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WELL POSEDNESS OF OPERATOR VALUED BACKWARD STOCHASTIC RICCATI EQUATIONS IN INFINITE DIMENSIONAL SPACES*

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Abstract. We prove existence and uniqueness of the mild solution of an infinite dimensional, operator valued, backward stochastic Riccati equation. We exploit the regularizing properties of the semigroup generated by the unbounded operator involved in the equation. Then the results will be applied to characterize the value function and optimal feedback law for an infinite dimensional, linear quadratic control problem with stochastic coefficients. Moreover we shall show that it covers the second variation equation arising in the optimal control of the stochastic heat equation in an interval (see [M. Fuhrman, Y. Hu, and G. Tessitore, *C. R. Acad. Sci. Paris Ser.* 1 *Math.*, 350 (2012), pp. 683–688] and [M. Fuhrman, Y. Hu, and G. Tessitore, *Appl. Math. Optim.*, 68 (2013), pp. 181–217]).

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1. Introduction. The present paper is concerned with the following infinite dimensional backward stochastic Riccati equation (BSRE)

(1.1)
$$\begin{cases} -dP_t = (AP_t + P_tA + C_t^*Q_t + Q_tC_t + C_t^*P_tC_t - P_tB_tB_t^*P_t + S_t)\,dt - Q_t dW_t, \\ \quad P_T = M, \end{cases}$$

where A is a self-adjoint operator on the Hilbert space H generating the analytic semigroup (e^{tA}); $(W_t)_{t\geq 0}$ is a real valued standard Brownian motion; (B_t), (C_t), (S_t) are operator valued adapted processes. The unknown of the equation is the couple (P, Q) of operator valued adapted processes.

As is well known (see [18]) the above equation represents the value function of a linear quadratic optimal control problem involving a Hilbert valued state equation with stochastic coefficients (in particular of a control problem with evolution modeled by a parabolic SPDE with stochastic coefficients). It is also well known that, as soon as the solution of the BSRE is obtained, then the synthesis of the optimal control easily follows with a clear applicative interest.

Moreover the special case in which $B_t \equiv 0$ (the so-called Lyapunov equation) turns out to be essential in the formulation of the Pontyagin maximum principle for controlled systems described by SPDEs (see [10] [12], [5], [6], [7], and section 5 here). This in particular happens in the so-called general case in which the space of controls is not convex and the control affects the diffusion term as well (see [16]). Indeed this is the case in which the second variation process, that satisfies an operator Lyapunov

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equation, has to be introduced. In this context the research on backward evolution equations in spaces of linear operators has recently gained a relevant interest.

The study of BSREs in finite dimensional spaces had quite a long history between the pioneering paper by Bismut and then Peng (see [3] and [15]) and the conclusive paper by Tang (see [17]) where existence and uniqueness is proved in the most general case.

On the contrary, the study of BSREs in infinite dimensional spaces adds specific new difficulties and few results are available. As far as the Lyapunov equation is concerned, in [10] the solution is obtained when the final condition M and the forcing term S are Hilbert–Schmidt operators (a condition that is rarely satisfied) while in [5], [7] the process P is characterized by an energy equality involving a suitable forward stochastic differential equation in H. Finally in [12] the concept of a *transposed* solution is given which again consists in a characterization of P and Q by a suitable *duality* relation that involves an infinite dimensional forward equation. We notice that in all the above cases no explicit differential or integral equation directly satisfied by P and Q is presented.

Regarding the Riccati equation (that, differently from the Lyapunov equation, is nonlinear), in [8] we proposed to characterize the P-part of the solution using the concept of strong solution which is of common use in PDE theory (see [2] or [13]). Roughly speaking we characterize the solution as the limit of a sequence of equations with regular (in this case Hilbert–Schmidt) data. This result is good enough to be applied to the corresponding linear quadratic control problem but has the drawback of not saying anything on the martingale term of the solution (the Q-term) and consequently not giving the representation through a (differential) equation.

The origin of the difficulties in dealing with stochastic backward Riccati (or even Lyapunov) equations in the infinite dimensional case is in the fact that the natural space in which it should be treated is the space $L(H)$ of bounded linear operators in H which is only a Banach space that does not enjoy any of the regularity properties (as UMD or M-type condition) allowing us to establish an analogue of the classical Hilbertian stochastic calculus. Moreover, although, as we have said above, different characterization of the solution have been recently proposed, it seems to us that the natural notion of solution is the one of *mild* solution introduced in the theory of infinite dimensional BSDEs since the seminal paper by [9]. We finally notice that this way both the P and the Q part of the unknown are characterized by a differential equation.

As far as we know this is the first paper in which existence and uniqueness of a mild solution of (1.1) is obtained. Indeed we show that (P, Q) is the unique couple of processes (with suitable regularity) verifying

$$
\begin{aligned}
P(t) &= e^{(T-t)A} M e^{(T-t)A} + \int_t^T e^{(s-t)A} S(s) e^{(s-t)A}\, ds \\
&\quad + \int_t^T e^{(s-t)A} \Big[C^*(s)P(s)C(s) + C^*(s)Q(s) + Q(s)C(s) \Big] e^{(s-t)A}\, ds \\
&\quad + \int_t^T e^{(s-t)A} P(s)B(s)B^*(s)P(se^{(s-t)A}\, ds \\
&\quad + \int_t^T e^{(s-t)A} Q(s) e^{(s-t)A}\, dW(s) \qquad \mathbb{P}-\text{a.s.},
\end{aligned}
\tag{1.2}
$$

where P is a predictable process with values in the space of bounded nonnegative, symmetric, linear operators in H which as we said is, in some sense, the natural space for the equation. On the contrary the identification of the right operators space for the evolution of Q is the main achievement of this work. We shall prove existence and uniqueness of Q as a square-integrable, adapted, process in a space $\mathcal{K}$ of Hilbert–Schimidt operators from suitable domains of the fractional powers of A (see (2.4)). This is a Hilbert space, large enough to contain all bounded operators. This choice will allow us to recover stochastic calculus tools. The price to pay is that the term $C_t^* Q_t + Q_t C + C_t^* P_t C_t$ becomes unbounded on $\mathcal{K}$. This difficulty will be handled by exploiting in a careful (and noncompletely standard) way the regularizing properties of the semigroup generated by A. By the way, we have to say that our results rely on the specific properties of A that we assume to be self-adjoint with rapidly increasing eigenvalues. Nevertheless our assumptions can cover important classes of strongly elliptic differential operators.

We think that the introduction of the space $\mathcal{K}$ gives a new framework that can be of general interest in relation to a wider class of operator valued BSDEs with the leading term $A^*P + PA$. On the other hand several techniques used here exploit the special structure of the Riccati equation so we have the impression that any extension requires new nontrivial work. In particular, the control dependent noise case that leads to a BSRE quadratic in Q (see [17] for the finite dimensional case) seems to be a challanging problem. Already in the finite dimensional case it requires ad hoc techniques (see again [17]) that involve properties of the evolution operator corresponding to the forward equation (namely, (1.3) with $u = 0$) which can not be easily transposed in the infinite dimensional case. This transposition and the harmonization with the function spaces techniques proposed here is the scope of further work.

The structure of the proof will be the following: first we introduce suitable approximations of the equation (see (3.30)) that can be treated by the standard Hilbert–Schmidt theory. Then showing the needed convergence estimates we prove existence and uniqueness of the solution to a simplified Lyapunov equation (see (3.44)). An a priori estimate (see (3.3)) helps to prove convergence and gives uniqueness. Consequently a fixed point argument yields existence and uniqueness of a solution to the Lyapunov equation. Moreover, in section 4, we exploit the interplay between the Riccati equation and the corresponding optimal control problem to obtain existence and uniqueness of the mild solution to the BSRE and the synthesis of optimal control.

We notice that the optimal control problem is given by the following *state equation*,

$$\begin{cases} dy(t) = (Ay(t) + B(t)u(t))\,dt + C(t)y(t)\,dW(t), \quad t \in [0,T], \\ y(0) = x, \end{cases} \tag{1.3}$$

where y is the *state* of the system and u is the *control*, y and u are adapted processes with values in H, and by the following quadratic *cost functional*

$$\mathbb{E}\int_0^T \left(|\sqrt{S}_s y_s|^2 + |u(s)|^2\right) ds + \mathbb{E}\langle My_T, y_T\rangle. \tag{1.4}$$

Finally in section 5 we show how the present results con be applied to the Lyapunov equation arizing in the maximum principle for parabolic SPDEs.

2. Main notation and assumptions.

Some classes of stochastic processes Let G be any separable Hilbert space. By $\mathcal{P}$ we denote the predictable σ-field on $\Omega \times [0,T]$ and by $\mathcal{B}(G)$ the Borel σ-field on G. The following classes of processes will be used in this work:

- $L^p_{\mathcal{P}}(\Omega \times [0,T];G)$, $p \in [1,+\infty]$ denotes a subset of $L^p(\Omega \times [0,T];G)$, given by all equivalence classes admitting a predictable version. This space is endowed with the natural norm.
- $C_{\mathcal{P}}([0,T];L^p(\Omega;G))$ denotes the space of G-valued processes Y such that $Y : [0,T] \to L^p(\Omega,G)$ is continuous and Y has a predictable modification, endowed with the norm

$$|Y|^p_{C_{\mathcal{P}}([0,T];L^p(\Omega;G))} = \sup_{t\in[0,T]} \mathbb{E}|Y_t|^p_G.$$

Elements of $C_{\mathcal{P}}([0,T];L^p(\Omega;G))$ are identified up to modification.
- $L^p_{\mathcal{P}}(\Omega;C([0,T];G))$ denotes the space of predictable processes Y with continuous paths in G, such that the norm

$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;C([0,T];G))} = \mathbb{E} \sup_{t\in[0,T]} |Y_t|^p_G$$

is finite. Elements of this space are defined up to indistiguishibility.

Now let us consider the space $L(G)$ of linear and bounded operators from G to G. This space, as long as G is infinite dimensional, is not separable (see [4, p. 23]) and therefore we introduce the following σ-field:

$$\mathcal{L}_S = \{T \in L(G) : Tu \in A\}, \text{ where } u \in G \text{ and } A \in \mathcal{B}(G).$$

Following again [4], the elements of $\mathcal{L}_S$ are called *strongly measurable*.

We notice that the maps $P \to |P|_{L(G)}$ and $(P,u) \to Pu$ are measurable from $(L(G),\mathcal{L}_S)$ to $\mathbb{R}$ and from $(L(G) \times G, \mathcal{L}_S \otimes \mathcal{B}(G))$ to $(G,\mathcal{B}(G))$, respectively.

Moreover $\mathcal{L}_S$ is equivalent to the weak σ-field

$$\mathcal{L}_S = \{T \in L(G) : (Tu,x) \in A\}, \text{ where } u,x \in G \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

We define the following spaces:

- $L^\infty_{\mathcal{P},S}((0,T)\times\Omega;L(G))$ is a space of predictable processes Y from $(0,T)$ to $L(G)$, endowed with the σ-field $\mathcal{L}_S$. For each element Y there exists a constant $C > 0$ such that:

$$|Y(t,\omega)|_{L(G)} \le C \qquad \mathbb{P}-\text{a.s. for a.e. } t \in (0,T).$$

- In the same way we define $L^\infty_S(\Omega,\mathcal{F}_T;L(G))$ as the space of maps Y from $(\Omega,\mathcal{F}_T)$ into $(L(G),\mathcal{L}_S)$ such that there exists a positive constant K such that

$$|Y(\omega)|_{L(G)} \le K \qquad \mathbb{P}-\text{a.s.}$$

Elements of this space are identified up to modification.

By $\Sigma(G)$ we denote the subspace of all symmetric operators and by $\Sigma^+(G)$ the convex subset of all positive semidefinite operators. We define identically the following spaces: $L^\infty_{\mathcal{P},S}((0,T)\times\Omega;\Sigma^+(G))$, $L^1_{\mathcal{P},S}((0,T);L^\infty(\Omega,\Sigma^+(G)))$, and $L^\infty_S(\Omega,\mathcal{F}_T;\Sigma^+(G))$.

Setting and general assumptions on the coefficients. We fix now a Hilbert space H, real and separable, and we are going to study the following Lyapunov equation

(2.1)
$$\begin{cases} -dP_t = (AP_t + P_tA + P_tB_tB_t^*P_t + C_t^*Q_t + Q_tC_t + C_t^*P_tC_t)\,dt + S_t\,dt - Q_t dW_t, \\ \quad P_T = M \end{cases}$$

in the space $L(H)$, where by L^* we denote the adjoint of the operator L.

The following assumptions on A, C, S, and M will be used throughout the paper.
HYPOTHESIS 2.1.

(A1) *A is a self-adjoint operator in H and there exists a complete orthonormal basis $\{e_k : k \geq 1\}$ in H (that we fix from now on), a sequence of real numbers $\{\lambda_k : k \geq 1\}$, and $\omega \in \mathbb{R}$, such that*

$$Ae_k = -\lambda_k e_k \quad \text{with} \quad \omega \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots . \tag{2.2}$$

Moreover we assume that for a suitable $\rho \in (\frac{1}{4}, \frac{1}{2})$, it holds

$$\sum_{k \geq 1} |\lambda_k|^{-2\rho} < +\infty. \tag{2.3}$$

Without weakening the generality of the problem we can, and will, assume that $\omega > 0$ (just multiply P and Q by an exponential weight).
As is well known (see [14]) in this case A generates an analytic semigroup $(e^{tA})_{t \geq 0}$ with $|e^{tA}|_{L(H)} \leq 1$.

