

A MULTIGRID ALGORITHM FOR THE p -VERSION OF THE VIRTUAL ELEMENT METHOD

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Abstract. We present a multigrid algorithm for the solution of the linear systems of equations stemming from the p -version of the virtual element discretization of a two-dimensional Poisson problem. The sequence of coarse spaces are constructed decreasing progressively the polynomial approximation degree of the virtual element space, as in standard p -multigrid schemes. The construction of the interspace operators relies on auxiliary virtual element spaces, where it is possible to compute higher order polynomial projectors. We prove that the multigrid scheme is uniformly convergent, provided the number of smoothing steps is chosen sufficiently large. We also demonstrate that the resulting scheme provides a uniform preconditioner with respect to the number of degrees of freedom that can be employed to accelerate the convergence of classical Krylov-based iterative schemes. Numerical experiments validate the theoretical results.

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1. INTRODUCTION

In recent years there has been a tremendous interest in developing numerical methods for the approximation of partial differential equations where the finite-dimensional space is built upon an underlying mesh composed by arbitrarily-shaped polygonal/polyhedral (polytopic, for short) elements. Examples of methods that have been proposed so far include mimetic finite differences [20, 35, 51, 52], polygonal finite element methods [58, 59], discontinuous-Galerkin finite element methods [4, 5, 9, 16, 37, 38, 40] hybridizable and hybrid high order methods [42, 43, 45], BEM-based FEM [56] and gradient schemes [46, 48], for example. Recently, in [17] the virtual element method (VEM) has been introduced, and further developed for elliptic and parabolic problems in [22, 24]. VEMs for linear and nonlinear elasticity have been developed in [18, 21, 50], whereas VEMs for plate bending, Cahn-Hilliard, Stokes, and Helmholtz problems have been addressed in [6, 8, 34, 55]. For discrete topology optimization and fracture networks problems we refer to [11] and [29], respectively. Moreover, several variants of the virtual element method, including mixed, $H(\text{div})$ and $H(\text{curl})$ -conforming, VEM on surfaces, serendipity and nonconforming VEMs have been proposed in [13, 14, 25–27, 36, 39, 49, 62].

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All the above mentioned contributions focus on the h -version of the virtual element method. The p -version VEM was introduced in [23] for the 2D Poisson problem, considering quasi-uniform meshes. It was shown that, analogously to the p -version finite element method (FEM) case, if the solution of the problem has fixed Sobolev regularity, then the convergence rate of the method in terms of p is algebraic, whereas if the solution is analytic then the convergence rate is p exponential. In [28], the full hp -version VEM was studied based on employing meshes geometrically graded towards the corners of the domain and properly choosing the distribution of polynomial approximation degree, so that the convergence rate of the method is exponential in terms of the number of degrees of freedom.

So far, the issue of developing efficient solution techniques for the linear systems of equations stemming from both the h -, p - or hp -versions of the VEM has not been addressed yet, with the exception of the work [31], where Domain Decomposition for VEM were studied. The main difficulty in the development of optimal (multilevel) solution techniques relies on the construction of consistent coarse solvers which are non-trivial on grids formed by general polyhedra. Recently, using the techniques of [7, 10] a multigrid algorithm for the hp -version Discontinuous Galerkin methods on agglomerated polygonal/polyhedral meshes has been analyzed in [12].

The aim of this paper is to develop efficient iterative solvers for the solution of the linear systems of equations stemming from the p -version of the virtual element discretization of a two-dimensional Poisson problem. We propose to employ a W -cycle p -multigrid algorithm, *i.e.* coarse levels are obtained by decreasing progressively the polynomial approximation degree up to the coarsest level which corresponds to the lowest (linear) virtual element (VE in short) space. The key point is the construction of suitable prolongation operators between the hierarchy of VE spaces. With the standard VE space such prolongation operators cannot be constructed based on employing only the degrees of freedom. For such a reason we introduce a suitable auxiliary VE space, which is identical to the standard VE space from the algebraic point of view and which allows to construct computable interspace operators but results into non-inherited sublevel solvers. This in turn complicates the analysis of the multigrid algorithms, since we need to account for non-inherited sublevel solvers. Employing a Richardson smoother and following the classical framework, see *e.g.* [33], we prove that the W -cycle algorithm converges uniformly provided the number of smoothing steps is chosen sufficiently large. We also demonstrate that the resulting multigrid algorithm provides a uniform preconditioner for the Preconditioned Conjugate Gradient method (PCG), *i.e.*, the number of PCG iterations needed to reduce the (relative) residual up to a (user-defined) tolerance is uniformly bounded independently of the number of degrees of freedom. Further, employing the Gauss-Seidel smoother in place of the Richardson one can improve the performance of our iterative scheme.

The extension of the present setting to h -multigrid methods, *i.e.* where the coarse levels are formed by geometric agglomeration of the underlying grid is currently under investigation.

The remaining part of the paper is organized as follows. In Section 2, we introduce the model problem, a virtual element method approximating its solution, the associated linear system and the multigrid algorithm; moreover, an auxiliary VE space, needed for the construction of the algorithm, is presented. In Section 3, we present the W -cycle p -VEM multigrid algorithm; its convergence analysis is the topic of Section 4. Finally, in Section 5, numerical results are shown.

Throughout the paper, we will adopt the standard notation for Sobolev spaces (see [2, 47]). In particular, given $\omega \subset \mathbb{R}^2$, $L^2(\omega)$ and $H^1(\omega)$ are the standard Lebesgue and Sobolev spaces over ω , respectively, and $\|\cdot\|_{0,\omega}$, $\|\cdot\|_{1,\omega}$, and $|\cdot|_{1,\omega}$, are the Lebesgue and the Sobolev (semi)norms, respectively. We will write $x \lesssim y$ and $x \approx y$ meaning that there exist positive constants c_1 , c_2 and c_3 independent of the discretization and multigrid parameters, such that $x \leq c_1 y$ and $c_2 y \leq x \leq c_3 y$, respectively. Constants c_1 , c_2 and c_3 may depend on the geometric properties of the polygonal mesh; we refer to Remarks 4.7 and 5.1 below for additional comments on this aspect. In addition, $\mathbb{P}_\ell(\omega)$, $\omega \subset \mathbb{R}^d$, $d = 1, 2$, denotes the space of polynomials of maximum degree $\ell \in \mathbb{N}$ over ω , with the convention $\mathbb{P}_{-1}(\omega) = \emptyset$. We will also employ the standard multi-index notation:

$$\mathbf{v} = (v_1, v_2), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2), \quad \mathbf{v}^\boldsymbol{\alpha} = v_1^{\alpha_1} v_2^{\alpha_2}. \quad (1.1)$$

2. THE MODEL PROBLEM AND THE p -VERSION VIRTUAL ELEMENT METHOD

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and $f \in L^2(\Omega)$ we consider the following model problem: find $u \in V = H_0^1(\Omega)$ such that:

$$a(u, v) = \int_{\Omega} f v, \quad \forall v \in V, \tag{2.1}$$

where $a(\cdot, \cdot) = (\nabla \cdot, \nabla \cdot)_{0, \Omega}$. Problem (2.1) is well-posed, cf. [33], for example. In the next section we introduce the p -version of the virtual element method and we discuss its implementation. In Section 2.2, we build an *auxiliary* VE space that will be instrumental to construct and analyse our multigrid algorithm.

2.1. The p -version virtual element method

In this section, we introduce the p -version virtual element method, based on polygonal meshes with straight edges for the discretization of problem (2.1). In particular, we construct the method on a fixed mesh only, disregarding in the analysis the explicit dependences on the geometry of the mesh. We refer to Remarks 4.7 and 5.1 for additional comments on this aspect.

Let \mathcal{T} be a *fixed* decomposition of Ω into non-overlapping polygonal elements E , and let \mathcal{V} and \mathcal{E} be the set of all vertices and edges of \mathcal{T} , respectively. We set $\mathcal{V}_b = \mathcal{V} \cap \partial\Omega$ and $\mathcal{E}_b = \mathcal{E} \cap \partial\Omega$. Given E generic polygon in \mathcal{T} , we also define $\mathcal{V}^E = \mathcal{V} \cap \partial E$ and $\mathcal{E}^E = \mathcal{E} \cap \partial E$ as the set of vertices and edges of polygon E , respectively. To each edge $e \in \mathcal{E}$, we associate $\boldsymbol{\tau}$ and \mathbf{n} , the tangential and normal unit vector (obtained by a counter-clockwise rotation of $\boldsymbol{\tau}$), respectively.

For future use, it is convenient to split the (continuous) bilinear form $a(\cdot, \cdot)$ defined in (2.1) into a sum of local contributions:

$$a(u, v) = \sum_{E \in \mathcal{T}} a^E(u, v), \quad \forall u, v \in V, \quad \text{where } a^E(\cdot, \cdot) = (\nabla \cdot, \nabla \cdot)_{0, E}. \tag{2.2}$$

In order to construct the p -VEM approximation of (2.1), we need the following ingredients:

- Finite dimensional subspaces $V_p(E)$ of $V(E) = V|_E$ for all $E \in \mathcal{T}$ and a finite dimensional subspace V_p of V , such that $V_p(E) = V_p|_E$;
- Local symmetric bilinear forms $a_p^E : V_p(E) \times V_p(E) \rightarrow \mathbb{R}$ for all $E \in \mathcal{T}$ so that:

$$a_p(u_p, v_p) = \sum_{E \in \mathcal{T}} a_p^E(u_p, v_p), \quad \forall u_p, v_p \in V_p; \tag{2.3}$$

- A duality pairing $\langle f_p, \cdot \rangle_p$, where $f_p \in V_p'$ and V_p' is the dual space of V_p .

The above ingredients must be built in such a way that the discrete version of (2.1):

$$\begin{cases} \text{find } u_p \in V_p \text{ such that,} \\ a_p(u_p, v_p) = \langle f_p, v_p \rangle_p, \quad \forall v_p \in V_p, \end{cases} \tag{2.4}$$

is well-posed and optimal a priori energy error estimates hold, cf. [23].

We begin by introducing the local space $V_p(E)$; given $E \in \mathcal{T}$ and $p \geq 1$, we set:

$$V_p(E) = \{v_p \in H^1(E) \mid \Delta v_p \in \mathbb{P}_{p-2}(E), v_p|_{\partial E} \in \mathbb{B}_p(\partial E)\}, \tag{2.5}$$

where

$$\mathbb{B}_p(\partial E) = \{v_p \in \mathcal{C}^0(\partial E) \mid v_p|_e \in \mathbb{P}_p(e), \quad \forall e \in \mathcal{E}^E\}. \quad (2.6)$$

We remark that the above definition coincides with the definition of the two dimensional VE space introduced in [17] for the Poisson equation, and that clearly $\mathbb{P}_p(E) \subseteq V_p(E)$, $p \geq 1$. The global space is then obtained by gluing continuously the local spaces:

$$V_p = \{v_p \in H_0^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) \mid v_p|_E \in V_p(E), \quad \forall E \in \mathcal{T}\}. \quad (2.7)$$

We note that for the sake of simplicity we are assuming a uniform p on each $E \in \mathcal{T}$. Nevertheless, it is possible to construct VEM with non-uniform degrees of accuracy over \mathcal{T} , see [28].

We endow the space (2.5) with the following set of degrees of freedom (dofs). To every $v_p \in V_p(E)$ we associate:

- the values of v_p at the vertices of E ;
- the values of v_p at $p - 1$ internal Gauss-Lobatto nodes on each edge $e \in \mathcal{E}^E$;
- the scaled internal moments:

$$\frac{1}{|E|} \int_E v_p m_\alpha, \quad (2.8)$$

where m_α is an $L^2(E)$ orthonormal basis of $\mathbb{P}_{p-2}(E)$.

Reasoning as in Proposition 4.1 from [17], it is easy to see that this is a unisolvent set of degrees of freedom. We observe that the basis $\{m_\alpha\}_{|\alpha|=0}^{p-2}$ introduced in (2.8) can be built by orthonormalizing (following, *e.g.*, [15]) for instance the monomial basis given by:

$$q_\alpha = \left(\frac{\mathbf{x} - \mathbf{x}_E}{\text{diam}(E)} \right)^\alpha, \quad \forall \alpha \in \mathbb{N}^2, \quad |\alpha| \leq p - 2, \quad (2.9)$$

where \mathbf{x}_E denotes the barycenter of the element E .

