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The Cotton Tensor and the Ricci Flow

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Abstract: We compute the evolution equation of the Cotton and the Bach tensor under the Ricci flow of a Riemannian manifold, with particular attention to the three dimensional case, and we discuss some applications.

Keywords: Ricci flow, Cotton tensor, Bach tensor, Ricci solitons

MSC: 53C21, 53C25

1 Preliminaries

The Riemann curvature operator of a Riemannian manifold (M^n, g) is defined, as in [6], by

$$\mathrm{Riem}(X, Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z\,.$$

In a local coordinate system the components of the $(3, 1)$–Riemann curvature tensor are given by $\mathrm{R}^l_{ijk}\frac{\partial}{\partial x^l} = \mathrm{Riem}\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big)\frac{\partial}{\partial x^k}$ and we denote by $\mathrm{R}_{ijkl} = g_{lm}\mathrm{R}^m_{ijk}$ its $(4, 0)$–version.

With the previous choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Riem}(v, w, v, w) = \mathrm{R}_{abcd}v^a w^b v^c w^d > 0$.

In all the paper the Einstein convention of summing over the repeated indices will be adopted.

The Ricci tensor is obtained by the contraction $\mathrm{R}_{ik} = g^{jl}\mathrm{R}_{ijkl}$ and $\mathrm{R} = g^{ik}\mathrm{R}_{ik}$ will denote the scalar curvature.

We recall the interchange of derivative formula,

$$\nabla^2_{ij}\omega_k - \nabla^2_{ji}\omega_k = \mathrm{R}_{ijkp}g^{pq}\omega_q\,,$$

and Schur lemma, which follows by the second Bianchi identity,

$$2g^{pq}\nabla_p\mathrm{R}_{qi} = \nabla_i\mathrm{R}\,.$$

They both will be used extensively in the computations that follows.

The so called Weyl tensor is then defined by the following decomposition formula (see [6, Chapter 3, Section K]) in dimension $n \geq 3$,

$$\mathrm{R}_{ijkl} = \frac{1}{n-2}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk} + \mathrm{R}_{jl}g_{ik} - \mathrm{R}_{jk}g_{il}) - \frac{\mathrm{R}}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) + \mathrm{W}_{ijkl}\,. \tag{1.1}$$

The Weyl tensor satisfies all the symmetries of the curvature tensor, moreover, all its traces with the metric are zero, as it can be easily seen by the above formula.

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In dimension three W is identically zero for every Riemannian manifold. It becomes relevant instead when $n \geq 4$ since its vanishing is a condition equivalent for (M^n, g) to be *locally conformally flat*, that is, around every point $p \in M^n$ there is a conformal deformation $\widetilde{g}_{ij} = e^f g_{ij}$ of the original metric g, such that the new metric is flat, namely, the Riemann tensor associated to $\widetilde{g}$ is zero in U_p (here $f : U_p \to \mathbb{R}$ is a smooth function defined in a open neighborhood U_p of p).
In dimension $n = 3$, instead, locally conformally flatness is equivalent to the vanishing of the following *Cotton tensor*

$$\mathrm{C}_{ijk} = \nabla_k \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{ik} - \frac{1}{2(n-1)}\big(\nabla_k \mathrm{R} g_{ij} - \nabla_j \mathrm{R} g_{ik}\big)\,, \tag{1.2}$$

which expresses the fact that the *Schouten* tensor

$$\mathrm{S}_{ij} = \mathrm{R}_{ij} - \frac{\mathrm{R} g_{ij}}{2(n-1)}$$

is a *Codazzi* tensor (see [1, Chapter 16, Section C]), that is, a symmetric bilinear form T_{ij} such that $\nabla_k \mathrm{T}_{ij} = \nabla_i \mathrm{T}_{kj}$.

By means of the second Bianchi identity, one can easily get (see [1]) that

$$\nabla^l \mathrm{W}_{lijk} = -\frac{n-3}{n-2}\mathrm{C}_{ijk}\,. \tag{1.3}$$

Hence, when $n \geq 4$, if we assume that the manifold is locally conformally flat (that is, $\mathrm{W} = 0$), the Cotton tensor is identically zero also in this case, but this is only a necessary condition.

By direct computation, we can see that the tensor C_{ijk} satisfies the following symmetries

$$\mathrm{C}_{ijk} = -\mathrm{C}_{ikj}, \qquad \mathrm{C}_{ijk} + \mathrm{C}_{jki} + \mathrm{C}_{kij} = 0\,, \tag{1.4}$$

moreover it is trace–free in any two indices,

$$g^{ij}\mathrm{C}_{ijk} = g^{ik}\mathrm{C}_{ijk} = g^{jk}\mathrm{C}_{ijk} = 0\,, \tag{1.5}$$

by its skew–symmetry and Schur lemma.

We suppose now that $(M^n, g(t))$ is a Ricci flow in some time interval, that is, the time–dependent metric $g(t)$ satisfies

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{R}_{ij}.$$

We have then the following evolution equations for the Christoffel symbols, the Ricci tensor and the scalar curvature (see for instance [7]),

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^k_{ij} &= -g^{ks}\nabla_i \mathrm{R}_{js} - g^{ks}\nabla_j \mathrm{R}_{is} + g^{ks}\nabla_s \mathrm{R}_{ij}\\
\frac{\partial}{\partial t}\mathrm{R}_{ij} &= \Delta \mathrm{R}_{ij} - 2\mathrm{R}^{kl}\mathrm{R}_{kijl} - 2g^{pq}\mathrm{R}_{ip}\mathrm{R}_{jq}\\
\frac{\partial}{\partial t}\mathrm{R} &= \Delta \mathrm{R} + 2|\mathrm{Ric}|^2\,.
\end{aligned} \tag{1.6}$$

All the computations which follow will be done in a fixed local frame, not in a moving frame.

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*Note*1.1. We remark that Huai-Dong Cao also, independently by us, worked out the computation of the evolution of the Cotton tensor in dimension three, in an unpublished note.

2 The Evolution Equation of the Cotton Tensor in 3D

The goal of this section is to compute the evolution equation under the Ricci flow of the Cotton tensor C_{ijk} in dimension three (see [5] for the evolution of the Weyl tensor), the general computation in any dimension is postponed to section 4.

In the special three–dimensional case we have,

$$R_{ijkl} = R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il} - \frac{R}{2}(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{2.1}$$

$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} - \frac{1}{4}\left(\nabla_k R g_{ij} - \nabla_j R g_{ik}\right), \tag{2.2}$$

hence, the evolution equations (1.6) become

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_{ij}^k &= -g^{ks}\nabla_i R_{js} - g^{ks}\nabla_j R_{is} + g^{ks}\nabla_s R_{ij}\\
\frac{\partial}{\partial t}R_{ij} &= \Delta R_{ij} - 6g^{pq}R_{ip}R_{jq} + 3RR_{ij} + 2|\mathrm{Ric}|^2 g_{ij} - R^2 g_{ij}\\
\frac{\partial}{\partial t}R &= \Delta R + 2|\mathrm{Ric}|^2 .
\end{aligned}$$

From these formulas we can compute the evolution equations of the derivatives of the curvatures assuming, from now on, to be in normal coordinates,

$$\frac{\partial}{\partial t}\nabla_l R = \nabla_l \Delta R + 2\nabla_l|\mathrm{Ric}|^2 ,$$

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_s R_{ij} &= \nabla_s\Delta R_{ij} - 6\nabla_s R_{ip}R_{jp} - 6R_{ip}\nabla_s R_{jp} + 3\nabla_s RR_{ij} + 3R\nabla_s R_{ij}\\
&\quad +2\nabla_s|\mathrm{Ric}|^2 g_{ij} - \nabla_s R^2 g_{ij} + (\nabla_i R_{sp} + \nabla_s R_{ip} - \nabla_p R_{is})R_{jp} + (\nabla_j R_{sp} + \nabla_s R_{jp} - \nabla_p R_{js})R_{ip}\\
&= \nabla_s\Delta R_{ij} - 5\nabla_s R_{ip}R_{jp} - 5R_{ip}\nabla_s R_{jp} + 3\nabla_s RR_{ij} + 3R\nabla_s R_{ij}\\
&\quad +2\nabla_s|\mathrm{Ric}|^2 g_{ij} - \nabla_s R^2 g_{ij} + (\nabla_i R_{sp} - \nabla_p R_{is})R_{jp} + (\nabla_j R_{sp} - \nabla_p R_{js})R_{ip}\\
&= \nabla_s\Delta R_{ij} - 5\nabla_s R_{ip}R_{jp} - 5R_{ip}\nabla_s R_{jp} + 3\nabla_s RR_{ij} + 3R\nabla_s R_{ij}\\
&\quad +2\nabla_s|\mathrm{Ric}|^2 g_{ij} - \nabla_s R^2 g_{ij} + C_{spi}R_{jp} + C_{spj}R_{ip} + R_{jp}[\nabla_i R g_{sp} - \nabla_p R g_{is}]/4 + R_{ip}[\nabla_j R g_{sp} - \nabla_p R g_{js}]/4 ,
\end{aligned}$$

where in the last passage we substituted the expression of the Cotton tensor.

We then compute,

$$\begin{aligned}
\frac{\partial}{\partial t}C_{ijk} &= \frac{\partial}{\partial t}\nabla_k R_{ij} - \frac{\partial}{\partial t}\nabla_j R_{ik} - \frac{\partial}{\partial t}\left(\nabla_k R g_{ij} - \nabla_j R g_{ik}\right)/4\\
&= \nabla_k\Delta R_{ij} - 5\nabla_k R_{ip}R_{jp} - 5R_{ip}\nabla_k R_{jp} + 3\nabla_k RR_{ij} + 3R\nabla_k R_{ij}\\
&\quad +2\nabla_k|\mathrm{Ric}|^2 g_{ij} - \nabla_k R^2 g_{ij} + C_{kpi}R_{jp} + C_{kpj}R_{ip}\\
&\quad +R_{jp}[\nabla_i R g_{kp} - \nabla_p R g_{ik}]/4 + R_{ip}[\nabla_j R g_{kp} - \nabla_p R g_{jk}]/4\\
&\quad -\nabla_j\Delta R_{ik} + 5\nabla_j R_{ip}R_{kp} + 5R_{ip}\nabla_j R_{kp} - 3\nabla_j RR_{ik} - 3R\nabla_j R_{ik}\\
&\quad -2\nabla_j|\mathrm{Ric}|^2 g_{ik} + \nabla_j R^2 g_{ik} - C_{jpi}R_{kp} - C_{jpk}R_{ip}\\
&\quad -R_{kp}[\nabla_i R g_{jp} - \nabla_p R g_{ij}]/4 - R_{ip}[\nabla_k R g_{jp} - \nabla_p R g_{kj}]/4\\
&\quad +(R_{ij}\nabla_k R - R_{ik}\nabla_j R)/2 - \left(\nabla_k\Delta R + 2\nabla_k|\mathrm{Ric}|^2\right)g_{ij}/4 + \left(\nabla_j\Delta R + 2\nabla_j|\mathrm{Ric}|^2\right)g_{ik}/4\\
&= \nabla_k\Delta R_{ij} - 5\nabla_k R_{ip}R_{jp} - 5R_{ip}\nabla_k R_{jp} + 3\nabla_k RR_{ij} + 3R\nabla_k R_{ij}\\
&\quad +3\nabla_k|\mathrm{Ric}|^2 g_{ij}/2 - \nabla_k R^2 g_{ij} + C_{kpi}R_{jp} + C_{kpj}R_{ip}\\
&\quad +R_{jk}\nabla_i R/4 - R_{jp}\nabla_p R g_{ik}/4 + R_{ik}\nabla_j R/4 - R_{ip}\nabla_p R g_{jk}/4\\
&\quad -\nabla_j\Delta R_{ik} + 5\nabla_j R_{ip}R_{kp} + 5R_{ip}\nabla_j R_{kp} - 3\nabla_j RR_{ik} - 3R\nabla_j R_{ik}\\
&\quad -3\nabla_j|\mathrm{Ric}|^2 g_{ik}/2 + \nabla_j R^2 g_{ik} - C_{jpi}R_{kp} - C_{jpk}R_{ip}
\end{aligned}$$

$$
\begin{aligned}
& -R_{kj}\nabla_i R/4 + R_{kp}\nabla_p R g_{ij}/4 - R_{ij}\nabla_k R/4 + R_{ip}\nabla_p R g_{kj}/4 \\
& +(R_{ij}\nabla_k R - R_{ik}\nabla_j R)/2 - \nabla_k \Delta R g_{ij}/4 + \nabla_j \Delta R g_{ik}/4 \\
= \; & \nabla_k \Delta R_{ij} - 5\nabla_k R_{ip} R_{jp} - 5R_{ip}\nabla_k R_{jp} + 13\nabla_k R R_{ij}/4 + 3R\nabla_k R_{ij} \\
& +3\nabla_k |\mathrm{Ric}|^2 g_{ij}/2 - \nabla_k R^2 g_{ij} + C_{kpi}R_{jp} + C_{kpj}R_{ip} - R_{jp}\nabla_p R g_{ik}/4 \\
& -\nabla_j \Delta R_{ik} + 5\nabla_j R_{ip} R_{kp} + 5R_{ip}\nabla_j R_{kp} - 13\nabla_j R R_{ik}/4 - 3R\nabla_j R_{ik} \\
& -3\nabla_j |\mathrm{Ric}|^2 g_{ik}/2 + \nabla_j R^2 g_{ik} - C_{jpi}R_{kp} - C_{jpk}R_{ip} \\
& +R_{kp}\nabla_p R g_{ij}/4 - \nabla_k \Delta R g_{ij}/4 + \nabla_j \Delta R g_{ik}/4
\end{aligned}
$$

and

$$
\Delta C_{ijk} = \Delta\nabla_k R_{ij} - \Delta\nabla_j R_{ik} - \Delta\nabla_k R g_{ij}/4 + \Delta\nabla_j R g_{ik}/4\,,
$$

hence,

$$
\begin{aligned}
\frac{\partial}{\partial t} C_{ijk} - \Delta C_{ijk} = \; & \nabla_k \Delta R_{ij} - \nabla_j \Delta R_{ik} - \Delta\nabla_k R_{ij} + \Delta\nabla_j R_{ik} \\
& -\nabla_k \Delta R g_{ij}/4 + \nabla_j \Delta R g_{ik}/4 + \Delta\nabla_k R g_{ij}/4 - \Delta\nabla_j R g_{ik}/4 \\
& -5\nabla_k R_{ip} R_{jp} - 5R_{ip}\nabla_k R_{jp} + 13\nabla_k R R_{ij}/4 + 3R\nabla_k R_{ij} \\
& +3\nabla_k |\mathrm{Ric}|^2 g_{ij}/2 - \nabla_k R^2 g_{ij} + C_{kpi}R_{jp} + C_{kpj}R_{ip} - R_{jp}\nabla_p R g_{ik}/4 \\
& +5\nabla_j R_{ip} R_{kp} + 5R_{ip}\nabla_j R_{kp} - 13\nabla_j R R_{ik}/4 - 3R\nabla_j R_{ik} \\
& -3\nabla_j |\mathrm{Ric}|^2 g_{ik}/2 + \nabla_j R^2 g_{ik} - C_{jpi}R_{kp} - C_{jpk}R_{ip} \\
& +R_{kp}\nabla_p R g_{ij}/4
\end{aligned}
$$