(A2) *We assume that $C \in L^\infty_{\mathcal{P},S}(\Omega \times [0,T]; L(H))$. We denote by M_C a positive constant such that:*

$$|C(t,\omega)|_{L(H)} < M_C \quad \mathbb{P} - a.s. \text{ and for a.e. } t \in (0,T).$$

(A3) *$S \in L^\infty_{\mathcal{P},S}((0,T) \times \Omega; \Sigma^+(H))$ and $M \in L^\infty_S(\Omega, \mathcal{F}_T; \Sigma^+(H))$.*

Remark 2.2. We notice that requirement (A1) in Hypothesis 2.1 is easily fulfilled in the case when A is the realization of the Laplace operator in $H = L^2([0,\pi])$ with Dirichlet boundary conditions. One has indeed

$$\begin{aligned} D(A) &= H^2([0,\pi]) \cap H^1_0([0,\pi]), \\ e_k(x) &= (2/\pi)^{1/2} \sin kx, \quad k = 1, 2, \ldots, \\ |\nabla e_k(x)| &\leq (2/\pi)^{1/2} k, \quad k = 1, 2, \ldots, \\ \lambda_k &= k^2, \quad k = 1, 2, \ldots . \end{aligned}$$

Requirement (A2) is fulfilled, for instance, as soon as $C(t,\omega)$ is defined on $L^2([0,\pi])$ by $(C(t,\omega)x)(\xi) := c(t,\omega,\xi)x(\xi)$ with c any bounded and progressive measurable map $[0,T] \times \Omega \times [0,\pi] \to \mathbb{R}$. The same holds for (A3); see also section 10 of [8].

Remark 2.3. Assumption (A1) is indeed restrictive but beside the Laplace operator on a bounded interval it holds as well for higher order operators in higher dimensional domains as well . As a matter of fact it is well known (see [11] and [1]) that if A is the realization of the Laplace operator with Dirichlet boundary conditions in a bounded domain of $\mathbb{R}^d$ then its eigenvalues satisfy $\lambda^A_k \geq ck^{2/d}$. Consequently the eigenvalues of the bi-Laplacian A^2 satisfy $\lambda^{A^2}_k = (\lambda^A_k)^2 \geq ck^{4/d}$ and $\sum_{k=1}^\infty (\lambda^{A^2}_k)^{-2\rho}$ converges if $8\rho/d > 1$. So we can conclude that there exists $\rho \in (1/4, 1/2)$ for which condition (2.3) holds, whenever $d \leq 3$.

The Hilbertian triple $V \hookrightarrow_d H \hookrightarrow_d V'$. In this paragraph we introduce the Hilbertian triple we will use to build the effective Hilbert space of operators where we are going to solve the Lyapunov equation. Let

$$V := D((-A)^\rho) = \left\{ x \in H : \sum_{n=1}^\infty \lambda_n^{2\rho} |\langle x, e_n \rangle|^2 := |x|^2_V < \infty \right\}. \tag{2.4}$$

By construction V is a Hilbert space endowed with its natural scalar product, in particular $\{\lambda_n^{-\rho}e_n\}_{n\geq 1}$ is a complete orthormal basis in V.

We can consider also its topological dual V' that has the following characterization:

$$V' := D((-A)^{-\rho}). \tag{2.5}$$

Notice that V' is the completion of H with the norm $|\cdot|_{V'}^2 = \sum_{n=1}^\infty \lambda_n^{-2\rho}|\langle x, e_n\rangle|^2$ and $\{\lambda_n^{\rho}e_n\}_{n\geq 1}$ and that is a complete orthormal basis in V'.

Once we make the usual identification $H \simeq H'$, we have the following dense inclusions:

$$V \hookrightarrow_d H \hookrightarrow_d V'. \tag{2.6}$$

We notice that both inclusion operators are Hilbert–Schmidt class

Remark 2.4. Under the previous Hypotheses 2.1 it is well known (see [14] and [13]) that for all $t > 0$

$$t^\rho |e^{tA}|_{L(H,V)} \leq 1, \;\; t^\rho |e^{tA}|_{L(V',H)} \leq 1, \tag{2.7}$$

$$|e^{tA}|_{L(V)} \leq 1, \;\; |e^{tA}|_{L(V')} \leq 1. \tag{2.8}$$

The Hilbert space $\mathcal{K}$. We set

$$\mathcal{K} := L_2(V;H) \cap L_2(H;V'), \tag{2.9}$$

where $L_2(V;H)$ denotes the Hilbert space of Hilbert–Schmidt operators form V to H, endowed with the Hilbert–Schmidt norm $|T|_{L_2(V;H)} = (\sum_{i=1}^\infty |Tf_i|_H^2)$ ($\{f_i : i \in \mathbb{N}\}$ being a complete orthonormal basis (c.o.b.) in V); see [4]. The obvious similar definition holds for $L_2(H;V')$. Space $\mathcal{K}$ will be endowed with the natural norm $|T|_{\mathcal{K}}^2 = |T|_{L_2(V;H)}^2 + |T|_{L_2(H;V')}^2$.

Finally, we introduce the following subspace of $\mathcal{K}$:

$$\mathcal{K}_s := \{G \in L_2(V;H) \cap L_2(H;V') \text{ such that } \langle Gx, y\rangle_H = \langle x, Gy\rangle_H \text{ for all } x, y \in V\}. \tag{2.10}$$

We summarize its main properties in the following lemma.

LEMMA 2.5. *The following hold:*

(i) *$\mathcal{K}$ is a separable Hilbert space.*

(ii) *$L(H) \subset \mathcal{K}$.*

(iii) *$T \in \mathcal{K}$ iff $T \in L(V;H) \cap L(H;V')$ and $|T|_{\mathcal{K}}^2 = \sum_{k=1}^\infty \lambda_k^{-2\rho}(|Te_k|_H^2 + |T^*e_k|_H^2) < \infty$, where $T^* \in L(V;H) \cap L(H;V')$ is the adjoint of T (in the sense that $\langle Tv, w\rangle = \langle v, T'w\rangle$ whenever $v \in V$ and $w \in H$ or $w \in V$ and $v \in H$).*

(iv) *If $T \in \mathcal{K}_s$ then $|T|_{\mathcal{K}_s}^2 = 2\sum_{k=1}^\infty \lambda_k^{-2\rho}|Te_k|_H^2$.*

Proof. We omit the proof of (i), it being obvious.

(ii) Let $G \in L(H)$; then since $\{\lambda_n^{-\rho}e_n\}_{n\geq 1}$ is a basis of V, we have

$$|G|_{L_2(V;H)} = \left(\sum_{n=1}^\infty \lambda_n^{-2\rho}|Ge_n|_H^2\right)^{1/2} \leq |G|_{L(H)} \left(\sum_{n=1}^\infty \lambda_n^{-2\rho}\right)^{1/2}. \tag{2.11}$$

Moreover, recalling that $\{e_n : n \geq 1\}$ is a c.o.b. of H, we have

$$|G|_{L_2(H;V')} = \left(\sum_{n=1}^{\infty} |Ge_n|^2_{V'}\right)^{1/2} \leq |G|_{L(H)} \left(\sum_{n=1}^{\infty}\sum_{h=1}^{\infty} \lambda_h^{-2\rho} |\langle e_n, e_h\rangle|^2_H\right)^{1/2}$$

$$= |G|_{L(H)} \left(\sum_{h=1}^{\infty} \lambda_h^{-2\rho} \sum_{n=1}^{\infty} |\langle e_n, e_h\rangle|^2_H\right)^{1/2} = |G|_{L(H)} \left(\sum_{h=1}^{\infty} \lambda_h^{-2\rho}\right)^{1/2}. \tag{2.12}$$

Thus $G \in \mathcal{K}$.

(iii) Notice that, for any c.o.b. $\{f_k : k \geq 1\}$ of H, we have

$$\sum_{k=1}^{\infty} |Tf_k|^2_{V'} = \sum_{k=1}^{\infty}\sum_{h=1}^{\infty} \lambda_h^{-2\rho} \langle f_k, T^* e_h\rangle^2_H \tag{2.13}$$

$$= \sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \lambda_h^{-2\rho} \langle f_k, T^* e_h\rangle^2_H = \sum_{h=1}^{\infty} \lambda_h^{-2\rho} |T^* e_h|^2_H. \qquad \square$$

3. Mild solutions of the Lyapunov equation. The natural space in which the deterministic Lyapunov equation is studied is the space $\Sigma(H)$ of bounded self-adjoint operators in H. Unfortunately this is not a Hilbert space and this fact causes serious difficulties when considering stochastic backward differential equations (for instance, the essential tool given by the martingale representation theorem does not hold). To overcome this difficulty we will work in the bigger space $\mathcal{K}$ that is a separable Hilbert space.

For convenience we rewrite the equation of interest:

$$\begin{cases} -dP_t = (AP_t + P_t A + C^* Q_t + Q_t C + C^* P_t C)\, dt + S_t\, dt - Q_t\, dW_t, \\ \quad P_T = M. \end{cases} \tag{3.1}$$

Definition 3.1. *A* mild solution *of problem* (3.1) *is a couple of processes*

$$(P, Q) \in L^2_{\mathcal{P},S}(\Omega, C([0,T]; \Sigma(H))) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathcal{K}_s)$$

that solves the following equation, for all $t \in [0,T]$:

$$P(t) = e^{(T-t)A} M e^{(T-t)A} + \int_t^T e^{(s-t)A} S(s) e^{(s-t)A}\, ds$$

$$+ \int_t^T e^{(s-t)A} \Big[C^*(s)P(s)C(s) + C^*(s)Q(s) + Q(s)C(s)\Big] e^{(s-t)A}\, ds \tag{3.2}$$

$$+ \int_t^T e^{(s-t)A} Q(s) e^{(s-t)A}\, dW(s) \qquad \mathbb{P} - a.s.$$

We first prove an a priori estimate for mild solutions.

Proposition 3.2. *Let (P, Q) be a mild solution to* (3.2). *Then there exists a $\delta_0 > 0$ just depending on T and the constants M_C, ρ introduced in Hypothesis* 2.1 *such that for every $0 \leq \delta \leq \delta_0$ the following holds:*

$$|P|^2_{L^2(\Omega; C([T-\delta,T]; L(H)))} + \mathbb{E}\int_{T-\delta}^T |Q(s)|^2_{\mathcal{K}}\, ds \leq c\left(\mathbb{E}|M|^2_{L(H)} + \delta\mathbb{E}\int_{T-\delta}^T |S(s)|^2_{L(H)}\, ds\right), \tag{3.3}$$

where c is a positive constant depending on $\delta_0, M_A, M_C, \rho,$ and T.

Proof. Let $(P,Q) \in L^2_{\mathcal{P},S}(\Omega, C([0,T];L(H))) \times L^2_{\mathcal{P}}(\Omega \times [0,T];\mathcal{K}_s)$ be any mild solution, hence we have that

$$P(t) = \mathbb{E}^{\mathcal{F}_t}\left[e^{(T-t)A}Me^{(T-t)A} + \int_t^T e^{(s-t)A}S(s)e^{(s-t)A}\,ds\right] + \mathbb{E}^{\mathcal{F}_t}\Big[\int_t^T e^{(s-t)A}\Big(C^*(s)P(s)C(s) + C^*(s)Q(s) + Q(s)C(s)\Big)e^{(s-t)A}\,ds\Big] \qquad \mathbb{P}-\text{a.s.} \tag{3.4}$$

We notice that if $(L(t))_{T\geq 0}$ is a Banach space valued process then by Doob's L^2 inequality

$$\mathbb{E}\sup_{t\in[r,T]}|\mathbb{E}^{\mathcal{F}_t}L(t)|^2 \leq \mathbb{E}\sup_{t\in[r,T]}\left[\mathbb{E}^{\mathcal{F}_t}\left(\sup_{s\in[r,T]}|L(s)|\right)\right]^2 \leq 4\mathbb{E}\sup_{t\in[r,T]}|L(t)|^2.$$

Moreover we have

$$\mathbb{E}\sup_{t\in[r,T]}|e^{(T-t)A}Me^{(T-t)A}|^2_{L(H)} \leq |M|^2_{L(H)}, \tag{3.5}$$

$$\mathbb{E}\sup_{t\in[r,T]}\left|\int_t^T e^{(s-t)A}C^*(s)P(s)C(s)e^{(s-t)A}\,ds\right|^2_{L(H)} \leq M_C^4(T-r)\mathbb{E}\int_r^T |P(u)|^2_{L(H)}\,ds, \tag{3.6}$$

$$\mathbb{E}\sup_{t\in[r,T]}\left|\int_t^T e^{(s-t)A}S(s)e^{(s-t)A}\,ds\right|^2_{L(H)} \leq (T-r)E\int_r^T |S(s)|^2_{L(H)}\,ds. \tag{3.7}$$

In estimating the latter terms we notice that, even if $G \in \mathcal{K}$, it is not true in general that $GC \in \mathcal{K}$, therefore we have to use the regularity properties of the semigroup (2.7):