Remark 2.1. The definition of the internal degrees of freedom in (2.8) differs from its classical counterpart in [17, 23] where the internal moments are defined with respect to the monomial basis (2.9). The new choice of the internal degrees of freedom will play a crucial role in the choice of the stabilization of the method, see Theorem 2.3, and in the choice of the space-dependent inner product associated with the multigrid algorithm, see Theorem 3.4 below.

We define the canonical basis $\{\varphi_j\}_{j=1}^{\dim(V_p(E))}$ as:

$$\text{dof}_i(\varphi_j) = \delta_{ij}, \quad i, j = 1, \dots, \dim(V_p(E)), \quad \text{where } \delta_{ij} \text{ is the Kronecker delta.} \quad (2.10)$$

Owing the definition (2.5) of the local VE space and the choice of the degrees of freedom, it is possible to compute the following operators:

- for $p \geq 2$, the $L^2(E)$ projection operator $\Pi_{p-2}^0 : V_p(E) \rightarrow \mathbb{P}_{p-2}(E)$:

$$(\Pi_{p-2}^0 v_p - v_p, q)_{0,E} = 0, \quad \forall v_p \in V_p(E), \quad \forall q \in \mathbb{P}_{p-2}(E); \quad (2.11)$$

- the $H^1(E)$ projector $\Pi_p^\nabla : V_p(E) \rightarrow \mathbb{P}_p(E)$:

$$\begin{cases} a^E(\Pi_p^\nabla v_p - v_p, q) = 0, & \forall q \in \mathbb{P}_p(E), \\ \int_E (\Pi_p^\nabla v_p - v_p) = 0, & \text{if } p \geq 2, \\ \int_{\partial E} (\Pi_p^\nabla v_p - v_p) = 0, & \text{if } p = 1, \end{cases} \quad \forall v_p \in V_p(E), \quad (2.12)$$

see [17, 19] for details. We observe that the last two conditions in (2.12) are needed in order to fix the constant part of the energy projector.

Next, we introduce the discrete right-hand side $f_p \in V_p'$ and the associated duality pairing:

$$\langle f_p, v_p \rangle_p = \sum_{E \in \mathcal{T}} \int_E \Pi_{\max(p-2,1)}^0 f \bar{v}_p, \quad (2.13)$$

where

$$\bar{v}_p = \begin{cases} \frac{1}{|\partial E|} \int_{\partial E} v_p, & \text{if } p = 1, \\ v_p, & \text{if } p \geq 2. \end{cases} \quad (2.14)$$

We observe that it is possible to compute up to machine precision the expression in (2.13), because the action of the projector $\Pi_{\max(p-2,1)}^0$ on all the elements of $V_p(E)$ is computable. For a deeper study concerning the approximation of the discrete loading term see [3, 18, 23].

Remark 2.2. If instead of considering homogeneous Dirichlet boundary conditions in continuous problem (2.1), mixed boundary conditions are considered, then an additional term of the form $(g, v)_{0, \Gamma_N}$ appears in the right-hand side of (2.1), where $\Gamma_n \subset \partial\Omega$ denotes the partition of boundary where Neumann condition g is enforced. In the VEM setting, one can deal with this extra term by resorting to a suitable L^2 -projection on the edges of Γ_N .

At this point, we turn our attention to the local and global discrete bilinear forms. We require that the local bilinear forms $a_p^E : V_p(E) \times V_p(E) \rightarrow \mathbb{R}$ satisfy, for all $E \in \mathcal{T}$, the two following assumptions.

(A1) p -consistency:

$$a^E(q, v_p) = a_p^E(q, v_p), \quad \forall q \in \mathbb{P}_p(E), \quad \forall v_p \in V_p(E); \quad (2.15)$$

(A2) Stability: there exist two positive constants $0 < \alpha_*(p) < \alpha^*(p) < +\infty$, possibly depending on p , such that:

$$\alpha_*(p) |v_p|_{1,E}^2 \leq a_p^E(v_p, v_p) \leq \alpha^*(p) |v_p|_{1,E}^2, \quad \forall v_p \in V_p(E). \quad (2.16)$$

Assumption **(A1)** guarantees that the method is exact whenever the solution of (2.1) is a polynomial of degree p , whereas assumption **(A2)** guarantees the well-posedness of problem (2.4). Let now Id_p be the identity operator on the space $V_p(E)$, we set:

$$a_p^E(u_p, v_p) = a^E(\Pi_p^\nabla u_p, \Pi_p^\nabla v_p) + S_p^E((\text{Id}_p - \Pi_p^\nabla)u_p, (\text{Id}_p - \Pi_p^\nabla)v_p), \quad \forall u_p, v_p \in V_p(E), \quad (2.17)$$

where Π_p^∇ is defined in (2.12) and the local bilinear form $S_p^E(\cdot, \cdot)$ as:

$$S_p^E(u_p, v_p) = \sum_{i=1}^{\dim(V_p(E))} \text{dof}_i(u_p) \text{dof}_i(v_p), \quad (2.18)$$

satisfies:

$$c_*(p)|v_p|_{1,E}^2 \leq S_p^E(v_p, v_p) \leq c^*(p)|v_p|_{1,E}^2, \quad \forall v_p \in \ker(\Pi_p^\nabla), \quad (2.19)$$

where $c_*(p)$ and $c^*(p)$ might depend on p . We underline that the local discrete bilinear form (2.17) satisfies (A1) and (A2) and, thanks to (2.19), the following bounds hold:

$$\alpha_*(p)|u_p|_{1,\Omega}^2 \lesssim a_p(u_p, u_p), \quad a_p(u_p, v_p) \lesssim \alpha^*(p)|u_p|_{1,\Omega}|v_p|_{1,\Omega}, \quad \forall u_p, v_p \in V_p,$$

with:

$$\alpha_*(p) = \min(1, c_*(p)), \quad \alpha^*(p) = \max(1, c^*(p)). \quad (2.20)$$

The following result provides bounds in terms of p for the constants $c_*(p)$ and $c^*(p)$ in (2.19).

Theorem 2.3. *Let $E \in \mathcal{T}$ and let $S_p^E(\cdot, \cdot)$ be the stabilizing bilinear form defined in (2.18). Then*

$$c_*(p)|v_p|_{1,E}^2 \leq S_p^E(v_p, v_p) \leq c^*(p)|v_p|_{1,E}^2, \quad \forall v_p \in \ker(\Pi_p^\nabla), \quad (2.21)$$

where $c_*(p) \gtrsim p^{-6}$ and $c^*(p) \lesssim p^4$.

Proof. The assertion follows by combining the forthcoming technical Lemmata 2.4 and 2.5. \square

An immediate consequence of Theorem 2.3 and (2.20) is that it holds:

$$\alpha_*(p) \gtrsim p^{-6}, \quad \alpha^*(p) \lesssim p^4, \quad (2.22)$$

where $\alpha_*(p)$ and $\alpha^*(p)$ are given in (2.16).

Lemma 2.4. *Let $S_{p,aux}^E(\cdot, \cdot)$ be the local auxiliary stabilization defined as:*

$$S_{p,aux}^E(u_p, v_p) = \frac{p}{h_E} (u_p, v_p)_{0,\partial E} + \frac{p^2}{h_E^2} (\Pi_{p-2}^0 u_p, \Pi_{p-2}^0 v_p)_{0,E}. \quad (2.23)$$

Then, it holds:

$$p^{-5}|v_p|_{1,E}^2 \lesssim S_{p,aux}^E(v_p, v_p) \lesssim p^2|v_p|_{1,E}^2, \quad \forall v_p \in \ker(\Pi_p^\nabla), \quad (2.24)$$

where Π_p^∇ is the energy projector defined in (2.12).

Proof. The assertion follows employing integration by parts, the properties of orthogonal projections and hp polynomial inverse estimates. It follows the lines of the proof of Theorem 2 from [28]; for the sake of brevity the details are not reported here. \square

Lemma 2.5. *Let $E \in \mathcal{T}$ and let S_p^E and $S_{p,aux}^E$ be defined as in (2.18) and (2.23), respectively. Then,*

$$p^{-1}S_p^E(v_p, v_p) \leq S_{p,aux}^E(v_p, v_p) \lesssim p^2S_p^E(v_p, v_p), \quad \forall v_p \in \ker(\Pi_p^\nabla). \quad (2.25)$$

Before showing the proof, we recall that given $\{\rho_j^{p+1}\}_{j=0}^p$ and $\{\xi_j\}_{j=0}^p$ the $p+1$ Gauss-Lobatto nodes and weights on $\widehat{T} = [-1, 1]$, respectively, it holds:

$$\sum_{j=0}^p q^2(\xi_j^{p+1})\rho_j^{p+1} \lesssim \|q\|_{0,\widehat{T}}^2 \leq \sum_{j=0}^p q^2(\xi_j^{p+1})\rho_j^{p+1} \quad \forall q \in \mathbb{P}_p(\widehat{T}), \quad (2.26)$$

cf. (2.14) from [30]. Moreover, it holds:

$$p^{-2} \lesssim \rho_j^{p+1} \lesssim 1, \quad \forall j = 0, \dots, p+1, \quad (2.27)$$

where the hidden constants are positive and independent of p , see [1], (25.4.32).

Proof of Lemma 2.5. Without loss of generality, we assume that $h_E = 1$. The assertion will follow from a scaling argument.

By using (2.26) and (2.27), we obtain:

$$p^{-1} \sum_{j=1}^{\text{card}(\mathcal{E}^E)} \text{dof}_{b,j}^2(v_p) \lesssim p \|v_p\|_{0,\partial E}^2 \lesssim p \sum_{j=1}^{\text{card}(\mathcal{E}^E)} \text{dof}_{b,j}^2(v_p), \quad (2.28)$$

where $\text{dof}_{b,j}$ denotes the j th boundary degree of freedom. This concludes the discussion concerning the boundary term. Next, we study the bulk term in (2.23), and consider the expansion of $\Pi_{p-2}^0 v_p$ into the $L^2(E)$ orthonormal basis $\{m_\alpha\}_{|\alpha|=0}^{p-2}$ introduced in (2.8):

$$\Pi_{p-2}^0 v_p = \sum_{|\alpha| \leq p-2} c_\alpha m_\alpha. \quad (2.29)$$

Testing (2.29) with m_β , $|\beta| \leq p-2$, we obtain:

$$|E| \text{dof}_\beta(v_p) = \int_E v_p m_\beta = \int_E \Pi_{p-2}^0 v_p m_\beta = c_\beta, \quad (2.30)$$

where $\text{dof}_\beta(\cdot)$ denotes the internal degrees of freedom associated with polynomial m_β . As a consequence:

$$\Pi_{p-2}^0 v_p = \sum_{|\alpha| \leq p-2} |E| \text{dof}_\alpha(v_p) m_\alpha. \quad (2.31)$$

Parseval identity implies:

$$p^2 (\Pi_{p-2}^0 v_p, \Pi_{p-2}^0 v_p)_{0,E} = p^2 \sum_{|\alpha| \leq p-2} |E|^2 \text{dof}_{i,\alpha}^2(v_p), \quad (2.32)$$

where $\text{dof}_{i,|\alpha|}(\cdot)$ denotes the internal degree of freedom associated with polynomial m_α . The assertion follows from (2.28) and (2.32). \square

Finally, from [17, 23] the following error bound in the energy norm holds:

$$|u - u_p|_{1,\Omega} \lesssim \frac{\alpha^*(p)}{\alpha_*(p)} \left\{ \mathcal{F}_p + \inf_{u_I \in V_p} |u - u_I|_{1,\Omega} + \sum_{E \in \mathcal{T}} \inf_{u_\pi \in \mathbb{P}_p(E)} |u - u_\pi|_{1,E} \right\}, \quad (2.33)$$

where u and u_p are the solution of (2.1) and (2.4), respectively, $\alpha_*(p)$ and $\alpha^*(p)$ are the stability constants given in (2.16) and \mathcal{F}_p is the smallest constant satisfying:

$$(f, v_p)_{0,\Omega} - \langle f_p, v_p \rangle_p \leq \mathcal{F}_p |v_p|_{1,\Omega}, \quad \forall v_p \in V_p. \quad (2.34)$$

From (2.33) and following [23], it is possible to prove p error bounds analogous to those in the p -FEM case, see [57].