Now to proceed, we need the following commutation rules for the derivatives of the Ricci tensor and of the scalar curvature, where we will employ the special form of the Riemann tensor in dimension three given by formula (2.1),

$$
\begin{aligned}
\nabla_k \Delta R_{ij} - \Delta\nabla_k R_{ij} = \; & \nabla^3_{kll} R_{ij} - \nabla^3_{lkl} R_{ij} + \nabla^3_{lkl} R_{ij} - \nabla^3_{llk} R_{ij} \\
= \; & -R_{kp}\nabla_p R_{ij} + R_{klip}\nabla_l R_{jp} + R_{kljp}\nabla_l R_{ip} + \nabla^3_{lkl} R_{ij} - \nabla^3_{llk} R_{ij} \\
= \; & -R_{kp}\nabla_p R_{ij} + R_{ik}\nabla_j R/2 + R_{jk}\nabla_i R/2 \\
& -R_{kp}\nabla_i R_{jp} - R_{kp}\nabla_j R_{ip} + R_{lp}\nabla_l R_{jp} g_{ik} + R_{lp}\nabla_l R_{ip} g_{jk} \\
& -R_{li}\nabla_l R_{jk} - R_{lj}\nabla_l R_{ik} - R\nabla_j R g_{ik}/4 - R\nabla_i R g_{jk}/4 \\
& +R\nabla_i R_{jk}/2 + R\nabla_j R_{ik}/2 + \nabla_l\big(R_{klip}R_{pj} + R_{kljp}R_{pi}\big) \\
= \; & -R_{kp}\nabla_p R_{ij} + R_{ik}\nabla_j R/2 + R_{jk}\nabla_i R/2 \\
& -R_{kp}\nabla_i R_{jp} - R_{kp}\nabla_j R_{ip} + R_{lp}\nabla_l R_{jp} g_{ik} + R_{lp}\nabla_l R_{ip} g_{jk} \\
& -R_{li}\nabla_l R_{jk} - R_{lj}\nabla_l R_{ik} - R\nabla_j R g_{ik}/4 - R\nabla_i R g_{jk}/4 \\
& +R\nabla_i R_{jk}/2 + R\nabla_j R_{ik}/2 \\
& +\nabla_l\big(R_{ik}R_{lj} - R_{il}R_{kj} + R_{pl}R_{pj}g_{ik} - R_{pk}R_{pj}g_{il} - g_{ik}RR_{lj}/2 + g_{il}RR_{jk}/2 \\
& +R_{jk}R_{li} - R_{jl}R_{ki} + R_{pl}R_{pi}g_{jk} - R_{pk}R_{pi}g_{jl} - g_{jk}RR_{li}/2 + g_{jl}RR_{ik}/2\big) \\
= \; & -R_{kp}\nabla_p R_{ij} + R_{ik}\nabla_j R/2 + R_{jk}\nabla_i R/2 \\
& -R_{kp}\nabla_i R_{jp} - R_{kp}\nabla_j R_{ip} + R_{lp}\nabla_l R_{jp} g_{ik} + R_{lp}\nabla_l R_{ip} g_{jk} \\
& -R_{li}\nabla_l R_{jk} - R_{lj}\nabla_l R_{ik} - R\nabla_j R g_{ik}/4 - R\nabla_i R g_{jk}/4 \\
& +R\nabla_i R_{jk}/2 + R\nabla_j R_{ik}/2 \\
& -\nabla_i R_{pk} R_{pj} + \nabla_i R R_{jk}/2 + g_{ik}R_{pl}\nabla_l R_{pj} \\
& -R_{pk}\nabla_i R_{pj} - g_{ik}R\nabla_j R/4 + R\nabla_i R_{jk}/2 \\
& -\nabla_j R_{pk} R_{pi} + \nabla_j R R_{ik}/2 + g_{jk}R_{pl}\nabla_l R_{pi} \\
& -R_{pk}\nabla_j R_{pi} - g_{jk}R\nabla_i R/4 + R\nabla_j R_{ik}/2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -R_{kp}\nabla_p R_{ij} + R_{ik}\nabla_j R + R_{jk}\nabla_i R \\
& -2R_{kp}\nabla_i R_{jp} - 2R_{kp}\nabla_j R_{ip} + 2R_{lp}\nabla_l R_{jp} g_{ik} + 2R_{lp}\nabla_l R_{ip} g_{jk} \\
& -R_{li}\nabla_l R_{jk} - R_{lj}\nabla_l R_{ik} - R_{pj}\nabla_i R_{pk} - R_{pi}\nabla_j R_{pk} \\
& -R\nabla_j R g_{ik}/2 - R\nabla_i R g_{jk}/2 + R\nabla_i R_{jk} + R\nabla_j R_{ik}
\end{aligned}
$$

and

$$\nabla_k \Delta R - \Delta \nabla_k R = R_{kllp}\nabla_p R = -R_{kp}\nabla_p R\,.$$

Then, getting back to the main computation, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}C_{ijk} - \Delta C_{ijk} \quad = \quad & -R_{kp}\nabla_p R_{ij} + R_{ik}\nabla_j R + R_{jk}\nabla_i R \\
& -2R_{kp}\nabla_i R_{jp} - 2R_{kp}\nabla_j R_{ip} + 2R_{lp}\nabla_l R_{jp} g_{ik} + 2R_{lp}\nabla_l R_{ip} g_{jk} \\
& -R_{li}\nabla_l R_{jk} - R_{lj}\nabla_l R_{ik} - R_{pj}\nabla_i R_{pk} - R_{pi}\nabla_j R_{pk} \\
& -R\nabla_j R g_{ik}/2 - R\nabla_i R g_{jk}/2 + R\nabla_i R_{jk} + R\nabla_j R_{ik} \\
& +R_{jp}\nabla_p R_{ik} - R_{ij}\nabla_k R - R_{kj}\nabla_i R \\
& +2R_{jp}\nabla_i R_{kp} + 2R_{jp}\nabla_k R_{ip} - 2R_{lp}\nabla_l R_{kp} g_{ij} - 2R_{lp}\nabla_l R_{ip} g_{kj} \\
& +R_{li}\nabla_l R_{kj} + R_{lk}\nabla_l R_{ij} + R_{pk}\nabla_i R_{pj} + R_{pi}\nabla_k R_{pj} \\
& +R\nabla_k R g_{ij}/2 + R\nabla_i R g_{kj}/2 - R\nabla_i R_{kj} - R\nabla_k R_{ij} \\
& +R_{kp}\nabla_p R g_{ij}/4 - R_{jp}\nabla_p R g_{ik}/4 \\
& -5\nabla_k R_{ip} R_{jp} - 5R_{ip}\nabla_k R_{jp} + 13\nabla_k R R_{ij}/4 + 3R\nabla_k R_{ij} \\
& +3\nabla_k |\mathrm{Ric}|^2 g_{ij}/2 - \nabla_k R^2 g_{ij} + C_{kpi}R_{jp} + C_{kpj}R_{ip} - R_{jp}\nabla_p R g_{ik}/4 \\
& +5\nabla_j R_{ip} R_{kp} + 5R_{ip}\nabla_j R_{kp} - 13\nabla_j R R_{ik}/4 - 3R\nabla_j R_{ik} \\
& -3\nabla_j |\mathrm{Ric}|^2 g_{ik}/2 + \nabla_j R^2 g_{ik} - C_{jpi}R_{kp} - C_{jpk}R_{ip} + R_{kp}\nabla_p R g_{ij}/4 \\
= \quad & C_{kpi}R_{jp} + C_{kpj}R_{ip} - C_{jpi}R_{kp} - C_{jpk}R_{ip} \\
& +[2R_{lp}\nabla_l R_{jp} + 3R\nabla_j R/2 - R_{jp}\nabla_p R/2 - 3\nabla_j |\mathrm{Ric}|^2/2] g_{ik} \\
& +[-2R_{lp}\nabla_l R_{kp} - 3R\nabla_k R/2 + R_{kp}\nabla_p R/2 + 3\nabla_k |\mathrm{Ric}|^2/2] g_{ij} \\
& -R_{kp}\nabla_i R_{jp} + R_{jp}\nabla_i R_{kp} - 3\nabla_k R_{ip} R_{jp} - 4R_{ip}\nabla_k R_{jp} + 9\nabla_k R R_{ij}/4 + 2R\nabla_k R_{ij} \\
& +3\nabla_j R_{ip} R_{kp} + 4R_{ip}\nabla_j R_{kp} - 9\nabla_j R R_{ik}/4 - 2R\nabla_j R_{ik}
\end{aligned}
$$

Now, by means of the very definition of the Cotton tensor in dimension three (2.2) and the identities (1.4), we substitute

$$
\begin{aligned}
C_{kpj} - C_{jpk} &= -C_{kjp} - C_{jpk} = C_{pkj} \\
\nabla_l R_{jp} &= \nabla_j R_{lp} + C_{pjl} + \frac{1}{4}\big(\nabla_l R g_{pj} - \nabla_j R g_{pl}\big) \\
\nabla_l R_{kp} &= \nabla_k R_{lp} + C_{pkl} + \frac{1}{4}\big(\nabla_l R g_{pk} - \nabla_k R g_{pl}\big) \\
\nabla_i R_{jp} &= \nabla_j R_{ip} + C_{pji} + \frac{1}{4}\big(\nabla_i R g_{jp} - \nabla_j R g_{ip}\big) \\
\nabla_i R_{kp} &= \nabla_k R_{ip} + C_{pki} + \frac{1}{4}\big(\nabla_i R g_{kp} - \nabla_k R g_{ip}\big)
\end{aligned}
$$

in the last expression above, getting

$$
\begin{aligned}
\frac{\partial}{\partial t}C_{ijk} - \Delta C_{ijk} \quad = \quad & R_{jp}C_{kpi} - R_{kp}C_{jpi} + R_{ip}C_{pkj} \\
& +\Big[2R_{lp}\big(\nabla_j R_{lp} + C_{pjl} + \nabla_l R g_{pj}/4 - \nabla_j R g_{pl}/4\big) \\
& \quad + 3R\nabla_j R/2 - R_{jp}\nabla_p R/2 - 3\nabla_j |\mathrm{Ric}|^2/2\Big] g_{ik} \\
& +\Big[-2R_{lp}\big(\nabla_k R_{lp} + C_{pkl} + \nabla_l R g_{pk}/4 - \nabla_k R g_{pl}/4\big)
\end{aligned}
$$

$$
\begin{aligned}
& - 3R\nabla_k R/2 + R_{kp}\nabla_p R/2 + 3\nabla_k|\mathrm{Ric}|^2/2\Big] g_{ij} \\
& -R_{kp}\left(\nabla_j R_{ip} + C_{pji} + \nabla_i R g_{jp}/4 - \nabla_j R g_{ip}/4\right) \\
& +R_{jp}\left(\nabla_k R_{ip} + C_{pki} + \nabla_i R g_{kp}/4 - \nabla_k R g_{ip}/4\right) \\
& -3\nabla_k R_{ip} R_{jp} - 4R_{ip}\nabla_k R_{jp} + 9\nabla_k R R_{ij}/4 + 2R\nabla_k R_{ij} \\
& +3\nabla_j R_{ip} R_{kp} + 4R_{ip}\nabla_j R_{kp} - 9\nabla_j R R_{ik}/4 - 2R\nabla_j R_{ik} \\
= \; & R_{jp}\left(C_{kpi} + C_{pki}\right) - R_{kp}\left(C_{jpi} + C_{pji}\right) + R_{ip}C_{pkj} \\
& +2R_{lp}C_{pjl}g_{ik} - 2R_{lp}C_{pkl}g_{ij} \\
& +\left[R\nabla_j R - \nabla_j|\mathrm{Ric}|^2/2\right] g_{ik} - \left[R\nabla_k R - \nabla_k|\mathrm{Ric}|^2/2\right] g_{ij} \\
& -2\nabla_k R_{ip} R_{jp} - 4R_{ip}\nabla_k R_{jp} + 2\nabla_k R R_{ij} + 2R\nabla_k R_{ij} \\
& +2\nabla_j R_{ip} R_{kp} + 4R_{ip}\nabla_j R_{kp} - 2\nabla_j R R_{ik} - 2R\nabla_j R_{ik}\,.
\end{aligned}
$$

then, we substitute again

$$
\begin{aligned}
\nabla_k R_{jp} &= \nabla_p R_{kj} + C_{jpk} + \frac{1}{4}\left(\nabla_k R g_{jp} - \nabla_p R g_{jk}\right) \\
\nabla_j R_{kp} &= \nabla_p R_{jk} + C_{kpj} + \frac{1}{4}\left(\nabla_j R g_{kp} - \nabla_p R g_{kj}\right) \\
\nabla_k R_{ij} &= \nabla_i R_{kj} + C_{jik} + \frac{1}{4}\left(\nabla_k R g_{ij} - \nabla_i R g_{jk}\right) \\
\nabla_j R_{ik} &= \nabla_i R_{jk} + C_{kij} + \frac{1}{4}\left(\nabla_j R g_{ik} - \nabla_i R g_{kj}\right),
\end{aligned}
$$

finally obtaining

$$
\begin{aligned}
\frac{\partial}{\partial t}C_{ijk} - \Delta C_{ijk} = \; & R_{jp}\left(C_{kpi} + C_{pki}\right) - R_{kp}\left(C_{jpi} + C_{pji}\right) + R_{ip}C_{pkj} \\
& +2R_{lp}C_{pjl}g_{ik} - 2R_{lp}C_{pkl}g_{ij} \\
& +\left[R\nabla_j R - \nabla_j|\mathrm{Ric}|^2/2\right] g_{ik} - \left[R\nabla_k R - \nabla_k|\mathrm{Ric}|^2/2\right] g_{ij} \\
& -2\nabla_k R_{ip} R_{jp} - 4R_{ip}\left(\nabla_p R_{kj} + C_{jpk} + \nabla_k R g_{jp}/4 - \nabla_p R g_{jk}/4\right) \\
& +2\nabla_k R R_{ij} + 2R\left(\nabla_i R_{kj} + C_{jik} + \nabla_k R g_{ij}/4 - \nabla_i R g_{jk}/4\right) \\
& +2\nabla_j R_{ip} R_{kp} + 4R_{ip}\left(\nabla_p R_{jk} + C_{kpj} + \nabla_j R g_{kp}/4 - \nabla_p R g_{kj}/4\right) \\
& -2\nabla_j R R_{ik} - 2R\left(\nabla_i R_{jk} + C_{kij} + \nabla_j R g_{ik}/4 - \nabla_i R g_{kj}/4\right) \\
= \; & R_{jp}\left(C_{kpi} + C_{pki}\right) - R_{kp}\left(C_{jpi} + C_{pji}\right) + R_{ip}C_{pkj} \\
& +4R_{ip}\left(C_{kpj} - C_{jpk}\right) + 2R\left(C_{jik} - C_{kij}\right) \\
& +2R_{lp}C_{pjl}g_{ik} - 2R_{lp}C_{pkl}g_{ij} \\
& +\left[R\nabla_j R/2 - \nabla_j|\mathrm{Ric}|^2/2\right] g_{ik} - \left[R\nabla_k R/2 - \nabla_k|\mathrm{Ric}|^2/2\right] g_{ij} \\
& -2\nabla_k R_{ip} R_{jp} + 2\nabla_j R_{ip} R_{kp} \\
& +\nabla_k R R_{ij} - \nabla_j R R_{ik} \\
= \; & R_{jp}\left(C_{kpi} + C_{pki}\right) - R_{kp}\left(C_{jpi} + C_{pji}\right) + 5R_{ip}C_{pkj} \\
& +2RC_{ijk} + 2R_{lp}C_{pjl}g_{ik} - 2R_{lp}C_{pkl}g_{ij} \\
& +\left[R\nabla_j R/2 - \nabla_j|\mathrm{Ric}|^2/2\right] g_{ik} - \left[R\nabla_k R/2 - \nabla_k|\mathrm{Ric}|^2/2\right] g_{ij} \\
& +2\nabla_j R_{ip} R_{kp} - 2\nabla_k R_{ip} R_{jp} \\
& +\nabla_k R R_{ij} - \nabla_j R R_{ik}\,,
\end{aligned}
$$

where in the last passage we used again the identities (1.4).

Hence, we can resume this long computation in the following proposition, getting back to a generic coordinate basis.