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[r,T]}\left|\int_t^T e^{(s-t)A}\Big[C^*(s)Q(s) + Q(s)C(s)\Big]e^{(s-t)A}\,ds\right|^2_{L(H)} \\
&\leq 2\mathbb{E}\left\{\sup_{t\in[r,T]}\left[\int_t^T |e^{(s-t)A}C^*(s)Q(s)e^{(s-t)A}|_{L(H)}\,ds\right]^2\right. \\
&\qquad\left. + \sup_{t\in[r,T]}\left[\int_t^T |e^{(s-t)A}Q(s)C(s)e^{(s-t)A}|_{L(H)}\,ds\right]^2\right\}.
\end{aligned}$$

Let us consider the first term:

$$\begin{aligned}
&\mathbb{E}\Big\{\sup_{t\in[r,T]}\left[\int_t^T |e^{(s-t)A}C^*(s)Q(s)e^{(s-t)A}|_{L(H)}\,ds\right]^2 \\
&\leq \mathbb{E}\sup_{t\in[r,T]}\left[\int_t^T |e^{(s-t)A}|_{L(H)}|C^*(s)|_{L(H)}|Q(s)|_{L(V,H)}|e^{(s-t)A}|_{L(V)}\,ds\right]^2 \\
&\leq M_C^2(T-r)\mathbb{E}\int_r^T |Q(s)|^2_{\mathcal{K}}\,ds.
\end{aligned}$$

Also consider the second one:

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[r,T]}\left[\int_t^T |e^{(s-t)A}Q(s)C(s)e^{(s-t)A}|_{L(H)}\,ds\right]^2\\
&\quad\le \mathbb{E}\sup_{t\in[r,T]}\left[\int_t^T |e^{(s-t)A}|_{L(V';H)}|Q(s)|_{L(H;V')}|C(s)|_{L(H)}|e^{(s-t)A}|_{L(H)}\,ds\right]^2\\
&\quad\le \mathbb{E}\sup_{t\in[r,T]}\left(\int_t^T \frac{M_C}{(s-t)^\rho}|Q(s)|_{\mathcal{K}}\,ds\right)^2 \le M_C^2(T-r)^{1-2\rho}\int_r^T |Q(s)|^2_{\mathcal{K}}\,ds.
\end{aligned}\tag{3.8}$$

Summing up all these estimates we obtain that, for $r = T-\delta$,

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[T-\delta,T]}|P(u)|^2_{L(H)}\\
&\quad\le C\Bigg(|M|^2_{L(H)}+\delta^2\mathbb{E}\sup_{u\in[T-\delta,T]}|P(u)|^2_{L(H)}\\
&\qquad\quad+\delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T|Q(s)|^2_{\mathcal{K}}\,ds+\delta\mathbb{E}\int_{T-\delta}^T|S(s)|^2_{L(H)}\,ds\Bigg),
\end{aligned}\tag{3.9}$$

where C depends only on M_C,ρ, and T and for δ small enough (changing the value of the constant C)

$$\begin{aligned}
&\mathbb{E}\sup_{u\in[T-\delta,T]}|P(u)|^2_{L(H)}\\
&\quad\le C\left(|M|^2_{L(H)}+\delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T|Q(s)|^2_{\mathcal{K}}\,ds+\delta\mathbb{E}\int_{T-\delta}^T|S(s)|^2_{L(H)}\,ds\right).
\end{aligned}\tag{3.10}$$

Now we have to recover an estimate for Q; this cannot be done in the same way because the term $Q(s)C(s)\notin\mathcal{K}$, and we can not follow the technique introduced in [9].

Therefore we exploit some duality relations. First of all we multiply both sides by the linear operators $J_n := n(nI-A)^{-1}$.

Such a family of operators has the following properties:

1. $J_n e_k = \frac{n}{(n+\lambda_k)}e_k$ for every $k\ge 1$, $\quad n\ge 1$,
2. $|J_n|_{L(H)}\le 1$, $\quad |J_n|_{L(V)}\le 1$, $\quad |J_n|_{L(V')}\le 1$ for every $n\ge 1$,
3. $|J_n|_{L(H,V)}\le n^\rho$, $\quad |J_n|_{L(V',H)}\le n^\rho$,
4. $\lim_{n\to+\infty} J_n x = x$ for every $x\in H$,
5. $J_n\in L_2(H)$ for every $n\ge 1$, and $|J_n|_{L_2(H)}\le |I_{V,H}|_{L_2(H)}|J_n|_{L(H,V)}$.

Hence (3.2), setting $P^n(s)=J_nP(s)J_n$ and $Q^n(s)=J_nQ(s)J_n$, becomes

$$\begin{aligned}
P^n(t) &= e^{(T-t)A}J_nMJ_ne^{(T-t)A}+\int_t^T e^{(s-t)A}J_nC^*(s)P(s)C(s)J_ne^{(s-t)A}\,ds\\
&\quad+\int_t^T e^{(s-t)A}J_nS(s)J_ne^{(s-t)A}\,ds\\
&\quad+\int_t^T e^{(s-t)A}\Big[J_nC^*(s)Q(s)J_n+J_nQ(s)C(s)J_n\Big]e^{(s-t)A}\,ds\\
&\quad+\int_t^T e^{(s-t)A}Q^n(s)e^{(s-t)A}\,dW(s)\qquad \mathbb{P}-\text{a.s.}
\end{aligned}\tag{3.11}$$

Notice that, thanks to the regularization property

$$J_n, (P^n, Q^n) \in L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(H)).$$

In particular,

$$|Q_n(s)|^2_{L_2(H)} \leq |J_n|^2_{L(V';H)} |Q(s)|^2_{\mathcal{K}}.$$

Moreover (P^n, Q^n) is also the unique mild solution of

$$\left\{ \begin{array}{l} -dP^n_t = (AP^n_t + P^n_t A)\, dt + \hat{S}^n_t\, dt - Q^n_t\, dW_t, \\ \quad P_T = M^n, \end{array} \right. \tag{3.12}$$

where $\hat{S}^n_s = J_n C^*_s P_s C_s J_n + J_n S_s J_n + J_n C^*_s Q_s J_n + J_n Q_s C_s J_n \in L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(H))$. We wish to apply Lemma 2.1 of [9]. Let us check that $\hat{S}^n$ has the required L_2 regularity:

$$\begin{aligned} &\mathbb{E} \int_0^T |J_n C^*(s) P(s) C(s) J_n|^2_{L_2(H)}\, ds \\ &\quad \leq \mathbb{E} \int_0^T |J_n|^2_{L(H)} |C^*(s)|^2_{L(H)} |P(s)|^2_{L(H)} |C(s)|^2_{L(H)} |J_n|^2_{L_2(H)}\, ds \\ &\quad \leq M^4_C |J_n|^2_{L_2(H)} |P|_{L^2(\Omega; C([T-\delta,T]; L(H)))}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\mathbb{E} \int_0^T |J_n Q(s) C(s) J_n|^2_{L_2(H)}\, ds \\ &\quad \leq \mathbb{E} \int_0^T |J_n|^2_{L(V',H)} |Q(s)|^2_{L_2(H;V')} |C(s)|^2_{L(H)} |J_n|^2_{L_2(H)}\, ds \\ &\quad \leq n^{2\rho} M^2_C \mathbb{E} \int_0^T |Q(s)|^2_{\mathcal{K}}\, ds, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\mathbb{E} \int_0^T |J_n C^*(s) Q(s) J_n|^2_{L_2(H)}\, ds \\ &\quad \leq \mathbb{E} \int_0^T |J_n|^2_{L(H)} |C^*(s)|^2_{L(H)} |Q(s)|^2_{L_2(V;H)} |J_n|^2_{L(H;V)}\, ds \\ &\quad \leq n^{2\rho} M^2_C \mathbb{E} \int_0^T |Q(s)|^2_{\mathcal{K}}\, ds. \end{aligned} \tag{3.15}$$

We seek for an estimate independent of n for the martingale term. We are going to use a duality argument; for this purpose we introduce an operator valued process defined as follows:

$$L^n(s) e_k := 2\lambda_k^{-2\rho} Q^n(s) e_k \qquad \text{for } k \geq 1. \tag{3.16}$$

Let us fix $\delta > 0$ then consider the following process

$$X^n_t = \int_{T-\delta}^t e^{(t-s)A} L^n(s) e^{(t-s)A}\, dW(s), \qquad t \in [T-\delta, T]. \tag{3.17}$$

It can be easily verified that $X^n \in C_{\mathcal{P}}([T-\delta,T]; L^2(\Omega; L_2(H)))$. Therefore, by standard regularization arguments (see, for instance, [4] for the forward equation and

[8] for the backward equation) we can prove that

(3.18)

$$\begin{aligned}\mathbb{E}\langle X^n(T), P^n(T)\rangle_{L_2(H)} = {} & \mathbb{E}\int_{T-\delta}^{T} \langle L^n(s), Q^n(s)\rangle_{L_2(H)}\, ds \\ & - \mathbb{E}\int_{T-\delta}^{T} \langle X^n(s), J_n S(s) J_n\rangle_{L_2(H)}\, ds \\ & - \mathbb{E}\int_{T-\delta}^{T} \langle X^n(s), J_n C^*(s)P(s)C(s)J_n + J_n C^*(s)Q(s)J_n \\ & \qquad + J_n Q(s) C(s) J_n\rangle_{L_2(H)}\, ds.\end{aligned}$$

First of all notice that $\langle L^n(s), Q^n(s)\rangle_{L_2(H)} = 2\sum_{k=1}^{\infty}\lambda_k^{-2\rho}|Q^n(s)e_k|_H^2$; such a quantity corresponds to $|Q^n|^2_{\mathcal{K}}$ being Q^n, a symmetric operator. Thus

$$\left|\mathbb{E}\int_{T-\delta}^{T}\langle L^n(s), Q^n(s)\rangle_{L_2(H)}\, ds\right| = \mathbb{E}\int_{T-\delta}^{T}|Q^n(s)|^2_{\mathcal{K}}\, ds. \tag{3.19}$$

Let us estimate the process X^n_T; we have, for every $t \in [T-\delta, T]$,

$$\begin{aligned}\mathbb{E}\sum_{k\geq 1}|X^n(t)e_k|_H^2\lambda_k^{2\rho} &= \sum_{k\geq 1}\mathbb{E}\left|\int_{T-\delta}^{t} e^{(t-s)A}L^n(s)e^{(t-s)A}e_k\, dW_s\right|_H^2\lambda_k^{2\rho} \\ &= \sum_{k\geq 1}\mathbb{E}\int_{T-\delta}^{t}\lambda_k^{2\rho}|e^{(t-s)A}L^n(s)e^{(t-s)A}e_k|_H^2\, ds \\ &\leq \mathbb{E}\int_{T-\delta}^{T}\sum_{k\geq 1}\lambda_k^{-2\rho}2|Q^n(s)e_k|^2_H\, ds = \mathbb{E}\int_{T-\delta}^{T}|Q^n(s)|^2_{\mathcal{K}}\, ds.\end{aligned} \tag{3.20}$$

Therefore, using (3.20) with $r = T-\delta$ we have

(3.21)

$$\begin{aligned}&|\mathbb{E}\langle X^n(T), P^n(T)\rangle_{L_2(H)}| \\ &= \left|\mathbb{E}\sum_{k=1}^{\infty}\langle X^n(T)e_k, P^n(T)e_k\rangle\right| \leq \left(\mathbb{E}\sum_{k=1}^{\infty}|X^n(T)e_k|^2\lambda_k^{2\rho}\right)^{\frac{1}{2}}\left(\mathbb{E}\sum_{k=1}^{\infty}|P^n(T)e_k|^2\lambda_k^{-2\rho}\right)^{\frac{1}{2}} \\ &\leq C\left(\mathbb{E}\int_{T-\delta}^{T}|Q^n_s|^2_{\mathcal{K}}\, ds\right)^{\frac{1}{2}}\left(\mathbb{E}|P^n(T)|^2_{L(H)}\right)^{\frac{1}{2}}.\end{aligned}$$

Moreover, thanks to (3.10) and $|P^n(T)|_{L(H)} \leq |P(T)|_{L(H)}$, we end up with

(3.22)

$$\begin{aligned}&|\mathbb{E}\langle X^n(T), P^n(T)\rangle_{L_2(H)}| \\ &\leq C\left(\mathbb{E}\int_{T-\delta}^{T}|Q^n_s|^2_{\mathcal{K}}\, ds\right)^{\frac{1}{2}}\left(|M|^2_{L(H)} + \delta\mathbb{E}\int_{T-\delta}^{T}|S(s)|^2 ds + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^{T}|Q(s)|^2_{\mathcal{K}}\, ds\right)^{\frac{1}{2}}.\end{aligned}$$

