65 + 0.51) \times 10^{-3} = 1.16 \times 10^{-3}\ \mathrm{m\,s^{-1}}\end{aligned} \tag{32b}$$

$$\frac{\overline{Q}^m}{S_o} = \overline{\phi_{Am}\,\{u\}^m}^m = 2.05 \times 10^{-3}\ \mathrm{m\,s^{-1}} \tag{33a}$$

$$\begin{aligned}\frac{\overline{Q}^m}{S_o} &= \overline{\phi_{Am}}^m\,\overline{\{u\}^m}^m + \overline{\phi'^m_{Am}\,\{u\}^{m'm}}^m \\ &= (0.0185 \times 110 + 0.02) \times 10^{-3} \\ &= (2.03 + 0.02) \times 10^{-3} = 2.05 \times 10^{-3}\ \mathrm{m\,s^{-1}}\end{aligned} \tag{33b}$$

$$\frac{\overline{Q}^s}{S_o} = \frac{\phi_{T\mu m}\overline{Q}^m}{S_o} = 0.57 \times 2.05 \times 10^{-3} = 1.16 \times 10^{-3}\ \mathrm{m\,s^{-1}} \tag{34}$$

The estimates above illustrate clear differences between the superficial and intrinsic time averages. Expressions (32a,b) describe the superficial sediment flux and employ the superficial time averages $\overline{\phi_{Am}}^s$ and $\overline{\{u\}^m}^s$, which are not directly related to the physical transport parameters, being around 40% lower. In contrast, expressions (33a,b) involve the intrinsic time averages only, making use of most "physical" quantities $\overline{\phi_{Am}}^m$ and $\overline{\{u\}^m}^m$. The net solid discharge $\overline{Q}^s/S_o$ in Eq. (34) is obtained by multiplying $\overline{Q}^m/S_o$ with the time porosity $\phi_{T\mu m}$ that illustrates how the time and spatial (areal) porosities are combined together to give a superficial flux. In this respect the product $\overline{\phi_{Am}}^m\phi_{T\mu m}$ can be viewed as a "global" sediment porosity.

Estimate (32b) shows almost equivalent contributions to the total superficial flux from the mean term $\overline{\phi_{Am}}^s\,\overline{\{u\}^m}^s$ and the correlation term $\overline{\phi'^s_{Am}\{u\}^{m's}}^s$, with the latter being basically imposed by the transport intermittency (as defined in Section 3.3). The occurrence of time intervals when no particles cross the control surface during the averaging time T_f (hence intermittency) leads to a non-zero term $\overline{\phi'^s_{Am}\{u\}^{m's}}^s$, even if both particle velocity and areal porosity are constant during transport periods. Thus, non-zero $\overline{\phi'^s_{Am}\{u\}^{m's}}^s$ should be interpreted as pseudo-correlation rather than a real correlation. Indeed, the corresponding term $\overline{\phi'^m_{Am}\{u\}^{m'm}}^m$ in relationship (33b) for the intrinsic sediment flux

Table 1 Relevant statistics of the experimental test

$\overline{\phi_{Am}}^s = 0.0105$	$\phi_{T\mu m} = 0.57$	$\overline{\{u\}^m}^s = 0.062\,\mathrm{m\,s^{-1}}$	$\overline{Q}^s/S_o = 1.16 \times 10^{-3}\,\mathrm{m\,s^{-1}}$
$\overline{\phi_{Am}}^m = 0.0185$		$\overline{\{u\}^m}^m = 0.110\,\mathrm{m\,s^{-1}}$	$\overline{Q}^m/S_o = 2.05 \times 10^{-3}\,\mathrm{m\,s^{-1}}$

is negligible meaning that the concentration and velocity of moving grains in reality are very weakly correlated. This effect was already noticed by Radice and Ballio (2008), who explained such behaviour by independency of mechanisms for sediment entrainment (linked to concentration of moving grains) and displacement (linked to velocity). As a consequence, the flux equation can be simplified as