Proposition 2.1. *During the Ricci flow of a 3–dimensional Riemannian manifold $(M^3, g(t))$, the Cotton tensor satisfies the following evolution equation*

$$
\begin{aligned}
(\partial_t - \Delta)\mathrm{C}_{ijk} \;=\;& g^{pq}\mathrm{R}_{pj}(\mathrm{C}_{kqi}+\mathrm{C}_{qki}) + 5g^{pq}\mathrm{R}_{ip}\mathrm{C}_{qkj} + g^{pq}\mathrm{R}_{pk}(\mathrm{C}_{jiq}+\mathrm{C}_{qij})\\
&+2\mathrm{R}\mathrm{C}_{ijk} + 2\mathrm{R}^{ql}\mathrm{C}_{qjl}g_{ik} - 2\mathrm{R}^{ql}\mathrm{C}_{qkl}g_{ij}\\
&+\frac{1}{2}\nabla_k|\mathrm{Ric}|^2 g_{ij} - \frac{1}{2}\nabla_j|\mathrm{Ric}|^2 g_{ik} + \frac{\mathrm{R}}{2}\nabla_j\mathrm{R}g_{ik} - \frac{\mathrm{R}}{2}\nabla_k\mathrm{R}g_{ij}\\
&+2g^{pq}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{qi} - 2g^{pq}\mathrm{R}_{pj}\nabla_k\mathrm{R}_{qi} + \mathrm{R}_{ij}\nabla_k\mathrm{R} - \mathrm{R}_{ik}\nabla_j\mathrm{R}\,.
\end{aligned} \tag{2.3}
$$

In particular if the Cotton tensor vanishes identically along the flow we obtain,

$$
\begin{aligned}
0 \;=\;& \nabla_k|\mathrm{Ric}|^2 g_{ij} - \nabla_j|\mathrm{Ric}|^2 g_{ik} + \mathrm{R}\nabla_j\mathrm{R}g_{ik} - \mathrm{R}\nabla_k\mathrm{R}g_{ij}\\
&+4g^{pq}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{qi} - 4g^{pq}\mathrm{R}_{pj}\nabla_k\mathrm{R}_{qi} + 2\mathrm{R}_{ij}\nabla_k\mathrm{R} - 2\mathrm{R}_{ik}\nabla_j\mathrm{R}\,.
\end{aligned} \tag{2.4}
$$

Corollary 2.2. *If the Cotton tensor vanishes identically along the Ricci flow of a 3–dimensional Riemannian manifold $(M^3, g(t))$, the following tensor*

$$|\mathrm{Ric}|^2 g_{ij} - 4\mathrm{R}_{pj}\mathrm{R}_{pi} + 3\mathrm{R}\mathrm{R}_{ij} - \frac{7}{8}\mathrm{R}^2 g_{ij}$$

is a Codazzi tensor (see [1, Chapter 16, Section C]).

Proof. We compute in an orthonormal basis,

$$
\begin{aligned}
&4\mathrm{R}_{pk}\nabla_j\mathrm{R}_{pi} - 4\,\mathrm{R}_{pj}\nabla_k\mathrm{R}_{pi} + 2\mathrm{R}_{ij}\nabla_k\mathrm{R} - 2\mathrm{R}_{ik}\nabla_j\mathrm{R}\\
&= 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) - 4\mathrm{R}_{pi}\nabla_j\mathrm{R}_{pk} + 4\mathrm{R}_{pi}\nabla_k\mathrm{R}_{pj} + 2\mathrm{R}_{ij}\nabla_k\mathrm{R} - 2\mathrm{R}_{ik}\nabla_j\mathrm{R}\\
&= 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) + \mathrm{R}_{pi}(4\mathrm{C}_{pjk} + \nabla_k\mathrm{R}g_{pj} - \nabla_j\mathrm{R}g_{pk}) + 2\mathrm{R}_{ij}\nabla_k\mathrm{R} - 2\mathrm{R}_{ik}\nabla_j\mathrm{R}\\
&= 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) + 3\mathrm{R}_{ij}\nabla_k\mathrm{R} - 3\mathrm{R}_{ik}\nabla_j\mathrm{R}\\
&= 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) + 3\nabla_k(\mathrm{R}\mathrm{R}_{ij}) - 3\nabla_j(\mathrm{R}\mathrm{R}_{ik}) - 3\mathrm{R}(\nabla_k\mathrm{R}_{ij} - \nabla_j\mathrm{R}_{ik})\\
&= 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) + 3\nabla_k(\mathrm{R}\mathrm{R}_{ij}) - 3\nabla_j(\mathrm{R}\mathrm{R}_{ik}) - 3\mathrm{R}(4\mathrm{C}_{ijk} + \nabla_k\mathrm{R}g_{ij} - \nabla_j\mathrm{R}g_{ik})/4\\
&= 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) + 3\nabla_k(\mathrm{R}\mathrm{R}_{ij}) - 3\nabla_j(\mathrm{R}\mathrm{R}_{ik}) - \frac{3}{8}\nabla_k\mathrm{R}^2 g_{ij} + \frac{3}{8}\nabla_j\mathrm{R}^2 g_{ik}\,.
\end{aligned}
$$

Hence, we have, by the previous proposition,

$$0 = \nabla_k|\mathrm{Ric}|^2 g_{ij} - \nabla_j|\mathrm{Ric}|^2 g_{ik} + 4\nabla_j(\mathrm{R}_{pk}\mathrm{R}_{pi}) - 4\nabla_k(\mathrm{R}_{pj}\mathrm{R}_{pi}) + 3\nabla_k(\mathrm{R}\mathrm{R}_{ij}) - 3\nabla_j(\mathrm{R}\mathrm{R}_{ik}) - \frac{7}{8}\nabla_k\mathrm{R}^2 g_{ij} + \frac{7}{8}\nabla_j\mathrm{R}^2 g_{ik}\,,$$

which is the thesis of the corollary. □

Remark 2.3. All the traces of the 3–tensor in the LHS of equation (2.4) are zero.

Remark 2.4. From the trace–free property (1.5) of the Cotton tensor and the fact that along the Ricci flow there holds

$$(\partial_t - \Delta)g^{ij} = 2\mathrm{R}^{ij}\,,$$

we conclude that the following relations have to hold

$$
\begin{aligned}
g^{ij}(\partial_t - \Delta)\mathrm{C}_{ijk} &= -2\mathrm{R}^{ij}\mathrm{C}_{ijk}\,,\\
g^{ik}(\partial_t - \Delta)\mathrm{C}_{ijk} &= -2\mathrm{R}^{ik}\mathrm{C}_{ijk}\,,\\
g^{jk}(\partial_t - \Delta)\mathrm{C}_{ijk} &= -2\mathrm{R}^{jk}\mathrm{C}_{ijk} = 0\,.
\end{aligned}
$$

They are easily verified for formula (2.3).

Corollary 2.5. *During the Ricci flow of a 3–dimensional Riemannian manifold $(M^3, g(t))$, the squared norm of the Cotton tensor satisfies the following evolution equation, in an orthonormal basis,*

$$
\begin{aligned}
(\partial_t - \Delta)|\mathrm{C}_{ijk}|^2 \;=\;& -2|\nabla\mathrm{C}_{ijk}|^2 - 16\mathrm{C}_{ipk}\mathrm{C}_{iqk}\mathrm{R}_{pq} + 24\mathrm{C}_{ipk}\mathrm{C}_{kqi}\mathrm{R}_{pq} + 4\mathrm{R}|\mathrm{C}_{ijk}|^2\\
&+8\mathrm{C}_{ijk}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{pi} + 4\mathrm{C}_{ijk}\mathrm{R}_{ij}\nabla_k\mathrm{R}\,.
\end{aligned}
$$

Proof.

$$\begin{aligned}
\left(\partial_t - \Delta\right)|\mathrm{C}_{ijk}|^2 &= -2|\nabla \mathrm{C}_{ijk}|^2 + 2\mathrm{C}^{ijk}\mathrm{R}_{ip}g^{pq}\mathrm{C}_{qjk} + 2\mathrm{C}^{ijk}\mathrm{R}_{jp}g^{pq}\mathrm{C}_{iqk} + 2\mathrm{C}^{ijk}\mathrm{R}_{kp}g^{pq}\mathrm{C}_{ijq} \\
&\quad +2\mathrm{C}^{ijk}\Big[g^{pq}\mathrm{R}_{pj}(\mathrm{C}_{kqi} + \mathrm{C}_{qki}) + 5g^{pq}\mathrm{R}_{ip}\mathrm{C}_{qkj} + g^{pq}\mathrm{R}_{pk}(\mathrm{C}_{jiq} + \mathrm{C}_{qij}) \\
&\quad +2\mathrm{R}\mathrm{C}_{ijk} + 2\mathrm{R}^{ql}\mathrm{C}_{qjl}g_{ik} - 2\mathrm{R}^{ql}\mathrm{C}_{qkl}g_{ij} \\
&\quad +\frac{1}{2}\nabla_k|\mathrm{Ric}|^2 g_{ij} - \frac{1}{2}\nabla_j|\mathrm{Ric}|^2 g_{ik} + \frac{\mathrm{R}}{2}\nabla_j\mathrm{R}g_{ik} - \frac{\mathrm{R}}{2}\nabla_k\mathrm{R}g_{ij} \\
&\quad +2g^{pq}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{qi} - 2g^{pq}\mathrm{R}_{pj}\nabla_k\mathrm{R}_{qi} + \mathrm{R}_{ij}\nabla_k\mathrm{R} - \mathrm{R}_{ik}\nabla_j\mathrm{R}\Big] \\
&= -2|\nabla \mathrm{C}_{ijk}|^2 + 2(\mathrm{C}^{kij} + \mathrm{C}^{jki})\mathrm{R}_{ip}g^{pq}(\mathrm{C}_{kqj} + \mathrm{C}_{jkq}) \\
&\quad +2\mathrm{C}^{ijk}\mathrm{R}_{jp}g^{pq}\mathrm{C}_{iqk} + 2\mathrm{C}^{ikj}\mathrm{R}_{kp}g^{pq}\mathrm{C}_{iqj} \\
&\quad +2\mathrm{C}^{ijk}\Big[2g^{pq}\mathrm{R}_{pj}(\mathrm{C}_{kqi} + \mathrm{C}_{qki}) + 5g^{pq}\mathrm{R}_{ip}\mathrm{C}_{qkj}\Big] \\
&\quad +4\mathrm{R}|\mathrm{C}_{ijk}|^2 + 8g^{pq}\mathrm{C}^{ijk}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{qi} + 4\mathrm{C}^{ijk}\mathrm{R}_{ij}\nabla_k\mathrm{R} \\
&= -2|\nabla \mathrm{C}_{ijk}|^2 - 16\mathrm{C}_{ipk}\mathrm{C}_{iqk}\mathrm{R}_{pq} + 24\mathrm{C}_{ipk}\mathrm{C}_{kqi}\mathrm{R}_{pq} + 4\mathrm{R}|\mathrm{C}_{ijk}|^2 \\
&\quad +8\mathrm{C}_{ijk}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{pi} + 4\mathrm{C}_{ijk}\mathrm{R}_{ij}\nabla_k\mathrm{R}
\end{aligned}$$

where in the last line we assumed to be in a orthonormal basis. □

3 Three–Dimensional Gradient Ricci Solitons

The structural equation of a gradient Ricci soliton $(M^n, g, \nabla f)$ is the following

$$\mathrm{R}_{ij} + \nabla_i\nabla_j f = \lambda g_{ij}\,, \tag{3.1}$$

for some $\lambda \in \mathbb{R}$.
The soliton is said to be *steady*, *shrinking* or *expanding* according to the fact that the constant λ is zero, positive or negative, respectively.

It follows that in dimension three, for $(M^3, g, \nabla f)$ there holds

$$\begin{aligned}
\Delta\mathrm{R}_{ij} &= \nabla_l\mathrm{R}_{ij}\nabla_l f + 2\lambda\mathrm{R}_{ij} - 2|\mathrm{Ric}|^2 g_{ij} + \mathrm{R}^2 g_{ij} - 3\mathrm{R}\mathrm{R}_{ij} + 4\mathrm{R}_{is}\mathrm{R}_{sj} && (3.2)\\
\Delta\mathrm{R} &= \nabla_l\mathrm{R}\nabla_l f + 2\lambda\mathrm{R} - 2|\mathrm{Ric}|^2 && (3.3)\\
\nabla_i\mathrm{R} &= 2\mathrm{R}_{li}\nabla_l f && (3.4)\\
\mathrm{C}_{ijk} &= \frac{\mathrm{R}_{lk}g_{ij}}{2}\nabla_l f - \frac{\mathrm{R}_{lj}g_{ik}}{2}\nabla_l f + \mathrm{R}_{ij}\nabla_k f - \mathrm{R}_{ik}\nabla_j f + \frac{\mathrm{R}g_{ik}}{2}\nabla_j f - \frac{\mathrm{R}g_{ij}}{2}\nabla_k f && (3.5)\\
&= \frac{\nabla_k\mathrm{R}}{4}g_{ij} - \frac{\nabla_j\mathrm{R}}{4}g_{ik} + \Big(\mathrm{R}_{ij} - \frac{\mathrm{R}}{2}g_{ij}\Big)\nabla_k f - \Big(\mathrm{R}_{ik} - \frac{\mathrm{R}}{2}g_{ik}\Big)\nabla_j f\,.
\end{aligned}$$

In the special case of a *steady* soliton the first two equations above simplify as follows,

$$\begin{aligned}
\Delta\mathrm{R}_{ij} &= \nabla_l\mathrm{R}_{ij}\nabla_l f - 2|\mathrm{Ric}|^2 g_{ij} + \mathrm{R}^2 g_{ij} - 3\mathrm{R}\mathrm{R}_{ij} + 4\mathrm{R}_{is}\mathrm{R}_{sj}\\
\Delta\mathrm{R} &= \nabla_l\mathrm{R}\nabla_l f - 2|\mathrm{Ric}|^2\,.
\end{aligned}$$

Remark 3.1. We notice that, by relation (3.5), we have

$$\begin{aligned}
\mathrm{C}_{ijk}\nabla_i f &= \frac{\nabla_k\mathrm{R}\nabla_j f}{4} - \frac{\nabla_j\mathrm{R}\nabla_k f}{4} + \mathrm{R}_{ij}\nabla_i f\nabla_k f - \frac{\mathrm{R}}{2}\nabla_j f\nabla_k f - \mathrm{R}_{ik}\nabla_i f\nabla_j f + \frac{\mathrm{R}}{2}\nabla_k f\nabla_j f\\
&= \frac{\nabla_j\mathrm{R}\nabla_k f}{4} - \frac{\nabla_k\mathrm{R}\nabla_j f}{4}\,,
\end{aligned}$$

where in the last passage we used relation (3.4).
It follows that

$$\mathrm{C}_{ijk}\nabla_i f\nabla_j f = \frac{\langle\nabla f, \nabla\mathrm{R}\rangle}{4}\nabla_k f - \frac{|\nabla f|^2}{4}\nabla_k\mathrm{R}\,.$$

Hence, if the Cotton tensor of a three–dimensional gradient Ricci soliton is identically zero, we have that at every point where ∇R is not zero, ∇f and ∇R are proportional.

This relation is a key step in (yet another) proof of the fact that a three–dimensional, locally conformally flat, steady or shrinking gradient Ricci soliton is locally a warped product of a constant curvature surface on a interval of $\mathbb{R}$, leading to a full classification, first obtained by H.-D. Cao and Q. Chen [4] for the steady case and H.-D. Cao, B.-L. Chen and X.-P. Zhu [3] for the shrinking case (actually this is the last paper of a series finally classifying, in full generality, all the three-dimensional gradient shrinking Ricci solitons, even without the LCF assumption).