Regarding $\mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n C^*(s)Q(s)J_n + J_n Q(s)C(s)J_n\rangle_{L_2(H)}\, ds$, we have

$$
\begin{aligned}
&\left|\mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n C^*(s)Q(s)J_n\rangle_{L_2(H)}\, ds\right| \\
&\quad\le M_C^2\mathbb{E}\int_{T-\delta}^T \left(\sum_{k\ge 1} |X^n(s)e_k|_H^2 \lambda_k^{2\rho}\right)^{\frac12} |Q(s)|_{\mathcal{K}}\, ds \\
&\quad\le C\mathbb{E}\int_{T-\delta}^T \left(\int_{T-\delta}^T |Q^n(s)|_{\mathcal{K}}^2\, ds\right)^{\frac12} |Q(s)|_{\mathcal{K}}\, ds \\
&\quad\le \frac14 \mathbb{E}\int_{T-\delta}^T |Q^n(s)|_{\mathcal{K}}^2\, ds + C\delta\mathbb{E}\int_{T-\delta}^T |Q(s)|_{\mathcal{K}}^2\, ds
\end{aligned}
$$

with $C > 0$ a constant that may change from line to line but always depends only on the ones introduced in Hypothesis 2.1. Notice that

$$
\begin{aligned}
&\mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n Q(s)C(s)J_n\rangle_{L_2(H)}\, ds \\
&= \mathbb{E}\int_{T-\delta}^T \sum_{k=1}^\infty \langle X^n(s)e_k, J_n Q(s)C(s)J_n e_k\rangle_H\, ds \\
&= \mathbb{E}\int_{T-\delta}^T \sum_{k=1}^\infty \sum_{h=1}^\infty \langle e_k, X^n(s)e_h\rangle \langle e_k, J_n C^*(s)Q(s)J_n e_h\rangle_H\, ds \\
&\le \mathbb{E}\int_{T-\delta}^T \sum_{h=1}^\infty |X^n(s)e_h||J_n C^*(s)Q(s)J_n e_h|\, ds \\
&\le \mathbb{E}\int_{T-\delta}^T (\sum_{h=1}^\infty \lambda_h^{2\rho}|X^n(s)e_h|^2)^{1/2} \left(\sum_{h=1}^\infty \lambda_h^{-2\rho}|Q(s)e_h|^2\right)^{1/2} ds.
\end{aligned}
\tag{3.23}
$$

Thus the same conclusion holds, so we have that, by (3.20),

$$
\begin{aligned}
&\left|\mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n C^*(s)Q(s)J_n + J_n Q(s)C(s)J_n\rangle_{L_2(H)}\, ds\right| \\
&\quad\le \frac12 \mathbb{E}\int_{T-\delta}^T |Q^n(s)|_{\mathcal{K}}^2\, ds + C\delta\mathbb{E}\int_{T-\delta}^T |Q(s)|_{\mathcal{K}}^2\, ds.
\end{aligned}
\tag{3.24}
$$

Moreover we have that

$$
\begin{aligned}
&\left|\mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n C^*(s)P(s)C(s)J_n\rangle_{L_2(H)}\, ds\right| \\
&\quad\le C\delta |P|_{L^2_{\mathcal{P}}(\Omega;C([T-\delta,T];L(H)))}^2 + \frac18 \mathbb{E}\int_{T-\delta}^T |Q^n(s)|_{\mathcal{K}}^2\, ds,
\end{aligned}
\tag{3.25}
$$

and that, similarly,

$$
\left|\mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n S(s)J_n\rangle_{L_2(H)}\, ds\right| \le C\mathbb{E}\int_{T-\delta}^T |S(s)|_{L(H)}^2 ds + \frac18 \mathbb{E}\int_{T-\delta}^T |Q^n(s)|_{\mathcal{K}}^2\, ds.
\tag{3.26}
$$

Taking into account (3.22), (3.24), (3.25), and (3.26) we have that there exists a positive constant C independent of n and δ such that

(3.27)
$$\mathbb{E}\int_{T-\delta}^{T}|Q^n(s)|^2_{\mathcal{K}}\,ds \le C\left(|M|^2_{L(H)}+\delta\mathbb{E}\int_{T-\delta}^{T}|S(s)|^2_{L(H)}ds+\delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^{T}|Q(s)|^2_{\mathcal{K}}\,ds\right).$$

From (3.10) and (3.27) the claim follows since $|Q^n(s)|^2_{\mathcal{K}}\nearrow|Q(s)|^2_{\mathcal{K}}$ choosing a δ small enough such that $C\delta^{1-2\rho}<\frac{1}{2}$. □

With an identical argument we get the estimate in the easier case in which the term C^*PC is not present.

Remark 3.3. Assume that $Q\in L^2_{\mathcal{P}}(\Omega\times[0,T];\mathcal{K}_s)$ and that P, given by

$$\begin{aligned}P(t)&=e^{(T-t)A}Me^{(T-t)A}+\int_t^T e^{(s-t)A}S(s)e^{(s-t)A}\,ds\\&\quad+\int_t^T e^{(s-t)A^*}\Big[C^*(s)Q(s)+Q(s)C(s)\Big]e^{(s-t)A}\,ds\\&\quad+\int_t^T e^{(s-t)A}Q(s)e^{(s-t)A}\,dW(s)\qquad\mathbb{P}-\text{a.s.},\end{aligned}\tag{3.28}$$

is an adapted $\mathcal{K}$-valued process.

Then there exists a $\delta_0>0$ just depending on T and the constants M_C and ρ introduced in Hypothesis 2.1 such that for every $0\le\delta\le\delta_0$ the following holds:

(3.29)
$$|P|^2_{L^2(\Omega;C([T-\delta,T];L(H)))}+\mathbb{E}\int_{T-\delta}^{T}|Q(s)|^2_{\mathcal{K}}\,ds\le c\left(\mathbb{E}|M|^2_{L(H)}+\delta\mathbb{E}\int_{T-\delta}^{T}|S(s)|^2_{L(H)}\,ds\right)$$

with c a positive constant depending on δ_0, M_C, ρ, and T.

We are now in a position to prove existence and uniqueness of the solution to the mild Lyapunov equation

THEOREM 3.4. *Under the assumptions of Hypothesis* 2.1, (3.1) *has a unique mild solution* (P,Q).

Proof. The idea is classical: we will buid a map Γ from the space $L^2_{\mathcal{P}}(\Omega,C([0,T];H))$ into itself and prove that is a contraction for small time.

In completing this program we follow three steps.

Step 1: *Regularization.* We introduce some regularizing processes in order to define $\hat{P}=\Gamma(P)$ for an arbitrary $P\in L^2_{\mathcal{P}}(\Omega,C([0,T];\Sigma(H)))$. So we fix P and for every $n\ge1$ we consider the following problem: find $\hat{P}^n,\hat{Q}^n$ such that

$$\begin{aligned}\hat{P}^n(t)&=e^{(T-t)A}J_nMJ_ne^{(T-t)A}+\int_t^T e^{(s-t)A}C^*(s)J_nP(s)J_nC(s)e^{(s-t)A}\,ds\\&\quad+\int_t^T e^{(s-t)A}J_nS(s)J_ne^{(s-t)A}\,ds\\&\quad+\int_t^T e^{(s-t)A}(C^*(s)\hat{Q}^n(s)+\hat{Q}^n(s)C(s))e^{(s-t)A}\,ds\\&\quad+\int_t^T e^{(s-t)A}\hat{Q}^n(s)e^{(s-t)A}\,dW(s)\qquad\mathbb{P}-\text{a.s.}\end{aligned}\tag{3.30}$$

Notice that for every $n \in \mathbb{N}$, we have that $C^* J_n P J_n C, J_n S J_n \in L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(H))$, $J_n M J_n \in L_2(H)$. Moreover for every $C \in L(H)$, the map $G \in L_2(H) \to C^* G + GC \in L_2(H)$ is Lipschitz continuous.

Thus Lemma 2.1 of [9] applies and we can deduce that there exists a unique solution $(\hat{P}^n, \hat{Q}^n) \in L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(H)) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(H))$ to (3.30). Moreover by Remark 3.3 there exists $\delta_0 < 1$ small enough and independent of n such that, for all $\delta \leq \delta_0$,

$$
\begin{aligned}
&\mathbb{E} \sup_{u \in [T-\delta, T]} |\hat{P}^n(u)|^2_{L(H)} + \mathbb{E} \int_{T-\delta}^{T} |\hat{Q}^n(s)|^2_{\mathcal{K}} \, ds \\
&\qquad \leq C \left(|M|^2_{L(H)} + \delta^2 \mathbb{E} \sup_{u \in [r,T]} |P(u)|^2_{L(H)} + \delta \int_r^T |S(s)|^2_{L(H)} \, ds \right)
\end{aligned} \tag{3.31}
$$

with C a constant depending only on M_C, T, and ρ but not on n.

We notice here that the operator $P \to C^* P C$ is Lipschitz from $L_2(H)$ to itself as well. We cannot treat it as the term $G \to C^* G + GC$ since we will then need to lower the regularity of P to the space $\mathcal{K}$ and, if P only belongs to $\mathcal{K}$, then the operator $e^{sA} C^* P C e^{sA}$ is not well defined while $G \to e^{sA}[C^* G + GC] e^{sA}$ is well defined from $\mathcal{K}$ to itself.

Step 2: *Limiting procedure.* Let us evaluate the difference $\hat{P}^n - \hat{P}^m$ for two integers m, n:

$$
\begin{aligned}
&\hat{P}^n(t) - \hat{P}^m(t) \\
&\quad = e^{(T-t)A} J_n M J_n e^{(T-t)A} - e^{(T-t)A} J_m M J_m e^{(T-t)A} \\
&\qquad + \int_t^T e^{(s-t)A} (J_n S(s) J_n - J_m S(s) J_m) e^{(s-t)A} \, ds \\
&\qquad + \int_t^T e^{(s-t)A} C^*(s) (J_n P(s) J_n - J_m P(s) J_m) C(s) e^{(s-t)A} \, ds \\
&\qquad + \int_t^T e^{(s-t)A} \Big[C^*(s)(\hat{Q}^n(s) - \hat{Q}^m(s)) + (\hat{Q}^n(s) - \hat{Q}^m(s)) C(s) \Big] e^{(s-t)A} \, ds \\
&\qquad + \int_t^T e^{(s-t)A} [\hat{Q}^n(s) - \hat{Q}^m(s)] e^{(s-t)A} \, dW(s) \qquad \mathbb{P}-\text{a.s.}
\end{aligned} \tag{3.32}
$$

We are going to show that

$$
\lim_{m,n \to \infty} \mathbb{E} \sup_{t \in [T-\delta, T]} |\hat{P}^n(t) - \hat{P}^m(t)|^2_{\mathcal{K}} = 0, \tag{3.33}
$$

$$
\lim_{m,n \to \infty} \mathbb{E} \int_{T-\delta}^{T} |\hat{Q}^n(s) - \hat{Q}^m(s)|^2_{\mathcal{K}} \, ds = 0. \tag{3.34}
$$

Let's begin to prove (3.33) by noticing that

$$
\begin{aligned}
&\hat{P}^n(t) - \hat{P}^m(t) \\
&\quad = \mathbb{E}^{\mathcal{F}_t}(e^{(T-t)A} J_n M J_n e^{(T-t)A} - e^{(T-t)A} J_m M J_m e^{(T-t)A}) \\
&\qquad + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{(s-t)A}(J_n S(s) J_n - J_m S(s) J_m) e^{(s-t)A}\, ds\right) \\
&\qquad + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{(s-t)A} C^*(s)(J_n P(s) J_n - J_m P(s) J_m) C(s) e^{(s-t)A}\, ds\right) \\
&\qquad + \mathbb{E}^{\mathcal{F}_t}\Bigg(\int_t^T e^{(s-t)A}\Big[C^*(s)(\hat{Q}^n(s) - \hat{Q}^m(s)) \\
&\qquad\qquad\qquad + (\hat{Q}^n(s) - \hat{Q}^m(s))C(s)\Big] e^{(s-t)A}\, ds\Bigg) \quad \mathbb{P}-\text{a.s.}
\end{aligned}
$$

M being a symmetric operator, we have that

$$
\begin{aligned}
&\left| e^{(T-t)A}(J_n M J_n - J_m M J_m) e^{(T-t)A} \right|_{\mathcal{K}}^2 \\
&\qquad\qquad = \sum_{k=1}^{\infty} \lambda_k^{-2\rho} \left| e^{(T-t)A}(J_n M J_n - J_m M J_m) e^{(T-t)A} e_k \right|_H^2 .
\end{aligned}
$$

For every fixed $k \geq 1$,

$$
\lim_{n,m\to\infty} |(J_n M (J_n - J_m) e_k|_H^2 = 0 \quad \forall t \in [0,T] \qquad \mathbb{P}-\text{a.s.}
$$

and

$$
\lim_{n,m\to\infty} |(J_n - J_m) M J_m e_k|_H^2 = 0 \quad \forall t \in [0,T] \qquad \mathbb{P}-\text{a.s.}
$$

Moreover

$$
\sum_{k=1}^{\infty} \lambda_k^{-2\rho} |(J_n M J_n - J_m M J_m) e_k|_H^2 \leq M_M^2 \sum_{k=1}^{\infty} \lambda_k^{-2\rho} < \infty \qquad \mathbb{P}-\text{a.s.}
$$

Hence by the dominated convergence theorem and the Doob inequality for martingales,

$$
\begin{aligned}
(3.35) \qquad &\lim_{n,m\to\infty} \mathbb{E}\left[\sup_{t\in[T-\delta,T]} |\mathbb{E}^{\mathcal{F}_t}(e^{(T-t)A}(J_n M J_n - J_m M J_m) e^{(T-t)A})|_{\mathcal{K}}^2\right] \\
&\leq \lim_{n,m\to\infty} 4\, \mathbb{E}|(J_n M J_n - J_m M J_m)|_{\mathcal{K}}^2 = 0.
\end{aligned}
$$

The second and the third terms are similar so we'll give the details only of the third.