Finally, we focus on the construction of the linear system of equations stemming from (2.4). By expanding the trial function u_p as a combination of the elements in the canonical basis, see (2.10),

$$u_p = \sum_{i=1}^{\dim(V_p)} \text{dof}_i(u_p) \varphi_i = \sum_{i=1}^{\dim(V_p)} (\mathbf{u}_p)_i \varphi_i, \quad (2.35)$$

where $\mathbf{u}_p \in \mathbb{R}^{\dim(V_p)}$ is the set of dofs of u_p , and selecting v_p as φ_j , $j = 1, \dots, \dim(V_p)$, we obtain:

$$\mathbf{A}_p \cdot \mathbf{u}_p = \mathbf{f}_p, \quad (2.36)$$

where:

$$(\mathbf{A}_p)_{i,j} = a_p(\varphi_j, \varphi_i), \quad (\mathbf{f}_p)_i = \langle f_p, \varphi_i \rangle_p, \quad i, j = 1, \dots, \dim(V_p). \quad (2.37)$$

Both the right-hand side and the coefficient matrix are computable exactly up to machine precision, see [19]. In the next section, we discuss the spectral condition number of the stiffness matrix \mathbf{A}_p .

2.1.1. The condition number of the stiffness matrix \mathbf{A}_p

In (2.8) we defined the internal degrees of freedom associated with space $V_p(E)$ defined in (2.5) as the scaled moments with respect to an $L^2(E)$ orthonormal basis of $\mathbb{P}_{p-2}(E)$. We observe that this choice is different from the usual choice adopted in standard VEM literature, see *e.g.* [3, 17, 19], where the internal dofs are defined as the (scaled) moments with respect to the monomial basis (2.9) of $\mathbb{P}_{p-2}(E)$. Our choice, which is a key ingredient in the proof of Lemma 2.5. also plays a fundamental role in the spectral properties of the stiffness matrix \mathbf{A}_p defined in (2.37). Indeed, in Table 1 we compare the spectral condition number $\kappa(\mathbf{A}_p)$ of the stiffness matrix \mathbf{A}_p as a function of the degree of accuracy of the method p , based on employing the two different sets of internal degrees of freedom, namely the scaled moments with respect to the $L^2(E)$ orthonormal basis of $\mathbb{P}_{p-2}(E)$ or with respect to the monomial basis (2.9).

Results reported in Table 1 have been obtained on the Voronoi-Lloyd polygonal mesh shown in Figure 1; the same kind of results have been obtained on meshes made of squares and of quasi-regular hexagons. For the sake of brevity these results have been omitted. From the results reported in Table 1 it is clear that \mathbf{A}_p grows, as for classical finite element methods, as p^4 whenever the interior dofs are defined with respect to an $L^2(E)$ orthonormal basis of $\mathbb{P}_{p-2}(E)$ whereas the condition number \mathbf{A}_p blows up exponentially if the scaled moments are defined with respect to the monomial basis (2.9). That is, the choice (2.8) for the internal degrees of freedom is the “right” choice as it damps the condition number of the stiffness matrix effectively and prevents round off errors, as those observed, for example, in [23] where the monomial basis (2.9) was employed.

A deeper investigation of the effects of the choice of internal moments on the condition number of the stiffness matrix in 2D and 3D VEM is carried out in [44, 53].

2.2. An auxiliary virtual element space

In this section, we introduce an auxiliary VE space which will be crucial for the construction of the multigrid algorithm in Section 3. Hence, following the spirit of [3], we consider a modification of $V_p(E)$ into a *diverse*

TABLE 1. Condition number $\kappa(\mathbf{A}_p)$ of the stiffness matrix \mathbf{A}_p as a function of p for two different sets of internal degrees of freedom: (*left*) scaled moments with respect to an $L^2(E)$ orthonormal basis of $\mathbb{P}_{p-2}(E)$ (orthogonalized basis); (*right*) scaled moments with respect to the monomial basis (2.9) of $\mathbb{P}_{p-2}(E)$ (monomial basis). Voronoi-Lloyd polygonal mesh.

p	$\kappa(\mathbf{A}_p)$	$\kappa(\mathbf{A}_p)$
1	1.3225e+01	7.2732e+01
2	4.9712e+02	1.0964e+03
3	7.5099e+02	1.2910e+05
4	1.1823e+03	1.5566e+07
5	1.9395e+03	1.6003e+09
6	3.5100e+03	1.7069e+11
7	6.0754e+03	1.7280e+13
8	1.0100e+04	1.7172e+15
Rate	p^4	$a \exp(bp)$, $a = 0.18$, $b = 4.59$

space on which we are able to compute a higher order L^2 projector. In particular, we set:

$$\tilde{V}_p(E) = \left\{ v_p \in H^1(E) \mid v_p|_{\partial E} \in \mathbb{B}_p(\partial E), \Delta v_p \in \mathbb{P}_{p-1}(E), \int_E (\Pi_{p-2}^0 v_p - v_p) m_\alpha = 0, |\alpha| = p-1 \right\}, \quad (2.38)$$

where we recall that $\alpha \in \mathbb{N}^2$ is a multi-index and $\{m_\alpha\}_{|\alpha|=0}^{p-1}$ is the $L^2(E)$ orthonormal basis employed also in the definition of the degrees of freedom (2.8) of the standard VE space introduced in (2.7).

Henceforth, we will denote by the expression *enhancing constraints* the following set of constraints employed in the definition of $\tilde{V}_p(E)$:

$$\int_E (\Pi_{p-2}^0 v_p - v_p) m_\alpha = 0, \quad |\alpha| = p-1, \quad \forall v_p \in \tilde{V}_p(E). \quad (2.39)$$

The definition of $\tilde{V}_p(E)$ is different from the one presented in [3]. Moreover, we observe that $\mathbb{P}_p(E) \not\subseteq \tilde{V}_p(E)$, but $\mathbb{P}_{p-2}(E) \subseteq \tilde{V}_p(E)$. To be more precise, owing to the $L^2(E)$ orthonormality of the m_α , it holds in fact:

$$\mathbb{P}_{p-2}(E) \oplus (\mathbb{P}_p(E)/\mathbb{P}_{p-2}(E))^{\perp_{\mathbb{P}_{p-1}(E)}} \subseteq \tilde{V}_p(E), \quad (2.40)$$

where $(\mathbb{P}_p(E)/\mathbb{P}_{p-2}(E))^{\perp_{\mathbb{P}_{p-1}(E)}}$ denotes the space of polynomials of degree p , not in the space of polynomials of degree $p-2$, orthogonal to all m_α with $|\alpha| = p-1$.

We endow \tilde{V}_p with the same degrees of freedom of the space V_p introduced in (2.5). Using the *auxiliary* local virtual space \tilde{V}_p introduced in (2.38), it is clear that we are able to compute the following operator:

- $\Pi_{p-1}^0 : \tilde{V}_p(E) \rightarrow \mathbb{P}_{p-1}(E)$, the L^2 projection onto the space of polynomials of degree $p-1$, defined as in (2.11).

We stress that there is no chance to be able to compute explicitly Π_{p-1}^0 as a map defined on $V_p(E)$, since the internal degrees of freedom are up to order $p-2$, whereas this is possible in the new space $\tilde{V}_p(E)$. We can do that since (2.39) allows to compute internal moments up to order $p-1$.

The global *auxiliary* VE space is obtained again by gluing continuously the local spaces as done in (2.7):

$$\tilde{V}_p = \left\{ v_p \in H_0^1(\Omega) \cap C^0(\bar{\Omega}) \mid v_p|_E \in \tilde{V}_p(E), \quad \forall E \in \mathcal{T} \right\}. \quad (2.41)$$

The choice of the discrete bilinear form a_p and of the right-hand side f_p in (2.4) are exactly the same as those in Section 2.1 for the space V_p .

Importantly, we emphasize that the choice of the stabilization when employing the new spaces $\tilde{V}_p(E)$ reads as in (2.18), although functions in local enhanced spaces $\tilde{V}_p(E)$ defined in (2.38) are different with respect to their counterparts in standard VE spaces defined in (2.5). In particular, the following spectral bounds for the stability are valid.

Theorem 2.6. *Given $E \in \mathcal{T}$ and S_p^E the stabilization defined in (2.18), the following holds true:*

$$c_*(p)|v_p|_{1,E}^2 \lesssim S_p^E(v_p, v_p) \lesssim c^*(p)|v_p|_{1,E}^2, \quad \forall v_p \in \ker(\Pi_p^\nabla) \subset \tilde{V}_p(E), \quad (2.42)$$

where $c_*(p)$ and $c^*(p)$ satisfy the same bounds of Theorem 2.3 and where Π_p^∇ is the energy projector defined in (2.12) applied to functions in $\tilde{V}_p(E)$.

Proof. It suffices to show that the counterparts of Lemmata 2.4 and 2.5 are valid for functions in $\ker(\Pi_p^\nabla) \subset \tilde{V}_p(E)$.

For what concerns Lemma 2.4, following Theorem 2 from [28], it is immediate to prove analogous bounds to those in (2.24) employing the auxiliary stabilization:

$$\tilde{S}_{p,\text{aux}}^E(u_p, v_p) = \frac{p}{h_E}(u_p, v_p)_{0,\partial E} + \frac{p^2}{h_E^2}(\Pi_{p-1}^0 u_p, \Pi_{p-1}^0 v_p)_{0,\partial E}. \quad (2.43)$$

However, we note that the definition of the enhancing constraints (2.39) entails that:

$$\Pi_{p-1}^0 v_p = \Pi_{p-2}^0 v_p, \quad \forall v_p \in \tilde{V}_p(E), \quad (2.44)$$

whence the counterpart of Lemma 2.4 for functions in $\ker(\Pi_p^\nabla) \subset \tilde{V}_p(E)$. In view of this, the counterpart of Lemma 2.5 for functions in $\ker(\Pi_p^\nabla) \subset \tilde{V}_p(E)$ immediately follows. \square

As a consequence of Theorem 2.6, we have:

$$\alpha_*(p)|v_p|_{1,E}^2 \leq a_p^E(v_p, v_p) \leq \alpha^*(p)|v_p|_{1,E}^2, \quad \forall v_p \in \tilde{V}_p(E), \forall E \in \mathcal{T}, \quad (2.45)$$

where $\alpha_*(p)$ and $\alpha^*(p)$ satisfy the same bounds as in (2.22).

It is crucial to remark that the linear systems stemming from the use of V_p and \tilde{V}_p are the same. In fact, it is clear from (2.17) that the construction of the local discrete bilinear forms depends uniquely on the choice of the set of the degrees of freedom (which we recall are the same for the two spaces) and the energy projector Π_p^∇ defined in (2.12), which is computed without the need of (2.39).

Also the construction of the discrete right-hand side (2.13) does not depend on the choice of the space since the L^2 projector $\Pi_{\max(1,p-2)}$ defined in (2.11) is built using the internal degrees of freedom only, while the enhancing constraints (2.39) are neglected.

Remark 2.7. The aforementioned equivalence between the two linear systems associated with spaces V_p and \tilde{V}_p is of great importance in order to design and analyse the multigrid algorithm in Section 3. However, V_p and \tilde{V}_p have significant differences.

The first issue we want to highlight is that the method associated with space \tilde{V}_p defined in (2.41) is not a “good” method from the point of view of the approximation property. It is possible to show p approximation results on the first and the third term on the right hand side of (2.33) following for instance Sections 4 and 5 from [23]. The problematic term is the second one, *i.e.* the best error term with respect to functions in the virtual space. The approach used in [23], which is the p -version of Proposition 4.2 from [54], does not hold anymore

in the enhanced version of VEM, or, better, still works following *e.g.* Theorem 11 from [41], providing yet an extremely suboptimal p error convergence rate. At the best of the authors knowledge, the p approximation of the “best virtual” error term in terms of functions in enhanced space \tilde{V}_p defined in (2.41) (as well as in terms of functions in the enhanced space introduced in [41]) is still an open problem. On the other hand, the error analysis with space V_p is available in [23, 28].

The second issue we underline, is that the space \tilde{V}_p defined in (2.41) is more suited for the construction of the multigrid algorithm than space V_p defined in (2.7), as will be clear from Section 3.

