A VARIATIONAL APPROACH TO COHESIVE-DAMAGING CRACK PROPAGATION IN A BAR

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ABSTRACT

The problem of inception and growth of a damaging-cohesive crack in an elastic bar is considered. It is shown that the position where the crack actually forms can the obtained from the minimality conditions of an energy functional, including the bulk energy and the surface energy, while the equilibrium of the system is obtained from the stationarity conditions. The progressive damage of the cohesive interface is taken into account by means of a step by step procedure. The finite step solution is also shown to make stationary a functional defined for each step.

1 INTRODUCTION

Cohesive-crack models, pioneered by Barenblatt [1], are commonly used for the simulation of fracture in quasi-brittle materials. In this area, among various other directions of recent research, we mention here: (a) development of efficient finite-element formulations such as extended finite elements [2], (b) coupling between continuum damage modelling and cohesive crack propagation [3] (c) energy characterization and variational formulations [4]-[9]. The present paper focuses on this last issue, in the line of what proposed in [5]. The major difference with respect to previous works is the nonholonomic nature of the cohesive crack model which is explicitly taken into account by means of a step by step procedure. A variational property of the finite step solution is established and illustrated with reference to a bar, endowed with non-homogeneous fracture properties, subject to body forces and imposed displacements. The actual location of crack initiation is also shown to be governed by the minimality of an energy functional.

2 DEFINITION OF THE PROBLEM

A bar constrained at both ends and subjected to a body force $b(x)$ directed along its axis is considered. The reference configuration of the bar is represented by the interval $I = [0, L]$. In order to account for fractures, the admissible configurations are assumed to belong to a space of (possibly) discontinuous functions u satisfying the boundary conditions $u(0) = 0$ and $u(L) = \eta$. The set of points where u is discontinuous, denoted by S_u, is not prescribed a priori and may contain the endpoints of the bar. Incompenetration is imposed by the constraint $[u] \geq 0$. A cohesive damaging model of the type shown in Fig. 1 is assumed for the opening crack.
In the bulk, the current state of the bar is governed by the following equations, where ε is the longitudinal strain, σ the axial stress and E the Young's modulus

$$\varepsilon = \frac{du}{dx} \quad \text{compatibility (in the bulk)} \tag{1}$$

$$\frac{d\sigma}{dx} + b = 0 \quad \text{equilibrium (in the bulk) with non-zero body forces} \tag{2}$$

$$\sigma = E\varepsilon \quad \text{elastic (bulk) behavior} \tag{3}$$

while for every crack point $z \in S_u$ we have

$$[u](z) = u^{+}(z) - u^{-}(z) \geq 0 \quad \text{compatibility at the interface with a crack} \tag{4}$$

$$\sigma^{+}(z) = \sigma^{-}(z) \quad \text{equilibrium across a crack} \tag{5}$$

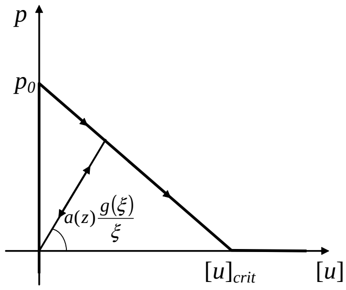


Figure 1: traction - displacement discontinuity law for cohesive damaging crack.

The bar behaves elastically as long as the axial stress $\sigma(x)$ is below a threshold $p_0(x)$, which is assumed to vary along the bar. The damage cumulated in the interface is taken into account by a non-decreasing internal variable ξ (Fig. 1). The maximum traction p which can be transmitted across the crack, is governed by a softening function $g(\xi)$ such that $a(x)\,g(0) = p_0(x)$, where $a(x)$ accounts for the variation of the resistance along the bar. For increasing opening displacement, the traction decreases following the softening branch $a(x)\,g(\xi)$. For $z \in S_u$, the traction-opening displacement cohesive law is expressed by the following set of relations