$$\frac{\overline{Q}^s}{S_o} = \phi_{T\mu m}\overline{\phi_{Am}\{u\}^m}^m \approx \left(\phi_{T\mu m}\overline{\phi_{Am}}^m\right)\overline{\{u\}^m}^m \tag{35a}$$

$$= (0.57 \times 0.185) \times 110 \times 10^{-3} = 0.105 \times 110 \times 10^{-3}$$

$$= 1.15 \times 10^{-3}\,\mathrm{m\,s^{-1}} \tag{35b}$$

where the two porosities are combined to give a global- (double) averaged concentration. Using Eqs. (16) and (30), Eq. (35) can be presented slightly differently as

$$\frac{\overline{Q}^s}{S_o} = \overline{\phi_{Am}\{u\}^m}^s \approx \frac{\overline{\phi_{Am}}^s\overline{\{u\}^m}^s}{\phi_{T\mu m}} = \frac{(0.0105 \times 62.5 \times 10^{-3})}{0.57}$$

$$= 1.15 \times 10^{-3}\,\mathrm{m\,s^{-1}} \tag{36}$$

Additional discussion of this issue can be found in Radice and Ballio (2008). Although the vanishing correlation between the particle concentration and velocity in our example is probably typical, we have to mention again that there may be situations when this correlation may not be neglected, as discussed in Section 3.3.

5 Comparisons with other formulations

In this section we compare the proposed approach with some currently available formulations with the aim of emphasizing the variety of possible approaches for the description of transport processes.

5.1 *Van Rijn (1984)*

Van Rijn (1984) proposed a model for bedload transport, whose structure is similar to Eq. (1) with the thickness of the bedload layer δ_{bl} identified as the saltation height. Although not explicitly indicated, the quantities of Van Rijn's model should be treated as mean values, where "mean" refers either to time or ensemble averages. It should be also noted that q_{bl} and c_{bl} are the Eulerian quantities defined over a space domain, whereas u_{bl} and δ_{bl} are Lagrangian properties of individual particles.

Following the nomenclature introduced in Section 3, we have $q_{bl} = \overline{Q}^s/B$, where B is the width of the flow. The thickness of the active layer is directly linked to a reference surface $S_o = B\delta_{bl}$ while the particle velocity u_{bl} defined as the mean velocity of particles along their trajectories (that is, when they are moving) can be approximated under uniformity and ergodicity conditions as $u_{bl} \simeq \overline{\{u\}^m}^m$; no explicit definition was given for c_{bl}. Combining Van Rijn's approach with our formulation, one can write:

$$c_{bl} = \frac{q_{bl}}{\delta_{bl}u_{bl}} \approx \frac{\overline{Q}^s}{S_o\overline{\{u\}^m}^m} = \frac{\phi_{T\mu m}S_o\overline{\phi_{Am}\{u\}^m}^m}{S_o\overline{\{u\}^m}^m}$$

$$= \phi_{T\mu m}\overline{\phi_{Am}}^m + \phi_{T\mu m}\frac{\overline{\phi'^m_{Am}\{u\}^{m'm}}^m}{\overline{\{u\}^m}^m} \tag{37}$$

For $\overline{\phi'^m_{Am}\{u\}^{m'm}}^m \ll \overline{\{u\}^m}^m$ the concentration c_{bl} is approximated as $c_{bl} = \phi_{T\mu m}\overline{\phi_{Am}}^m$. In other words, together with the expected space porosity it also includes the time porosity of the transport process. For relatively intense transport conditions and/or large reference surfaces, one can expect that the time porosity $\phi_{T\mu m}$ is close to unity and thus the conventional concept of concentration is recovered, i.e. $c_{bl} = \overline{\phi_{Am}}^m$. However, if $\overline{\phi'^m_{Am}\{u\}^{m'm}}^m \approx \overline{\{u\}^m}^m$ then the quantity c_{bl} in Eq. (37) does not really represent a physical concentration as it also contains information on kinematics of the transport process (diffusion).