Let us summarize the strategy we will follow. We consider a discretization of (2.1) by means of the virtual element method (2.4) employing as an approximation space V_p defined in (2.7). The associated linear system (2.36) coincides algebraically with the one arising by employing the VE space \tilde{V}_p defined in (2.41). For this reason, we can solve system (2.36) by means of a multigrid algorithm based on the sequences of spaces \tilde{V}_p defined in (2.41).

Having the vector of degrees of freedom \mathbf{u}_p , one can reconstruct functions in two different spaces: either in space V_p defined in (2.7), or in space \tilde{V}_p defined in (2.41). The discrete solution in the former space is the one to be taken into account, since it has the proper p approximation properties.

3. MULTIGRID ALGORITHM WITH NON-INHERITED SUBLEVEL SOLVERS

In this section, we present a p -VEM multigrid algorithm and the key ingredients for its formulation.

In the construction of our multigrid algorithm, we will make use of two key ingredients. The first one are suitable (computable) interspace operators, *i.e.* prolongation/restriction operators between two VE spaces. These operators will be constructed by employing the properties of the following space-dependent inner product:

$$(w_p, v_p)_p = \sum_{i=1}^{\dim(\tilde{V}_p)} \text{dof}_i(w_p) \text{dof}_i(v_p), \quad \forall w_p, v_p \in \tilde{V}_p. \quad (3.1)$$

The second ingredient is a suitable smoothing scheme B_p , which aims at reducing the high frequency components of the error.

We aim at introducing a multigrid iterative method for the solution of the linear system in (2.36), which we recall is given by:

$$\mathbf{A}_p \cdot \mathbf{u}_p = \mathbf{f}_p, \quad (3.2)$$

where the coefficient matrix \mathbf{A}_p and the right-hand side \mathbf{f}_p are the matrix representations with respect to the their expansion in the canonical basis of space \tilde{V}_p , defined in (2.41), of the operators:

$$(A_p w_p, v_p)_p = a_p(w_p, v_p), \quad (\mathbf{f}_p, v_p)_p = \langle \mathbf{f}_p, v_p \rangle_p, \quad \forall w_p, v_p \in \tilde{V}_p, \quad (3.3)$$

cf. (2.17) and (2.13), respectively.

In order to introduce our p -multigrid method, we assume that $p \geq 2$ and we consider a sequence of VE spaces given by $\tilde{V}_p, \tilde{V}_{p-1}, \dots, \tilde{V}_1$, where the ℓ th level is given by \tilde{V}_ℓ , $\ell = 1, \dots, p - 1$. Let now consider the linear system of equations on level ℓ : $\mathbf{A}_\ell \cdot \mathbf{z}_\ell = \mathbf{g}_\ell$. We denote by $\text{MG}(\ell, \mathbf{g}_\ell, \mathbf{z}_\ell^{(0)}, m_2)$ one iteration obtained by applying the ℓ th level iteration of our MG scheme to the above linear system, with initial guess $\mathbf{z}_\ell^{(0)}$ and using m_2 post-smoothing steps, respectively. For $\ell = 1$, (coarsest level) the solution is computed with a direct method, that is $\text{MG}(1, \mathbf{g}_1, \mathbf{z}_1^{(0)}, m_2) = \mathbf{A}_1^{-1} \mathbf{g}_1$, while for $\ell > 1$ we adopt the recursive procedure described in Algorithm 3.1.

Algorithm 3.1. ℓ th level of the p -multigrid algorithm.

Coarse grid correction:
 $\mathbf{r}_{\ell-1} = \mathbf{I}_{\ell}^{\ell-1} \cdot (\mathbf{g}_{\ell} - \mathbf{A}_{\ell} \cdot \mathbf{z}_{\ell}^{(0)});$ (restriction of the residual)
 $\bar{\mathbf{e}}_{\ell-1} = \text{MG}(\ell-1, \mathbf{r}_{\ell-1}, \mathbf{0}_{\ell-1}, m_2);$ (approximation of the residual equation ...)
 $\mathbf{e}_{\ell-1} = \text{MG}(\ell-1, \mathbf{r}_{\ell-1}, \bar{\mathbf{e}}_{\ell-1}, m_2);$ ($\dots \mathbf{A}_{p-1} \cdot \mathbf{z}_{p-1} = \mathbf{r}_{p-1}$)
 $\mathbf{z}_{\ell}^{(1)} = \mathbf{z}_{\ell}^{(0)} + \mathbf{I}_{\ell-1}^{\ell} \cdot \mathbf{e}_{\ell-1};$ (error correction step)
Post-smoothing:
for $i = 2 : m_2 + 1$
 $\mathbf{z}_{\ell}^{(i)} = \mathbf{z}_{\ell}^{(i-1)} + \mathbf{B}_{\ell}^{-1} \cdot (\mathbf{g}_{\ell} - \mathbf{A}_{\ell} \cdot \mathbf{z}_{\ell}^{(i-1)});$
end for
 $\text{MG}(\ell, \mathbf{g}_{\ell}, \mathbf{z}_{\ell}^{(0)}, m_2) = \mathbf{z}_{\ell}^{(m_2+1)}.$

In presenting Algorithm 3.1, we used some objects that are not defined so far. In particular, $\mathbf{I}_{\ell-1}^{\ell}$ and $\mathbf{I}_{\ell}^{\ell-1}$ denote the matrix representation of the interspace operators defined in Section 3.2, while \mathbf{B}_p denotes the matrix representation of the smoothing operator defined in Section 3.3.

For a given, user defined tolerance tol and a given initial guess $\mathbf{u}_p^{(0)}$, the full p -multigrid algorithm employed to solve (3.2) is summarized in Algorithm 3.2; its analysis is presented in the forthcoming Section 4.

Algorithm 3.2. p -multigrid algorithm: $\tilde{\mathbf{u}}_p = \text{MG}(p, \mathbf{f}_p, \tilde{\mathbf{u}}_p^{(0)}, m_2).$

$\mathbf{r}_p^{(0)} = \mathbf{f}_p - \mathbf{A}_p \cdot \tilde{\mathbf{u}}_p^{(0)};$
while $\|\mathbf{r}_p^{(i)}\| \leq \text{tol} \|\mathbf{f}_p\|$ **do**
 $\tilde{\mathbf{u}}_p^{(i+1)} = \text{MG}(p, \mathbf{f}_p, \tilde{\mathbf{u}}_p^{(i)}, m_2);$
 $\mathbf{r}_p^{(i+1)} = \mathbf{f}_p - \mathbf{A}_p \cdot \tilde{\mathbf{u}}_p^{(i+1)};$
 $i \rightarrow i + 1;$
end while

Remark 3.3. As a byproduct, we underline that it is possible to employ multigrid algorithms where two “adjacent” levels, associated to spaces V_{p_1} and V_{p_2} , respectively, satisfy $|p_1 - p_2| \geq 2$. In such cases, to build the interspace operators, it suffices to modify the definition (2.38) by using a “larger” enhancing technique and imposing that the laplacian of functions in the virtual space is a polynomial of higher degree, and then reduce the space with additional constraints on the L^2 -projectors.

3.1. Space-dependent inner products

The aim of this section is to prove the following result on the space-dependent inner product (3.1), which will be useful for the forthcoming analysis.

Theorem 3.4. *Let $(\cdot, \cdot)_p$ be defined as in (3.1). Then, the following holds true:*

$$\beta_*(p) |v_p|_{1,\Omega}^2 \lesssim (v_p, v_p)_p \lesssim \beta^*(p) |v_p|_{1,\Omega}^2, \quad \forall v_p \in \tilde{V}_p, \quad (3.4)$$

where $\beta_*(p) \gtrsim p^{-8}$ and $\beta^*(p) \lesssim 1$.

In order to prove Theorem 3.4, it suffices to combine the forthcoming technical results. The first one makes use of the following *auxiliary* space-dependent inner product defined as:

$$(u_p, v_p)_{p,\text{aux}} = \sum_{E \in \mathcal{T}} (u_p, v_p)_{p,\text{aux};E}, \quad \forall u_p, v_p \in \tilde{V}_p, \quad (3.5)$$

where the local contributions read:

$$(u_p, v_p)_{p, \text{aux}; E} = h_E^{-1} (u_p, v_p)_{0, \partial E} + h_E^{-2} (\Pi_{p-1}^0 u_p, \Pi_{p-1}^0 v_p)_{0, E}, \quad \forall u_p, v_p \in \tilde{V}_p(E), \quad \forall E \in \mathcal{T}. \quad (3.6)$$

Lemma 3.5. *Let $(\cdot, \cdot)_{p, \text{aux}}$ be defined in (3.5). Then, it holds:*

$$\tilde{\beta}_*(p) |v_p|_{1, \Omega}^2 \lesssim (v_p, v_p)_{p, \text{aux}} \lesssim \tilde{\beta}^*(p) |v_p|_{1, \Omega}^2, \quad \forall v_p \in \tilde{V}_p, \quad (3.7)$$

where $\tilde{\beta}_*(p) \gtrsim p^{-6}$ and $\tilde{\beta}^*(p) \lesssim 1$.

Before showing the proof, we recall that from Theorem 5 in [28] the following p polynomial inverse inequality on polygons holds:

$$\|q\|_{0, E} \lesssim (p+1)^2 \|q\|_{-1, E}, \quad \forall q \in \mathbb{P}_p(E), \quad (3.8)$$

where:

$$\|\cdot\|_{-1, E} = \sup_{\Phi \in H_0^1(E) \setminus \{0\}} \frac{(\cdot, \Phi)_{0, E}}{|\Phi|_{1, E}}. \quad (3.9)$$

Proof of Lemma 3.5. The proof is slightly different from the one for the stability bounds (2.24); in fact, here we work on the complete virtual space and not on $\ker(\Pi_p^\nabla)$, being Π_p^∇ defined in (2.12). In the following, we neglect the dependence on the size of the elements since we are assuming that the mesh is fixed; the general case follows from a scaling argument.

The upper bound follows from a trace inequality and the stability of orthogonal projection

$$(v_p, v_p)_{p, \text{aux}; E} = \|v_p\|_{0, \partial E}^2 + \|\Pi_{p-1}^0 v_p\|_{0, E}^2 \lesssim \|v_p\|_{1, E}, \quad \forall v_p \in \tilde{V}_p, \quad (3.10)$$

and summing up on all the mesh elements and applying the Poincarè inequality.

For the lower bound, by using an integration by parts and the definition of the local *auxiliary* space (2.38), we have:

$$|v_p|_{1, E}^2 = \int_E \nabla v_p \cdot \nabla v_p = \int_E -\Delta v_p \Pi_{p-1}^0 v_p + \int_{\partial E} \frac{\partial v_p}{\partial \mathbf{n}} v_p. \quad (3.11)$$

Owing to (3.8) and recalling that $\Delta v_p \in \mathbb{P}_{p-1}(E)$, we deduce:

$$\|\Delta v_p\|_{0, E} \lesssim p^2 \|\Delta v_p\|_{-1, E} = p^2 \sup_{\Phi \in H_0^1(E) \setminus \{0\}} \frac{(\Delta v_p, \Phi)_{0, E}}{|\Phi|_{1, E}} = p^2 \sup_{\Phi \in H_0^1(E) \setminus \{0\}} \frac{(\nabla v_p, \nabla \Phi)_{0, E}}{|\Phi|_{1, E}} \lesssim p^2 |v_p|_{1, E}. \quad (3.12)$$

We bound now the two terms appearing on the right-hand side of (3.11). Applying (3.12), we have:

$$\int_E \Delta v_p \Pi_{p-1}^0 v_p \leq \|\Delta v_p\|_{0, E} \|\Pi_{p-1}^0 v_p\|_{0, E} \lesssim p^2 \|\Pi_{p-1}^0 v_p\|_{0, E} |v_p|_{1, E}. \quad (3.13)$$

Applying next a Neumann trace inequality [57], Theorem A.33 a one dimensional hp inverse inequality, the interpolation estimates [60, 61] and (3.12), we get:

$$\int_{\partial E} \frac{\partial v_p}{\partial \mathbf{n}} v_p \leq \left\| \frac{\partial v_p}{\partial \mathbf{n}} \right\|_{-\frac{1}{2}, \partial E} \|v_p\|_{\frac{1}{2}, \partial E} \lesssim (|v_p|_{1, E} + \|\Delta v_p\|_{0, E}) p \|v_p\|_{0, \partial E} \lesssim p^3 \|v_p\|_{0, \partial E} |v_p|_{1, E}. \quad (3.14)$$