$$\begin{cases} p = p([u], \xi, z) = a(z)\,\frac{g(\xi)}{\xi}\,[u] & \text{for } \xi > 0 \\ p(0,0,z) \in [0, p_0(z)] & \text{for } \xi = 0 \end{cases} \quad \text{traction-crack opening displacement relation} \tag{6}$$

$$p - a(z)\,g(\xi) \leq 0 \qquad (p - a(z)\,g(\xi))\,\dot{\xi} = 0 \quad \dot{\xi} \geq 0, \quad \text{loading-unloading conditions} \tag{7}$$

$$g(\xi) = \frac{dG(\xi)}{d\xi} \text{ with } G(\xi) \text{ inelastic potential of internal variable } \xi \tag{8}$$

These definitions imply that, for a crack at $z \in S_u$

$$\xi(t,z) = \max_{\tau \leq t} [u](\tau, z) \tag{9}$$

The fracture energy, i.e. the energy necessary to create the discontinuity, is defined as

$$G_f(x) = \lim_{[u] \to [u]_{crit}} a(x)\,G([u]) < \infty \tag{10}$$

Note that cohesive models with $[u]_{crit} \to \infty$ are also feasible as long as $G_f(x)$ remains bounded.

3 FINITE-STEP PROBLEM

In view of the non-reversible nature of the crack evolution, the analysis of the bar response to an assigned history $\eta(t)$, $t \in [0,T]$, of the imposed displacement, requires the definition of a step-by-step time marching procedure. Let $0 = t_0 < t_1 < \ldots < t_k = T$ be a discretization of the time interval with a finite increment

$\Delta t > 0$. The structural response u_{n+1} at time t_{n+1} satisfies

$$\varepsilon_{n+1} = \frac{du_{n+1}}{dx} \qquad \sigma_{n+1} = E\varepsilon_{n+1} \qquad \frac{d\sigma_{n+1}}{dx} + b = 0 \quad \text{in the bulk} \tag{11}$$

$$[u]_{n+1}(z) = u^+_{n+1}(z) - u^-_{n+1}(z) \geq 0, \quad \sigma^+_{n+1}(z) = \sigma^-_{n+1}(z) \quad \text{for } z \in S_u \tag{12}$$

Knowing the configuration at time t_n the following stepwise-reversible behavior is assumed for the cohesive cracks

$$\xi_{n+1} = \xi_n + \Delta\xi \tag{13}$$

$$p_{n+1} = p\left([u]_{n+1}, \xi_{n+1}, z\right) = a(z)\,\frac{g(\xi_{n+1})}{\xi_{n+1}}\,[u]_{n+1} \quad \text{for } \xi_{n+1} > 0$$

traction-crack opening displacement relation (14)

$$p_{n+1} - a(z)\,g(\xi_{n+1}) \leq 0 \quad (p_{n+1} - a(z)\,g(\xi_{n+1}))\,\Delta\xi = 0 \quad \Delta\xi \geq 0$$

loading-unloading conditions (15)

Note that the above defined finite-step problem can be conceived as resulting from a backward-difference integration of the incremental problem, while the original (continuous) problem wil be recovered for $\Delta t \to 0$.

The above defined behavior is reversible in the finite step because $\dot{\xi} < 0$ is allowed as long as a final nonnegative increment $\Delta\xi$ of the internal variable is attained. It is therefore possible to define an energy $\tilde{G}([u], \xi_n)$ associated to the reversible finite-step law

$$\tilde{G}([u], \xi_n) = \begin{cases} \frac{g(\xi_n)}{2\xi_n}[u]^2 & \text{for} \quad [u] \leq \xi_n \\ G([u]) - G(\xi_n) + \frac{\xi_n g(\xi_n)}{2} & \text{for} \quad [u] \geq \xi_n \end{cases} \tag{16}$$

For $\xi_n = 0$, one has $\tilde{G}([u], \xi_n) = G([u])$.

