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Superimposition of the atmosphere density for fast and accurate semi-analytical propagation

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Introduction

- Overall project
  - density based cloud propagation
  - of fragments originating from explosions or collisions
  - in all orbital regions
  - using semi-analytical methods
- Problems
  - most common atmosphere models are either
    - non-smooth or
    - expensive in terms of density evaluations
  - inaccurate results due to approximations especially for eccentric orbits

Overview

Density Models

Models
- Jacchia
- CIRA
- NRLMSIS
- …

Derivations
- Non-smooth exponential
- …
- Smooth exponential

Contraction Methods

Numerical
- Full numerical
- Gaussian-Legendre
- Simpson's
- …

Analytical
- King-Hele
- …
- Superimposed King-Hele

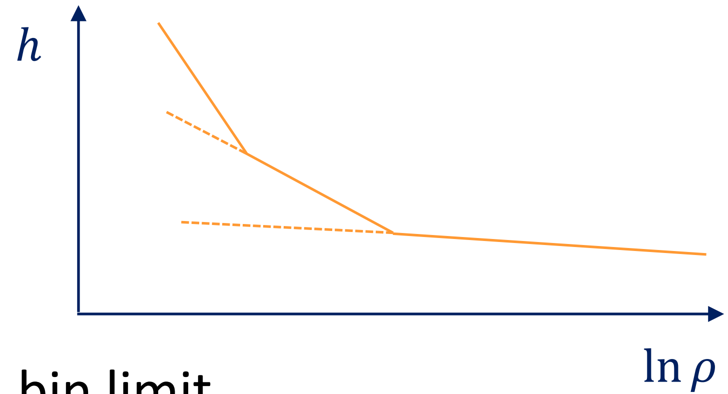
Background

Non-Smooth Atmosphere Model

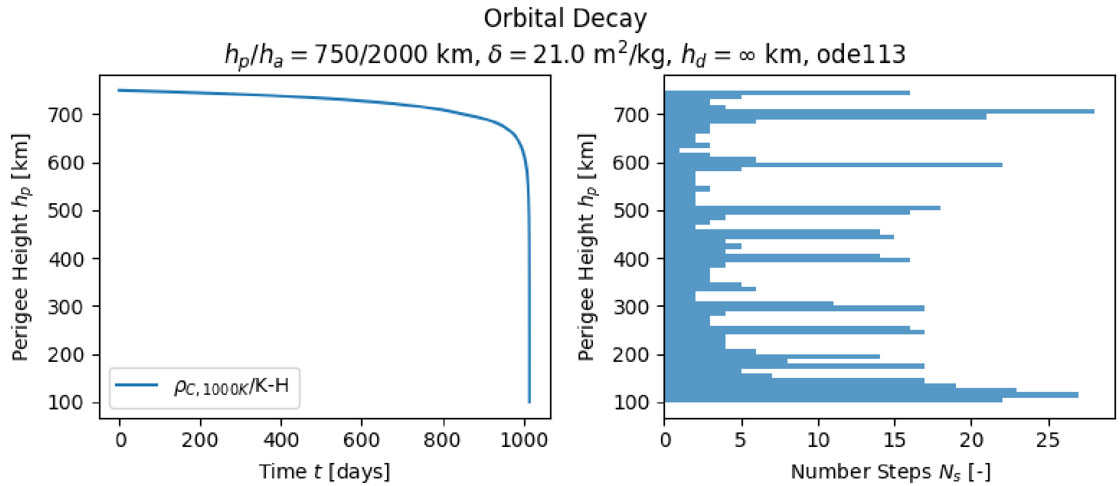
- Definition: for each altitude bin

$$\rho_C(h) = \rho_{0,i}\exp(-\frac{h-h_i}{H_i})$$

$$h_i < h < h_{i+1}$$



- Problems
  - non-smooth transition at each bin limit
  - forces the variable step size integrator to increase number of steps