Substituting (3.13) and (3.14) in (3.11), we obtain:

$$|v_p|_{1,E} \lesssim p^3(\|v_p\|_{0,\partial E} + \|\Pi_{p-1}^0 v_p\|_{0,E}), \quad (3.15)$$

whence

$$|v_p|_{1,E}^2 \lesssim p^6(v_p, v_p)_{p,\text{aux};E}. \quad (3.16)$$

The assertion follows summing on all the elements. \square

Lemma 3.6. *Let $(\cdot, \cdot)_{p,\text{aux}}$ and $(\cdot, \cdot)_p$ be defined as in (3.5) and (3.1), respectively. Then it holds:*

$$p^{-2}(v_p, v_p)_{p,\text{aux}} \lesssim (v_p, v_p)_p \lesssim (v_p, v_p)_{p,\text{aux}}, \quad \forall v_p \in \tilde{V}_p. \quad (3.17)$$

Proof. The proof is a straightforward modification of the one of Lemma 2.5. \square

Remark 3.7. The choice (3.1) for the space-dependent inner product is crucial for the construction of the interspace operators, see Section 3.2. Moreover, we point out that it coincides with the usual choice for the space-dependent inner product in the hp dG-finite element framework, see [7, 12]. The finite element counterpart of Theorem 3.4 is much less technical, since it suffices to choose an L^2 orthonormal basis of polynomials as canonical basis; via Parseval identity, the (scaled) L^2 norm is spectrally equivalent to the space-dependent inner product (3.1); thus, the employment of polynomial inverse inequality implies a straightforward relation with the H^1 seminorm. In the VEM framework, it is not possible to proceed similarly for two reasons. The first one is that, at the best of the authors knowledge, inverse inequalities for functions in virtual spaces are not available; the second reason is that an L^2 orthonormal basis of functions in the virtual space is not computable, since such functions are not known explicitly.

3.2. Interspace operators

In this section, we introduce and construct suitable prolongation and restriction operators acting between the VE spaces $V_{\ell-1}$ and V_ℓ , $\ell = p, p-1, \dots, 2$. First of all, we stress that the sequence of local spaces $\tilde{V}_p(E)$, and thus the associated sequence of global spaces \tilde{V}_p , are not nested. As a consequence, we cannot define the prolongation interspace operator simply as the natural injection, as done for instance in [7, 12, 32, 33]. In our context, the prolongation operator:

$$I_{p-1}^p : \tilde{V}_{p-1} \rightarrow \tilde{V}_p, \quad (3.18)$$

associates to a function v_{p-1} in \tilde{V}_{p-1} a function $I_{p-1}^p v_{p-1}$ in \tilde{V}_p , having the same values as v_{p-1} for all the dofs that are in common with space \tilde{V}_{p-1} , while the remaining values of the dofs (*i.e.* the internal higher order ones) are computed using the enhancing constraints presented in definition (2.38). More precisely, we define $I_{p-1}^p : \tilde{V}_{p-1} \rightarrow \tilde{V}_p$ for $p \geq 2$ as:

$$\begin{cases} I_{p-1}^p v_{p-1} = v_{p-1}, & \text{on } \partial E, \\ \int_E I_{p-1}^p v_{p-1} m_\alpha = \int_E v_{p-1} m_\alpha = \text{dof}_\alpha(v_{p-1}) & \text{if } |\alpha| \leq p-3 \\ \int_E I_{p-1}^p v_{p-1} m_\alpha = \int_E \Pi_{p-3}^0 v_{p-1} m_\alpha = 0 & \text{if } |\alpha| = p-2 \end{cases} \quad (3.19)$$

since m_α are the elements of an $L^2(E)$ -orthonormal basis of $\mathbb{P}_p(E)$. We recall that the third equation in (3.19) follows from the enhancing constraints in the definition of local spaces (2.38). The restriction operator I_p^{p-1} is

defined as the adjoint of I_{p-1}^p with respect to the space-dependent inner product defined in (3.1), *i.e.*:

$$(I_p^{p-1}v_p, w_{p-1})_{p-1} = (v_p, I_{p-1}^p w_{p-1})_p, \quad \forall v_p \in \tilde{V}_p, \quad \forall w_{p-1} \in \tilde{V}_{p-1}. \quad (3.20)$$

We remark that, thanks to definition (3.1) of the space-dependent inner product, the matrix associated with I_p^{p-1} is the transpose of the matrix associated with the operator I_{p-1}^p .

3.3. Smoothing scheme and spectral bounds

In this section, we introduce and discuss the smoothing scheme entering in the multigrid algorithm. To this aim, we introduce the following space-dependent norms:

$$\|v_p\|_{s,p} = \sqrt{(A_p^s v_p, v_p)_p}, \quad \forall v_p \in \tilde{V}_p, \quad s \in \mathbb{R}^+. \quad (3.21)$$

We highlight that it holds:

$$\|v_p\|_{1,p}^2 = a_p(v_p, v_p). \quad (3.22)$$

Since the matrix \mathbf{A}_p is a symmetric positive definite matrix, there exists an orthonormal (with respect to the inner product $(\cdot, \cdot)_p$) basis of eigenvectors of \mathbf{A}_p , and the associated eigenvalues are real and strictly positive. Let $\{\psi_i, \lambda_i\}_{i=1}^{\dim(\tilde{V}_p)}$ be the related set of eigenpairs. We show now a bound of the spectrum of \mathbf{A}_p in terms of p .

Lemma 3.8. *The following upper bound Λ_p for the spectrum of \mathbf{A}_p holds true:*

$$\Lambda_p \lesssim \frac{\alpha^*(p)}{\beta_*(p)}, \quad (3.23)$$

where $\alpha^*(p)$ and $\beta_*(p)$ are introduced in (2.45) and (3.4), respectively.

Proof. Let λ_i be an eigenvalue of \mathbf{A}_p and let ψ_i be the associated normalized eigenvector. Then:

$$\mathbf{A}_p \cdot \psi_i = \lambda_i \psi_i \implies (A_p \psi_i, \psi_i)_p = \lambda_i (\psi_i, \psi_i)_p. \quad (3.24)$$

Owing to (2.16) and (3.4):

$$\lambda_i = \frac{(A_p \psi_i, \psi_i)_p}{(\psi_i, \psi_i)_p} = \frac{a_p(\psi_i, \psi_i)}{(\psi_i, \psi_i)_p} \lesssim \alpha^*(p) \frac{|\psi_i|_{1,\Omega}^2}{(\psi_i, \psi_i)_p} \lesssim \frac{\alpha^*(p)}{\beta_*(p)}. \quad (3.25)$$

□

As a smoothing scheme, we choose a Richardson scheme, which is given by:

$$B_p = \tilde{\Lambda}_p \cdot \text{Id}_p, \quad (3.26)$$

where $\tilde{\Lambda}_p \leq \Lambda_p$. A numerical study concerning the (sharp) dependence of Λ_p on p of the spectral bound Λ_p is presented in Section 5.

3.4. Error propagator operator

As in the classical analysis of the multigrid algorithms [33], in this section we introduce and analyze the error propagator operator. To this aim, we firstly consider a “projection” operator $P_p^{p-1} : \tilde{V}_p \rightarrow \tilde{V}_{p-1}$, defined as the adjoint of I_{p-1}^p with respect to inner product $a_p(\cdot, \cdot)$, i.e.:

$$a_{p-1}(v_{p-1}, P_p^{p-1} w_p) = a_p(I_{p-1}^p v_{p-1}, w_p), \quad v_{p-1} \in \tilde{V}_{p-1}, w_p \in \tilde{V}_p. \quad (3.27)$$

The following auxiliary result holds.

Lemma 3.9. *Let $q_{p-1} \in \tilde{V}_{p-1}$ be such that*

$$A_{p-1} q_{p-1} = r_{p-1}, \quad \text{with} \quad r_{p-1} = I_p^{p-1}(g_p - A_p z_p^{(0)}), \quad (3.28)$$

where I_p^{p-1} is defined in (3.20), while $z_p^{(0)}$ is the initial guess of the algorithm and A_p and A_{p-1} are defined in (3.3). Then, it holds:

$$q_{p-1} = P_p^{p-1}(z_p - z_p^{(0)}), \quad (3.29)$$

where P_p^{p-1} is defined in (3.27) and where we recall that z_p is the solution of $A_p z_p = g_p$.

Proof. As the proof is very similar to its analogous version in Lemma 6.4.2 from [33], here we briefly sketch it. For all $v_{p-1} \in \tilde{V}_{p-1}$:

$$\begin{aligned} a_{p-1}(q_{p-1}, v_{p-1}) &= (A_{p-1} q_{p-1}, v_{p-1})_{p-1} = (r_{p-1}, v_{p-1})_{p-1} = (I_p^{p-1}(g_p - A_p z_p^{(0)}), v_{p-1})_{p-1} \\ &= (A_p(z_p - z_p^{(0)}), I_{p-1}^p v_{p-1})_p = a_p(z_p - z_p^{(0)}, I_{p-1}^p v_{p-1}) = a_{p-1}(P_p^{p-1}(z_p - z_p^{(0)}), v_{p-1}). \end{aligned} \quad (3.30)$$

□

We now introduce the error propagator operator:

$$\begin{cases} \mathbb{E}_{1, m_2} v_p = 0, \\ \mathbb{E}_{p, m_2} v_p = [G_p^{m_2} (\text{Id}_p - I_{p-1}^p (\text{Id}_{p-1} - \mathbb{E}_{p-1, m_2}^2) P_p^{p-1})] v_p, \end{cases} \quad (3.31)$$

where the relaxation operator G_p is defined as:

$$G_p = \text{Id}_p - B_p^{-1} A_p, \quad B_p \text{ being introduced in (3.26)}. \quad (3.32)$$

The following result holds.

Theorem 3.10. *Let z_p and $z_p^{(m_2+1)}$ be the exact and the multigrid solutions associated with system (3.2), respectively. Then, given $z_p^{(0)}$ initial guess of the algorithm, it holds:*

$$z_p - z_p^{(m_2+1)} = \mathbb{E}_{p, m_2}(z_p - z_p^{(0)}), \quad (3.33)$$

where \mathbb{E}_{p, m_2} is defined in (3.31).

Proof. We follow the guidelines of Lemma 6.6.2 from [33] and proceed by induction. The initial step of the induction is straightforward since the system is solved exactly at the coarsest level. Therefore, we assume (3.33) true up to $p-1$ and we prove the claim for p .

Let q_{p-1} , e_{p-1} and \bar{e}_{p-1} be introduced in Algorithm 3.1; owing to the induction hypothesis applied to the residual equation, we have:

$$q_{p-1} - e_{p-1} = \mathbb{E}_{p-1, m_2}(q_{p-1} - \bar{e}_{p-1}) = \mathbb{E}_{p-1, m_2}^2(q_{p-1} - 0) = \mathbb{E}_{p-1, m_2}^2(q_{p-1}), \quad (3.34)$$

whence:

$$e_{p-1} = q_{p-1} - \mathbb{E}_{p-1, m_2}^2(q_{p-1}) = (\text{Id}_{p-1} - \mathbb{E}_{p-1, m_2}^2)q_{p-1}. \quad (3.35)$$

Thus:

$$\begin{aligned} z_p - z_p^{(m_2+1)} &= z_p - z_p^{(m_2)} - B_p^{-1}(g_p - A_p z_p^{(m_2)}) = (\text{Id}_p - B_p^{-1}A_p)(z_p - z_p^{(m_2)}) \\ &= (\text{Id}_p - B_p^{-1}A_p)^{m_2}(z_p - z_p^{(1)}) = G_p^{m_2}(z_p - z_p^{(1)}) = G_p^{m_2}(z_p - z_p^{(0)} - I_{p-1}^p e_{p-1}). \end{aligned} \quad (3.36)$$

Inserting (3.29) and (3.35) in (3.36), we get:

$$\begin{aligned} z_p - z_p^{(m_2+1)} &= G_p^{m_2} \left(z_p - z_p^{(0)} - I_{p-1}^p (\text{Id}_{p-1} - \mathbb{E}_{p-1, m_2}^2) P_p^{p-1} (z_p - z_p^{(0)}) \right) \\ &= G_p^{m_2} (\text{Id}_p - I_{p-1}^p (\text{Id}_{p-1} - \mathbb{E}_{p-1, m_2}^2) P_p^{p-1}) (z_p - z_p^{(0)}). \end{aligned}$$

□

4. CONVERGENCE ANALYSIS OF THE MULTIGRID ALGORITHM

We prove in Section 4.5 the convergence of the multigrid algorithm presented in Section 3. For the purpose, we preliminarily introduce some technical tools. In Section 4.1, we discuss the so-called smoothing property associated with the Richardson scheme (3.26). In Section 4.2, we show bounds related to the prolongation operator I_{p-1}^p defined in (3.18) and its adjoint with respect to the space-dependent inner product defined in (3.1). Bounds concerning the error correction steps are the topic of Section 4.3. Finally, in Sections 4.4 and 4.5, we treat the convergence of the two-level and multilevel algorithm, respectively.