As before we have that for every $k \geq 1$,

$$\lim_{n,m\to\infty} |(C^*(s)(J_nP(s)J_n - J_mP(s)J_m)C(s)e_k|_H^2 = 0 \qquad \mathbb{P}-\text{a.s. and for a.e. } s \in [T-\delta, T];$$

and $\mathbb{P}$-a.s. and for a.e. $s \in [T-\delta, T]$,

$$\sum_{k\geq 1} \lambda_k^{-2\rho} |(C^*(s)(J_nP(s)J_n - J_mP(s)J_m)C(s)e_k|_H^2 \, ds \leq M_C^4 \sum_{k\geq 1} \lambda_k^{-2\rho} < \infty.$$

Therefore again by the dominated convergence theorem and the Doob inequality for martingales,

(3.36)

$$\begin{aligned}
&\lim_{n,m\to\infty} \mathrm{E} \sup_{t\in[T-\delta,T]} \left| \mathbb{E}^{\mathcal{F}_t} \left(\int_t^T e^{(s-t)A} [(C^*(s)(J_nP(s)J_n - J_mP(s)J_m)C(s)] e^{(s-t)A} \, ds \right) \right|_{\mathcal{K}}^2 \\
&\leq \delta \lim_{n,m\to\infty} 4\,\mathbb{E} \int_{T-\delta}^T \sum_{k\geq 1} \lambda_k^{-2\rho} |(C^*(s)(J_nP(s)J_n - J_mP(s)J_m)C(s)e_k|_H^2 \, ds = 0.
\end{aligned}$$

At last let us consider the term

$$\begin{aligned}
\mathbb{E} \sup_{t\in[T-\delta,T]} \Bigg| \mathbb{E}^{\mathcal{F}_t} \Bigg(\int_t^T e^{(s-t)A} &[C^*(s)(\hat{Q}^n(s) - \hat{Q}^m(s)) \\
&+ (\hat{Q}^n(s) - \hat{Q}^m(s))C(s)] e^{(s-t)A} \, ds \Bigg) \Bigg|_{\mathcal{K}}^2 .
\end{aligned}$$

First of all,

$$\begin{aligned}
&\left(\int_t^T |e^{(s-t)A} (\hat{Q}^n(s) - \hat{Q}^m(s)) C(s) e^{(s-t)A}|_{\mathcal{K}} \, ds \right)^2 \\
&= \left[\int_t^T \left(\sum_{k\geq 1} \lambda_k^{-2\rho} |e^{(s-t)A} (\hat{Q}^n(s) - \hat{Q}^m(s)) C(s) e^{(s-t)A} e_k|_H^2 \right)^{1/2} ds \right]^2 \\
&\leq \Bigg(\int_t^T |e^{(s-t)A}|_{L(V';H)} |(\hat{Q}^n(s) - \hat{Q}^m(s))|_{L_2(H;V')} \\
&\qquad \left(\sum_{k\geq 1} \lambda_k^{-2\rho} |C(s) e^{(s-t)A} e_k|_H^2 \right)^{1/2} ds \Bigg)^2 \\
&\leq M_C^2 \left(\sum_{k\geq 1} \lambda_k^{-2\rho} \right) \int_t^T (s-t)^{-2\rho} \, ds \int_t^T |\hat{Q}^n(s) - \hat{Q}^m(s)|_{\mathcal{K}}^2 \, ds \\
&\leq C\delta^{1-2\rho} \int_{T-\delta}^T |\hat{Q}^n(s) - \hat{Q}^m(s)|_{\mathcal{K}}^2 \, ds.
\end{aligned}$$

Similarily,

$$\left[\int_t^T \left(\sum_{k\geq 1}\lambda_k^{-2\rho}|e^{(s-t)A}C^*(s)(\hat{Q}^n(s)-\hat{Q}^m(s))e^{(s-t)A}e_k|_H^2\right)^{1/2} ds\right]^2$$
$$\leq \left(\int_t^T |e^{(s-t)A}|_{L(H)}|C^*(s)|^2_{L(H)}\right.$$
$$\left.\times\left(\sum_{k\geq 1}\lambda_k^{-2\rho}e^{-2(s-t)\lambda_k}|(\hat{Q}^n(s)-\hat{Q}^m(s))e_k|_H^2\right)^{1/2} ds\right)^2$$
$$\leq M_C^2\delta\int_t^T |\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\, ds$$
$$\leq C\delta^{1-2\rho}\int_{T-\delta}^T |\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\, ds.$$

Hence,

$$\begin{aligned}(3.37)\qquad &\mathbb{E}\sup_{t\in[T-\delta,T]}|\hat{P}^n(t)-\hat{P}^m(t)|^2_{\mathcal{K}}\\ &\leq C\left[\delta^{1-2\rho}\int_{T-\delta}^T \mathbb{E}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\, ds+\mathbb{E}|(J_nMJ_n-J_mMJ_m)|^2_{\mathcal{K}}\right.\\ &\quad+\delta\,\mathbb{E}\int_{T-\delta}^T |C^*(s)(J_nP(s)J_n-J_mP(s)J_m)C(s)|^2_{\mathcal{K}}\, ds\\ &\quad\left.+\,\mathbb{E}\int_{T-\delta}^T |J_nS(s)J_n-J_mS(s)J_m)|^2_{\mathcal{K}}\, ds\right]\\ &\leq C\delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\, ds+R(m,n)\end{aligned}$$

with $R(m,n)\to 0$ as $m,n\to+\infty$.

The duality relation between $\hat{P}^n-\hat{P}^m$ and $\hat{X}^n-\hat{X}^m$ yields to

(3.38)

$$\begin{aligned}&\mathbb{E}\langle \hat{X}^n(T)-\hat{X}^m(T),\hat{P}^n(T)-\hat{P}^m(T)\rangle_{L_2(H)}\\ &=\mathbb{E}\int_{T-\delta}^T\langle \hat{L}^n(s)-\hat{L}^m(s),\hat{Q}^n(s)-\hat{Q}^m(s)\rangle_{L_2(H)}\, ds\\ &\quad-\mathbb{E}\int_{T-\delta}^T\langle \hat{X}^n(s)-\hat{X}^m(s),J_nS(s)J_n-J_mS(s)J_m\rangle_{L_2(H)}\, ds\\ &\quad-\mathbb{E}\int_{T-\delta}^T\langle \hat{X}^n(s)-\hat{X}^m(s),C^*(s)J_nP(s)J_nC(s)-C^*(s)J_mP(s)J_mC(s)\rangle_{L_2(H)}\, ds\\ &\quad-\mathbb{E}\int_{T-\delta}^T\langle X^n(s)-X^m(s),C^*(s)(\hat{Q}^n(s)-\hat{Q}^m(s))\\ &\qquad\qquad+(\hat{Q}^n(s)-\hat{Q}^m(s))C(s)\rangle_{L_2(H)}\, ds,\end{aligned}$$

where

$$\mathbb{E}\int_{T-\delta}^{T}\langle \hat{L}^n(s)-\hat{L}^m(s),\hat{Q}^n(s)-\hat{Q}^m(s)\rangle_{L_2(H)}\,ds=\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds.$$

As in (3.20) we have

$$\mathbb{E}\sum_{k\geq 1}|(\hat{X}^n(t)-\hat{X}^m(t))e_k|_H^2\lambda_k^{2\rho}\leq \mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds, \tag{3.39}$$

where $\hat{X}^n$ and $\hat{X}^m$ are defined as in (3.17) with Q_n replaced by $\hat{Q}^n$ and we get, noticing that $|\langle X,Z\rangle_{L_2(H)}|\leq (\sum_{k=1}^{\infty}|Xe_k|^2\lambda_k^{2\rho})^{1/2}|Z|_{\mathcal{K}}$,

(3.40)

$$\begin{aligned}
&\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds\\
&\quad\leq \mathbb{E}\sup_{t\in[T-\delta,T]}|\langle \hat{X}^n(T)-\hat{X}^m(T),\hat{P}^n(T)-\hat{P}^m(T)\rangle_{L_2(H)}|\\
&\qquad+\mathbb{E}\int_{T-\delta}^{T}|\langle J_nS(s)J_n-J_mS(s)J_m,\hat{X}^n(s)-\hat{X}^m(s)\rangle_{L_2(H)}|\,ds\\
&\qquad+\mathbb{E}\int_{T-\delta}^{T}|\langle C^*(s)[J_nP(s)J_n-J_mP(s)J_m]C(s),\hat{X}^n(s)-\hat{X}^m(s)\rangle_{L_2(H)}|\,ds\\
&\qquad+\mathbb{E}\int_{T-\delta}^{T}|\langle X^n(s)-X^m(s),C^*(s)(\hat{Q}^n(s)-\hat{Q}^m(s))\rangle_{L_2(H)}|\,ds\\
&\qquad+\mathbb{E}\int_{T-\delta}^{T}|\langle X^n(s)-X^m(s),(\hat{Q}^n(s)-\hat{Q}^m(s))C(s)\rangle_{L_2(H)}|\,ds\Big)\\
&\quad=I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

We have

$$\begin{aligned}
I_1&\leq \mathbb{E}\left(\sum_{k=1}^{\infty}\lambda_k^{2\rho}|(\hat{X}^n(T)-\hat{X}^m(T))e_k|^2\right)^{1/2}|\hat{P}^n(T)-\hat{P}^m(T)|_{\mathcal{K}}\\
&\leq \frac{l}{2}\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds+\frac{1}{2l}|\hat{P}^n(T)-\hat{P}^m(T)|^2_{\mathcal{K}},\\
I_2+I_3&\leq \frac{l}{2}\mathbb{E}\int_{T-\delta}^{T}\sum_{k=1}^{\infty}\lambda_k^{2\rho}|(\hat{X}^n(s)-\hat{X}^m(s))e_k|^2\,ds\\
&\quad+\frac{1}{2l}\mathbb{E}\int_{T-\delta}^{T}|\langle J_nS(s)J_n-J_mS(s)J_m|^2_{\mathcal{K}}\,ds\\
&\quad+\frac{1}{2l}\mathbb{E}\int_{T-\delta}^{T}|C^*(s)[J_nP(s)J_n-J_mP(s)J_m]C(s)|^2_{\mathcal{K}}\,ds,\\
I_4&\leq \frac{1}{2l}\mathbb{E}\int_{T-\delta}^{T}\sum_{k=1}^{\infty}\lambda_k^{2\rho}|(\hat{X}^n(s)-\hat{X}^m(s))e_k|^2\,ds+\frac{l}{2}\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds\\
&\leq \frac{\delta}{2l}\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds+\frac{l}{2}\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds.
\end{aligned}$$

I_5 can be treated as I_4, following (3.23). Summarizing and choosing l small enough (depending only on the constants introduced in Hypothesis 2.1), we finally get

$$\begin{aligned}(3.41)\qquad & \mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds \\ &\le C\Bigg(\mathbb{E}|\hat{P}^n(T)-\hat{P}^m(T)|^2_{\mathcal{K}}+\mathbb{E}\int_{T-\delta}^{T}|J_nS(s)J_n-J_mS(s)J_m|^2_{\mathcal{K}}\,ds \\ &\quad+\mathbb{E}\int_{T-\delta}^{T}|C^*(s)J_nP(s)J_nC(s)-C^*(s)J_mP(s)J_mC(s)|^2_{\mathcal{K}}\,ds \\ &\quad+\delta\,\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds\Bigg).\end{aligned}$$

Putting together (3.37) and (3.41) we then prove that, for a small enough δ,

$$\lim_{m,n\to\infty}\mathbb{E}\sup_{t\in[T-\delta,T]}|\hat{P}^n(t)-\hat{P}^m(t)|^2_{\mathcal{K}}=0,$$
$$\lim_{m,n\to\infty}\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}^m(s)|^2_{\mathcal{K}}\,ds=0.$$

Therefore there exist the limit $\hat{P}\in L^2_{\mathcal{P}}(\Omega;C([T-\delta,T];\mathcal{K}))$ and $\hat{Q}\in L^2_{\mathcal{P}}(\Omega\times[T-\delta,T];\mathcal{K}))$ such that

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[T-\delta,T]}|\hat{P}^n(t)-\hat{P}(t)|^2_{\mathcal{K}}=0, \tag{3.42}$$

$$\lim_{m,\to\infty}\mathbb{E}\int_{T-\delta}^{T}|\hat{Q}^n(s)-\hat{Q}(s)|^2_{\mathcal{K}}\,ds=0. \tag{3.43}$$

Step 3: *Construction of* Γ. The equation being linear, thanks to (3.42) and (3.43), we obtain the following relation:

(3.44)

$$\begin{aligned}\hat{P}(t)&=e^{(T-t)A}Me^{(T-t)A}+\int_t^T e^{(s-t)A}C^*(s)P(s)C(s)e^{(s-t)A}\,ds \\ &\quad+\int_t^T e^{(s-t)A}S(s)e^{(s-t)A}\,ds+\int_t^T e^{(s-t)A}(C^*(s)\hat{Q}(s)+\hat{Q}(s)C(s))e^{(s-t)A}\,ds \\ &\quad+\int_t^T e^{(s-t)A}\hat{Q}(s)e^{(s-t)A}\,dW(s)\qquad \mathbb{P}-\text{a.s.}\end{aligned}$$

The fact that $\hat{P}\in L^2_{\mathcal{P},S}(\Omega;C([T-\delta,T];L(H)))$ follows from Remark 3.3. So far we have that the map Γ such that $\Gamma(P)=\hat{P}$ is actually defined from the space $L^2_{\mathcal{P},S}(\Omega;C([T-\delta,T];L(H))$ into itself.