Proposition 3.2. *Let (M^3, g, f) be a three–dimensional gradient Ricci soliton. Then,*

$$\begin{aligned}\Delta|C_{ijk}|^2 &= \nabla_l|C_{ijk}|^2\nabla_l f + 2|\nabla C_{ijk}|^2 - 2R|C_{ijk}|^2\\ &\quad -6C_{ijk}R_{ij}\nabla_k R + 8C_{jsk}C_{jik}R_{si} - 16C_{jsk}C_{kij}R_{si} - 8C_{ijk}R_{lk}\nabla_j R_{il}\,.\end{aligned}$$

Proof. First observe that

$$\Delta|C_{ijk}|^2 = 2C_{ijk}\Delta C_{ijk} + 2|\nabla C_{ijk}|^2.$$

Using relations (3.5), (3.2) and, repeatedly, the trace–free property (1.5) of the Cotton tensor, we have that

$$\begin{aligned}C_{ijk}\Delta C_{ijk} &= \Delta(R_{ij}\nabla_k f - R_{ik}\nabla_j f)C_{ijk}\\ &= (\Delta R_{ij}\nabla_k f + R_{ij}\Delta\nabla_k f + 2\nabla_l R_{ij}\nabla_l\nabla_k f)C_{ijk}\\ &\quad -(\Delta R_{ik}\nabla_j f + R_{ik}\Delta\nabla_j f + 2\nabla_l R_{ik}\nabla_l\nabla_j f)C_{ijk}\\ &= (\nabla_s R_{ij}\nabla_s f - 3RR_{ij} + 4R_{is}R_{sj})\nabla_k f C_{ijk}\\ &\quad +R_{ij}\Delta\nabla_k f C_{ijk} + 2\nabla_l R_{ij}\nabla_l\nabla_k f C_{ijk}\\ &\quad -(\nabla_s R_{ik}\nabla_s f - 3RR_{ik} + 4R_{is}R_{sk})\nabla_j f C_{ijk}\\ &\quad -R_{ik}\Delta\nabla_j f C_{ijk} - 2\nabla_l R_{ik}\nabla_l\nabla_j f C_{ijk}\\ &= (\nabla_s R_{ij}\nabla_k f - \nabla_s R_{ik}\nabla_j f)\nabla_s f C_{ijk}\\ &\quad -3R(R_{ij}\nabla_k f - R_{ik}\nabla_j f)C_{ijk}\\ &\quad +4R_{is}(R_{sj}\nabla_k f - R_{sk}\nabla_j f)C_{ijk}\\ &\quad +(R_{ij}\nabla_l\nabla_l\nabla_k f - R_{ik}\nabla_l\nabla_l\nabla_j f)C_{ijk}\\ &\quad +2(\nabla_l R_{ij}\nabla_l\nabla_k f - \nabla_l R_{ik}\nabla_l\nabla_j f)C_{ijk}\\ &= (\nabla_s R_{ij}\nabla_k f - \nabla_s R_{ik}\nabla_j f)\nabla_s f C_{ijk}\\ &\quad +(-3R)|C_{ijk}|^2\\ &\quad +4R_{is}(R_{sj}\nabla_k f - R_{sk}\nabla_j f)C_{ijk}\\ &\quad +(R_{ij}\nabla_l\nabla_l\nabla_k f - R_{ik}\nabla_l\nabla_l\nabla_j f)C_{ijk}\\ &\quad +2(\nabla_l R_{ij}\nabla_l\nabla_k f - \nabla_l R_{ik}\nabla_l\nabla_j f)C_{ijk}\,,\end{aligned}$$

where we used the identity

$$(R_{ij}\nabla_k f - R_{ik}\nabla_j f)C_{ijk} = |C_{ijk}|^2 \tag{3.6}$$

which follows easily by equation (3.5) and the fact that every trace of the Cotton tensor is zero.
Using now equations (3.1), (3.5), (1.5), (1.4), and (3.4), we compute

$$\begin{aligned}(\nabla_s R_{ij}\nabla_k f - \nabla_s R_{ik}\nabla_j f)\nabla_s f C_{ijk} &= (\nabla_s(R_{ij}\nabla_k f) - R_{ij}\nabla_s\nabla_k f)\nabla_s f C_{ijk}\\ &\quad -(\nabla_s(R_{ik}\nabla_j f) - R_{ik}\nabla_s\nabla_j f)\nabla_s f C_{ijk}\\ &= (\nabla_s(R_{ij}\nabla_k f - R_{ik}\nabla_j f) + R_{ij}(R_{sk}))\nabla_s f C_{ijk}\\ &\quad -(R_{ik}(R_{sj}))\nabla_s f C_{ijk}\\ &= \nabla_s C_{ijk}C_{ijk}\nabla_s f + R_{ij}R_{sk}\nabla_s f C_{ijk} - R_{ik}R_{sj}\nabla_s f C_{ijk}\end{aligned}$$

$$= \frac{1}{2}\nabla_s|C_{ijk}|^2\nabla_s f + \frac{1}{2}R_{ij}\nabla_k RC_{ijk} - \frac{1}{2}R_{ik}\nabla_j RC_{ijk}$$

$$\begin{aligned}4R_{is}(R_{sj}\nabla_k f - R_{sk}\nabla_j f)C_{ijk} &= 4R_{is}(C_{sjk} - \frac{1}{4}\nabla_k Rg_{sj} + \frac{1}{4}\nabla_j Rg_{sk} + \frac{R}{2}\nabla_k fg_{sj} - \frac{R}{2}\nabla_j fg_{sk})C_{ijk}\\ &= 4R_{is}(-C_{jks} - C_{ksj})(-C_{jki} - C_{kij}) - R_{ij}\nabla_k RC_{ijk}\\ &\quad +R_{ik}\nabla_j RC_{ijk} + 2RR_{ij}\nabla_k fC_{ijk} - 2RR_{ik}\nabla_j fC_{ijk}\\ &= 8R_{is}C_{jsk}C_{jik} - 8R_{is}C_{jsk}C_{kij}\\ &\quad -R_{ij}\nabla_k RC_{ijk} + R_{ik}\nabla_j RC_{ijk} + 2R|C_{ijk}|^2\end{aligned}$$

$$\begin{aligned}(R_{ij}\nabla_l\nabla_l\nabla_k f - R_{ik}\nabla_l\nabla_l\nabla_j f)C_{ijk} &= (R_{ij}\nabla_l(-R_{lk}) - R_{ik}\nabla_l(-R_{lj}))C_{ijk}\\ &= -\frac{1}{2}R_{ij}\nabla_k RC_{ijk} + \frac{1}{2}R_{ik}\nabla_j RC_{ijk}\end{aligned}$$

$$\begin{aligned}2(\nabla_l R_{ij}\nabla_l\nabla_k f - \nabla_l R_{ik}\nabla_l\nabla_j f)C_{ijk} &= 2((C_{ijl} + \nabla_j R_{il} + \frac{1}{4}\nabla_l Rg_{ij} - \frac{1}{4}\nabla_j Rg_{il})(-R_{lk}))C_{ijk}\\ &\quad -2((C_{ikl} + \nabla_k R_{il} + \frac{1}{4}\nabla_l Rg_{ik} - \frac{1}{4}\nabla_k Rg_{il})(-R_{lj}))C_{ijk}\\ &= -2C_{ijl}C_{ijk}R_{lk} - 2C_{ijk}R_{lk}\nabla_j R_{il} + \frac{1}{2}C_{ijk}R_{ik}\nabla_j R\\ &\quad +2C_{ikl}C_{ijk}R_{lj} + 2C_{ijk}R_{lj}\nabla_k R_{il} - \frac{1}{2}C_{ijk}R_{ij}\nabla_k R\\ &= -2C_{ilj}C_{ikj}R_{lk} - 2C_{ijk}R_{lk}\nabla_j R_{il} + \frac{1}{2}C_{ijk}R_{ik}\nabla_j R\\ &\quad -2C_{ilk}C_{ijk}R_{lj} + 2C_{ijk}R_{lj}\nabla_k R_{il} - \frac{1}{2}C_{ijk}R_{ij}\nabla_k R.\end{aligned}$$

Hence, getting back to the main computation and using again the symmetry relations (1.4), we finally get

$$\begin{aligned}C_{ijk}\Delta C_{ijk} &= \frac{1}{2}\nabla_s|C_{ijk}|^2\nabla_s f - R|C_{ijk}|^2\\ &\quad -\frac{3}{2}C_{ijk}R_{ij}\nabla_k R + \frac{3}{2}C_{ijk}R_{ik}\nabla_j R\\ &\quad +4C_{jsk}C_{jik}R_{si} - 8C_{jsk}C_{kij}R_{si}\\ &\quad -2C_{ijk}R_{lk}\nabla_j R_{il} + 2C_{ijk}R_{lj}\nabla_k R_{il}\\ &= \frac{1}{2}\nabla_s|C_{ijk}|^2\nabla_s f - R|C_{ijk}|^2\\ &\quad -3C_{ijk}R_{ij}\nabla_k R + 4C_{jsk}C_{jik}R_{si} - 8C_{jsk}C_{kij}R_{si} - 4C_{ijk}R_{lk}\nabla_j R_{il}\end{aligned}$$

where in the last passage we applied the skew–symmetry of the Cotton tensor in its last two indexes. The thesis follows. □

4 The Evolution Equation of the Cotton Tensor in any Dimension

In this section we will compute the evolution equation under the Ricci flow of the Cotton tensor C_{ijk}, for every n–dimensional Riemannian manifold $(M^n, g(t))$ evolving by Ricci flow.

Among the evolution equations (1.6) we expand the one for the Ricci tensor,

$$\begin{aligned}\frac{\partial}{\partial t}R_{ij} = \Delta R_{ij} - \frac{2n}{n-2}g^{pq}R_{ip}R_{jq} + \frac{2n}{(n-1)(n-2)}RR_{ij} + \frac{2}{n-2}|\mathrm{Ric}|^2 g_{ij}\\ - \frac{2}{(n-1)(n-2)}R^2 g_{ij} - 2R^{pq}W_{pijq}\,.\end{aligned}$$

Then, we compute the evolution equations of the derivatives of the curvatures assuming, from now on, to be in normal coordinates,

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla_l \mathrm{R} &= \nabla_l \Delta \mathrm{R} + 2\nabla_l |\mathrm{Ric}|^2\,, \\
\frac{\partial}{\partial t}\nabla_s \mathrm{R}_{ij} &= \nabla_s \Delta \mathrm{R}_{ij} - \frac{2n}{n-2}\nabla_s \mathrm{R}_{ip}\mathrm{R}_{jp} - \frac{2n}{n-2}\mathrm{R}_{ip}\nabla_s \mathrm{R}_{jp} + \frac{2n}{(n-1)(n-2)}\nabla_s \mathrm{R}\mathrm{R}_{ij} \\
&\quad + \frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_s \mathrm{R}_{ij} + \frac{2}{n-2}\nabla_s |\mathrm{Ric}|^2 g_{ij} - \frac{2}{(n-1)(n-2)}\nabla_s \mathrm{R}^2 g_{ij} \\
&\quad -2\nabla_s \mathrm{R}_{kl}\mathrm{W}_{kijl} - 2\mathrm{R}_{kl}\nabla_s \mathrm{W}_{kijl} + (\nabla_i \mathrm{R}_{sp} + \nabla_s \mathrm{R}_{ip} - \nabla_p \mathrm{R}_{is})\mathrm{R}_{jp} \\
&\quad +(\nabla_j \mathrm{R}_{sp} + \nabla_s \mathrm{R}_{jp} - \nabla_p \mathrm{R}_{js})\mathrm{R}_{ip} \\
&= \nabla_s \Delta \mathrm{R}_{ij} - \frac{n+2}{n-2}\nabla_s \mathrm{R}_{ip}\mathrm{R}_{jp} - \frac{n+2}{n-2}\mathrm{R}_{ip}\nabla_s \mathrm{R}_{jp} + \frac{2n}{(n-1)(n-2)}\nabla_s \mathrm{R}\mathrm{R}_{ij} \\
&\quad + \frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_s \mathrm{R}_{ij} + \frac{2}{n-2}\nabla_s |\mathrm{Ric}|^2 g_{ij} - \frac{2}{(n-1)(n-2)}\nabla_s \mathrm{R}^2 g_{ij} \\
&\quad -2\nabla_s \mathrm{R}_{kl}\mathrm{W}_{kijl} - 2\mathrm{R}_{kl}\nabla_s \mathrm{W}_{kijl} + (\nabla_i \mathrm{R}_{sp} - \nabla_p \mathrm{R}_{is})\mathrm{R}_{jp} + (\nabla_j \mathrm{R}_{sp} - \nabla_p \mathrm{R}_{js})\mathrm{R}_{ip} \\
&= \nabla_s \Delta \mathrm{R}_{ij} - \frac{n+2}{n-2}\nabla_s \mathrm{R}_{ip}\mathrm{R}_{jp} - \frac{n+2}{n-2}\mathrm{R}_{ip}\nabla_s \mathrm{R}_{jp} + \frac{2n}{(n-1)(n-2)}\nabla_s \mathrm{R}\mathrm{R}_{ij} \\
&\quad + \frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_s \mathrm{R}_{ij} + \frac{2}{n-2}\nabla_s |\mathrm{Ric}|^2 g_{ij} - \frac{2}{(n-1)(n-2)}\nabla_s \mathrm{R}^2 g_{ij} \\
&\quad -2\nabla_s \mathrm{R}_{kl}\mathrm{W}_{kijl} - 2\mathrm{R}_{kl}\nabla_s \mathrm{W}_{kijl} + \mathrm{C}_{spi}\mathrm{R}_{jp} + \mathrm{C}_{spj}\mathrm{R}_{ip} \\
&\quad + \frac{1}{2(n-1)}\mathrm{R}_{jp}[\nabla_i \mathrm{R} g_{sp} - \nabla_p \mathrm{R} g_{is}] + \frac{1}{2(n-1)}\mathrm{R}_{ip}[\nabla_j \mathrm{R} g_{sp} - \nabla_p \mathrm{R} g_{js}]\,,
\end{aligned}
$$

where in the last passage we substituted the expression of the Cotton tensor.