4.1. Smoothing property

Lemma 4.1 (Smoothing property). *For any $v_p \in \tilde{V}_p$, it holds that:*

$$\begin{aligned} \| \| G_p^{m_2} v_p \| \|_{1,p} &\leq \| \| v_p \| \|_{1,p}, \\ \| \| G_p^{m_2} v_p \| \|_{s,p} &\lesssim \left(\frac{\alpha^*(p)}{\beta_*(p)} \right)^{\frac{s-t}{2}} (1 + m_2)^{\frac{t-s}{2}} \| \| v_p \| \|_{t,p}, \end{aligned} \quad (4.1)$$

for some $0 \leq t \leq s \leq 2$, $m_2 \in \mathbb{N} \setminus \{0\}$, where $\alpha^*(p)$ and $\beta_*(p)$ are defined in (2.45) and in (3.4), respectively.

Proof. The proof is analogous to that in Lemma 4.3 from [7]. For the sake of clarity, we report the details. To start with, we rewrite v_p in terms of the orthonormal basis of eigenvectors $\{\psi_i\}_{i=1}^{\dim(\tilde{V}_p)}$ of \mathbf{A}_p as follows:

$$v_p = \sum_{i=1}^{\dim(\tilde{V}_p)} v_i \psi_i, \quad \forall v_p \in \tilde{V}_p. \quad (4.2)$$

As a consequence,

$$G_p^{m_2} v_p = \left(\text{Id}_p - \frac{1}{\Lambda_p} A_p \right)^{m_2} v_p = \sum_{i=1}^{\dim(\tilde{V}_p)} \left(1 - \frac{\lambda_i}{\Lambda_p} \right)^{m_2} v_i \psi_i, \quad (4.3)$$

where Λ_p is the upper bound for the spectrum of \mathbf{A}_p presented in Lemma 3.8. Then, owing to the orthonormality of ψ_i with respect to the inner product $(\cdot, \cdot)_p$, we have:

$$\begin{aligned} \left\| \left\| G_p^{m_2} v_p \right\|_{s,p} \right\|_2^2 &= \sum_{i=1}^{\dim(\tilde{V}_p)} \left(1 - \frac{\lambda_i}{\Lambda_p} \right)^{2m_2} v_i^2 \lambda_i^s = \Lambda_p^{s-t} \sum_{i=1}^{\dim(\tilde{V}_p)} \left(1 - \frac{\lambda_i}{\Lambda_p} \right)^{2m_2} \frac{\lambda_i^{s-t}}{\Lambda_p^{s-t}} \lambda_i^t v_i^2 \\ &\leq \Lambda_p^{s-t} \max_{x \in [0,1]} (x^{s-t} (1-x)^{2m_2}) \left\| v_p \right\|_{t,p}^2 \lesssim \left(\frac{\alpha^*(p)}{\beta_*(p)} \right)^{s-t} (1+m_2)^{t-s} \left\| v_p \right\|_{t,p}^2, \end{aligned} \quad (4.4)$$

where in the last inequality we used Lemma 4.2 from [7] and (3.23). \square

4.2. Prolongation and projection operators

In this section, we prove bounds in the $\left\| \cdot \right\|_{1,p}$ norm of the prolongation and the projection operators defined in (3.18) and (3.27), respectively. We stress that this set of results deeply relies on the *new* enhancing strategy presented in the definition of the virtual space (2.38).

We start with a bound on the prolongation operator.

Theorem 4.2 (bound on the prolongation operator). *There exists c_{STAB} , positive constant independent of the discretization and multigrid parameters, such that:*

$$\left\| \left\| I_{p-1}^p v_{p-1} \right\|_{1,p} \right\| \leq c_{STAB} \sqrt{\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}} \left\| v_{p-1} \right\|_{1,p-1}, \quad \forall v_{p-1} \in \tilde{V}_{p-1}, \quad (4.5)$$

where $\alpha_*(p)$ and $\alpha^*(p)$ are introduced in (2.45) whereas $\beta_*(p)$ and $\beta^*(p)$ are introduced in (3.4).

Proof. Recalling bounds (2.16), (3.7) and the definition of the auxiliary space-dependent inner product (3.5), we have:

$$\begin{aligned} \left\| \left\| I_{p-1}^p v_{p-1} \right\|_{1,p} \right\|_2^2 &= \sum_{E \in \mathcal{T}} \left\| \left\| I_{p-1}^p v_{p-1} \right\|_{1,p;E} \right\|_2^2 = \sum_{E \in \mathcal{T}} a_p^E(I_{p-1}^p v_{p-1}, I_{p-1}^p v_{p-1}) \\ &\lesssim \alpha^*(p) a(I_{p-1}^p v_{p-1}, I_{p-1}^p v_{p-1}) \lesssim \frac{\alpha^*(p)}{\beta_*(p)} (I_{p-1}^p v_{p-1}, I_{p-1}^p v_{p-1})_p. \end{aligned} \quad (4.6)$$

We recall that:

$$(I_{p-1}^p v_{p-1}, I_{p-1}^p v_{p-1})_p = \sum_{j=1}^{\dim(\tilde{V}_p)} \text{dof}_j^2(I_{p-1}^p v_{p-1}). \quad (4.7)$$

Since $\{\mathbb{B}_p(\partial E)\}_{p=1}^{+\infty}$ defined in (2.6) is a sequence of *nested* space for all $E \in \mathcal{T}$, we directly have:

$$\text{dof}_{b,j}^2(I_{p-1}^p v_{p-1}) = \text{dof}_{b,j}^2(v_{p-1}), \quad (4.8)$$

where $\text{dof}_{b,j}(\cdot)$ denotes the j th boundary dof.

Now, we deal with the internal degrees of freedom. We cannot use the above nestedness argument since the sequence $\{\tilde{V}_p\}_{p=1}^{\dim(\tilde{V}_p)}$ is made of non-nested spaces. In order to overcome this hindrance, recalling the definition of the prolongation operator (3.19), we write:

$$\text{dof}_{i,j}(I_{p-1}^p v_{p-1}) = \frac{1}{|E|} \int_E I_{p-1}^p v_{p-1} m_\alpha = \begin{cases} \frac{1}{|E|} \int_E v_{p-1} m_\alpha, & \text{if } |\alpha| \leq p-3 \\ 0, & \text{if } |\alpha| = p-2 \end{cases}, \quad (4.9)$$

where $\text{dof}_{i,j}(\cdot)$ denotes the j th internal dof. As a consequence, it holds:

$$(I_{p-1}^p v_{p-1}, I_{p-1}^p v_{p-1})_p = (v_{p-1}, v_{p-1})_{p-1} = \|v_{p-1}\|_{0,p-1}^2. \quad (4.10)$$

Next, we relate $\|\cdot\|_{0,p-1}$ with $\|\cdot\|_{1,p-1}$. We note that:

$$\|v_{p-1}\|_{0,p-1}^2 \lesssim \beta^*(p) |v_{p-1}|_{1,E}^2 \lesssim \frac{\beta^*(p)}{\alpha_*(p)} \|v_{p-1}\|_{1,p-1}^2, \quad (4.11)$$

where we used in the last but one and in the last inequalities (3.4) and (2.16), respectively.

Combining (4.6), (4.10) and (4.11), we get the claim. \square

We show an analogous bound for the ‘‘projection’’ operator P_p^{p-1} introduced in (3.27).

Theorem 4.3 (bound on the ‘‘projection’’ operator). *There exists c_{STAB} , positive constant independent of the discretization and multigrid parameters, such that:*

$$\|P_p^{p-1} v_p\|_{1,p-1} \leq c_{STAB} \sqrt{\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}} \|v_p\|_{1,p}, \quad \forall v_p \in \tilde{V}_p, \quad (4.12)$$

where $\alpha_*(p)$ and $\alpha^*(p)$ are introduced in (2.45) whereas $\beta_*(p)$ and $\beta^*(p)$ are introduced in (3.4). The constant c_{STAB} is the same constant appearing in the statement of Theorem 4.2.

Proof. It suffices to note that:

$$\|P_p^{p-1} v_p\|_{1,p-1} = \max_{w_{p-1} \in \tilde{V}_{p-1} \setminus \{0\}} \frac{a_{p-1}(P_p^{p-1} v_p, w_{p-1})}{\|w_{p-1}\|_{1,p-1}} = \max_{w_{p-1} \in \tilde{V}_{p-1} \setminus \{0\}} \frac{a_p(v_p, I_{p-1}^p w_{p-1})}{\|w_{p-1}\|_{1,p-1}} \quad (4.13)$$

and then apply Theorem 4.2 along with a Cauchy-Schwarz inequality. \square

4.3. Error correction step

In this section, we prove a bound for the error correction step discussed in the multigrid algorithm, see Algorithm 3.2.

Theorem 4.4 (bound on the error correction step). *There exists a positive constant c independent of the discretization parameters such that:*

$$\|(Id_p - I_{p-1}^p P_p^{p-1})v_p\|_{0,p} \leq c \frac{\alpha^*(p)}{\alpha_*(p)^{\frac{3}{2}}} \frac{\beta^*(p)^{\frac{3}{2}}}{\beta_*(p)} \|v_p\|_{1,p}, \quad \forall v_p \in \tilde{V}_p, \quad (4.14)$$

where $\alpha_*(p)$ and $\alpha^*(p)$ are introduced in (2.45) whereas $\beta_*(p)$ and $\beta^*(p)$ are introduced in (3.4).

Proof. Applying (2.16) and (3.4), we have:

$$\begin{aligned} \left\| (\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p \right\|_{0,p}^2 &\lesssim \beta^*(p) \left| (\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p \right|_{1,\Omega}^2 \\ &\lesssim \beta^*(p) \alpha_*(p)^{-1} \sum_{E \in \mathcal{T}} \left\{ a_p^E \left((\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p, (\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p \right) \right\}. \end{aligned} \quad (4.15)$$

Therefore, we deduce:

$$\begin{aligned} &\left\| (\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p \right\|_{0,p}^2 \\ &\lesssim \beta^*(p) \alpha_*(p)^{-1} \sum_{E \in \mathcal{T}} \left\{ a_p^E(v_p, v_p) + a_p^E(I_{p-1}^p P_p^{p-1} v_p, I_{p-1}^p P_p^{p-1} v_p) - 2a_p^E(v_p, I_{p-1}^p P_p^{p-1} v_p) \right\} \\ &= \beta^*(p) \alpha_*(p)^{-1} \sum_{E \in \mathcal{T}} \left\{ a_p^E(v_p, v_p) + a_p^E(I_{p-1}^p P_p^{p-1} v_p, I_{p-1}^p P_p^{p-1} v_p) - 2a_{p-1}^E(P_p^{p-1} v_p, P_p^{p-1} v_p) \right\} \\ &\lesssim \beta^*(p) \alpha_*(p)^{-1} \sum_{E \in \mathcal{T}} \left\{ a_p^E(v_p, v_p) + \frac{\alpha^*(p) \beta^*(p)}{\alpha_*(p) \beta_*(p)} a_p^E(P_p^{p-1} v_p, P_p^{p-1} v_p) \right\}, \end{aligned} \quad (4.16)$$

where in the last inequality we applied Theorem 4.2 and we dropped the third term since it is negative. Finally, applying Theorem 4.3, we obtain:

$$\left\| (\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p \right\|_{0,p}^2 \lesssim \frac{\alpha^*(p)^2 \beta^*(p)^3}{\alpha_*(p)^3 \beta_*(p)^2} \|v_p\|_{1,p}^2, \quad (4.17)$$

whence the claim. \square

4.4. Convergence of the two-level algorithm

In this section, we prove the convergence of the two-level algorithm.