Step 4: Γ *is a contraction for a suitable* δ. Let P^1 and P^2 be two elements of $L^2_{\mathcal{P},S}(\Omega;C([T-\delta,T];L(H))$, then we can evaluate the difference between $\Gamma(P^1)$ and

$\Gamma(P^2)$. Indeed we have

(3.45)
$$(\hat{P}^1 - \hat{P}^2)(t) = \int_t^T e^{(s-t)A} C^*(s)(P^1 - P^2)(s)C(s)e^{(s-t)A}\,ds$$
$$+ \int_t^T e^{(s-t)A}[C^*(s)(\hat{Q}^1 - \hat{Q}^2)(s) + (\hat{Q}^1 - \hat{Q}^2)(s)C(s)]e^{(s-t)A}\,ds$$
$$+ \int_t^T e^{(s-t)A}(\hat{Q}^1 - \hat{Q}^2)(s)e^{(s-t)A}\,dW(s) \qquad \mathbb{P} - \text{a.s.}$$

Clearly (3.10) and (3.27) hold also in this case:

(3.46)
$$\mathbb{E} \sup_{u\in[T-\delta,T]} |(\bar{P}^1 - \bar{P}^2)(u)|^2_{L(H)}$$
$$\leq C\left(\delta\,\mathbb{E} \sup_{u\in[T-\delta,T]} |(P^1 - P^2)(u)|^2_{L(H)} + \delta^{1-2\rho}(\mathbb{E}\int_{T-\delta}^T |(\hat{Q}^1 - \hat{Q}^2)(u)|^2_{\mathcal{K}}\,du\right)$$

with the constant C depending on the constants M_C and T but not on δ. And the same holds for $\hat{Q}^1 - \hat{Q}^2$:

(3.47)
$$\mathbb{E}\int_{T-\delta}^T |(\hat{Q}^1 - \hat{Q}^2)(s)|^2_{\mathcal{K}}\,ds$$
$$\leq C\left(\delta|P^1 - P^2|^2_{L^2_{\mathcal{P}}(\Omega;C([T-\delta,T];L(H)))} + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |(\hat{Q}^1 - \hat{Q}^2)(s)|^2_{\mathcal{K}}\,ds\right).$$

So we can find a δ small enough such that Γ is a contraction and there's a fixed point P . The couple $(P, \hat{Q})$, where $\hat{Q}$ is defined in (3.44), is the mild solution in $[T-\delta, T]$.

Step 5: *Construction of the mild solution.* Since the problem is linear and the value of δ depends only on the constants introduced in Hypothesis 2.1, can restart on $[T-2\delta, T-\delta]$ with final datum $P(T-\delta)$. Proceeding backwards we are able to cover the whole interval $[0, T]$.

Step 6: *Uniqueness.* From Proposition 3.2 we have that there is local uniqueness for the mild solution. δ_0 being independent of the data, we can deduce global uniqueness. □

We end the section by proving the following stability results for the approximants processes $\hat{P}^n$.

PROPOSITION 3.5. *Under the hypotheses of the previous theorem, let $\hat{P}^n$ be defined by* (3.2) *and P the mild solution just obtained, then the following holds: there exists a $\delta > 0$ such that, for every $\varepsilon < \delta$,*

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[T-\delta,T-\varepsilon]} |P(t) - \hat{P}^n(t)|^2_{L(H)} = 0. \tag{3.48}$$

Proof. For every $t \in [0,T]$ we have

(3.49)

$$
\begin{aligned}
P(t) - \hat{P}^n(t) &= \mathbb{E}^{\mathcal{F}_t}\Big\{ e^{(T-t)A}(M - J_n M J_n) e^{(T-t)A} \\
&\quad + \int_t^T e^{(s-t)A}(S(s) - J_n S(s) J_n) e^{(s-t)A}\, ds \\
&\quad + \int_t^T e^{(s-t)A}[C^*(s)(P(s) - J_n P(s) J_n) C(s) \\
&\qquad + C^*(s)(Q(s) - \hat{Q}^n(s)) + (Q(s) - \hat{Q}^n(s))C(s)] e^{(s-t)A}\, ds \Big\};
\end{aligned}
$$

thus, assume that $\delta < 1$,

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[T-\delta,T-\varepsilon]} \left| \mathbb{E}^{\mathcal{F}_t} e^{(T-t)A}(M - J_n M J_n) e^{(T-t)A} \right|^2_{L(H)} \\
&\quad = \mathbb{E} \sup_{t\in[T-\delta,T-\varepsilon]} \left| \mathbb{E}^{\mathcal{F}_t} e^{(T-\varepsilon-t)A} e^{\varepsilon A}(M - J_n M J_n) e^{\varepsilon A} e^{(T-\varepsilon-t)A} \right|^2_{L(H)} \\
&\quad \le 4\varepsilon^{-2\rho}\, \mathbb{E}|M - J_n M J_n|^2_{\mathcal{K}}, \\
&\mathbb{E} \sup_{t\in[T-\delta,T-\varepsilon]} \left| \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A} C^*(s)(P(s) - J_n P(s) J_n) C(s) e^{(s-t)A}\, ds \right|^2_{L(H)} \\
&\quad \le 4\delta^{1-2\rho} \mathbb{E} \int_{T-\delta}^T |C^*(s)(P(s) - J_n P(s) J_n) C(s)|^2_{\mathcal{K}}\, ds, \\
&\mathbb{E} \sup_{t\in[T-\delta,T-\varepsilon]} \Bigg| \mathbb{E}^{\mathcal{F}_t} \int_t^T e^{(s-t)A}[C^*(s)(Q(s) - \hat{Q}^n(s)) \\
&\qquad\qquad + (Q(s) - \hat{Q}^n(s))C(s)] e^{(s-t)A}\, ds \Bigg|^2_{L(H)} \\
&\quad \le 2\, \mathbb{E} \sup_{t\in[T-\delta,T]} \left(\mathbb{E}^{\mathcal{F}_t} \int_t^T \frac{M_C}{(s-t)^\rho} |(Q(s) - \bar{Q}^n(s)|_{\mathcal{K}}\, ds \right)^2 \\
&\quad \le 8M_C^2\, \delta^{1-2\rho} \mathbb{E} \int_{T-\delta}^T |(Q(s) - \bar{Q}^n(s)|^2_{\mathcal{K}}\, ds, \\
&\mathbb{E} \sup_{t\in[T-\delta,T]} \left| \int_t^T e^{(s-t)A}(J_n S(s) J_n - S(s)) e^{(s-t)A}\, ds \right|^2_{L(H)} \\
&\quad \le \delta^{1-2\rho} \mathbb{E} \int_r^T |S(s) - J_n S(s) J_n|^2_{\mathcal{K}}\, ds.
\end{aligned}
$$

Summing up all these estimates we deduce that there exists a constant C depending only on M_C, ρ such that

(3.50)

$$\begin{aligned}&\mathbb{E}\sup_{t\in[T-\delta,T-\varepsilon]}|P(t)-\hat{P}^n(t)|^2_{L(H)}\\&\quad\leq C\Bigg(\varepsilon^{-2\rho}\mathbb{E}|M-J_nMJ_n|^2_{\mathcal{K}}+\delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T|P(s)-J_nP(s)J_n|^2_{\mathcal{K}}\,ds\\&\qquad\delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T|Q(s)-\hat{Q}^n(s)|^2_{\mathcal{K}}\,ds+\delta^{1-2\rho}\mathbb{E}\int_r^T|S(s)-J_nS(s)J_n|^2_{\mathcal{K}}\,ds\Bigg).\end{aligned}$$

Thanks to previous considerations, in particular (3.43), and recalling that by the dominated convergence theorem $\mathbb{E}\int_{T-\delta}^T|P(s)-J_nP(s)J_n|^2_{\mathcal{K}}\to 0$, we deduce the thesis. □

4. Backward stochastic Riccati equations and LQ optimal control. Besides Hypothesis 2.1, let us fix $T>S>0$ and consider the following infinite dimensional stochastic control problem, with the *state equation* given by

$$\begin{cases} dy(t)=(Ay(t)+B(t)u(t))\,dt+C(t)y(t)\,dW(t), & S\leq r\leq t\leq T,\\ y(r)=x,\end{cases}\tag{4.1}$$

where u is the *control* and takes values in another Hilbert space U.

Besides Hypothesis 2.1 we assume the following.

(A4) We assume that $B\in L^\infty_{\mathcal{P},S}(\Omega\times[0,T];L(U;H))$. We denote by M_B a positive constant such that

$$|B(t,\omega)|_{L(U;H)}<M_B\quad \mathbb{P}-\text{a.s. and for a.e. } t\in(0,T).$$

We recall the definition of a *mild solution.*

Definition 4.1. *Given $x\in H$ and $u\in L^2_{\mathcal{P}}(\Omega\times[t,T];U)$, a mild solution of* (4.1) *is a process $y\in L^2_{\mathcal{P}}(\Omega\times[t,T];H)$ such that, almost everywhere in $\Omega\times[t,T]$,*

$$y(s)=e^{(s-t)A}x+\int_t^s e^{(s-\sigma)A}B(\sigma)u(\sigma)\,d\sigma+\int_t^s e^{(s-\sigma)A}C(\sigma)y(\sigma)\,dW(\sigma).$$

The following existence and uniqueness results hold.

Theorem 4.2. *Assume Hypothesis* 2.1. *Given any $x\in H$ and $u\in L^2_{\mathcal{P}}(\Omega\times[t,T];U)$ problem* (4.1) *has a unique mild solution $y\in C_{\mathcal{P}}([t,T];L^2(\Omega;H))$. Moreover,*

$$\sup_{s\in[t,T]}\mathbb{E}|y(s)|^2\leq C_2\left[|x|^2+\mathbb{E}\int_t^T|u(s)|^2\,ds\right]\tag{4.2}$$

for a suitable constant C_2 depending only on T, M_B, M_C (notice that $C_2\geq 1$).

Finally if $p>2$ and

$$\mathbb{E}\left(\int_t^T|u(s)|^2\,ds\right)^{\frac{p}{2}}<\infty$$

then we have that $y\in L^p_{\mathcal{P}}(\Omega;C([t,T];H))$ and

$$\mathbb{E}\sup_{s\in[t,T]}|y(s)|^p\leq C_p\left[|x|^p+\mathbb{E}\left(\int_t^T|u(s)|^2\,ds\right)^{\frac{p}{2}}\right]\tag{4.3}$$

for some positive constant C_p depending on p, T, M_B, M_C. The *cost functional* to minimize over all processes taking values in $L^2_{\mathcal{P}}(\Omega \times [0,T], U)$—the space of *admissible controls*—is

$$\mathbb{E}\int_0^T \left(|\sqrt{S}(s)y(s)|_H^2 + |u(s)|_H^2\right) ds + \mathbb{E}\langle My(T), y(T)\rangle_H. \tag{4.4}$$

Associated with this *linear and quadratic* control problem we have the following BSRE (see [3, 15] and [8] for the present infinite dimensional version):

$$\begin{cases} -dP(t) = (AP(t) + P(t)A + C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t))\,dt \\ \qquad\qquad - (P(t)B(t)B^*(t)P(t) - S(t))\,dt + Q(t)\,dW(t), \qquad t \in [0,T], \\ P(T) = M. \end{cases} \tag{4.5}$$

In this section we will prove that such an equation has a unique mild solution, in the sense of Definition 3.1, improving the result obtained in [8]. To be more specific we have the following definition.

Definition 4.3. *A* mild solution *of problem* (4.5) *is a couple of processes*

$$(P,Q) \in L^2_{\mathcal{P},S}(\Omega, C([0,T];\Sigma(H))) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; \mathcal{K}_s)$$

that solves the following equation, for all $t \in [0,T]$:

$$\begin{aligned} P(t) &= e^{(T-t)A}Me^{(T-t)A} + \int_t^T e^{(s-t)A}S(s)e^{(s-t)A}\,ds \\ &\quad + \int_t^T e^{(s-t)A}\Big[C^*(s)P(s)C(s) - P)(s)B(s)B^*(S)P(s) \\ &\qquad\qquad + C^*(s)Q(s) + Q(s)C(s)\Big]e^{(s-t)A}\,ds \\ &\quad + \int_t^T e^{(s-t)A}Q(s)e^{(s-t)A}\,dW(s) \qquad \mathbb{P}-a.s. \end{aligned} \tag{4.6}$$

We have indeed the following theorem.

Theorem 4.4. *Assume that Hypotheses* 2.1 *hold true and that B verifies assumption* (A4). *Then there exists a unique mild solution (P,Q) of* (4.5) *in $[0,T]$. Moreover $P \in L^\infty_{\mathcal{P},S}(\Omega \times (0,T); \Sigma^+(H))$. Moreover, fix $T > 0$ and $x \in H$, then*

1. *there exists a unique control $\overline{u} \in L^2_{\mathcal{P}}(\Omega \times [0,T]; U)$ such that*
$$J(0,x,\overline{u}) = \inf_{u \in L^2_{\mathcal{P}}(\Omega\times[0,T];U)} J(0,x,u);$$
2. *if $\overline{y}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is, the optimal state) then $\overline{y}$ is the unique mild solution to the* closed loop *equation*
$$\begin{cases} d\overline{y}(r) = [A\overline{y}(r) - B(r)B^*(r)P(r)\overline{y}(r)]\,dr + C\overline{y}(r)\,dW(r), \\ \overline{y}(0) = x; \end{cases} \tag{4.7}$$
3. *the following feedback law holds $\mathbb{P}$-a.s. for almost every s:*
$$\overline{u}(s) = -B^*(s)P(s)\overline{y}(s); \tag{4.8}$$
4. *the optimal cost is given by $J(0,x,\overline{u}) = \langle P(0)x, x\rangle_H$.*

Before going into the details of the proof, we establish the following a priori estimate.