Background

Dynamical System

- Modified Lagrange's planetary equations in
- Ignoring all perturbations, apart from tangential drag force

$$f_T = \frac{1}{2}\rho v^2 \delta$$

- In this case, the dynamics become

$$\dot{a} = -\frac{a^2 \rho \delta v^3}{\mu}$$

$$\dot{e} = -\frac{a \rho \delta v}{r}(1 - e^2)\cos E$$

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}}$$

- This system propagated numerically is the baseline

where

a	semi-major axis
e	eccentricity
E	eccentric anomaly
μ	gravitational parameter
r	radial distance
v	velocity
δ^{-1}	ballistic coefficient
ρ	density

Background

Dynamical System

- Over full period, the dynamics become (assuming *a* and *e* to be fixed)

$$\Delta a = -a^2 \delta \int_0^{2\pi} \rho \frac{(1 + e\cos E)^{3/2}}{(1 - e\cos E)^{1/2}} dE$$

$$\Delta e = -a \delta \int_0^{2\pi} \rho \frac{(1 + e\cos E)^{1/2}}{(1 - e\cos E)^{1/2}} \cos E \, (1 - e^2) dE$$

- This system needs approximation, if full numerical integration is to be avoided
- Then the following semi-analytical propagation can be performed

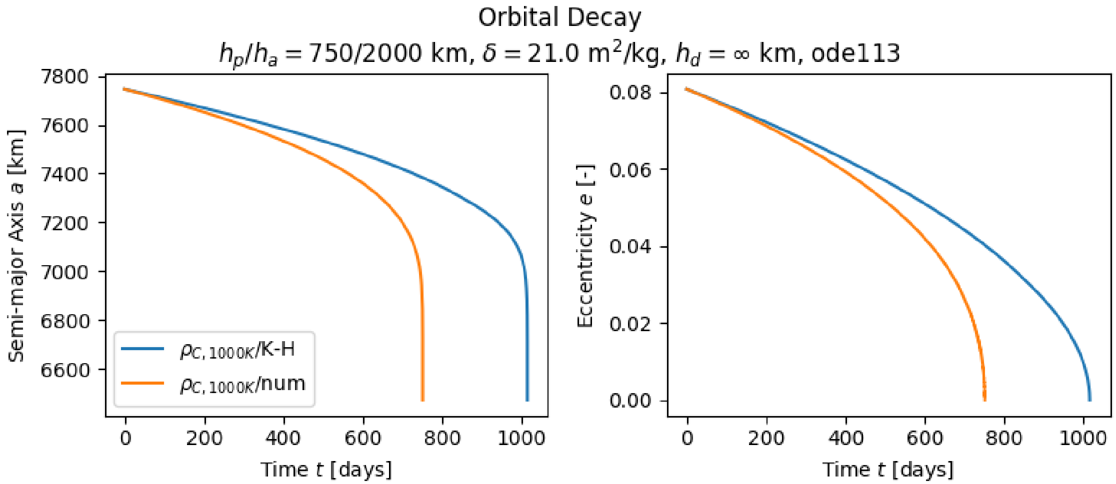
$$\dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{\Delta x}{P} \qquad x \in \{a, e\}$$

where P is the period

Background

Approximation of Integral: King-Hele

- Analytical approximation of the integral, using series expansion, assuming H to be constant above perigee*
- Dis-/Advantages
  - \- constant H assumptions introduces large errors for eccentric orbits
  - \- discontinuity between transition of low/high eccentric approximation
  - \+ retain analytic formulation



*another approach is to linearly fit $H = H_0 + \eta(h - h_0)$, but the slope parameter η depends on the orbit configuration