5.2 *Coleman and Nikora (2009)*

These authors proposed an expression for the sediment mass balance as

$$\frac{\partial\phi_V}{\partial t} + \frac{\partial\phi_V\langle u\rangle}{\partial x} = 0 \tag{38}$$

where the notations of Eqs. (8) and (12) are used. This is a differential form of the instantaneous mass balance where quantities ϕ_V and $\langle u\rangle$ are averaged over a thin volume V_o parallel to the bottom, allowing the analysis of vertical distributions of the transport variables; note that their equation also holds if the integration volume covers large extensions in the vertical direction. Equation (38) is the differential (along direction x) counterpart of Eq. (18b). The main difference between the two expressions is that all variables of Eq. (38) are regularized over the same averaging volume while in Eq. (18b) a mixture of volume- and area-averaged quantities is used. However, if the divergence theorem (in its one-dimensional form) is applied to the second term of Eq. (38) we obtain:

$$\frac{\partial\phi_V\langle u\rangle}{\partial x} = \frac{\partial\langle u\rangle^s}{\partial x} = \frac{\partial}{\partial x}\frac{1}{V_o}\int_{V_o} u\gamma\,\mathrm{d}V = \frac{1}{V_o}\int_{V_o}\frac{\partial u\gamma}{\partial x}\,\mathrm{d}V$$

$$= \frac{1}{V_o}\left(\int_{S_{o,out}} u\gamma\,\mathrm{d}S - \int_{S_{o,in}} u\gamma\,\mathrm{d}S\right)$$

$$= \frac{S_o}{V_o}\left(\{u\}^s_{out} - \{u\}^s_{in}\right) = \frac{S_o}{V_o}\left((\phi_A\{u\})_{out} - (\phi_A\{u\})_{in}\right)$$

Thus, the mass balance (38) can also be expressed as

$$\frac{\mathrm{d}\phi_V}{\mathrm{d}t} + \frac{S_o}{V_o}\left((\phi_A\{u\})_{out} - (\phi_A\{u\})_{in}\right) = 0 \tag{39}$$

which is identical to Eq. (18b). In other words, the differential mass balance employing volume-averaged variables coincides with the integral mass balance over the same volume: the spatial differentiation "cancels" all information inside the volume so that only values at its borders are related to the flux of sediments, as explicitly expressed by Eqs. (18) and (39).

5.3 *Furbish et al. (2012)*

Furbish *et al.* (2012) present different approaches for defining sediment fluxes, mainly focusing on the links between Lagrangian and Eulerian properties of variables describing sediment motion. Their basic expression for the sediment discharge (Eq. 1 in their paper) is essentially equivalent to our Eq. (13), also involving sediment velocity at a reference surface. However, due to practical difficulties in measuring particle velocities and the focus on linking Eulerian distributed properties to Lagrangian properties of individual particles Furbish *et al.* (2012) propose an approximate expression (Eq. 8 in the paper) equivalent to

$$Q(t) = \langle S_i(t)\rangle \langle u_i(t)\rangle = S_o\hat{C}\,\hat{u} \tag{40}$$

where S_i and u_i are, respectively, the cross section and the velocity of the ith particle, crossing the reference surface, angular brackets $\langle\cdot\rangle$ here indicate average over N particles, $\hat{C}$ and $\hat{u}$ are corresponding averaged values of solid concentration and velocity. Equation (40) is similar to Eq. (17b), if we note that $\hat{C} \cong \phi_{Am}$ and $\hat{u} \cong \{u\}^m$. It has some advantages compared with Eq. (17b) in that it is operationally more straightforward and its parameters are directly linked to the properties of individual grains, making Eq. (40) a convenient starting point for linking the Lagrangian and Eulerian frameworks. However, for Eq. (40) to be valid it is required that $\langle S_i u_i\rangle = \langle S_i\rangle \langle u_i\rangle$, which is (approximately) true when N is large enough and S_i and u_i are uncorrelated in space. As indicated by the authors, the latter condition is satisfied if, for example, all particles have the same size. In other words, advantages of expression (40) come with some loss of generality compared with Eq. (17b), which accounts for spatial correlations independently of the scale of consideration.