The following functional is defined for the current time-step

$$U^\eta(u, \xi_n) = \frac{1}{2}\int_I E\left(\frac{du}{dx}\right)^2 dx + \sum_{z\in S_u} a(z)\tilde{G}([u](z), \xi_n(z)) - \int_I bu\,dx \tag{17}$$

Let $B(x)$ be a primitive of $b(x)$ vanishing in zero. Integrating by parts the body force integral, the functional $U^\eta(u, \xi_n)$ can be rewritten as

$$U^\eta(u, \xi_n) = \int_I \frac{du}{dx}\left(\frac{1}{2}E\frac{du}{dx} + B\right) dx + \sum_{z\in S_u}\Big(a(z)\tilde{G}([u](z), \xi_n(z)) + B(z)[u](z)\Big) - B(L)\eta \tag{18}$$

4 CRACK INITIATION AND OPENING PROBLEMS

The following crack initiation problem is considered. The bar is subjected to an assigned body force $b(x)$ whose intensity is such that the strength is not exceeded in any point of the bar. Then a growing positive displacement η is imposed at $x = L$ until the threshold value η_0 is reached for which, at a position $x = \bar{x}$, the stress σ reaches its limit value $p_0(\bar{x})$. The value of η_0 depends on the strength of the bar, which is governed by the function $p_0(x)$, and on the body force $b(x)$. For the sake of simplicity we will assume that

the function $p_0(x) + B(x)$ has a unique minimizer in the interval I.
For $\eta \to \eta_0^+$, (i.e. for $[u] \to 0^+$) and $\xi \equiv 0$, in view of the concavity of $G([u])$, the functional $U^{\eta\to\eta_0^+}(u, 0)$ can be shown to be minimized by a configuration with only one crack activated. We will denote by $x = \bar{x}$ the position of this crack. The following proposition holds.
Proposition. Starting from an elastic state ($\xi \equiv 0$), for an assigned value $\eta \to \eta_0^+$ of the imposed displacement, the value of u satisfying the governing equations can be obtained from the stationarity conditions of $U^{\eta\to\eta_0^+}(u, 0)$. The position $\bar{x}$ of the activated crack can be obtained from the minimality conditions of $U^{\eta\to\eta_0^+}(u, 0)$.
The proof of the first part of the proposition follows closely the path of reasoning proposed by Braides, Dal Maso and Garroni [5].
As for the position of the crack, one can note that in correspondence of the crack initiation $G([u])$ behaves like $g(0)[u]$. Thus the energy (18) can be written as

$$U^{\eta\to\eta_0^+}(u, 0) = \int_I \frac{du}{dx}\left(\frac{1}{2}E\frac{du}{dx} + B\right) dx + \sum_{z\in S_u}(p_0(z) + B(z))[u] \tag{19}$$

As the elastic energy does not depend on the position of cracks, it is clear that the minimizer will concentrate the jump $[u]$ in the point where $(p_0(z) + B(z))$ reaches its minimum. Therefore, the minimality condition imply that the position of the first crack, for $\eta \to \eta_0$, is given by the solution of the following minimization problem

$$\min_{x\in[0.L]} \{p_0(x) + B(x)\} \tag{20}$$

The crack opening problem is now considered at time t_n for an imposed displacement $\eta_n > \eta_0$. This time the crack is assumed to remain fixed in the position $\bar{x}$ of the first activation for $\eta = \eta_0$. A load step is considered where the imposed displacement is incremented by a quantity $\Delta\eta$. The functional defined in (17) is considered for $\xi_n > 0$. The following proposition holds.
Proposition. For fixed crack position $\bar{x}$, considering an evolutionary problem, discretized in time steps with finite increments Δt, the displacement u_{n+1} at time $t_n + \Delta t$, solution of the finite-step problem (11)-(15), is obtained as the (local) minimizer of the energy $U^\eta(u, \xi_n)$ with the boundary conditions $u(0) = 0$ and $u(L) = \eta(t_n + \Delta t)$ and ξ freezed at ξ_n.
Note that the incremental problem is explicit with respect to the internal variable ξ, therefore this variable can be updated indepently at the end of the step, when u_{n+1} is known.