Background

Approximation of Integral: Quadrature

- Numerical approximation of the integral using quadrature:
  - Evaluate the integrand at nodes n and sum them weighted with w_i
  - Here, Gauss-Legendre quadrature was chosen (with $n = 33$)

$$\int_0^{2\pi} f(E)dE \approx \pi \sum_{i=1}^{n} w_i f(E_i)$$

- Dis-/Advantages
  - \+ independent of atmosphere model
  - \+ no series expansion necessary; valid across all conditions (however care needs to be taken for low perigees)
  - \- not an analytical solution; cannot directly derive Jacobian or find closed form solution for lifetime

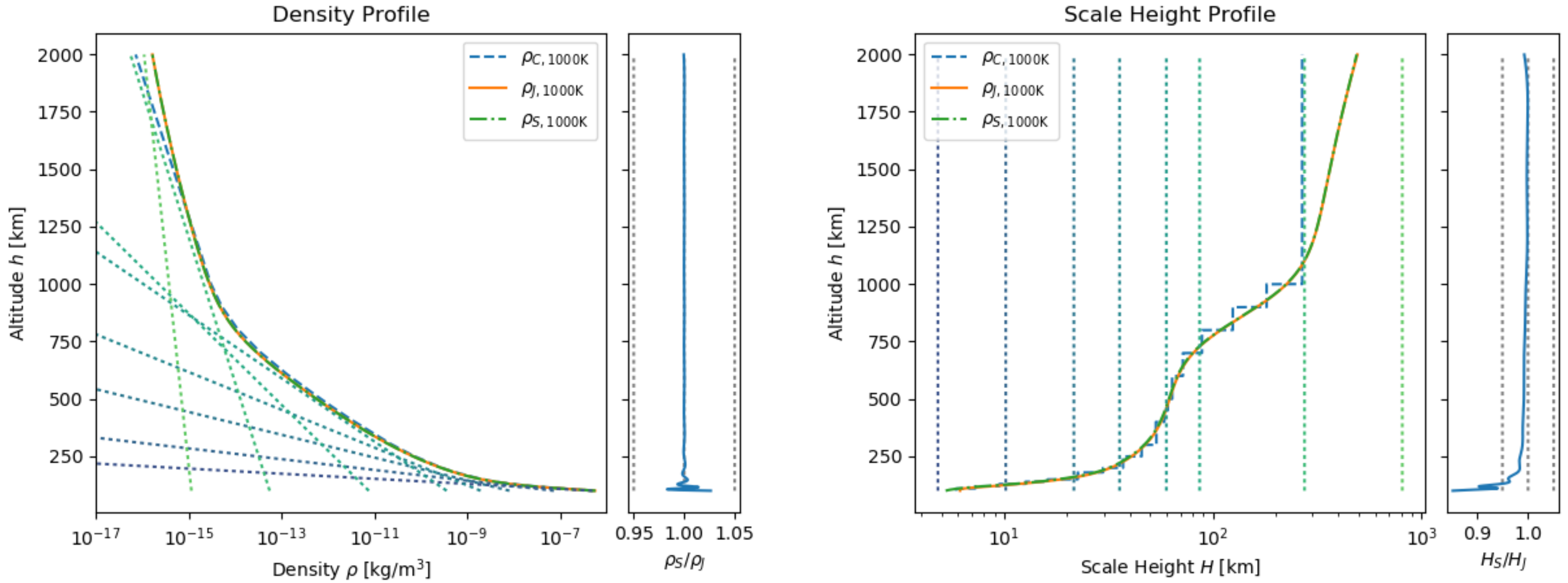
Extension

Smooth Exponential Atmosphere Model

- Atmosphere as sum of exponentials (idea is not new, older Jacchia models have one exponential per constituent)