We then compute,

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathrm{C}_{ijk} &= \frac{\partial}{\partial t}\nabla_k \mathrm{R}_{ij} - \frac{\partial}{\partial t}\nabla_j \mathrm{R}_{ik} - \frac{1}{2(n-1)}\frac{\partial}{\partial t}\big(\nabla_k \mathrm{R} g_{ij} - \nabla_j \mathrm{R} g_{ik}\big) \\
&= \nabla_k \Delta \mathrm{R}_{ij} - \frac{n+2}{n-2}\nabla_k \mathrm{R}_{ip}\mathrm{R}_{jp} - \frac{n+2}{n-2}\mathrm{R}_{ip}\nabla_k \mathrm{R}_{jp} + \frac{2n}{(n-1)(n-2)}\nabla_k \mathrm{R}\mathrm{R}_{ij} \\
&\quad + \frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_k \mathrm{R}_{ij} + \frac{2}{n-2}\nabla_k |\mathrm{Ric}|^2 g_{ij} - \frac{2}{(n-1)(n-2)}\nabla_k \mathrm{R}^2 g_{ij} \\
&\quad -2\nabla_k \mathrm{R}_{pl}\mathrm{W}_{pijl} - 2\mathrm{R}_{pl}\nabla_k \mathrm{W}_{pijl} + \mathrm{C}_{kpi}\mathrm{R}_{jp} + \mathrm{C}_{kpj}\mathrm{R}_{ip} \\
&\quad + \frac{\mathrm{R}_{jp}}{2(n-1)}[\nabla_i \mathrm{R} g_{kp} - \nabla_p \mathrm{R} g_{ik}] + \frac{\mathrm{R}_{ip}}{2(n-1)}[\nabla_j \mathrm{R} g_{kp} - \nabla_p \mathrm{R} g_{jk}] \\
&\quad -\nabla_j \Delta \mathrm{R}_{ik} + \frac{n+2}{n-2}\nabla_j \mathrm{R}_{ip}\mathrm{R}_{kp} + \frac{n+2}{n-2}\mathrm{R}_{ip}\nabla_j \mathrm{R}_{kp} - \frac{2n}{(n-1)(n-2)}\nabla_j \mathrm{R}\mathrm{R}_{ik} \\
&\quad - \frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_j \mathrm{R}_{ik} - \frac{2}{n-2}\nabla_j |\mathrm{Ric}|^2 g_{ik} + \frac{2}{n-1}\nabla_j \mathrm{R}^2 g_{ik} \\
&\quad -2\nabla_k \mathrm{R}_{pl}\mathrm{W}_{pijl} - 2\mathrm{R}_{pl}\nabla_k \mathrm{W}_{pijl} - \mathrm{C}_{jpi}\mathrm{R}_{kp} - \mathrm{C}_{jpk}\mathrm{R}_{ip} \\
&\quad - \frac{\mathrm{R}_{kp}}{2(n-1)}[\nabla_i \mathrm{R} g_{jp} - \nabla_p \mathrm{R} g_{ij}] - \frac{\mathrm{R}_{ip}}{2(n-1)}[\nabla_k \mathrm{R} g_{jp} - \nabla_p \mathrm{R} g_{kj}] \\
&\quad + \frac{1}{n-1}(\mathrm{R}_{ij}\nabla_k \mathrm{R} - \mathrm{R}_{ik}\nabla_j \mathrm{R}) \\
&\quad -\big(\nabla_k \Delta \mathrm{R} + 2\nabla_k |\mathrm{Ric}|^2\big)\frac{g_{ij}}{2(n-1)} + \big(\nabla_j \Delta \mathrm{R} + 2\nabla_j |\mathrm{Ric}|^2\big)\frac{g_{ik}}{2(n-1)} \\
&= \nabla_k \Delta \mathrm{R}_{ij} - \frac{n+2}{n-2}\nabla_k \mathrm{R}_{ip}\mathrm{R}_{jp} - \frac{n+2}{n-2}\mathrm{R}_{ip}\nabla_k \mathrm{R}_{jp} \\
&\quad + \frac{5n-2}{2(n-1)(n-2)}\nabla_k \mathrm{R}\mathrm{R}_{ij} + \frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_k \mathrm{R}_{ij} \\
&\quad + \frac{n}{(n-1)(n-2)}\nabla_k |\mathrm{Ric}|^2 g_{ij} - \frac{2}{(n-1)(n-2)}\nabla_k \mathrm{R}^2 g_{ij} \\
&\quad + \mathrm{C}_{kpi}\mathrm{R}_{jp} + \mathrm{C}_{kpj}\mathrm{R}_{ip} - 2\nabla_k \mathrm{R}_{pl}\mathrm{W}_{pijl} - 2\mathrm{R}_{pl}\nabla_k \mathrm{W}_{pijl}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2(n-1)}\mathrm{R}_{pj}\nabla_p\mathrm{R}g_{ik}\\
&-\nabla_j\Delta\mathrm{R}_{ik}+\frac{n+2}{n-2}\nabla_j\mathrm{R}_{ip}\mathrm{R}_{kp}+\frac{n+2}{n-2}\mathrm{R}_{ip}\nabla_j\mathrm{R}_{kp}\\
&-\frac{5n-2}{2(n-1)(n-2)}\nabla_j\mathrm{R}\mathrm{R}_{ik}-\frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_j\mathrm{R}_{ik}\\
&-\frac{n}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2g_{ik}+\frac{2}{(n-1)(n-2)}\nabla_j\mathrm{R}^2g_{ik}\\
&-\mathrm{C}_{jpi}\mathrm{R}_{kp}-\mathrm{C}_{jpk}\mathrm{R}_{ip}+2\nabla_j\mathrm{R}_{pl}\mathrm{W}_{pikl}+2\mathrm{R}_{pl}\nabla_j\mathrm{W}_{pikl}\\
&+\frac{1}{2(n-1)}\nabla_l\mathrm{R}\mathrm{R}_{lk}g_{ij}-\frac{1}{2(n-1)}\nabla_k\Delta\mathrm{R}g_{ij}+\frac{1}{2(n-1)}\nabla_j\Delta\mathrm{R}g_{ik}
\end{aligned}
$$

and

$$
\Delta\mathrm{C}_{ijk}=\Delta\nabla_k\mathrm{R}_{ij}-\Delta\nabla_j\mathrm{R}_{ik}-\frac{1}{2(n-1)}\Delta\nabla_k\mathrm{R}g_{ij}+\frac{1}{2(n-1)}\Delta\nabla_j\mathrm{R}g_{ik}\,,
$$

hence,

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathrm{C}_{ijk}-\Delta\mathrm{C}_{ijk} \;=\;& \nabla_k\Delta\mathrm{R}_{ij}-\nabla_j\Delta\mathrm{R}_{ik}-\Delta\nabla_k\mathrm{R}_{ij}+\Delta\nabla_j\mathrm{R}_{ik}\\
&-\frac{1}{2(n-1)}(\nabla_k\Delta\mathrm{R}g_{ij}-\nabla_j\Delta\mathrm{R}g_{ik}-\Delta\nabla_k\mathrm{R}g_{ij}+\Delta\nabla_j\mathrm{R}g_{ik})\\
&-\frac{n+2}{n-2}(\nabla_k\mathrm{R}_{ip}\mathrm{R}_{jp}+\mathrm{R}_{ip}\nabla_k\mathrm{R}_{jp})+\frac{5n-2}{2(n-1)(n-2)}\nabla_k\mathrm{R}\mathrm{R}_{ij}\\
&+\frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_k\mathrm{R}_{ij}\\
&+\frac{n}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2g_{ij}-\frac{2}{(n-1)(n-2)}\nabla_k\mathrm{R}^2g_{ij}\\
&+\mathrm{C}_{kpi}\mathrm{R}_{jp}+\mathrm{C}_{kpj}\mathrm{R}_{ip}-2\nabla_k\mathrm{R}_{pl}\mathrm{W}_{pijl}-2\mathrm{R}_{pl}\nabla_k\mathrm{W}_{pijl}\\
&-\frac{1}{2(n-1)}\mathrm{R}_{jp}\nabla_p\mathrm{R}g_{ik}\\
&+\frac{n+2}{n-2}(\nabla_j\mathrm{R}_{ip}\mathrm{R}_{kp}+\mathrm{R}_{ip}\nabla_j\mathrm{R}_{kp})-\frac{5n-2}{2(n-1)(n-2)}\nabla_j\mathrm{R}\mathrm{R}_{ik}\\
&-\frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_j\mathrm{R}_{ik}\\
&-\frac{n}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2g_{ik}+\frac{2}{(n-1)(n-2)}\nabla_j\mathrm{R}^2g_{ik}\\
&-\mathrm{C}_{jpi}\mathrm{R}_{kp}-\mathrm{C}_{jpk}\mathrm{R}_{ip}+2\nabla_j\mathrm{R}_{pl}\mathrm{W}_{pikl}+2\mathrm{R}_{pl}\nabla_j\mathrm{W}_{pikl}\\
&+\frac{1}{2(n-1)}\mathrm{R}_{kp}\nabla_p\mathrm{R}g_{ij}
\end{aligned}
$$

Now to proceed, we need the following commutation rules for the derivatives of the Ricci tensor and of the scalar curvature, where we will employ the decomposition formula of the Riemann tensor (1.1).

$$
\begin{aligned}
\nabla_k\Delta\mathrm{R}_{ij}-\Delta\nabla_k\mathrm{R}_{ij} \;=\;& \nabla^3_{kll}\mathrm{R}_{ij}-\nabla^3_{lkl}\mathrm{R}_{ij}+\nabla^3_{lkl}\mathrm{R}_{ij}-\nabla^3_{llk}\mathrm{R}_{ij}\\
=\;& -\mathrm{R}_{kp}\nabla_p\mathrm{R}_{ij}+\mathrm{R}_{klip}\nabla_l\mathrm{R}_{jp}+\mathrm{R}_{kljp}\nabla_l\mathrm{R}_{ip}\\
&+\nabla^3_{lkl}\mathrm{R}_{ij}-\nabla^3_{llk}\mathrm{R}_{ij}\\
=\;& -\mathrm{R}_{kp}\nabla_p\mathrm{R}_{ij}+\frac{1}{2(n-2)}(\mathrm{R}_{ik}\nabla_j\mathrm{R}+\mathrm{R}_{jk}\nabla_i\mathrm{R})\\
&-\frac{1}{n-2}(\mathrm{R}_{kp}\nabla_i\mathrm{R}_{jp}+\mathrm{R}_{kp}\nabla_j\mathrm{R}_{ip}-\mathrm{R}_{lp}\nabla_l\mathrm{R}_{jp}g_{ik}-\mathrm{R}_{lp}\nabla_l\mathrm{R}_{ip}g_{jk})\\
&-\frac{1}{n-2}(\mathrm{R}_{li}\nabla_l\mathrm{R}_{jk}+\mathrm{R}_{lj}\nabla_l\mathrm{R}_{ik})-\frac{1}{2(n-1)(n-2)}(\mathrm{R}\nabla_j\mathrm{R}g_{ik}+\mathrm{R}\nabla_i\mathrm{R}g_{jk})\\
&+\frac{1}{(n-1)(n-2)}(\mathrm{R}\nabla_i\mathrm{R}_{jk}+\mathrm{R}\nabla_j\mathrm{R}_{ik})\\
&+\nabla_l\big(\mathrm{R}_{klip}\mathrm{R}_{pj}+\mathrm{R}_{kljp}\mathrm{R}_{pi}\big)\\
&+\mathrm{W}_{kljp}\nabla_l\mathrm{R}_{ip}+\mathrm{W}_{klip}\nabla_j\mathrm{R}_{jp}
\end{aligned}
$$

$$
\begin{aligned}
= \; & -R_{kp}\nabla_p R_{ij} + \frac{1}{2(n-2)}(R_{ik}\nabla_j R + R_{jk}\nabla_i R) \\
& -\frac{1}{n-2}(R_{kp}\nabla_i R_{jp} + R_{kp}\nabla_j R_{ip} - R_{lp}\nabla_l R_{jp} g_{ik} - R_{lp}\nabla_l R_{ip} g_{jk}) \\
& -\frac{1}{n-2}(R_{li}\nabla_l R_{jk} + R_{lj}\nabla_l R_{ik}) - \frac{1}{2(n-1)(n-2)}(R\nabla_j R g_{ik} + R\nabla_i R g_{jk}) \\
& +\frac{1}{(n-1)(n-2)}(R\nabla_i R_{jk} + R\nabla_j R_{ik}) \\
& +\nabla_l\Big(\frac{1}{n-2}(R_{ki} g_{pl} R_{pj} + R_{pl} g_{ki} R_{pj} - R_{li} g_{kp} R_{pj} - R_{kp} g_{li} R_{pj}) \\
& -\frac{1}{(n-1)(n-2)}(RR_{pj} g_{ki} g_{lp} - RR_{pj} g_{kp} g_{il}) + W_{klip} R_{pj} \\
& +\frac{1}{n-2}(R_{kj} g_{lp} R_{pi} + R_{lp} g_{kj} R_{pi} - R_{lj} g_{kp} R_{pi} - R_{kp} g_{lj} R_{pi}) \\
& -\frac{1}{(n-1)(n-2)}(RR_{pi} g_{kj} g_{pl} - RR_{pi} g_{kp} g_{lj}) + W_{kljp} R_{pi}\Big) \\
& +W_{kljp}\nabla_l R_{ip} + W_{klip}\nabla_j R_{jp} \\
= \; & -R_{kp}\nabla_p R_{ij} + \frac{1}{2(n-2)}(R_{ik}\nabla_j R + R_{jk}\nabla_i R) \\
& -\frac{1}{n-2}(R_{kp}\nabla_i R_{jp} + R_{kp}\nabla_j R_{ip} - R_{lp}\nabla_l R_{jp} g_{ik} - R_{lp}\nabla_l R_{ip} g_{jk}) \\
& -\frac{1}{n-2}(R_{li}\nabla_l R_{jk} + R_{lj}\nabla_l R_{ik}) - \frac{1}{2(n-1)(n-2)}(R\nabla_j R g_{ik} + R\nabla_i R g_{jk}) \\
& +\frac{1}{(n-1)(n-2)}(R\nabla_i R_{jk} + R\nabla_j R_{ik}) \\
& +\frac{1}{n-2}(\nabla_p R_{ki} R_{pj} + R_{ki}\nabla_j R/2 + \nabla_p R g_{ki} R_{pj}/2 \\
& +R_{lp}\nabla_l R_{pj} g_{ik} - \nabla_i RR_{jk}/2 - R_{pi}\nabla_p R_{kj} - \nabla_i R_{kp} R_{pj} \\
& -R_{kp}\nabla_i R_{pj}) \\
& -\frac{1}{(n-1)(n-2)}(\nabla_p RR_{pj} g_{ik} + R\nabla_j R g_{ik}/2 - \nabla_i RR_{kj} - R\nabla_i R_{jk}) \\
& +\frac{n-3}{n-2}C_{kip} R_{pj} + W_{klip}\nabla_l R_{pj} \\
& +\frac{1}{n-2}(\nabla_p R_{kj} R_{pi} + R_{kj}\nabla_i R/2 + \nabla_p R g_{kj} R_{pi}/2 \\
& +R_{lp} g_{kj}\nabla_l R_{pi} - \nabla_j RR_{ki}/2 - R_{pj}\nabla_p R_{ki} - \nabla_j R_{kp} R_{pi} - R_{kp}\nabla_j R_{pi}) \\
& -\frac{1}{(n-1)(n-2)}(\nabla_p RR_{pi} g_{kj} + R\nabla_i R g_{jk}/2 - \nabla_j RR_{ki} - R\nabla_j R_{ki}) \\
& +\frac{n-3}{n-2}C_{kjp} R_{pi} + W_{kljp}\nabla_l R_{pi} \\
& +W_{kljp}\nabla_l R_{ip} + W_{klip}\nabla_j R_{jp} \\
= \; & -R_{kp}\nabla_p R_{ij} + \frac{n+1}{2(n-1)(n-2)}R_{kj}\nabla_i R - \frac{2}{n-2}R_{kp}\nabla_j R_{ip} \\
& +\frac{2}{n-2}R_{lp}\nabla_l R_{pi} g_{jk} - \frac{1}{n-2}R_{pj}\nabla_p R_{ik} - \frac{1}{(n-1)(n-2)}R\nabla_i R g_{jk} \\
& +\frac{2}{(n-1)(n-2)}R\nabla_j R_{ik} + \frac{n+1}{2(n-1)(n-2)}\nabla_j RR_{ki} - \frac{2}{n-2}R_{kp}\nabla_i R_{jp} \\
& +\frac{2}{n-2}R_{lp}\nabla_l R_{pj} g_{ik} - \frac{1}{n-2}R_{pi}\nabla_p R_{jk} - \frac{1}{(n-1)(n-2)}R\nabla_j R g_{ik} \\
& +\frac{2}{(n-1)(n-2)}R\nabla_i R_{jk} + 2W_{kljp}\nabla_l R_{pi} + 2W_{klip}\nabla_l R_{pj} \\
& +\frac{n-3}{2(n-1)(n-2)}(\nabla_p R g_{ik} R_{pj} + \nabla_p R g_{jk} R_{pi}) \\
& +\frac{n-3}{n-2}(C_{kip} R_{pj} + C_{kjp} R_{pi}) - \frac{1}{n-2}(\nabla_i R_{kp} R_{pj} + \nabla_j R_{kp} R_{pi})
\end{aligned}
$$

and

$$\nabla_k \Delta \mathrm{R} - \Delta \nabla_k \mathrm{R} = \mathrm{R}_{kllp} \nabla_p \mathrm{R} = -\mathrm{R}_{kp} \nabla_p \mathrm{R}\,.$$