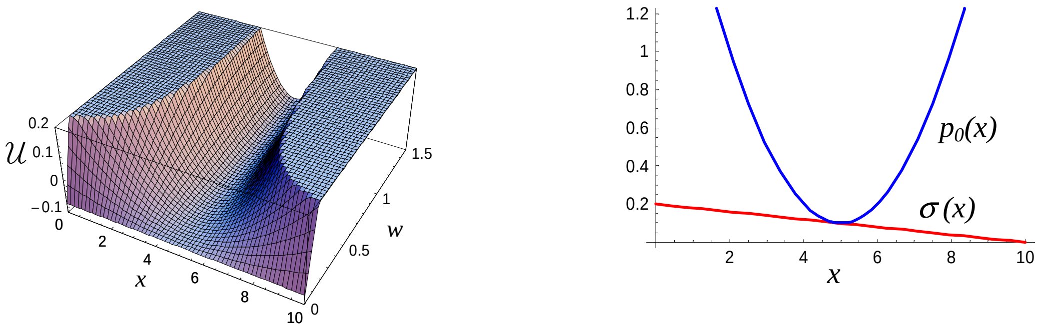
5 APPLICATION TO A BAR IN TENSION WITH CONSTANT AXIAL BODY FORCE

Consider a bar of length $L = 10$ mm and uniform elastic modulus $E = 1$ MPa, subject to a constant body force $b = 0.2E/L$ and a monotonically increasing imposed displacement $\eta(t)$. The bar is assumed to have a fracture strength $p_0(x) = a(x)\, g(0)$ varying along the bar with $a(x) = 1 + (x - L/2)^2$ and $p_0(L/2) = 0.1E$. Denoting by w the displacement discontinuity, a linear cohesive crack model is considered

$$g(w) = \begin{cases} g_0(1 - \frac{1}{w_{crit}}w) & \text{for } w \leq w_{crit} \\ 0 & \text{for } w \geq w_{crit} \end{cases}$$

with $w_{crit} = 0.15L$. The potential $G(w)$ is then given by

$$G(w) = \begin{cases} g_0(w - \frac{1}{2w_{crit}}w^2) & \text{for } w \leq w_{crit} \\ g_0\frac{w_{crit}}{2} & \text{for } w \geq w_{crit} \end{cases}$$



(a) (b)

Figure 2: (a) energy function for $\eta = \eta_0$; (b) crack position.

Let $\mathcal{A}\,(\eta, w)$ be the set of displacements such that $u(0) = 0$, $u(L) = \eta$ and $[u] = w$. Assuming a holonomic process ($\xi = 0$) and imposing equilibrium in the bulk, one can express the displacements in terms of the imposed η, the crack opening w and the crack position $\bar{x}$. Equivalently one can minimize U^{η} with respect to $u \in \mathcal{A}\,(\eta, w)$, thus obtaining the function $\mathcal{U}(\eta, w, \bar{x})$

$$\mathcal{U}(\eta, w, \bar{x}) = \min\{U^{\eta}(u, 0)\ :\ u \in \mathcal{A}\,(\eta, w)\}$$

For this example the solution $u(x)$ can be explicitly computed and the energy function can be obtained

$$\mathcal{U}(\eta, w, \bar{x}) = \frac{E\,\left(\eta - w\right)^2}{2\,L} + \frac{b\,(2\,w\bar{x} - L\,\left(\eta + w\right))}{2} - \frac{b^2 L^3}{24E} + \left[1 + \left(\bar{x} - \frac{L}{2}\right)^2\right] G(w) \tag{21}$$