$$\rho_S(h) = \sum_p \rho_{0,p} \exp(-\frac{h}{H_p})$$

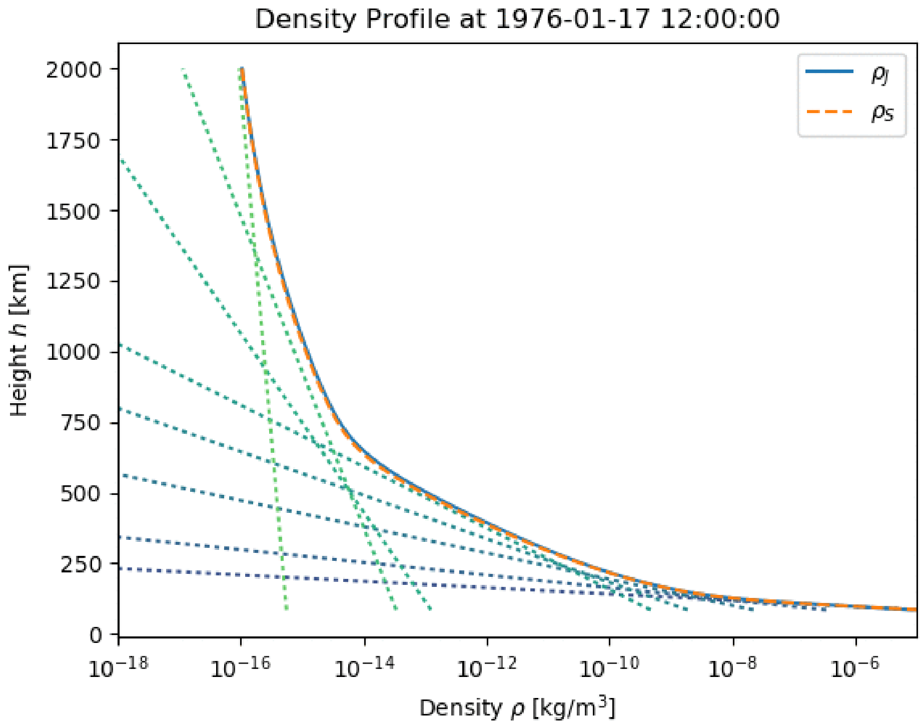
- Possible to fit to any of the atmosphere models, given H increases



Extension

Smooth Exponential Atmosphere Model

- Can be extended to contain time-dependencies (daily, annual, solar-cycle), or latitude-dependency
- E.g. solar cycles 21-23