This example, once more, confirms that no absolute preference can be given to any of the many possible alternative expressions for the sediment discharge, as the "best" choice is linked to physical and statistical properties of the variables involved in different expressions.

6 Conclusions

The key objectives of this paper were to accentuate the importance of unambiguous quantitative definitions for sediment transport variables that are often poorly defined, and to propose alternative definitions that may serve as a sound conceptual basis for phenomenological considerations. General forms for integral sediment balances and sediment fluxes through the boundaries have been defined, analysed, and compared with existing formulations. The proposed expressions refer to the Eulerian quantities (sediment fluxes, concentrations, and averaged velocities) that can be applied at sub-particle resolution and upscaled to larger scales through averaging. The link between the proposed Eulerian quantities and the Lagrangian descriptors (position and velocity) of the individual grains is straightforward; the suggested equations, therefore, constitute a possible base for quantitative interrelations between Lagrangian and Eulerian kinematic descriptions. The validity and generality of the proposed approaches and the relative advantages of the different strategies for the transport description have been discussed using experimental data and by comparison with formulations of other researchers.

Our analysis highlights the necessity for unambiguous definitions of transport variables in research publications (e.g. areal vs. volumetric averages, instantaneous/deterministic vs. mean values, etc.), as in many publications these issues are given little attention, if at all, making comparisons among studies difficult.

Acknowledgements

The authors thank Alessio Radice and Jenny Campagnol for providing access to experimental data used in this research, Andrea Marion for fruitful discussions, and Aurora Luzzi for editorial work with the manuscript. The insightful comments and suggestions of Christopher Ancey, David Jon Furbish and two anonymous reviewers have been gratefully incorporated in the final version.

Funding

The research was partly supported by the FP7-People-2012-ITN grant HYTECH [GA-2012-316546] and EPSRC, UK [EP/G056404/1].

Notations

bl	=	bed load
in, out	=	inflow, outflow section of the reference volume
s	=	superficial averaging
m	=	moving particles
b	=	bed (still) particles
$\langle\theta\rangle^s$, $\langle\theta\rangle$	=	superficial and intrinsic volumetric averages of variable θ, respectively

$\langle\theta\rangle^m$	=	intrinsic volumetric average of variable θ over moving particles
$\{\theta\}^s, \{\theta\}$	=	superficial and intrinsic areal averages of variable θ, respectively
$\{\theta\}^{sm}, \{\theta\}^m$	=	superficial and intrinsic areal averages of variable θ over moving particles, respectively
$\overline{\theta}^s, \overline{\theta}$	=	superficial and intrinsic time averages of variable θ, respectively
$\overline{\theta}^m$	=	intrinsic time average of variable θ over moving particles
c	=	volumetric sediment concentration (–)
d	=	grain size (m)
e_r, d_r	=	entrainment, deposition rates within a reference volume V_o ($m^3\ s^{-1}$)
N	=	number of moving particles per unit area (particle m^{-2})
q	=	volume sediment transport rate per unit width ($m^2\ s^{-1}$)
Q	=	volume sediment transport rate through surface S_o ($m^3\ s^{-1}$)
S	=	surface of sediments intersected by the control surface S_o (m^2)
$S_{o\,(in,out)}$	=	reference control surface for integral balance (inflow, outflow) (m^2)
t	=	time (s)
T_o	=	reference time lag for sediment balance (s)
T_f	=	averaging "finite" time (s)
T_ε	=	time scale assumed as "infinitesimal" (s)
$T_{\mu m}$	=	fraction of T_f during which surface S_o is crossed by moving particles (s)
T_μ	=	fraction of T_f during which surface S_o is crossed by particles (s)
T_p	=	time scale for particle crossing, $T_p = d/u$ (s)
T_i	=	inter arrival time scale (s)
T_{bf}	=	bedforms time scale (s)
u	=	longitudinal velocity of the solid phase ($m\ s^{-1}$)
V	=	volume of sediments within the control volume V_o (m^3)
V_o	=	reference control volume for integral balance (m^3)
w	=	particle volume (m^3)
$x, \boldsymbol{x}$	=	space coordinate (scalar, vector) (m)
δ	=	thickness of the sediment transport layer (m)
ϕ_A	=	areal concentration of sediments over surface S_o (–)
ϕ_V	=	volumetric concentration of sediments over volume V_o (–)
$\phi_{T\mu}, \phi_{T\mu m}$	=	time porosities (–)
$\gamma, \gamma_m, \gamma_b$	=	clipping functions (–)