PROPOSITION 4.5. *Let* $(\bar{P}, \bar{Q})$ *be a mild solution of* (4.5) *in* $[\tau, T] \subset [0, T]$ *such that* $\bar{P} \in L^{\infty}_{\mathcal{P},S}(\Omega \times [\tau, T], \Sigma(H))$*, then the following holds for every* $t \in [\tau, T]$*:*

(i) *for all* $t \in [\tau, T]$*,* $\bar{P}(t) \in \Sigma^+(H)$ $\quad \mathbb{P}-a.s.$*;*

(ii) *for all* $t \in [\tau, T]$*,*

(4.9)
$$|\bar{P}(t)|_{L(H)} \leq C_2(|M|_{L^{\infty}_{\mathcal{P},S}(\Omega, \mathcal{F}_T; L(H))} + (T-\tau)|S|_{L^{\infty}_{\mathcal{P},S}(\Omega \times [\tau, T], L(H))}) \qquad \mathbb{P}-a.s.,$$

where C_2 *is given in* (4.2)*.*

Proof. Step 1: *Fundamental relation for the Lyapunov equation.* Let (P, Q) be the unique mild solution to the Lyapunov equation (3.2) and let $y^{t,x}$ be the mild solution to (4.1); we claim that for all $t \in [0, T]$, $x \in H$, it holds,

$$\begin{aligned} \langle P(t)x, x\rangle_H &= \mathbb{E}^{\mathcal{F}_t}\langle My^{t,x}(T), y^{t,x}(T)\rangle + \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle S(s)y^{t,x}(s), y^{t,x}(s)\rangle_H \, ds \\ &\quad - 2\,\mathbb{E}^{\mathcal{F}_t}\int_t^T \langle P(s)B^*(s)y^{t,x}(s), u(s)\rangle \, ds \qquad \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.10}$$

Let us prove the claim. We will use again the approximants processes $(\hat{P}^n, \hat{Q}^n)$ introduced in the proof of Theorem 3.4. From proposition 3.5 we know that there's a δ small enough such that, for every $\varepsilon < \delta$,

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[T-\delta, T-\varepsilon]} |P(t) - \hat{P}^n(t)|^2_{L(H)} = 0. \tag{4.11}$$

On the other hand we have already noticed that $(\hat{P}^n, \hat{Q}^n)$ is a solution in the sense of Proposition 2.1 of [9]; therefore, by Theorem 5.6 of [8], we have that, for all $t \in [0, T]$, $x \in H$, it holds, $\mathbb{P}$-a.s. that

(4.12)
$$\begin{aligned} &\langle \hat{P}^n(t)x, x\rangle_H \\ &\quad = \mathbb{E}^{\mathcal{F}_t}\langle \hat{P}^n(T-\varepsilon)y^{t,x}(T-\varepsilon), y^{t,x}(T-\varepsilon)\rangle_H \\ &\qquad + \mathbb{E}^{\mathcal{F}_t}\int_t^{T-\varepsilon} \langle S(s)y^{t,x,u}(s), y^{t,x,u}(s)\rangle_H \, ds \\ &\qquad + \mathbb{E}^{\mathcal{F}_t}\int_t^{T-\varepsilon} \langle [C^*(s)\hat{P}^n(s)C(s) - C^*(s)J_nP(s)J_nC(s)]y^{t,x,u}(s), y^{t,x,u}(s)\rangle_H \, ds \\ &\qquad - 2\,\mathbb{E}^{\mathcal{F}_t}\int_t^{T-\varepsilon} \langle \hat{P}^n(s)B^*(s)y^{t,x,u}(s), u(s)\rangle_H \, ds. \end{aligned}$$

By (4.11) and recalling that $y \in L^p_{\mathcal{P}}(\Omega; C([t, T]; H))$, $p \geq 2$ (see (4.3)), we get that

$$\begin{aligned} &\int_t^{T-\varepsilon} \langle \hat{P}^n(s)C(s)y^{t,x,u}(s), C(s)y^{t,x,u}(s)\rangle ds \\ &\qquad\qquad \to \int_t^{T-\varepsilon} \langle P(s)C(s)y^{t,x,u}(s), C(s)y^{t,x,u}(s)\rangle ds \end{aligned}$$

in L^1 norm. Moreover, since $\mathbb{E}\sup_{t\in[0,T]}|P(t)|^2_{L(H)}<+\infty$, by the dominated convergence theorem we obtain that

$$\int_t^{T-\varepsilon}\langle P(s)J_nC(s)y^{t,x,u}(s),J_nC(s)y^{t,x,u}(s)\rangle ds \to \int_t^{T-\varepsilon}\langle P(s)C(s)y^{t,x,u}(s),C(s)y^{t,x,u}(s)\rangle ds$$

again in L^1 norm.

Thus letting n tend to ∞ in (4.12), we obtain that for every $t\in[T-\delta,T]$ $\mathbb{P}$-a.s.,

$$\begin{aligned}\langle P(t)x,x\rangle_H &= \mathbb{E}^{\mathcal{F}_t}\langle P(T-\varepsilon)y^{t,x}(T-\varepsilon),y^{t,x}(T-\varepsilon)\rangle_H\\ &\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^{T-\varepsilon}\langle S(s)y^{t,x,u}(s),y^{t,x,u}(s)\rangle\,ds\\ &\quad-2\,\mathbb{E}^{\mathcal{F}_t}\int_t^{T-\varepsilon}\langle P(s)B^*(s)y^{t,x,u}(s),u(s)\rangle\,ds.\end{aligned}\tag{4.13}$$

Now, thanks again to $\mathbb{E}\sup_{t\in[0,T]}|P(t)|^2_{L(H)}<+\infty$, we can let ε go to 0 and get that for every $x\in H$, and every $t\in[T-\delta,T]$ $\mathbb{P}$-a.s.,

$$\begin{aligned}\langle P(t)x,x\rangle_H &= \mathbb{E}^{\mathcal{F}_t}\langle My^{t,x}(T),y^{t,x}(T)\rangle_H+\mathbb{E}^{\mathcal{F}_t}\int_t^{T}\langle S(s)y^{t,x,u}(s),y^{t,x,u}(s)\rangle\,ds\\ &\quad-2\,\mathbb{E}^{\mathcal{F}_t}\int_t^{T}\langle P(s)B^*(s)y^{t,x,u}(s),u(s)\rangle\,ds.\end{aligned}\tag{4.14}$$

Choosing $u=0$ then (see, also, Theorem 5.6 of [8]) we get that

$$\begin{aligned}&\sup_{x\in H,\,|x|_H=1}|\langle P(t)x,x\rangle_H|\\ &\qquad\le C_2(|M|_{L^\infty_{\mathcal{S}}(\Omega,\mathcal{F}_T,P)}+T|S|_{L^\infty_{\mathcal{P},S}(\Omega\times(0,T);L(H))})\quad\forall t\in[T-\delta,T].\end{aligned}\tag{4.15}$$

We can prove relation (4.13) on the interval $[T-2\delta,T-\delta]$ (notice that δ does not depend on M) and so on to cover the whole interval $[0,T]$, because $P(T-k\delta)\in L(H)$ for every $k=0,1,2,3,\dots$ and thus we can extend (4.15) to the whole $[0,T]$.

Step 2: *Upper bound.* Let $(\bar P,\bar Q)$ be the mild solution of the BSRE (4.5) in $[\tau,T]$; we can see a couple of such processes as the mild solution to the following Lyapunov equation, for $t\in[\tau,T]$:

$$\left\{\begin{aligned}-d\bar P(t)&=(A\bar P(t)+\bar P(t)A+C^*(t)\bar P(t)C(t)+C^*(t)\bar Q(t)+\bar Q(t)C(t)+\bar S(t))\,dt\\ &\qquad+\bar Q(t)\,dW(t),\\ \bar P(T)&=M\end{aligned}\right.\tag{4.16}$$

with $\bar S=-B^*\bar P\bar PB+S$; thus from (4.14) and completing the square, we obtain

$$\begin{aligned}\langle\bar P(t)x,x\rangle_H &= \mathbb{E}^{\mathcal{F}_t}\langle My^{t,x}(T),y^{t,x}(T)\rangle_H+\mathbb{E}^{\mathcal{F}_t}\int_t^T|u(s)|^2\,ds\\ &\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^{T}\langle S(s)y^{t,x,u}(s),y^{t,x,u}(s)\rangle\,ds\\ &\quad-\mathbb{E}^{\mathcal{F}_t}\int_t^{T}|\bar P(s)B^*(s)y^{t,x,u}(s)+u(s)|^2\,ds.\end{aligned}\tag{4.17}$$

So, choosing the admissible control $u = 0$, we get

(4.18)
$$\langle \bar{P}(t)x, x\rangle_H = \mathbb{E}^{\mathcal{F}_t}\langle My^{t,x,0}(T), y^{t,x,0}(T)\rangle_H + \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle S(s)y^{t,x,0}(s), y^{t,x,0}(s)\rangle\, ds$$
$$- \mathbb{E}^{\mathcal{F}_t}\int_t^T |\bar{P}(s)B^*(s)y^{t,x,0}(s)|^2\, ds,$$

from which we deduce the following upper bound

$$\langle \bar{P}(t)x, x\rangle_H \le C_2(|M|_{L^\infty_S(\Omega,\mathcal{F}_T,P)} + T|S|_{L^\infty_{\mathcal{P},S}(\Omega\times(0,T);L(H))}) \quad \forall t \in [\tau, T]. \tag{4.19}$$

Step 3: *Lower bound.* Let us consider the following equations for initial time $t \in [\tau, T]$ and initial state x:

$$\begin{cases} d\overline{y}(s) = [A\overline{y}(s) - B(s)B^*(s)\bar{P}(s)\overline{y}(s)]\, ds + C\overline{y}(s)\, dW(s) & s \in [t, T], \\ \overline{y}(t) = x. \end{cases} \tag{4.20}$$

Notice that, thanks to the regularity of $\bar{P}$, Theorem 3.2 of [8] applies and in particular the following estimates holds true for the solution $\bar{y}^{t,x}$, for every $t \in [\tau, T]$:

$$\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]} |\bar{y}(s)|^p \le C_p|x|^p \qquad \forall p \ge 2, \tag{4.21}$$

where C_p depends also on the L^∞ norm of $\bar{P}$. Therefore $\bar{u}(s) = B^*(s)\bar{P}(s)\bar{y}^{t,x}(s)$ is an admissible control, i.e., $\bar{u} \in L^2_{\mathcal{P}}(\Omega \times [t, T], U)$, and (4.17) corresponds to

(4.22)
$$\langle \bar{P}(t)x, x\rangle_H = \mathbb{E}^{\mathcal{F}_t}\Bigg[\langle M\bar{y}^{t,x}((T), \bar{y}^{t,x}((T)\rangle_H$$
$$+ \int_t^T (|B^*(s)\bar{P}(s)\overline{y}^{t,x}((s)|^2 + |\sqrt{S(s)}\bar{y}^{t,x}(s)|^2)\, ds\Bigg]\ \mathbb{P} - a.s.$$

Consequently from (4.22) holding for every $t \in [\tau, T]$ we get (i). Eventually (4.19) and (4.22) imply (ii). □

We are now in the position to prove Theorem 4.4.

Proof of Theorem 4.4. *Step* 1: *Local existence and uniqueness.* In order to be able to follow the same argument not only on $[T-\delta, T]$ but also on $[T-2\delta, T-\delta]$ and so on (with the same δ) we prove the existence of a solution (for notational convenience, on $[T-\delta, T]$) with generic final condition $\tilde{M} \in L^\infty_{\mathcal{P},S}(\Omega, \mathcal{F}_T; L(H))$ with

$$|\widetilde{M}|_{L^\infty_{\mathcal{P},S}(\Omega,\mathcal{F}_T;L(H))} < C_2(|M|_{L^\infty_{\mathcal{P},S}(\Omega,\mathcal{F}_T;L(H))} + T|S|_{L^\infty_{\mathcal{P},S}(\Omega\times[0,T],L(H))}).$$

We fix a number r with

$$r > C_2^2|M|_{L^\infty_S(\Omega,\mathcal{F}_T,P)} + 2C_2^2T|S|_{L^\infty_{\mathcal{P},S}(\Omega\times(0,T);L(H))},$$

where C_2 is the constant obtained in Proposition 4.5:

$$\mathcal{B}(r) = \left\{P \in L^2_{\mathcal{P},S}(\Omega; C([T-\delta, T]; L(H))) : \sup_{t\in[T-\delta,T]} |P(t,\omega)|_{L(H)} \le r \quad \mathbb{P}\text{-a.s.}\right\},$$

where $\delta > 0$ will be fixed later on. On $\mathcal{B}(r)$ we construct the map $\Lambda : \mathcal{B}(r) \to \mathcal{B}(r)$, letting $\Lambda(K) = P$, where (P, Q) is the unique mild solution to (3.2) (in $[T-\delta, T]$) with S replaced by $S - KBB^*K$ and M by $\widetilde{M}$ that verifies

$$\begin{aligned} P(t) = e^{(T-t)A}\widetilde{M}e^{(T-t)A} &+ \int_t^T e^{(s-t)A}[C^*(s)P(s)C(s) + C^*(s)Q(s) + Q(s)C(s)]e^{(s-t)A}\,ds \\ &+ \int_t^T e^{(s-t)A}S(s)e^{(s-t)A}\,ds + \int_t^T e^{(s-t)A}K(s)B(s)B^*(s)K(s)e^{(s-t)A}\,ds \\ &+ \int_t^T e^{(s-t)A}Q(s)e^{(s-t)A}\,dW(s). \end{aligned}$$