Extension

Superimposition of King-Hele Method

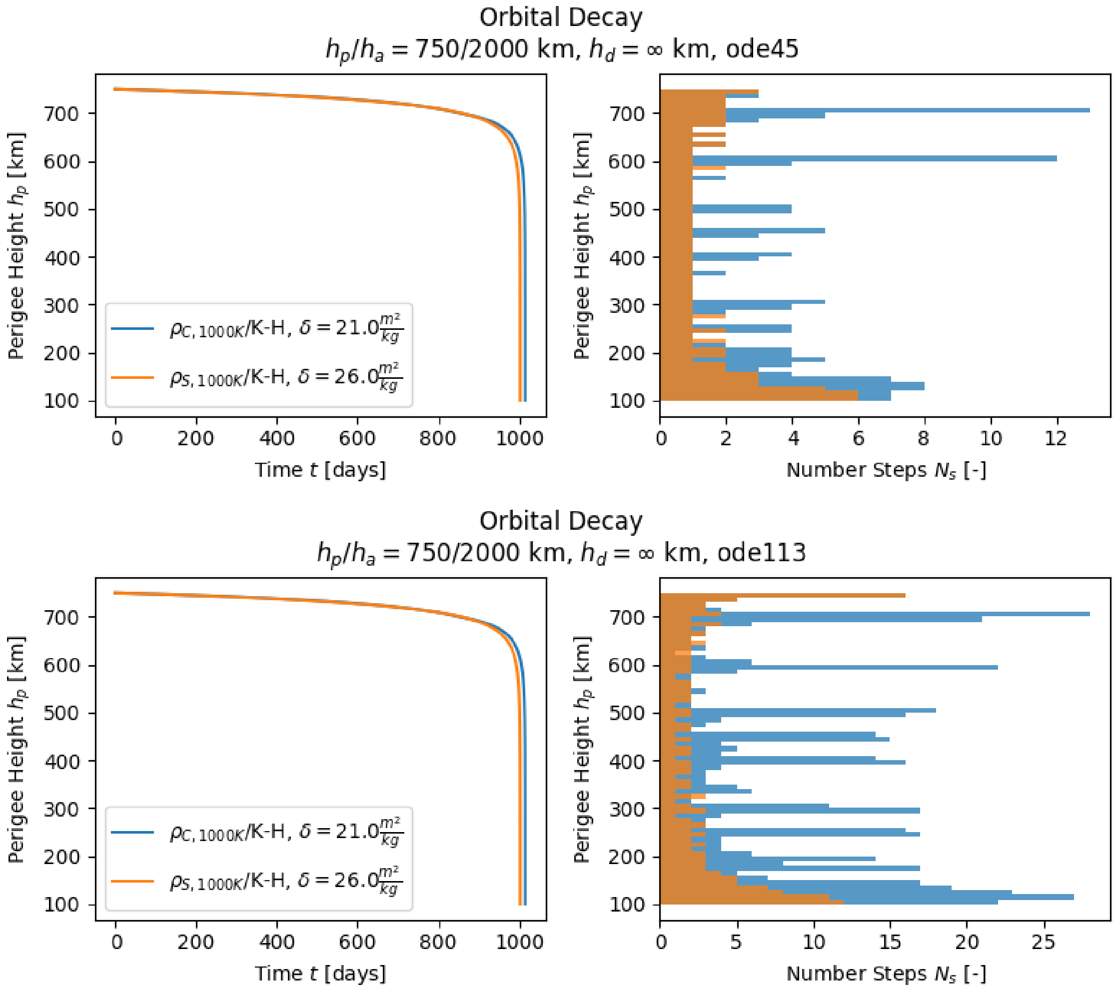
- Given the sum of exponentials with truly fixed H, King-Hele can be summed for each partial atmosphere

$$\Delta a = \sum_{p} \Delta a_p \qquad \Delta e = \sum_{p} \Delta e_p$$

- No error due to changing scale heights H
- Further adaptions made on method:
  - Change directly in e rather than $x = ae$
  - Up to fifth order retained to address large H, i.e. not so small z^{-1}
  - Bridge function introduced to mitigate discontinuity

Results

Non-smooth vs Smooth Atmosphere



Number of function evaluations N_f needed for the integration using MATALB's ode solvers:

Model	ode45	ode113
Non-smooth	1555	1240
Smooth	781	430
	-50%	**-65%**

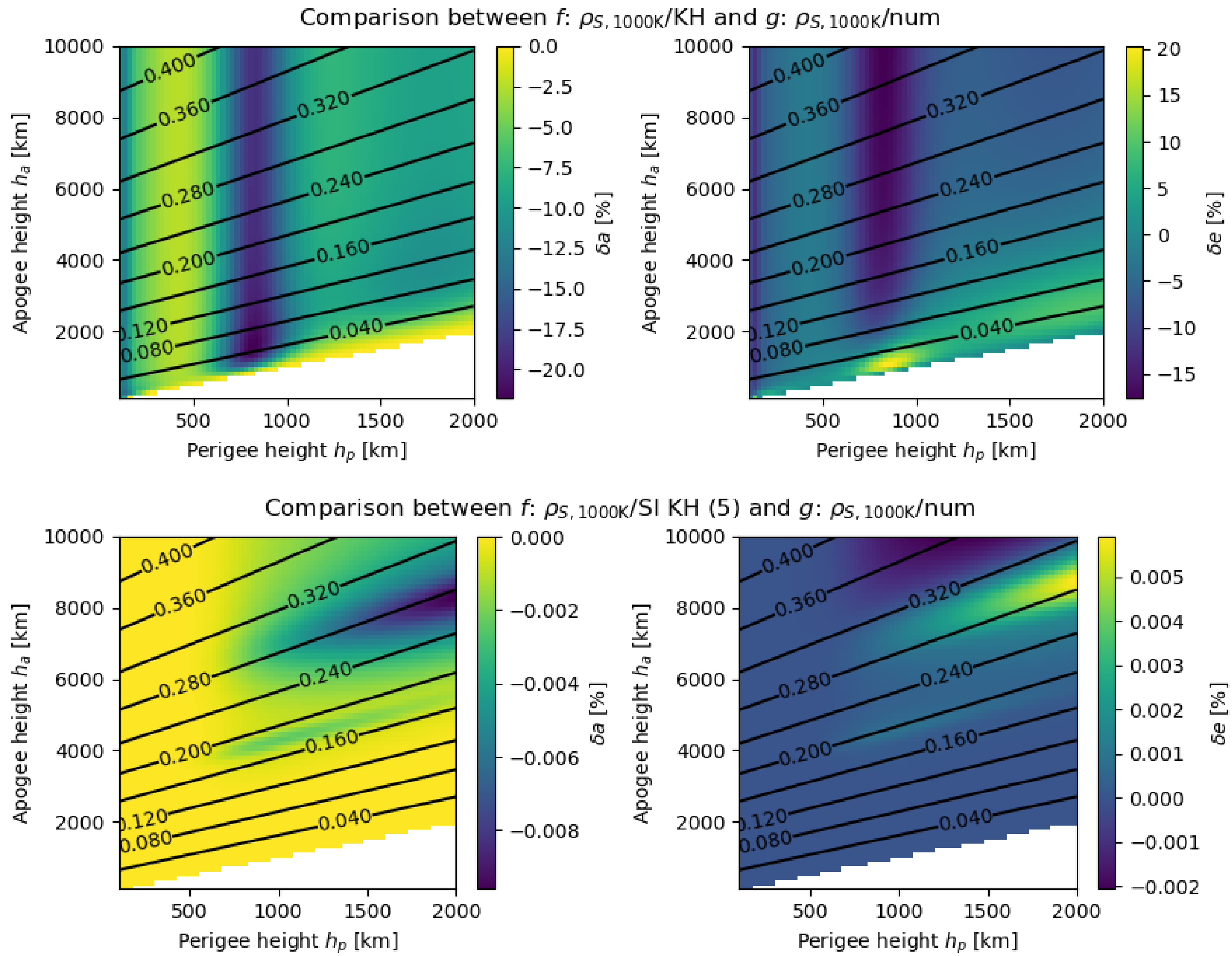
$h_p = 750$ km
$h_a = 2000$ km

Results

Classical vs Superimposed King-Hele

$$\delta a = \frac{\Delta a^{KH} - \Delta a^{n}}{\Delta a^{n}}$$

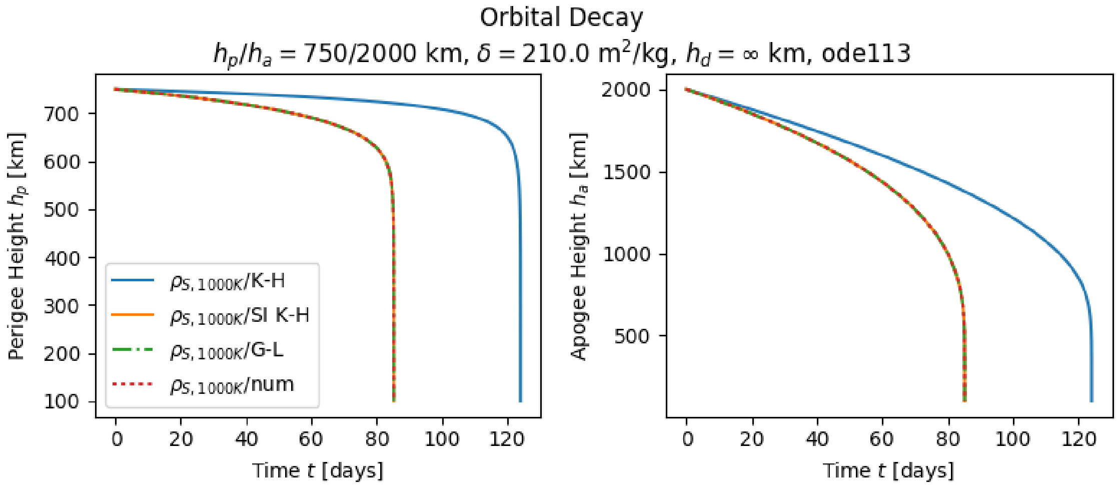
$$\delta e = \frac{\Delta e^{KH} - \Delta e^{n}}{\Delta e^{n}}$$