Then, getting back to the main computation, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{C}_{ijk} - \Delta \mathrm{C}_{ijk} \;=\;& -\mathrm{R}_{kp}\nabla_p \mathrm{R}_{ij} + \frac{n+1}{2(n-1)(n-2)}\mathrm{R}_{kj}\nabla_i \mathrm{R}\\
&-\frac{2}{n-2}\mathrm{R}_{kp}\nabla_j \mathrm{R}_{ip} + \frac{2}{n-2}\mathrm{R}_{lp}\nabla_l \mathrm{R}_{pi} g_{jk}\\
&-\frac{1}{n-2}\mathrm{R}_{jp}\nabla_p \mathrm{R}_{ik} - \frac{1}{(n-1)(n-2)}\mathrm{R}\nabla_i \mathrm{R} g_{jk}\\
&+\frac{2}{(n-1)(n-2)}\mathrm{R}\nabla_j \mathrm{R}_{ik} + \frac{n+1}{2(n-1)(n-2)}\nabla_j \mathrm{R}\mathrm{R}_{ki}\\
&-\frac{2}{n-2}\mathrm{R}_{kp}\nabla_i \mathrm{R}_{pj} + \frac{2}{n-2}\mathrm{R}_{lp}\nabla_l \mathrm{R}_{pj} g_{ik}\\
&-\frac{1}{n-2}\mathrm{R}_{pi}\nabla_p \mathrm{R}_{kj} - \frac{1}{(n-1)(n-2)}\mathrm{R}\nabla_j \mathrm{R} g_{ik}\\
&+\frac{2}{(n-1)(n-2)}\mathrm{R}\nabla_i \mathrm{R}_{jk} + 2\mathrm{W}_{kljp}\nabla_l \mathrm{R}_{pi} + 2\mathrm{W}_{klip}\nabla_l \mathrm{R}_{pj}\\
&+\frac{n-3}{2(n-1)(n-2)}(\nabla_p \mathrm{R} g_{ik}\mathrm{R}_{pj} + \nabla_p \mathrm{R} g_{jk}\mathrm{R}_{pi})\\
&+\frac{n-3}{n-2}(\mathrm{C}_{kip}\mathrm{R}_{pj} + \mathrm{C}_{kjp}\mathrm{R}_{pi})\\
&-\frac{1}{n-2}(\nabla_i \mathrm{R}_{kp}\mathrm{R}_{jp} + \nabla_j \mathrm{R}_{kp}\mathrm{R}_{pi})\\
&+\mathrm{R}_{jp}\nabla_p \mathrm{R}_{ik} - \frac{n+1}{2(n-1)(n-2)}\mathrm{R}_{kj}\nabla_i \mathrm{R} + \frac{2}{n-2}\mathrm{R}_{jp}\nabla_k \mathrm{R}_{ip}\\
&-\frac{2}{n-2}\mathrm{R}_{lp}\nabla_l \mathrm{R}_{pi} g_{kj} + \frac{1}{n-2}\mathrm{R}_{pk}\nabla_p \mathrm{R}_{ij}\\
&+\frac{1}{(n-1)(n-2)}\mathrm{R}\nabla_i \mathrm{R} g_{jk} - \frac{2}{(n-1)(n-2)}\mathrm{R}\nabla_k \mathrm{R}_{ij}\\
&-\frac{n+1}{2(n-1)(n-2)}\nabla_k \mathrm{R}\mathrm{R}_{ij} + \frac{2}{n-2}\mathrm{R}_{jp}\nabla_i \mathrm{R}_{kp}\\
&-\frac{2}{n-2}\mathrm{R}_{lp}\nabla_p \mathrm{R}_{pk} g_{ij} + \frac{1}{n-2}\mathrm{R}_{pi}\nabla_p \mathrm{R}_{kj}\\
&+\frac{1}{(n-1)(n-2)}\mathrm{R}\nabla_k \mathrm{R} g_{ij} - \frac{2}{(n-1)(n-2)}\mathrm{R}\nabla_i \mathrm{R}_{kj}\\
&-2\mathrm{W}_{jlkp}\nabla_l \mathrm{R}_{pi} - 2\mathrm{W}_{jlip}\nabla_l \mathrm{R}_{pk}\\
&-\frac{n-3}{2(n-2)(n-2)}(\nabla_p \mathrm{R} g_{ij}\mathrm{R}_{pk} + \nabla_p \mathrm{R} g_{jk}\mathrm{R}_{pi})\\
&-\frac{n-3}{n-2}(\mathrm{C}_{jip}\mathrm{R}_{pk} + \mathrm{C}_{jkp}\mathrm{R}_{pi}) + \frac{1}{n-2}(\nabla_i \mathrm{R}_{pj}\mathrm{R}_{pk} + \nabla_k \mathrm{R}_{jp}\mathrm{R}_{pi})\\
&+\frac{1}{2(n-1)}(\mathrm{R}_{kp}\nabla_p \mathrm{R} g_{ij} - \mathrm{R}_{jp}\nabla_p \mathrm{R} g_{ki}) - \frac{n+2}{n-2}(\nabla_k \mathrm{R}_{pi}\mathrm{R}_{pj} + \mathrm{R}_{pi}\nabla_k \mathrm{R}_{pj})\\
&+\frac{n}{(n-1)(n-2)}\nabla_k |\mathrm{Ric}|^2 g_{ij} + \frac{5n-2}{2(n-1)(n-2)}\nabla_k \mathrm{R}\mathrm{R}_{ij}\\
&+\frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_k \mathrm{R}_{ij} - \frac{2}{(n-1)(n-2)}\nabla_k \mathrm{R}^2 g_{ij}\\
&-2\nabla_k \mathrm{R}_{pl}\mathrm{W}_{pijl} - 2\mathrm{R}_{pl}\nabla_k \mathrm{W}_{pijl}\\
&+\mathrm{C}_{kli}\mathrm{R}_{lj} - \frac{1}{2(n-1)}\nabla_l \mathrm{R}\mathrm{R}_{lj} g_{ik} + \mathrm{C}_{klj}\mathrm{R}_{li}\\
&+\frac{n+2}{n-2}(\nabla_j \mathrm{R}_{pi}\mathrm{R}_{pk} + \mathrm{R}_{pi}\nabla_j \mathrm{R}_{pk})\\
&-\frac{n}{(n-1)(n-2)}\nabla_j |\mathrm{Ric}|^2 g_{ki} - \frac{5n-2}{2(n-1)(n-2)}\nabla_j \mathrm{R}\mathrm{R}_{ik}\\
&-\frac{2n}{(n-1)(n-2)}\mathrm{R}\nabla_j \mathrm{R}_{ik} + \frac{2}{(n-1)(n-2)}\nabla_j \mathrm{R}^2 g_{ik}
\end{aligned}$$

$$
\begin{aligned}
&+2\nabla_j R_{pl} W_{pikl} + 2R_{pl}\nabla_j W_{pikl}\\
&-C_{jli}R_{lk} + \frac{1}{2(n-1)}\nabla_l R R_{lk} g_{ij} - C_{jlk}R_{li}\\
= &\ \frac{1}{n-2}(R_{pi}C_{jkp} + R_{pk}C_{jip} - C_{kip}R_{pj} - C_{kjp}R_{pi})\\
&+\Big[\frac{2}{n-2}R_{lp}\nabla_l R_{pj} + \frac{3}{2(n-1)(n-2)}\nabla_j R^2\\
&-\frac{1}{2(n-2)}\nabla_p R R_{pj} - \frac{n}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2\Big] g_{ik}\\
&-\Big[\frac{2}{n-2}R_{lp}\nabla_l R_{pk} + \frac{3}{2(n-1)(n-2)}\nabla_k R^2\\
&-\frac{1}{2(n-2)}\nabla_p R R_{pk} - \frac{n}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2\Big] g_{ij}\\
&-\frac{n-3}{n-2}R_{kp}\nabla_p R_{ij} + \frac{n-3}{n-2}R_{pj}\nabla_p R_{ik}\\
&+\frac{n}{n-2}R_{kp}\nabla_j R_{pi} + \frac{n+1}{n-2}\nabla_j R_{pk}R_{pi} - \frac{2}{n-2}R\nabla_j R_{ik}\\
&-\frac{4n-3}{2(n-1)(n-2)}\nabla_j R R_{ik} - \frac{1}{n-2}R_{kp}\nabla_i R_{pj} + \frac{1}{n-2}R_{pj}\nabla_i R_{pk}\\
&-\frac{n}{n-2}R_{jp}\nabla_k R_{ip} - \frac{n+1}{n-2}\nabla_k R_{jp}R_{ip} + \frac{2}{n-2}R\nabla_k R_{ij}\\
&+\frac{4n-3}{2(n-1)(n-2)}\nabla_k R R_{ij} + 2W_{klip}\nabla_l R_{pj} + 2W_{kljp}\nabla_l R_{pi} - 2W_{jlkp}\nabla_l R_{pi}\\
&-2W_{jlip}\nabla_l R_{pk} - 2\nabla_k R_{pl}W_{pijl} - 2R_{pl}\nabla_k W_{pijl} + 2\nabla_j R_{pl}W_{pikl} + 2R_{pl}\nabla_j W_{pikl}
\end{aligned}
$$

Now, by means of the very definition of the Cotton tensor (1.2), the identities (1.4), and the symmetries of the Weyl tensor, we substitute

$$
\begin{aligned}
C_{kpj} - C_{jpk} &= -C_{kjp} - C_{jpk} = C_{pkj}\\
\nabla_l R_{jp} &= \nabla_j R_{lp} + C_{pjl} + \frac{1}{2(n-1)}\left(\nabla_l R g_{pj} - \nabla_j R g_{pl}\right)\\
\nabla_l R_{kp} &= \nabla_k R_{lp} + C_{pkl} + \frac{1}{2(n-1)}\left(\nabla_l R g_{pk} - \nabla_k R g_{pl}\right)\\
\nabla_i R_{jp} &= \nabla_j R_{ip} + C_{pji} + \frac{1}{2(n-1)}\left(\nabla_i R g_{jp} - \nabla_j R g_{ip}\right)\\
\nabla_i R_{kp} &= \nabla_k R_{ip} + C_{pki} + \frac{1}{2(n-1)}\left(\nabla_i R g_{kp} - \nabla_k R g_{ip}\right)\\
\nabla_p R_{ij} &= \nabla_j R_{pi} + C_{ijp} + \frac{1}{2(n-1)}\left(\nabla_p R g_{ji} - \nabla_j R g_{pi}\right)\\
\nabla_p R_{ik} &= \nabla_k R_{pi} + C_{ikp} + \frac{1}{2(n-1)}\left(\nabla_p R g_{ki} - \nabla_k R g_{pi}\right)
\end{aligned}
$$

in the last expression above, getting

$$
\begin{aligned}
\frac{\partial}{\partial t}C_{ijk} - \Delta C_{ijk} = &\ \frac{1}{n-2}(R_{pi}C_{pkj} + R_{pk}C_{jip} - C_{kip}R_{pj})\\
&+\Big[\frac{2}{n-2}R_{lp}\Big(\nabla_j R_{lp} + C_{pjl} + \frac{1}{2(n-1)}\nabla_l R g_{pj}\\
&-\frac{1}{2(n-1)}\nabla_j R g_{pl}\Big)\Big) + \frac{3}{2(n-1)(n-2)}\nabla_j R^2\\
&-\frac{1}{2(n-2)}\nabla_p R R_{pj} - \frac{n}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2\Big] g_{ik}\\
&-\Big[\frac{2}{n-2}R_{lp}\Big(\nabla_k R_{pl} + C_{pkl} + \frac{1}{2(n-1)}\nabla_l R g_{pk}\\
&-\frac{1}{2(n-1)}\nabla_k R g_{pl}\Big) + \frac{3}{2(n-1)(n-2)}\nabla_k R^2\\
&-\frac{1}{2(n-2)}\nabla_p R R_{pk} - \frac{n}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2\Big] g_{ij}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{n-3}{n-2}\mathrm{R}_{kp}\Big(\mathrm{C}_{ijp}+\nabla_j\mathrm{R}_{ip}+\frac{1}{2(n-1)}(\nabla_p\mathrm{R}g_{ij}-\nabla_j\mathrm{R}g_{ip})\Big)\\
&+\frac{n-3}{n-2}\mathrm{R}_{pj}\Big(\mathrm{C}_{ikp}+\nabla_k\mathrm{R}_{ip}+\frac{1}{2(n-1)}(\nabla_p\mathrm{R}g_{ik}-\nabla_k\mathrm{R}g_{ip})\Big)\\
&+\frac{n}{n-2}\mathrm{R}_{kp}\nabla_j\mathrm{R}_{pi}+\frac{n+1}{n-2}\nabla_j\mathrm{R}_{pk}\mathrm{R}_{pi}-\frac{2}{n-2}\mathrm{R}\nabla_j\mathrm{R}_{ik}\\
&-\frac{4n-3}{2(n-1)(n-2)}\nabla_j\mathrm{R}\mathrm{R}_{ik}\\
&-\frac{1}{n-2}\mathrm{R}_{kp}\Big(\nabla_j\mathrm{R}_{ip}+\mathrm{C}_{pji}+\frac{1}{2(n-1)}(\nabla_i\mathrm{R}g_{jp}-\nabla_j\mathrm{R}g_{ip})\Big)\\
&+\frac{1}{n-2}\mathrm{R}_{pj}\Big(\nabla_k\mathrm{R}_{ip}+\mathrm{C}_{kpi}+\frac{1}{2(n-1)}(\nabla_i\mathrm{R}g_{kp}-\nabla_k\mathrm{R}g_{ip})\Big)\\
&-\frac{n}{n-2}\mathrm{R}_{jp}\nabla_k\mathrm{R}_{ip}-\frac{n+1}{n-2}\nabla_k\mathrm{R}_{jp}\mathrm{R}_{ip}+\frac{2}{n-2}\mathrm{R}\nabla_k\mathrm{R}_{ij}\\
&+\frac{4n-3}{2(n-1)(n-2)}\nabla_k\mathrm{R}\mathrm{R}_{ij}\\
&+2\mathrm{C}_{plj}\mathrm{W}_{pikl}-2\mathrm{C}_{plk}\mathrm{W}_{pijl}-2\mathrm{C}_{pil}\mathrm{W}_{jklp}\\
&-2\mathrm{W}_{jklp}\nabla_i\mathrm{R}_{pl}-2\mathrm{R}_{pl}\nabla_k\mathrm{W}_{pijl}+2\mathrm{R}_{pl}\nabla_j\mathrm{W}_{pikl}\\
=\;&\frac{1}{n-2}\left(\mathrm{R}_{pi}\mathrm{C}_{pkj}+\mathrm{R}_{pk}(\mathrm{C}_{jip}-\mathrm{C}_{pji}-(n-3)\mathrm{C}_{ijp})+\mathrm{R}_{pj}(\mathrm{C}_{pki}-\mathrm{C}_{kip}+(n-3)\mathrm{C}_{ikp})\right)\\
&+\frac{2}{n-2}\mathrm{C}_{pjl}\mathrm{R}_{pl}g_{ik}-\frac{2}{n-2}\mathrm{C}_{pkl}\mathrm{R}_{pl}g_{ij}-2\mathrm{C}_{pjl}\mathrm{W}_{pikl}+2\mathrm{C}_{pkl}\mathrm{W}_{pijl}-2\mathrm{C}_{pil}\mathrm{W}_{jklp}\\
&+g_{ik}\Big[\frac{\nabla_j\mathrm{R}^2}{(n-1)(n-2)}-\frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2\Big]\\
&-g_{ij}\Big[\frac{\nabla_k\mathrm{R}^2}{(n-1)(n-2)}-\frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2\Big]\\
&-\frac{2}{n-2}\mathrm{R}_{jp}\nabla_k\mathrm{R}_{ip}-\frac{n+1}{n-2}\nabla_k\mathrm{R}_{jp}\mathrm{R}_{ip}+\frac{3n-1}{2(n-1)(n-2)}\nabla_k\mathrm{R}\mathrm{R}_{ij}+\frac{2}{n-2}\mathrm{R}\nabla_k\mathrm{R}_{ij}\\
&+\frac{2}{n-2}\mathrm{R}_{kp}\nabla_j\mathrm{R}_{ip}+\frac{n+1}{n-2}\nabla_j\mathrm{R}_{kp}\mathrm{R}_{ip}-\frac{3n-1}{2(n-1)(n-2)}\nabla_j\mathrm{R}\mathrm{R}_{ik}-\frac{2}{n-2}\mathrm{R}\nabla_j\mathrm{R}_{ik}\\
&-2\mathrm{W}_{jklp}\nabla_i\mathrm{R}_{lp}-2\mathrm{R}_{lp}\nabla_k\mathrm{W}_{pijl}+2\mathrm{R}_{pl}\nabla_j\mathrm{W}_{pikl}\,.
\end{aligned}
$$