References

Ancey, C. (2010). Stochastic modeling in sediment dynamics: Exner equation for planar bed incipient bed load transport conditions. *J. Geophys. Res. Earth Surface* 115, F00A11.

Ancey, C., Heyman, J. (2013). A microstructural approach to bedload transport: Mean behaviour and fluctuations of particle transport rates. *J. Fluid Mech.* (in print).

Böhm, T., Ancey, C., Frey, P., Reboud, J.L., Ducottet, C. (2004). Fluctuations of the solid discharge of gravity-driven particle flows in a turbulent stream. *Phys. Rev. E* 69(6), 061307.

Campagnol, J., Radice, A., Ballio, F. (2012). Scale-based statistical analysis of sediment fluxes. *Acta Geophys.* 60(6), 1744–1777.

Coleman, S.E., Nikora, V. (2009). The Exner equation: A continuum approximation of a discrete granular system. *Water Resour. Res.* 45, W09421.

Fernandez Luque, R., Van Beek, R. (1976). Erosion and transport of bed-load sediment. *J. Hydraulic Res.* 14(2), 127–144.

Furbish, D.J., Haff, P.K., Roseberry, J.C., Schmeeckle, M.W. (2012). A probabilistic description of the bed load sediment flux: 1. Theory. *J. Geophys. Res.* 117(F3), F03031.

Garcia, M. (2008). Sediment transport and morphodynamics. In *Sedimentation engineering: Processes, measurements, modeling, and practice*. M.H. Garcia, ed. American Society of Civil Engineers, Manuals and Reports on Engineering Practice 110, ASCE, Reston, USA, 21–164.

Hurther, D., Lemmin, U. (2003). Turbulent particle flux and momentum flux statistics in suspension flow. *Water Resour. Res.* 39(5), 1139.

Lhuillier, D. (1992). Ensemble averaging in slightly non-uniform suspensions. *Eur. J. Mech., B/Fluids* 11(6), 649–661.

Nikora, V., Ballio, F., Coleman, S., Pokrajac, D. (2013). Spatially-averaged flows over mobile rough beds: Definitions, averaging theorems, and conservation equations. *J. Hydraulic Eng.* 139(8), 803–811.

Nokes, R. (2007). FluidStream, version 7.0: System theory and design, Department of Civil Engineering, University of Canterbury, New Zealand. http://ir.canterbury.ac.nz/handle/10092/437.

Radice, A. Ballio, F. (2008). Double-average characteristics of sediment motion in one-dimensional bed load. *Acta Geophysica* 56(3), 654–668.

van Rijn, L.C. (1984). Sediment transport, part I: Bed load transport. *J. Hydraulic Eng.* 110(10), 1431–1456.

Singh, A., Fienberg, K., Jerolmack, D.J., Marr, J., Foufoula-Georgiou, E. (2009). Experimental evidence for statistical scaling and intermittency in sediment transport rates. *J. Geophys. Res. – Earth Surface* 114(F1), F01025.

Zhang, D.Z., Prosperetti, A. (1994). Averaged equations for inviscid disperse two-phase flow. *J. Fluid Mech.* 267, 185–219.