Theorem 4.5. *There exists a positive constant c_{2lvl} independent of the discretization and multilevel parameters, such that:*

$$\left\| \mathbb{E}_{p,m_2}^{2lvl} v_p \right\|_{1,p} \leq c_{2lvl} \Sigma_{p,m_2} \|v_p\|_{1,p}, \quad \forall v_p \in \tilde{V}_p, \quad (4.18)$$

where

$$\Sigma_{p,m_2} = \left(\frac{\alpha^*(p) \beta^*(p)}{\alpha_*(p) \beta_*(p)} \right)^{\frac{3}{2}} \cdot \frac{1}{\sqrt{1+m_2}} \quad (4.19)$$

and $\mathbb{E}_{p,m_2}^{2lvl}$ is the two-level error propagator operator:

$$\mathbb{E}_{p,m_2}^{2lvl} v_p = [G_p^{m_2} (\text{Id}_p - I_{p-1}^p P_p^{p-1})] v_p. \quad (4.20)$$

The constants $\alpha_*(p)$ and $\alpha^*(p)$ are introduced in (2.45), whereas the constants $\beta_*(p)$ and $\beta^*(p)$ are introduced in (3.4).

Proof. Using the smoothing property (4.1) and Theorem 4.4, we get:

$$\begin{aligned} \|\mathbb{E}_{p,m_2}^{2lv_1} v_p\|_{1,p} &= \|G_p^{m_2}(\text{Id}_p - I_{p-1}^p P_p^{p-1})v_p\|_{1,p} \lesssim \frac{1}{\sqrt{1+m_2}} \cdot \sqrt{\frac{\alpha^*(p)}{\beta_*(p)}} \|\text{Id}_p - I_{p-1}^p P_p^{p-1}\|_{0,p} \\ &\lesssim \frac{1}{\sqrt{1+m_2}} \cdot \sqrt{\frac{\alpha^*(p)}{\beta_*(p)}} \cdot \frac{\alpha^*(p)}{\alpha_*(p)^{\frac{3}{2}}} \cdot \frac{\beta^*(p)^{\frac{3}{2}}}{\beta_*(p)} \|v_p\|_{1,p} = \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}\right)^{\frac{3}{2}} \cdot \frac{1}{\sqrt{1+m_2}} \|v_p\|_{1,p}. \end{aligned} \quad (4.21)$$

□

As a consequence of Theorem 4.5, we deduce that taking m_2 , number of post-smoothing iterations large enough, the two-level algorithm converges, since the two-level error propagator operator $\mathbb{E}_{p,m_2}^{2lv_1}$ is a contraction. We point out that a sufficient condition for the convergence of the two-level algorithm is that the number of post-smoothing iterations m_2 must satisfy:

$$\sqrt{1+m_2} > c_{2lv_1}^{-1} \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}\right)^{\frac{3}{2}}, \quad (4.22)$$

see Remark 4.7 for more details. We stress that (4.22) is a sufficient condition only, in practice the number of post-smoothing steps needed for the convergence of the algorithm is much smaller; see numerical results in Section 5.

4.5. Convergence of the multilevel algorithm

In this section, we prove the main result of the paper, namely the convergence of our p -VEM multigrid algorithm.

Theorem 4.6. *Let Σ_{p,m_2} and c_{2lv_1} be defined as in Theorem 4.5. Let c_{STAB} be defined as in Theorem 4.3. Let $\alpha_*(p)$ and $\alpha^*(p)$ be defined in (2.45) and $\beta_*(p)$ and $\beta^*(p)$ be defined in (3.4). Then, there exists $\hat{c} > c_{2lv_1}$ such that, if the number of post-smoothing iterations satisfies:*

$$\sqrt{1+m_2} > \frac{c_{STAB}^2 \hat{c}^2}{\hat{c} - c_{2lv_1}} \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}\right)^{\frac{5}{2}}, \quad (4.23)$$

it holds:

$$\|\mathbb{E}_{p,m_2} v_p\|_{1,p} \leq \hat{c} \Sigma_{p,m_2} \|v_p\|_{1,p}, \quad \forall v_p \in \tilde{V}_p, \quad (4.24)$$

with $\hat{c} \Sigma_{p,m_2} < 1$. As a consequence, this implies that the multilevel algorithm converges uniformly with respect to the discretization parameters and the number of levels provided that m_2 satisfies (4.23), since \mathbb{E}_{p,m_2} is a contraction.

Proof. We proceed by induction in p . For $p = 1$, the assertion is trivially true, owing to (3.31). Assume next that the following induction hypothesis is valid:

$$\|\mathbb{E}_{p-1,m_2}^2 v_{p-1}\|_{1,p-1} \leq \hat{c} \Sigma_{p-1,m_2} \|v_{p-1}\|_{1,p-1}, \quad \forall v_{p-1} \in \tilde{V}_{p-1}. \quad (4.25)$$

We want to prove the assertion for induction step p .

From Theorem 4.5, we have that:

$$\Sigma_{p,m_2} = \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}\right)^{\frac{3}{2}} \cdot \frac{1}{\sqrt{1+m_2}}. \quad (4.26)$$

Recalling (3.31), we decompose the error propagator operator as:

$$\mathbb{E}_{p,m_2} v_p = G_p^{m_2} (\text{Id}_p - I_{p-1}^p P_p^{p-1}) v_p + G_p^{m_2} I_{p-1}^p \mathbb{E}_{p-1,m_2}^2 P_p^{p-1} v_p = \mathbb{E}_{p,m_2}^{2\text{lv}1} v_p + G_p^{m_2} I_{p-1}^p \mathbb{E}_{p-1,m_2}^2 P_p^{p-1} v_p. \quad (4.27)$$

Thus:

$$\|\mathbb{E}_{p,m_2} v_p\|_{1,p} \leq \|\mathbb{E}_{p,m_2}^{2\text{lv}1} v_p\|_{1,p} + \|G_p^{m_2} I_{p-1}^p \mathbb{E}_{p-1,m_2}^2 P_p^{p-1} v_p\|_{1,p} = I + II. \quad (4.28)$$

We bound the two terms separately. The first one is estimated directly applying the two-level error result, namely Theorem 4.5:

$$I \leq c_{2\text{lv}1} \Sigma_{p,m_2} \|v_p\|_{1,p}. \quad (4.29)$$

On the other hand, the second term can be bounded applying the smoothing property Lemma 4.1, the bounds regarding the interspace operator Theorem 4.2, the induction hypothesis and Theorem 4.3. We can write:

$$\begin{aligned} II &\leq \|I_{p-1}^p \mathbb{E}_{p-1,m_2}^2 P_p^{p-1} v_p\|_{1,p} \leq c_{\text{STAB}} \sqrt{\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}} \|\mathbb{E}_{p-1,m_2}^2 P_p^{p-1} v_p\|_{1,p-1} \\ &\leq c_{\text{STAB}} \sqrt{\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)}} \widehat{c}^2 \Sigma_{p-1,m_2}^2 \|P_p^{p-1} v_p\|_{1,p-1} \leq c_{\text{STAB}}^2 \widehat{c}^2 \frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)} \Sigma_{p-1,m_2}^2 \|v_p\|_{1,p}. \end{aligned} \quad (4.30)$$

We note that owing to (2.45) and (3.4), the following holds true:

$$\Sigma_{p-1,m_2}^2 = \left(\frac{\alpha^*(p-1)\beta^*(p-1)}{\alpha_*(p-1)\beta_*(p-1)} \right)^3 \cdot \frac{1}{1+m_2} \approx \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)} \right)^{\frac{3}{2}} \cdot \frac{1}{\sqrt{1+m_2}} \Sigma_{p,m_2}. \quad (4.31)$$

We deduce:

$$\|\mathbb{E}_{p,m_2} v_p\|_{1,p} \leq \underbrace{\left(c_{2\text{lv}1} + c_{\text{STAB}}^2 \widehat{c}^2 \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)} \right)^{\frac{5}{2}} \cdot \frac{1}{\sqrt{1+m_2}} \right)}_{\zeta} \Sigma_{p,m_2} \|v_p\|_{1,p}. \quad (4.32)$$

We want that ζ is such that $\zeta < \widehat{c} \Sigma_{p,m_2}$. In particular, we require:

$$c_{2\text{lv}1} + c_{\text{STAB}}^2 \widehat{c}^2 \left(\frac{\alpha^*(p)\beta^*(p)}{\alpha_*(p)\beta_*(p)} \right)^{\frac{5}{2}} \cdot \frac{1}{\sqrt{1+m_2}} < \widehat{c}, \quad (4.33)$$

which is in fact equivalent to (4.23). \square

Remark 4.7. We briefly comment on equations (4.22) and (4.23) highlighting the origin of the different terms:

- * the term $\frac{\alpha^*(p)}{\alpha_*(p)} \lesssim p^{10}$ originates from the spectral property (2.45) of the stabilization term S^E ; if it were possible to provide a discrete bilinear form (2.16) with continuity and coercivity constants provably independent of p , then $\frac{\alpha^*(p)}{\alpha_*(p)} \approx 1$;
- * the term $\frac{\beta^*(p)}{\beta_*(p)} \lesssim p^6$ is related to (3.4) which is not p robust; again, if it were possible to provide space-dependent inner products spectrally equivalent to the H^1 seminorm, then $\frac{\beta^*(p)}{\beta_*(p)} \approx 1$.

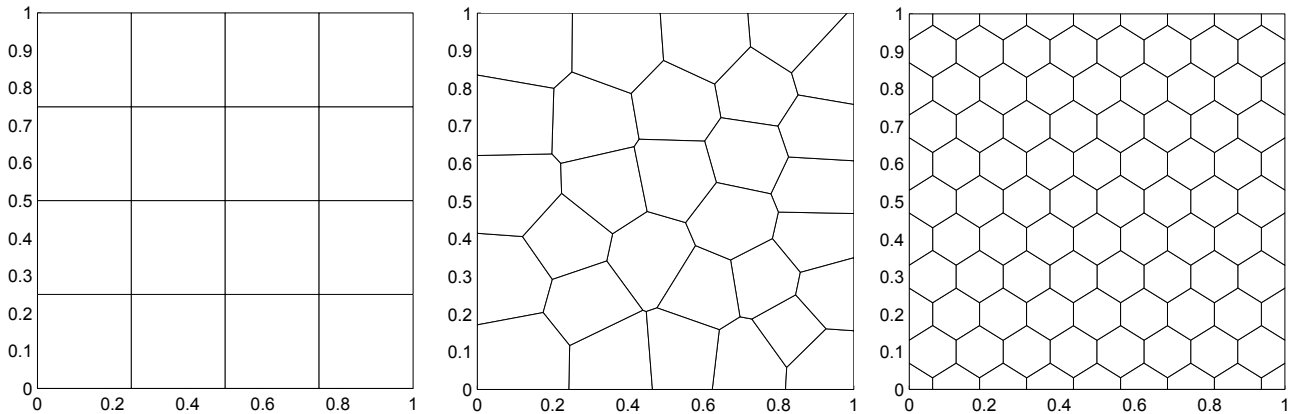


FIGURE 1. Meshes made of: squares (*left*), Voronoi-Lloyd polygons (*centre*), quasi-regular hexagons (*right*).

The existence of a p independent stabilization of the method and the existence of a computable virtual L^2 -orthonormal basis is still, at the best of the authors knowledge, an open issue.

In addition, we highlight that the ratios $\frac{\alpha^*(p)}{\alpha_*(p)}$ and $\frac{\beta^*(p)}{\beta_*(p)}$ depend also on the shape of the elements of the decomposition. More precisely, from the proofs of Lemmas 2.4, 2.5 and 3.5 and Theorem 3.4, we observe that such dependence appears when applying Neumann and trace inequalities. If the shape of the elements is allowed to be very general (small edges, collapsing bulks, ...), then the constants appearing in Neumann and trace inequalities may be very large. As a consequence, one expects that also the above mentioned ratios may get larger, see Remark 5.1.