then, we substitute again

$$
\begin{aligned}
\nabla_k\mathrm{R}_{jp}&=\nabla_p\mathrm{R}_{kj}+\mathrm{C}_{jpk}+\frac{1}{2(n-1)}\big(\nabla_k\mathrm{R}g_{jp}-\nabla_p\mathrm{R}g_{jk}\big)\\
\nabla_j\mathrm{R}_{kp}&=\nabla_p\mathrm{R}_{jk}+\mathrm{C}_{kpj}+\frac{1}{2(n-1)}\big(\nabla_j\mathrm{R}g_{kp}-\nabla_p\mathrm{R}g_{kj}\big)\\
\nabla_k\mathrm{R}_{ij}&=\nabla_i\mathrm{R}_{kj}+\mathrm{C}_{jik}+\frac{1}{2(n-1)}\big(\nabla_k\mathrm{R}g_{ij}-\nabla_i\mathrm{R}g_{jk}\big)\\
\nabla_j\mathrm{R}_{ik}&=\nabla_i\mathrm{R}_{jk}+\mathrm{C}_{kij}+\frac{1}{2(n-1)}\big(\nabla_j\mathrm{R}g_{ik}-\nabla_i\mathrm{R}g_{kj}\big)\,,
\end{aligned}
$$

finally obtaining

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathrm{C}_{ijk}-\Delta\mathrm{C}_{ijk}=\;&\frac{1}{n-2}\left(\mathrm{R}_{pi}\mathrm{C}_{pkj}+\mathrm{R}_{pk}(\mathrm{C}_{jip}-\mathrm{C}_{pji}-(n-3)\mathrm{C}_{ijp})+\mathrm{R}_{pj}(\mathrm{C}_{pki}-\mathrm{C}_{kip}+(n-3)\mathrm{C}_{ikp})\right)\\
&+\frac{2}{n-2}\mathrm{C}_{pjl}\mathrm{R}_{pl}g_{ik}-\frac{2}{n-2}\mathrm{C}_{pkl}\mathrm{R}_{pl}g_{ij}-2\mathrm{C}_{pjl}\mathrm{W}_{pikl}+2\mathrm{C}_{pkl}\mathrm{W}_{pijl}-2\mathrm{C}_{pil}\mathrm{W}_{jklp}\\
&+g_{ik}\Big[\frac{\nabla_j\mathrm{R}^2}{(n-1)(n-2)}-\frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2\Big]\\
&-g_{ij}\Big[\frac{\nabla_k\mathrm{R}^2}{(n-1)(n-2)}-\frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2\Big]\\
&-\frac{2}{n-2}\mathrm{R}_{jp}\nabla_k\mathrm{R}_{ip}-\frac{n+1}{n-2}\mathrm{R}_{ip}\nabla_p\mathrm{R}_{kj}-\frac{n+1}{n-2}\mathrm{R}_{ip}\mathrm{C}_{jpk}\\
&-\frac{n+1}{2(n-1)(n-2)}\mathrm{R}_{ij}\nabla_k\mathrm{R}+\frac{n+1}{2(n-1)(n-2)}\mathrm{R}_{ip}\nabla_p\mathrm{R}g_{jk}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3n-1}{2(n-1)(n-2)}\nabla_k \mathrm{R}\mathrm{R}_{ij}+\frac{2}{n-2}\mathrm{R}(\nabla_i \mathrm{R}_{jk}+\mathrm{C}_{jik}+\frac{1}{2(n-1)}(\nabla_k \mathrm{R}g_{ij}-\nabla_i \mathrm{R}g_{jk}))\\
&+\frac{2}{n-2}\mathrm{R}_{kp}\nabla_j \mathrm{R}_{ip}+\frac{n+1}{n-2}\mathrm{R}_{ip}\nabla_p \mathrm{R}_{kj}+\frac{n+1}{n-2}\mathrm{R}_{ip}\mathrm{C}_{kpj}+\frac{n+1}{2(n-1)(n-2)}\nabla_j \mathrm{R}\mathrm{R}_{ik}\\
&-\frac{n+1}{2(n-1)(n-2)}\mathrm{R}_{ip}\nabla_p \mathrm{R}g_{jk}-\frac{3n-1}{2(n-1)(n-2)}\nabla_j \mathrm{R}\mathrm{R}_{ik}\\
&-\frac{2}{n-2}\mathrm{R}(\nabla_i \mathrm{R}_{jk}+\mathrm{C}_{kij}+\frac{1}{2(n-1)}(\nabla_j \mathrm{R}g_{ik}-\nabla_i \mathrm{R}g_{jk}))\\
&-2\mathrm{W}_{jklp}\nabla_i \mathrm{R}_{lp}-2\mathrm{R}_{lp}\nabla_k \mathrm{W}_{pijl}+2\mathrm{R}_{pl}\nabla_j \mathrm{W}_{pikl}\\
=\;&\frac{1}{n-2}(\mathrm{R}_{pk}(\mathrm{C}_{jip}-\mathrm{C}_{pji}-(n-3)\mathrm{C}_{ijp})-\mathrm{R}_{pj}(\mathrm{C}_{kip}-\mathrm{C}_{pki}-(n-3)\mathrm{C}_{ikp})\\
&+(n+2)\mathrm{R}_{pi}\mathrm{C}_{pkj})+\frac{2}{n-2}(\mathrm{C}_{pjl}\mathrm{R}_{pl}g_{ik}-\mathrm{C}_{pkl}\mathrm{R}_{pl}g_{ij})+\frac{2}{n-2}\mathrm{R}\mathrm{C}_{ijk}\\
&-2\mathrm{W}_{pikl}\mathrm{C}_{pjl}+2\mathrm{W}_{pijl}\mathrm{C}_{pkl}-2\mathrm{C}_{pil}\mathrm{W}_{jklp}\\
&+g_{ik}\Big[\frac{\nabla_j \mathrm{R}^2}{2(n-1)(n-2)}-\frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2\Big]\\
&-g_{ij}\Big[\frac{\nabla_k \mathrm{R}^2}{2(n-1)(n-2)}-\frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2\Big]\\
&-\frac{2}{n-2}\mathrm{R}_{jp}\nabla_k \mathrm{R}_{ip}+\frac{1}{n-2}\nabla_k \mathrm{R}\mathrm{R}_{ij}\\
&+\frac{2}{n-2}\mathrm{R}_{kp}\nabla_j \mathrm{R}_{ip}-\frac{1}{n-2}\nabla_j \mathrm{R}\mathrm{R}_{ik}\\
&+2\mathrm{R}_{lp}\nabla_j \mathrm{W}_{pikl}-2\mathrm{R}_{lp}\nabla_k \mathrm{W}_{pijl}\,,
\end{aligned}
$$

where in the last passage we used again the identities (1.4) and the fact that

$$
\mathrm{W}_{jklp}\nabla_i \mathrm{R}_{lp}=\mathrm{W}_{jkpl}\nabla_i \mathrm{R}_{pl}=\mathrm{W}_{jkpl}\nabla_i \mathrm{R}_{lp}=-\mathrm{W}_{jklp}\nabla_i \mathrm{R}_{lp}\,.
$$

Hence, we can resume this long computation in the following proposition, getting back to a generic coordinate basis.

Proposition 4.1. *During the Ricci flow of a n–dimensional Riemannian manifold $(M^n, g(t))$, the Cotton tensor satisfies the following evolution equation*

$$
\begin{aligned}
(\partial_t-\Delta)\mathrm{C}_{ijk}\;=\;&\frac{1}{n-2}\big[g^{pq}\mathrm{R}_{pj}(\mathrm{C}_{kqi}+\mathrm{C}_{qki}+(n-3)\mathrm{C}_{ikq})\\
&+(n+2)g^{pq}\mathrm{R}_{ip}\mathrm{C}_{qkj}-g^{pq}\mathrm{R}_{pk}(\mathrm{C}_{jqi}+\mathrm{C}_{qji}+(n-3)\mathrm{C}_{ijq})\big]\\
&+\frac{2}{n-2}\mathrm{R}\mathrm{C}_{ijk}+\frac{2}{n-2}\mathrm{R}^{ql}\mathrm{C}_{qjl}g_{ik}-\frac{2}{n-2}\mathrm{R}^{ql}\mathrm{C}_{qkl}g_{ij}\\
&+\frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2 g_{ij}-\frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2 g_{ik}\\
&+\frac{\mathrm{R}}{(n-1)(n-2)}\nabla_j \mathrm{R}g_{ik}-\frac{\mathrm{R}}{(n-1)(n-2)}\nabla_k \mathrm{R}g_{ij}\\
&+\frac{2}{n-2}g^{pq}\mathrm{R}_{pk}\nabla_j \mathrm{R}_{qi}-\frac{2}{n-2}g^{pq}\mathrm{R}_{pj}\nabla_k \mathrm{R}_{qi}+\frac{1}{n-2}\mathrm{R}_{ij}\nabla_k \mathrm{R}-\frac{1}{n-2}\mathrm{R}_{ik}\nabla_j \mathrm{R}\\
&-2g^{pq}\mathrm{W}_{pikl}\mathrm{C}_{qjl}+2g^{pq}\mathrm{W}_{pijl}\mathrm{C}_{qkl}-2g^{pq}\mathrm{W}_{jklp}\mathrm{C}_{qil}+2g^{pq}\mathrm{R}_{pl}\nabla_j \mathrm{W}_{qikl}-2g^{pq}\mathrm{R}_{pl}\nabla_k \mathrm{W}_{qijl}\,.
\end{aligned}
$$

In particular if the Cotton tensor vanishes identically along the flow we obtain,

$$
\begin{aligned}
0\;=\;&\frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2 g_{ij}-\frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2 g_{ik}\\
&+\frac{\mathrm{R}}{(n-1)(n-2)}\nabla_j \mathrm{R}g_{ik}-\frac{\mathrm{R}}{(n-1)(n-2)}\nabla_k \mathrm{R}g_{ij}\\
&+\frac{2}{n-2}g^{pq}\mathrm{R}_{pk}\nabla_j \mathrm{R}_{qi}-\frac{2}{n-2}g^{pq}\mathrm{R}_{pj}\nabla_k \mathrm{R}_{qi}+\frac{1}{n-2}\mathrm{R}_{ij}\nabla_k \mathrm{R}-\frac{1}{n-2}\mathrm{R}_{ik}\nabla_j \mathrm{R}\\
&+2g^{pq}\mathrm{R}_{pl}\nabla_j \mathrm{W}_{qikl}-2g^{pq}\mathrm{R}_{pl}\nabla_k \mathrm{W}_{qijl}\,,
\end{aligned}
$$

while, in virtue of relation (1.3), *if the Weyl tensor vanishes along the flow we obtain (compare with [5, Proposition 1.1 and Corollary 1.2])*

$$\begin{aligned} 0 &= \frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2 g_{ij} - \frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2 g_{ik} \\ &+\frac{\mathrm{R}}{(n-1)(n-2)}\nabla_j \mathrm{R} g_{ik} - \frac{\mathrm{R}}{(n-1)(n-2)}\nabla_k \mathrm{R} g_{ij} \\ &+\frac{2}{n-2}g^{pq}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{qi} - \frac{2}{n-2}g^{pq}\mathrm{R}_{pj}\nabla_k\mathrm{R}_{qi} + \frac{1}{n-2}\mathrm{R}_{ij}\nabla_k\mathrm{R} - \frac{1}{n-2}\mathrm{R}_{ik}\nabla_j\mathrm{R}\,. \end{aligned}$$

Corollary 4.2. *During the Ricci flow of a n–dimensional Riemannian manifold $(M^n, g(t))$, the squared norm of the Cotton tensor satisfies the following evolution equation, in an orthonormal basis,*

$$\begin{aligned} (\partial_t - \Delta)|\mathrm{C}_{ijk}|^2 &= -2|\nabla\mathrm{C}_{ijk}|^2 - \frac{16}{n-2}\mathrm{C}_{ipk}\mathrm{C}_{iqk}\mathrm{R}_{pq} + \frac{24}{n-2}\mathrm{C}_{ipk}\mathrm{C}_{kqi}\mathrm{R}_{pq} \\ &+\frac{4}{n-2}\mathrm{R}|\mathrm{C}_{ijk}|^2 + \frac{8}{n-2}\mathrm{C}_{ijk}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{pi} + \frac{4}{n-2}\mathrm{C}_{ijk}\mathrm{R}_{ij}\nabla_k\mathrm{R} \\ &+8\mathrm{C}_{ijk}\mathrm{R}_{lp}\nabla_j\mathrm{W}_{pikl} - 8\mathrm{C}_{ijk}\mathrm{C}_{pjl}\mathrm{W}_{pikl} - 4\mathrm{C}_{jpi}\mathrm{C}_{ljk}\mathrm{W}_{pikl}\,. \end{aligned}$$

$$\begin{aligned} (\partial_t - \Delta)|\mathrm{C}_{ijk}|^2 &= -2|\nabla\mathrm{C}_{ijk}|^2 + 2\mathrm{C}^{ijk}\mathrm{R}_{ip}g^{pq}\mathrm{C}_{qjk} + 2\mathrm{C}^{ijk}\mathrm{R}_{jp}g^{pq}\mathrm{C}_{iqk} + 2\mathrm{C}^{ijk}\mathrm{R}_{kp}g^{pq}\mathrm{C}_{iqk} \\ &+2\mathrm{C}_{ijk}\Big[\frac{1}{n-2}\big[(\mathrm{R}_{pj}(\mathrm{C}_{kpi} + \mathrm{C}_{pki} + (n-3)\mathrm{C}_{ikp}) \\ &+(n+2)\mathrm{R}_{pi}\mathrm{C}_{pkj} - \mathrm{R}_{pk}(\mathrm{C}_{jpi} + \mathrm{C}_{pji} + (n-3)\mathrm{C}_{ijp})) \\ &+\frac{2}{n-2}\mathrm{R}\mathrm{C}_{ijk} + \frac{2}{n-2}\mathrm{R}_{ql}\mathrm{C}_{qjl}g_{ik} - \frac{2}{n-2}\mathrm{R}_{ql}\mathrm{C}_{qkl}g_{ij} \\ &+\frac{1}{(n-1)(n-2)}\nabla_k|\mathrm{Ric}|^2 g_{ij} - \frac{1}{(n-1)(n-2)}\nabla_j|\mathrm{Ric}|^2 g_{ik} \\ &+\frac{\mathrm{R}}{(n-1)(n-2)}\nabla_j\mathrm{R}g_{ik} - \frac{\mathrm{R}}{(n-1)(n-2)}\nabla_k\mathrm{R}g_{ij} \\ &+\frac{2}{n-2}\mathrm{R}_{qk}\nabla_j\mathrm{R}_{qi} - \frac{2}{n-2}\mathrm{R}_{qj}\nabla_k\mathrm{R}_{qi} + \frac{1}{n-2}\mathrm{R}_{ij}\nabla_k\mathrm{R} - \frac{1}{n-2}\mathrm{R}_{ik}\nabla_j\mathrm{R} \\ &-2\mathrm{W}_{pikl}\mathrm{C}_{pjl} + 2\mathrm{W}_{pijl}\mathrm{C}_{pkl} - 2\mathrm{W}_{jklp}\mathrm{C}_{pil} + 2\mathrm{R}_{pl}\nabla_j\mathrm{W}_{pikl} - 2\mathrm{R}_{pl}\nabla_k\mathrm{W}_{pikl}\Big] \\ &= -2|\nabla\mathrm{C}_{ijk}|^2 - \frac{16}{n-2}\mathrm{C}_{ipk}\mathrm{C}_{iqk}\mathrm{R}_{pq} + \frac{24}{n-2}\mathrm{C}_{ipk}\mathrm{C}_{kqi}\mathrm{R}_{pq} \\ &+\frac{4}{n-2}\mathrm{R}|\mathrm{C}_{ijk}|^2 + \frac{8}{n-2}\mathrm{C}_{ijk}\mathrm{R}_{pk}\nabla_j\mathrm{R}_{pi} + \frac{4}{n-2}\mathrm{C}_{ijk}\mathrm{R}_{ij}\nabla_k\mathrm{R} \\ &+8\mathrm{C}_{ijk}\mathrm{R}_{lp}\nabla_j\mathrm{W}_{pikl} - 8\mathrm{C}_{ijk}\mathrm{C}_{pjl}\mathrm{W}_{pikl} - 4\mathrm{C}_{jpi}\mathrm{C}_{ljk}\mathrm{W}_{pikl}\,. \end{aligned}$$

Remark 4.3. Notice that if $n = 3$ the two formulas in Proposition 4.1 and Corollary 4.2 become the ones in Proposition 2.1 and Corollary 2.5.