Note that $\mathcal{U}(\eta, w, \bar{x})$ is differentiable with respect to w and $\bar{x}$. Local minimizers are found from

$$\begin{aligned}\frac{\partial \mathcal{U}}{\partial w} &= -\frac{E\,\left(\eta - w\right)}{L} + \frac{b\,(2\,\bar{x} - L\,)}{2} + \left[1 + \left(\bar{x} - \frac{L}{2}\right)^2\right] g(w)\ = 0 \\ \frac{\partial \mathcal{U}}{\partial \bar{x}} &= bw + (2\bar{x} - L)\,G(w) = 0\end{aligned} \tag{22}$$

The value η_0 of η at crack initiation and the crack position $\bar{x}$ can be obtained by solving (22) for η and $\bar{x}$ with $w = 0$. One obtains $\eta_0 = 0.099L$ and $\bar{x} = 0.49L$. Fig. 2a shows the plot of $\mathcal{U}$ as a function of the crack position and opening displacement for $\eta = \eta_0$ (for representation convenience values of $\mathcal{U}$ >0.2 have been cut in this plot). It should be noted that for $\eta = \eta_0$, $\bar{x} = 0.49L$ is the position of the point where the curve representing the stresses along the bar is tangent to the curve representing the fracture strength $p_0\,(x)$, see Fig. 2b.
For fixed crack position $\bar{x} = 0.49L$, the optimal value of energy $\mathcal{U}$ as a function of the imposed displacement is plotted in Fig. 3a. The red line represents the bulk energy, while the blue line corresponds to the sum of bulk and surface energy, defined only for $\eta \geq \eta_0$. For $\eta \geq \eta_0$ the minimum is obtained by activating the crack.
For $\eta > \eta_0$ the minimizers of (21) give a position of the crack different from $\bar{x} = 0.49L$ which is not feasible for the real problem. The contour plot of $\mathcal{U}$ for $\eta = 0.14L$ is shown in Fig. 3b. The optimal value

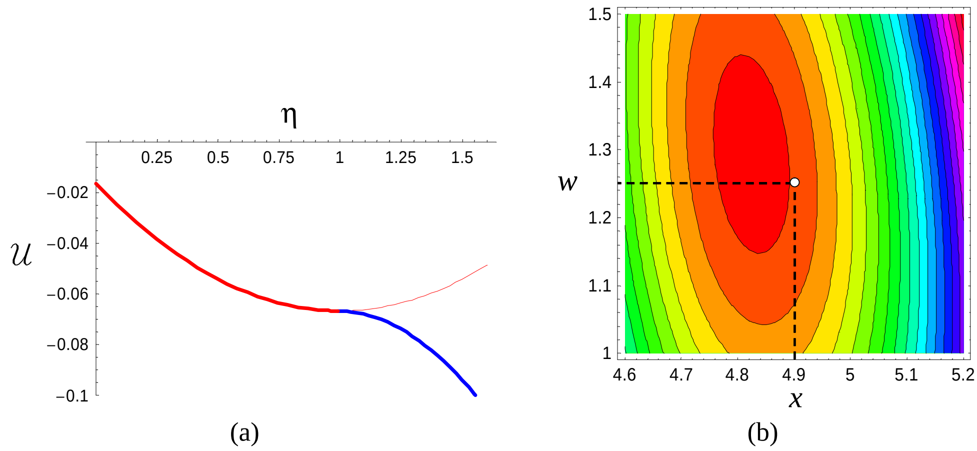


Figure 3: (a) energy vs imposed displacement; (b) contour plot of the energy for $\eta = 0.14L$.

is for $\bar{x} = 0.482L$, $w = 0.129$; the correct solution marked by a white dot in 3b is obtained minimizing $\mathcal{U}$ with respect to w for $\bar{x} = 0.49L$.

ACKNOWLEDGMENTS. This work has been carried out within the context of MIUR-Cofin2003 programme "Interfacial damage failure in structural systems: applications to civil engineering and emerging research fields".

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