Results

Propagation with Different Approximations

$h_p = 750$ km
$h_a = 2000$ km
Smooth atmosphere



Model	Δt	N_f
Classical King-Hele	124.1	571
Superimposed King-Hele	85.3	552
Gauss-Legendre	85.3	536
Full numerical	85.3	2607424

Conclusion

And future work

- The two major problem of the King-Hele + non-smooth atmospheric model have been addressed
- The smooth atmosphere cuts the number of function evaluations in half
- The superimposed King-Hele method is very exact and as fast as quadrature
- Semi-analytical propagation with air-drag shown to be as exact as numerical

- Future work
  - find analytical solution for lifetime
  - compare method to real observations

- Questions?

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Modified King Hele Method

- Circular

$$\Delta a = -2\pi\delta a^2 \rho(h)$$

$$\Delta e = 0$$

- Low eccentric

$$k = -2\pi\delta\rho(h_p)\exp^{(-z)}$$

$$\vec{e} = \begin{pmatrix} 1 & e & e^2 & e^3 & e^4 & e^5 \end{pmatrix}$$

$$\vec{I} = \begin{pmatrix} I_0 & I_1 & I_2 & I_3 & I_4 & I_5 & I_6 \end{pmatrix}^T$$

$$\Delta a = ka^2[\vec{e}K_a\vec{I} + \mathcal{O}(e^6)]$$

$$\Delta e = ka[\vec{e}K_e\vec{I} + \mathcal{O}(e^6)]$$

- High eccentric

$$k = -2\delta(\frac{2\pi}{z})^{\frac{1}{2}}\rho(h_p)$$

$$l = z(1 - e^2)$$

$$\vec{e} = \begin{pmatrix} 1 & e & e^2 & e^3 & e^4 & e^5 & e^6 & e^7 & e^8 & e^9 & e^{10} \end{pmatrix}^T$$

$$\vec{r} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}\frac{1}{4l} & \frac{3}{8}\frac{1}{32l^2} & \frac{15}{16}\frac{1}{128l^3} & \frac{105}{32}\frac{1}{2048l^4} & \frac{945}{64}\frac{1}{8192l^5} \end{pmatrix}$$

$$\Delta a = ka^2\frac{(1+e)^{\frac{3}{2}}}{(1-e)^{\frac{1}{2}}}[\vec{e}K_a\vec{r} + \mathcal{O}\left(\frac{1}{z^6}\right)]$$

$$\Delta e = ka\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}(1-e^2)[\vec{e}K_e\vec{r} + \mathcal{O}\left(\frac{1}{z^6}\right)]$$

- Boundary

$$e_b = \left(\frac{H}{a}\right)^{\frac{1}{2}}$$

Bridge function

- Only necessary for high H
- Bridge function: 3rd order polynomial
- Idea
  - keep all variables constant, except e
  - evaluate low eccentric approximation at $e_L = (1 + \Delta y_L)\, e_L$
  - evaluate high eccentric approximation at $e_H = (1 + \Delta y_H)\, e_H$
  - solve linear problem for 4 variables, 4 unknowns, with boundary functions to ensure continuous and smooth transition