First of all we check that it maps $\mathcal{B}(r)$ into itself. It is enough to show that for all $t \in [T-\delta, T]$ it holds $|\Lambda(K)(t)|_{L(H)} \le r$ $\mathbb{P}$-a.s. Thanks to (4.9) we have that $\mathbb{P}$-a.s.

$$\begin{aligned} |\Lambda(K)(t)|_{L(H)} &\le C_2\Big[|\widetilde{M}|_{L^\infty_S(\Omega,\mathcal{F}_T;L(H))} + \delta|KBB^*K|_{L^\infty_{\mathcal{P},S}(\Omega\times[T-\delta,T];L(H))} \\ &\qquad + \delta|S|_{L^\infty_{\mathcal{P},S}(\Omega\times[T-\delta,T];L(H))})\,ds\Big] \\ &\le C_2^2|M|_{L^\infty} + C_2r^2\delta M_B^2 + 2C_2^2T|S|_{L^\infty_{\mathcal{P},S}(\Omega\times[0,T];L(H))} < r \end{aligned}$$

as soon as we choose

$$\delta < \frac{r - (C_2^2|M|_{L^\infty} + 2C_2^2T|S|_{L^\infty_{\mathcal{P},S}(\Omega\times[0,T];L(H))})}{C_2^2M_B^2r^2}.$$

Let K_1 and K_2 be in $B(r)$, then by (4.10) evaluated at $u = 0$ we have

$$\begin{aligned} &\langle (P^1(t) - P^2(t))x, x\rangle_H \\ &= \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle K^1(s)B(s)B^*(s)(K^1(s) - K^2(s))y^{t,x,0}(s), y^{t,x,0}(s)\rangle\,ds \\ &\quad - \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle K^2(s)B(s)B^*(s)(K^1(s) - K^2(s))y^{t,x,0}(s), y^{t,x,0}(s)\rangle\,ds, \end{aligned} \tag{4.23}$$

and thus, by the Hölder inequality,

(4.24)

$$\begin{aligned} &|\langle (P^1(t) - P^2(t))x, x\rangle_H| \\ &\le 2\mathbb{E}^{\mathcal{F}_t}\int_t^T rM_B^2|K^1(s) - K^2(s)|_{L(H)}|y^{t,x,0}(s)|^2\,ds \\ &\le 2rM_B^2\int_t^T (\mathbb{E}^{\mathcal{F}_t}|K^1(s) - K^2(s)|^2_{L(H)})^{1/2}(\mathbb{E}^{\mathcal{F}_t}|y^{t,x,0}(s)|^4)^{1/2}\,ds \\ &\le 2rM_B^2\delta^2\left(\sup_{t\in[T-\delta,T]}\mathbb{E}^{\mathcal{F}_t}|K^1(s) - K^2(s)|^2_{L(H)}\right)^{1/2}\left(\sup_{t\in[T-\delta,T]}\mathbb{E}^{\mathcal{F}_t}|y^{t,x,0}(s)|^4\right)^{1/2} \end{aligned}$$

using again Doob inequality and (4.21) which we deduce:

$$\mathbb{E} \sup_{t\in[T-\delta,T]} |P^1(t) - P^2(t)|^2_{L(H)} \leq 16r^2 M_B^4 \delta^4 C_4 \mathbb{E} \sup_{t\in[T-\delta,T]} |K^1(t) - K^2(t)|^2_{L(H)}, \tag{4.25}$$

where $C_4 = C_4(r)$ is given in (4.21). Therefore reducing if necessary the value of δ, we obtain that Λ is a contraction.

Step 2: *Global existence and uniqueness.* We notice that the choice of δ depends only on r and the constants introduced in Hypothesis 2.1. Therefore we can repeat the previous step to cover the whole interval $[0, T]$.

Final step: Synthesis of the optimal control. So far we have proved the existence and uniqueness of the mild solution for the BSRE and, thanks to Proposition 4.5 we also have that the first component of the solution $P \in L^\infty_{\mathcal{P},S}(\Omega \times [0, T]; L(H))$. Consequently the closed loop equation (4.7) is well posed and the associated feedback control is admissible, hence the rest of the claims of the theorem easily follow. ☐

5. The Lyapunov equation of the maximum principle. In this section we extend Proposition 3.2 and Theorem 3.4 in order to cover the Lyapunov equation arising in the maximum principle for a class of SPDE; see [6], [7, eq. (4.22)]. We rewrite such an equation with our notation

$$\begin{cases} -dP(t) = -Q(t)\, dW(t) + [AP(t) + P(t)A + A_\sharp(t)P(t) + P(t)A_\sharp]\, dt \\ \qquad\qquad +[C'(t)P(t)C(t) + C(t)Q(t) + Q(t)C(t) + S(t)]\, dt \\ P(T) = M, \end{cases} \tag{5.1}$$

where $A_\sharp \in L^\infty_{\mathcal{P},S}((0, T) \times \Omega; L(H))$.

The presence of the bounded term $A_\sharp$ is completely irrelevant and we will not consider it in the following.

On the contrary it is not suitable, in this context, to require assumption (A3). Indeed, assumption (A3) has to be replaced by this following weaker one.

Hypothesis 5.1.

(A3′) $S \in L^2_{\mathcal{P},S}((0, T) \times \Omega; \mathcal{K}))$ *and* $M \in L^\infty_S(\Omega, \mathcal{F}_T; L(H))$.

Notice that the assumption on M is unchanged.

Under (A3′) the statement of the a priori estimate in Proposition 3.2 becomes the following.

Proposition 5.2. *Let (P, Q) be a mild solution to* (5.1). *Then there exists a $\delta_0 > 0$ just depending on T and constants M_C, ρ introduced in* (A1)–(A2) *such that for every $0 \leq \delta \leq \delta_0$ the following holds:*

$$|P|^2_{L^2(\Omega;C([T-\delta,T];L(H)))} + \mathbb{E}\int_{T-\delta}^T |Q(s)|^2_{\mathcal{K}}\, ds \leq c\left(\mathbb{E}|M|^2_{L(H)} + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |S(s)|^2_{\mathcal{K}}\, ds\right), \tag{5.2}$$

where c is a positive constant depending on $\delta_0, M_A, M_C, \rho,$ and T.

Proof. Let us reestimate (3.7). We have (by the Cauchy inequality)

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[r,T]} \left| \int_t^T e^{(s-t)A} S(s) e^{(s-t)A}\, ds \right|^2_{L(H)} \\
&\quad\leq \mathbb{E} \sup_{t\in[r,T]} \left(\int_t^T (s-t)^{-2\rho}\, ds \int_t^T |S(s)|^2_{\mathcal{K}}\, ds \right) \\
&\quad\leq (T-r)^{1-2\rho} \int_r^T |S(s)|^2_{\mathcal{K}}\, ds \qquad \forall t\in[T-\delta,T].
\end{aligned} \tag{5.3}
$$

Therefore (3.9) becomes

$$
\begin{aligned}
\mathbb{E} \sup_{u\in[T-\delta,T]} |P(u)|^2_{L(H)} &\leq C\Bigg(|M|^2_{L(H)} + \delta^2 \mathbb{E} \sup_{u\in[T-\delta,T]} |P(u)|^2_{L(H)} \\
&\qquad + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |Q(s)|^2_{\mathcal{K}}\, ds + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |S(s)|^2_{\mathcal{K}}\, ds \Bigg).
\end{aligned} \tag{5.4}
$$

From which we deduce

$$
\begin{aligned}
&\mathbb{E} \sup_{u\in[T-\delta,T]} |P(u)|^2_{L(H)} \\
&\quad\leq C\left(|M|^2_{L(H)} + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |Q(s)|^2_{\mathcal{K}}\, ds + \delta^{1-2\rho}\mathbb{E}\int_{T-\delta}^T |S(s)|^2_{\mathcal{K}}\, ds \right).
\end{aligned} \tag{5.5}
$$

Regarding the duality argument used to estimate $\mathbb{E}\int_{T-\delta}^T |Q(s)|^2_{\mathcal{K}} ds$, the only thing to check is that (3.26) still holds:

$$
\begin{aligned}
&\left| \mathbb{E}\int_{T-\delta}^T \langle X^n(s), J_n S(s) J_n\rangle_{L_2(H)}\, ds \right| \\
&\quad= \left| \mathbb{E}\int_{T-\delta}^T \sum_{k\geq 1} \langle X^n(s)e_k, J_n S(s) J_n e_k\rangle_H\, ds \right| \\
&\quad\leq \left(\int_{T-\delta}^T \mathbb{E}\sum_{k\geq 1} \lambda_k^{2\rho} |X^n(s)e_k|^2\, ds \right)^{1/2} \left(\int_{T-\delta}^T \mathbb{E}\sum_{k\geq 1} \lambda_k^{-2\rho} |J_n S(s) J_n e_k|^2\, ds \right)^{1/2} \\
&\quad\leq \left(\delta \int_{T-\delta}^T |Q^n(s)|^2_{\mathcal{K}}\, ds \right)^{1/2} \left(\int_{T-\delta}^T \mathbb{E}\sum_{k\geq 1} \lambda_k^{-2\rho} |S(s)e_k|^2\, ds \right)^{1/2} \\
&\quad\leq \delta^{1/2} \left(\int_{T-\delta}^T |Q^n(s)|^2_{\mathcal{K}}\, ds \right)^{1/2} \left(\int_{T-\delta}^T |S(s)|^2_{\mathcal{K}}\, ds \right)^{1/2} \\
&\quad\leq 2\delta \mathbb{E}\int_{T-\delta}^T |S(s)|^2_{L(H)} ds + \frac{1}{8}\mathbb{E}\int_{T-\delta}^T |Q^n(s)|^2_{\mathcal{K}}\, ds.
\end{aligned} \tag{5.6}
$$

Thus we again deduce (3.27), that together with (5.5) leads to the proof of (5.2). ∎

We also have the following theorem.

THEOREM 5.3. *Under assumptions* (A1)–(A2)–(A3′), (5.1) *has a unique mild solution* (P,Q).

Proof. The only thing to check is that the following still holds:

$$\lim_{n,m\to+\infty} \mathbb{E}\int_{T-\delta}^{T} |J_n S(s) J_n - J_m S(s) J_m|^2_{\mathcal{K}}\, ds = 0. \tag{5.7}$$

Recalling that $e_k \in V$, for every $k \geq 1$, we have

$$\lim_{n,m\to+\infty} |J_n S(s) J_n e_k - J_m S(s) J_m e_k|^2_{\mathcal{K}}\, ds = 0 \qquad \forall k \geq 1. \tag{5.8}$$

Moreover

$$\mathbb{E}\int_{T-\delta}^{T} |J_n S(s) J_n - J_m S(s) J_m|^2_{\mathcal{K}}\, ds \leq 2\mathbb{E}\int_{T-\delta}^{T} |S(s)|^2_{\mathcal{K}}\, ds. \tag{5.9}$$

Thus by the dominated convergence theorem we get that (5.7). The rest of the proof follows then identically to Theorem 3.4. ☐

Example 5.4. Notice that in the mentioned papers [6] and [7], $H = L^2([0,1])$ and the operator $S(t)$ is the multiplication operator by an adapted stochastic random field $H : (\Omega \times [0,T] \times [0,1]) \to \mathbb{R}$, namely,

$$[S(t)e](\xi) = H(t,x)e(x)\ \forall e \in L^\infty([0,1])\ \forall x \in [0,1] \quad \text{with} \quad \mathbb{E}\int_0^t \int_0^1 H(t,x)^2 dt dx$$

(notice that in thes case $S(t)$ is not even defined on the whole H).

Moreover the infinitesimal generator A is the realization of the Laplacian in $L^2([0,1])$ with Dirichlet boundary conditions.

Thus we have, choosing the basis, $\{e_m\}_{m\in\mathbb{N}}$, of eigenvectors of A:

(a) $\sup_{m\geq 1} |e_m|_{L^\infty([0,1])} < \infty$;

(b) S is self-adjoint and

$$|S(s)e_m|^2 = \int_0^1 (H^2(s,x) e_m^2(x)\, dx \leq \sup_{m\geq 1} |e_m|^2_{L^\infty([0,1])} |H(s,\cdot)|_{L^2([0,1])};$$

(c)

$$\begin{aligned}
|S(s)|_{\mathcal{K}} &= \sum_{k\geq 1}\sum_{m\geq 1} \lambda_m^{-2\rho} |\langle S(s)e_k, e_m\rangle_{L^2([0,1])}|^2 \\
&= \sum_{m\geq 1} \lambda_m^{-2\rho} \sum_{k\geq 1} |\langle e_k, S(s)e_m\rangle_{L^2([0,1])}|^2 \\
&= \sum_{m\geq 1} \lambda_m^{-2\rho} |S(s)e_m|^2_{L^2([0,1])} \\
&\leq |H^2(s,\cdot)|_{L^2([0,1])} \cdot \sum_{m\geq 1} \lambda_m^{-2\rho} \leq \text{cost}|H^2(s,\cdot)|_{L^2([0,1])};
\end{aligned}$$

and assumptions (A1), (A2), (A3′) hold.

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