5. NUMERICAL RESULTS

In this section, we test the performance of the multigrid solver for the p -version of the VEM by studying the behaviour of the convergence factor:

$$\rho = \exp\left(\frac{1}{N} \ln\left(\frac{\|r_N\|_2}{\|r_0\|_2}\right)\right), \quad (5.1)$$

where N denotes the iteration counts needed to reduce the residual below a given tolerance of 10^{-8} and r_N , r_0 are the final and the initial residuals, respectively. We also show that our multigrid algorithm can be employed as a preconditioner for the PCG method. Throughout the section we fix the maximum number of iterations to 1000 and consider three different kind of decompositions: meshes made of squares, Voronoi-Lloyd polygons and quasi-regular hexagons; *cf.* Figure 1. In Section 5.1, we present some tests aiming at assessing the performance of our multigrid scheme with different smoothers. In Section 5.2 we show that our multigrid method can be successfully employed as a preconditioner for the Conjugate Gradient (CG) iterative scheme, more precisely we consider a single iteration of the multigrid algorithm as a preconditioner to accelerate the Preconditioned CG method.

5.1. The p -multigrid algorithm as an iterative solver

In this section we investigate the performance of our multigrid scheme with different smoothers. We consider both the Richardson scheme (3.26) as well as a symmetrized Gauss-Seidel scheme as a smoother.

The first set of numerical experiment has been obtained based on employing a Richardson smoother. Before presenting the computed estimates of the convergence factor, we investigate numerically the behaviour of the

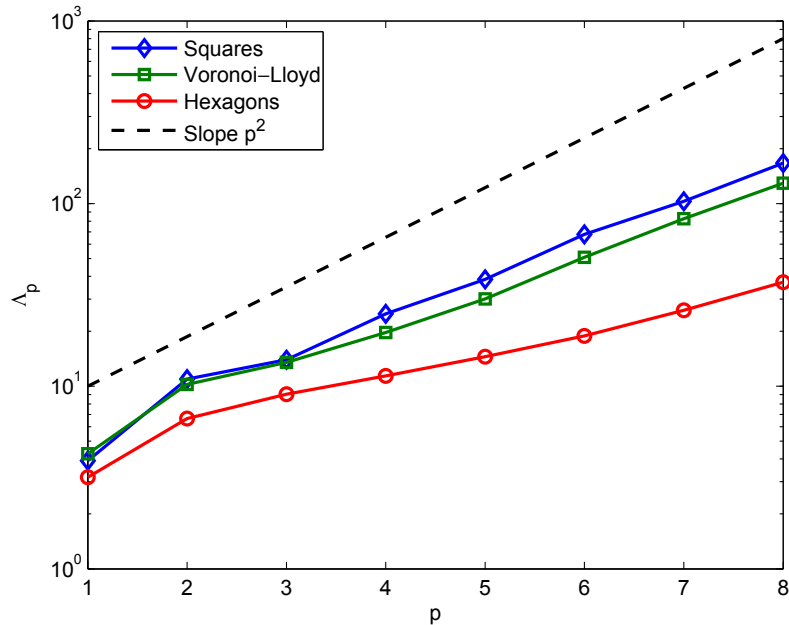


FIGURE 2. Maximum eigenvalue Λ_p of \mathbf{A}_p as a function of p .

TABLE 2. Convergence factor ρ of the p -multigrid scheme as a function of K (number of levels), p (“finest” level) and m_2 (number of post-smoothing steps). Richardson smoother. Mesh of squares.

K	$p = 2$	$p = 3$		$p = 4$		$p = 5$	
	2	2	3	3	4	3	4
$m_2 = 2$	0.99	x	0.97	x	0.97	x	x
$m_2 = 4$	0.97	x	0.95	x	0.92	x	x
$m_2 = 6$	0.96	0.93	0.92	0.79	0.88	x	0.85
$m_2 = 8$	0.95	0.69	0.89	0.74	0.84	0.98	0.82

smoothing parameter Λ_p associated with the Richardson scheme (3.26), for which a far-from-being-sharp bound is given in Lemma 3.8. As shown in Figure 2, where Λ_p as a function of p is shown, the maximum eigenvalue of \mathbf{A}_p seems to behave even better than p^2 , which is the expected behaviour in standard finite elements. The numerical tests presented in the following have been obtained with an approximation of Λ_p obtained (in a off line stage) with ten iterations of the power method.

Next, we numerically investigate the behaviour of the multigrid algorithm using a Richardson smoother. The results reported in Table 2 shows the computed convergence factor defined in ρ (5.1) as a function of the number of level K , the number of post-smoothing steps $m_2 = m$, and the degree of accuracy p employed at the “finest level” on a mesh made of squares, *cf.* Figure 1. Analogous results have been obtained on the other decompositions; such results are not reported here for the sake of brevity. As expected, increasing the number of post-smoothing m_2 implies a decreasing of the convergence factor ρ . Moreover, a minimum number of smoothing steps is required to guarantee the convergence of the underlying solver. We also observe that, even though both two-level and multilevel solvers converge for a fixed value of m , the number of iterations required to reduce the relative residual below the given tolerance grows with increasing p . A numerical estimate of the minimum

TABLE 3. Minimum number of post-smoothing steps needed to guarantee convergence.

K	$p = 2$		$p = 3$		$p = 4$			$p = 5$			$p = 6$		
	2		2	3	2	3	4	2	3	4	2	3	4
Squares	1		6	1	10	5	1	14	8	5	42	15	8
Voronoi-Lloyd	7		14	5	12	11	5	14	10	11	36	24	9
Hexagons	7		25	6	12	20	5	9	10	19	17	7	9

TABLE 4. Convergence factor ρ of the p -multigrid scheme as a function of K (number of levels), P (“finest” level) and m_2 (number of post-smoothing steps). Gauss-Seidel smoother. Mesh of squares.

K	$p = 2$		$p = 3$		$p = 4$		$p = 5$		
	2		2	3	3	4	3	4	
$m_2 = 2$	0.96		0.90	0.92	x		0.75	0.97	x
$m_2 = 4$	0.92		0.69	0.85	0.57		0.57	0.72	x
$m_2 = 6$	0.88		0.60	0.78	0.43		0.44	0.60	0.85
$m_2 = 8$	0.84		0.53	0.72	0.34		0.35	0.53	0.82

TABLE 5. Convergence factor ρ of the p -multigrid scheme as a function of K (number of levels), p (“finest” level) and m_2 (number of post-smoothing steps). Gauss-Seidel smoother. Mesh of Voronoi-Lloyd polygons.

K	$p = 2$		$p = 3$		$p = 4$		$p = 5$		
	2		2	3	3	4	3	4	
$m_2 = 8$	0.91		0.63	0.81	0.45		0.61	0.49	0.46
$m_2 = 10$	0.89		0.57	0.77	0.37		0.54	0.44	0.43
$m_2 = 12$	0.87		0.52	0.73	0.31		0.47	0.40	0.40
$m_2 = 14$	0.86		0.48	0.69	0.25		0.42	0.37	0.37

number of post-smoothing steps needed in practice to achieve convergence is reported in Table 3 for all the meshes depicted in Figure 1. This represents a *practical* indication for (4.23).

As expected, such a minimum number depends on the polynomial degree employed in the finest level.

We next investigate the behaviour of our MG algorithm whenever a symmetrized Gauss-Seidel scheme as a smoother is employed. We recall that the smoothing matrix B_p associated with the symmetrized Gauss-Seidel operator now reads:

$$B_p = \begin{cases} L_p & \text{if the post-smoothing iteration is odd,} \\ L_p^T & \text{if the post-smoothing iteration is even,} \end{cases} \tag{5.2}$$

where L_p is the lower triangular part of A_p . We have repeated the set of experiments carried out before employing the same set of parameters: the results are shown in Tables 4 and 5. As expected, employing a symmetrized Gauss-Seidel smoother yields to an iterative scheme whose convergence factor is smaller than in the analogous cases with the Richardson smoother. In Table 5 we report the same results obtained on a mesh of Voronoi-Lloyd polygonal elements keep on increasing the number of post-smoothing steps: as expected the performance of the algorithm improves further. The same kind of results have been obtained on a regular hexagonal grid; for the sake of brevity these results have been omitted.

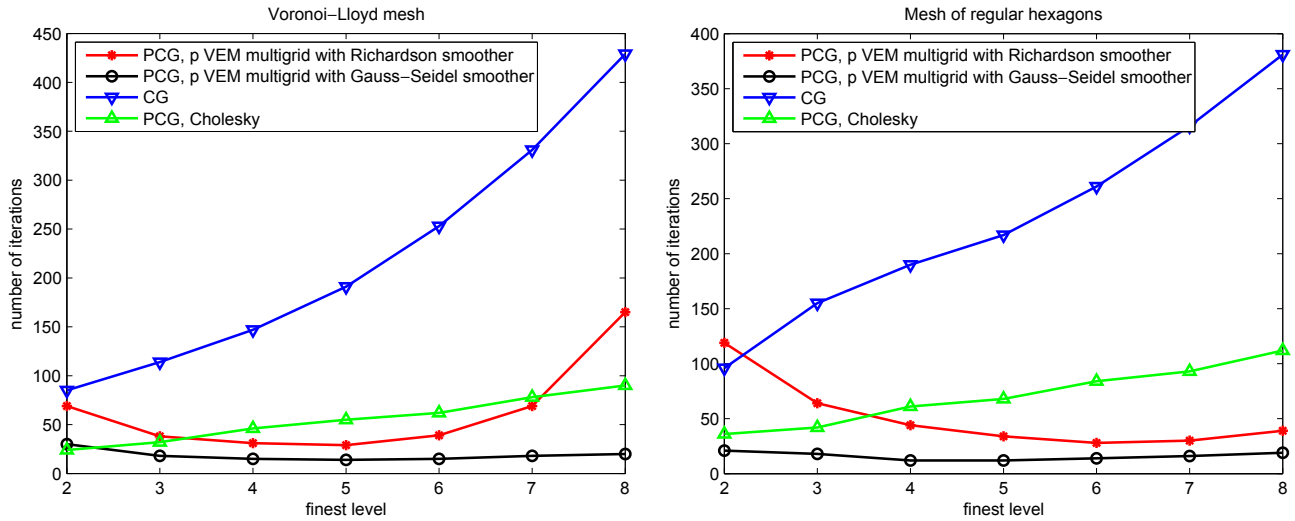


FIGURE 3. PCG iteration counts as a function of p with p -multigrid preconditioner (with either Richardson or Gauss-Seidel smoothers). For the sake of comparison the CG iteration counts without preconditioning and with an incomplete Cholesky preconditioner are also shown. For the p -multigrid preconditioner, the coarsest level is $p = 1$ and the number of post-smoothing steps is 8. Meshes made of: Voronoi-Lloyd polygons (*left*), quasi-regular hexagons (*right*).

Remark 5.1. We also carried out numerical tests employing meshes characterized by progressively increasing aspect ratios. We observed that the number of post-smoothing steps required for achieving convergence grows with such aspect ratio. This is consistent with what observed in Remark 4.7.

5.2. The p -multigrid algorithm as a preconditioner for the PCG method

In this set of experiments we aim at demonstrating that *a single iteration* of the p -multigrid algorithm can be successfully employed to precondition the CG method. In this set of experiments, the coarsest level is given by $p = 1$. In all the test cases, we have employed as a stopping criterion in order to reduce the (relative) residual below a tolerance of 10^{-6} , with a maximum number of iterations set equal to 1000. In Figure 3, we compare the PCG iteration counts with our multigrid preconditioner, which is constructed employing either a Richardson or a Gauss-Seidel smoother and $m = 8$ post-smoothing steps. For the sake of comparison, we report the same quantities computed with the unpreconditioned CG method and with the PCG method with preconditioner given by an incomplete Cholesky factorization. As before, the results reported in Figure 3 have been obtained on the computational grids depicted in Figure 1. From Figure 3, we infer that PCG iteration counts needed to reduce the residual below a given tolerance seems to be almost constant whenever the multigrid preconditioner with Gauss-Seidel smoother is employed, even for a relatively small number of smoothing steps. In contrast, as expected, the incomplete Cholesky preconditioner does not provide a uniform preconditioner. Also, the multigrid preconditioner with Richardson smoother seems to perform well at least on regular hexagonal grids.

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