5 The Bach Tensor

The Bach tensor in dimension three is given by

$$\mathrm{B}_{ik} = \nabla_j\mathrm{C}_{ijk}\,.$$

Let $\mathrm{S}_{ij} = \mathrm{R}_{ij} - \frac{1}{2(n-1)}\mathrm{R}g_{ij}$ be the Schouten tensor, then

$$\mathrm{B}_{ik} = \nabla_j\mathrm{C}_{ijk} = \nabla_j(\nabla_k\mathrm{S}_{ij} - \nabla_j\mathrm{S}_{ik}) = \nabla_j\nabla_k\mathrm{S}_{ij} - \Delta\mathrm{S}_{ik}\,. \tag{5.1}$$

We compute, in generic dimension n,

$$\nabla_j\mathrm{C}_{ijk} = \nabla_j\nabla_k\mathrm{R}_{ij} - \frac{1}{2(n-1)}\nabla_j\nabla_k\mathrm{R}g_{ij} - \Delta\mathrm{S}_{ik}$$

$$
\begin{aligned}
&= +R_{jkil}R_{jl} + R_{jkjl}R_{il} + \nabla_k\nabla_j R_{ij} - \frac{1}{2(n-1)}\nabla_k\nabla_j R g_{ij} - \Delta S_{ik} \\
&= +\frac{1}{n-2}\left(R_{ij}g_{kl} - R_{jl}g_{ki} + R_{kl}g_{ij} - R_{ki}g_{jl} - \frac{R}{(n-1)}(g_{ij}g_{kl} - g_{jl}g_{ki})\right)R_{jl} + W_{jkil}R_{jl} \\
&\quad +R_{kl}R_{il} + \frac{1}{2}\nabla_k\nabla_i R - \frac{1}{2(n-1)}\nabla_k\nabla_i R - \Delta S_{ik} \\
&= +\frac{1}{n-2}(R_{ji}R_{jk} - |\mathrm{Ric}|^2 g_{ik} + R_{kl}R_{il} - RR_{ik}) - \frac{R}{(n-1)(n-2)}R_{ik} + \frac{R^2}{(n-1)(n-2)}g_{ik} \\
&\quad +W_{jkil}R_{jl} + R_{kl}R_{il} + \frac{n-2}{2(n-1)}\nabla_k\nabla_i R - \Delta S_{ik} \\
&= \frac{n}{n-2}R_{ij}R_{kj} - \frac{n}{(n-1)(n-2)}RR_{ik} - \frac{1}{n-2}|\mathrm{Ric}|^2 g_{ik} + \frac{R^2}{(n-1)(n-2)}g_{ik} \\
&\quad +W_{jkil}R_{jl} + \frac{n-2}{2(n-1)}\nabla_k\nabla_i R - \Delta S_{ik}\,.
\end{aligned}
$$

From this last expression, it is easy to see that the Bach tensor in dimension 3 is symmetric, i.e. $B_{ik} = B_{ki}$. Moreover, it is trace–free, that is, $g^{ik}B_{ik} = 0$ as $g^{ik}\nabla C_{ijk} = 0$.

Remark 5.1. In higher dimension, the Bach tensor is given by

$$
B_{ik} = \frac{1}{n-2}(\nabla_j C_{ijk} - R_{jl}W_{ijkl})\,.
$$

We note that, since $R_{jl}W_{ijkl} = R_{jl}W_{klij} = R_{jl}W_{kjil}$, from the above computation we get that the Bach tensor is symmetric in any dimension; finally, as the Weyl tensor is trace-free in every pair of indexes, there holds $g^{ik}B_{ik} = 0$.

We recall that Schur lemma yields the following equation for the divergence of the Schouten tensor

$$
\nabla_j S_{ij} = \frac{n-2}{2(n-1)}\nabla_i R\,. \tag{5.2}
$$

We write

$$
\nabla_k\nabla_j C_{ijk} = \nabla_k\nabla_j\nabla_k S_{ij} - \nabla_k\nabla_j\nabla_j S_{ik} = [\nabla_j, \nabla_k]\nabla_j S_{ik}\,,
$$

therefore,

$$
\begin{aligned}
\nabla_k\nabla_j C_{ijk} &= R_{jkjl}\nabla_l S_{ik} + R_{jkil}\nabla_j S_{lk} + R_{jkkl}\nabla_j S_{li} \\
&= R_{kl}\nabla_l S_{ik} + R_{jkil}\nabla_j S_{lk} - R_{jl}\nabla_j S_{li} \\
&= \left[\frac{1}{n-2}(R_{ij}g_{kl} - R_{jl}g_{ik} + R_{kl}g_{ij} - R_{ik}g_{jl}) - \frac{1}{(n-1)(n-2)}R(g_{ij}g_{kl} - g_{ik}g_{jl}) + W_{jkil}\right]\nabla_j S_{lk} \\
&= \frac{1}{n-2}(-R_{jl}\nabla_j S_{il} + R_{kl}\nabla_i S_{kl}) + W_{jkil}\nabla_j S_{lk} \\
&= \frac{1}{n-2}R_{jl}(\nabla_i S_{lj} - \nabla_j S_{il}) + W_{jkil}\nabla_j R_{kl} \\
&= \frac{1}{n-2}R_{jl}C_{lji} + W_{iljk}\nabla_j R_{kl}\,,
\end{aligned}
$$

where we repeatedly used equation (5.2), the trace–free property of the Weyl tensor and the definition of the Cotton tensor.

Recalling that

$$
\nabla_k W_{ijkl} = \nabla_k W_{klij} = -\frac{n-3}{n-2}C_{lij} = \frac{n-3}{n-2}C_{lji}\,,
$$

the divergence of the Bach tensor is given by

$$
\begin{aligned}
\nabla_k B_{ik} &= \frac{1}{n-2}\nabla_k(\nabla_j C_{ijk} - R_{jl}W_{ijkl}) = \frac{1}{(n-2)^2}R_{jl}C_{jli} - \frac{n-3}{(n-2)^2}C_{jli}R_{jl} \\
&= -\frac{n-4}{(n-2)^2}C_{jli}R_{jl}\,.
\end{aligned}
$$

In particular, for $n = 3$, we obtain $\nabla_k B_{ik} = \nabla_k B_{ki} = R_{jl}C_{jli}$ and, for $n = 4$, we get the classical result $\nabla_k B_{ik} = \nabla_k B_{ki} = 0$.

5.1 The Evolution Equation of the Bach Tensor in 3D

We turn now our attention to the evolution of the Bach tensor along the Ricci flow in dimension three. In order to obtain its evolution equation, instead of calculating directly the time derivative and the Laplacian of the Bach tensor, we employ the following equation

$$(\partial_t - \Delta)\mathrm{B}_{ik} = \nabla_j(\partial_t - \Delta)\mathrm{C}_{ijk} - [\Delta, \nabla_j]\mathrm{C}_{ijk} + 2\mathrm{R}_{pj}\nabla_p\mathrm{C}_{ijk} + [\partial_t, \nabla_j]C_{ijk}\,, \tag{5.3}$$

which relates the quantity we want to compute with the evolution of the Cotton tensor, the evolution of the Christoffel symbols and the formulas for the exchange of covariant derivatives. We will work on the various terms separately.

By the commutations formulas for derivatives, we have

$$\nabla_l\nabla_l\nabla_q\mathrm{C}_{ijk} - \nabla_l\nabla_q\nabla_l\mathrm{C}_{ijk} = \nabla_l(\mathrm{R}_{lqip}\mathrm{C}_{pjk} + \mathrm{R}_{lqjp}\mathrm{C}_{ipk} + \mathrm{R}_{lqkp}\mathrm{C}_{ijp})$$

$$\nabla_l\nabla_q\nabla_s\mathrm{C}_{ijk} - \nabla_q\nabla_l\nabla_s\mathrm{C}_{ijk} = \mathrm{R}_{lqsp}\nabla_p\mathrm{C}_{ijk} + \mathrm{R}_{lqip}\nabla_s\mathrm{C}_{pjk} + \mathrm{R}_{lqjp}\nabla_s\mathrm{C}_{ipk} + \mathrm{R}_{lqkp}\nabla_s\mathrm{C}_{ijp}\,,$$

and putting these together with $q = j$ and $l = s$, we get

$$\begin{aligned}
[\Delta, \nabla_j]\mathrm{C}_{ijk} &= \nabla_l(\mathrm{R}_{ljip}\mathrm{C}_{pjk} - \mathrm{R}_{lp}\mathrm{C}_{ipk} + \mathrm{R}_{ljkp}\mathrm{C}_{ijp}) \\
&\quad + \mathrm{R}_{jp}\nabla_p\mathrm{C}_{ijk} + \mathrm{R}_{ljip}\nabla_l\mathrm{C}_{pjk} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{ipk} + \mathrm{R}_{ljkp}\nabla_l\mathrm{C}_{ijp} \\
&= \nabla_l\Big[\Big(\mathrm{R}_{li}g_{jp} - \mathrm{R}_{lp}g_{ji} + \mathrm{R}_{jp}g_{li} - \mathrm{R}_{ji}g_{lp} - \frac{\mathrm{R}}{2}(g_{li}g_{jp} - g_{lp}g_{ji})\Big)\mathrm{C}_{pjk} \\
&\quad - \mathrm{R}_{lp}\mathrm{C}_{ipk} + \Big(\mathrm{R}_{lk}g_{jp} - \mathrm{R}_{lp}g_{jk} + \mathrm{R}_{jp}g_{lk} - \mathrm{R}_{jk}g_{lp} - \frac{\mathrm{R}}{2}(g_{lk}g_{jp} - g_{lp}g_{jk})\Big)\mathrm{C}_{ijp}\Big] \\
&\quad + \mathrm{R}_{jp}\nabla_p\mathrm{C}_{ijk} + \mathrm{R}_{ljip}\nabla_l\mathrm{C}_{pjk} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{ipk} + \mathrm{R}_{ljkp}\nabla_l\mathrm{C}_{ijp} \\
&= -\frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{pik} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{pik} + \nabla_i\mathrm{R}_{jp}\mathrm{C}_{pjk} + \mathrm{R}_{jp}\nabla_i\mathrm{C}_{pjk} - \nabla_p\mathrm{R}_{ji}\mathrm{C}_{pjk} - \mathrm{R}_{ji}\nabla_p\mathrm{C}_{pjk} \\
&\quad + \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{pik} + \frac{\mathrm{R}}{2}\nabla_p\mathrm{C}_{pik} - \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{ipk} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{ipk} - \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{ikp} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{ikp} \\
&\quad + \nabla_k\mathrm{R}_{jp}\mathrm{C}_{ijp} + \mathrm{R}_{jp}\nabla_k\mathrm{C}_{ijp} - \nabla_p\mathrm{R}_{jk}\mathrm{C}_{ijp} - \mathrm{R}_{jk}\nabla_p\mathrm{C}_{ijp} + \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{ikp} + \frac{\mathrm{R}}{2}\nabla_p\mathrm{C}_{ikp} \\
&\quad + \mathrm{R}_{jp}\nabla_p\mathrm{C}_{ijk} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{pik} + \mathrm{R}_{jp}\nabla_i\mathrm{C}_{pjk} - \mathrm{R}_{ji}\nabla_p\mathrm{C}_{pjk} + \frac{\mathrm{R}}{2}\nabla_p\mathrm{C}_{pik} \\
&\quad - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{ipk} - \mathrm{R}_{lp}\nabla_l\mathrm{C}_{ikp} + \mathrm{R}_{jp}\nabla_k\mathrm{C}_{ijp} - \mathrm{R}_{jk}\nabla_p\mathrm{C}_{ijp} + \frac{\mathrm{R}}{2}\nabla_p\mathrm{C}_{ikp} \\
&= \nabla_i\mathrm{R}_{jp}\mathrm{C}_{pjk} - \nabla_p\mathrm{R}_{ji}\mathrm{C}_{ijp} - \nabla_p\mathrm{R}_{jk}\mathrm{C}_{ijp} - \nabla_p\mathrm{R}_{jk}\mathrm{C}_{ijp} - 2\mathrm{R}_{lp}\nabla_l\mathrm{C}_{pik} \\
&\quad + 2\mathrm{R}_{lp}\nabla_i\mathrm{C}_{plk} - 2\mathrm{R}_{ji}\nabla_p\mathrm{C}_{pjk} + \mathrm{R}\nabla_p\mathrm{C}_{pik} + \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{ikp} + 2\mathrm{R}_{jp}\nabla_k\mathrm{C}_{ijp} \\
&\quad - 2\mathrm{R}_{jk}\nabla_p\mathrm{C}_{ijp} + \mathrm{R}\nabla_p\mathrm{C}_{ikp} + \mathrm{R}_{jp}\nabla_p\mathrm{C}_{ijk} \\
&= \nabla_i\mathrm{R}_{lp}\mathrm{C}_{plk} - \nabla_p\mathrm{R}_{li}\mathrm{C}_{plk} + \nabla_k\mathrm{R}_{lp}\mathrm{C}_{ilp} - \nabla_p\mathrm{R}_{lk}\mathrm{C}_{ilp} \\
&\quad - 2\mathrm{R}_{lp}\nabla_l\mathrm{C}_{pik} + 2\mathrm{R}_{lp}\nabla_i\mathrm{C}_{plk} + 2\mathrm{R}_{li}\mathrm{B}_{kl} - 2\mathrm{R}_{li}\mathrm{B}_{lk} + 2\mathrm{R}_{lp}\nabla_k\mathrm{C}_{ilp} \\
&\quad + 2\mathrm{R}_{lk}\mathrm{B}_{il} + \mathrm{R}_{lp}\nabla_p\mathrm{C}_{ilk} - \mathrm{R}\mathrm{B}_{ik} + \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{ikp} + \mathrm{R}\mathrm{B}_{ik} - \mathrm{R}\mathrm{B}_{ik} \\
&= \nabla_i\mathrm{R}_{lp}\mathrm{C}_{plk} - \nabla_p\mathrm{R}_{li}\mathrm{C}_{plk} + \nabla_k\mathrm{R}_{lp}\mathrm{C}_{ilp} - \nabla_p\mathrm{R}_{lk}\mathrm{C}_{ilp} \\
&\quad + \mathrm{R}_{lp}\nabla_p\mathrm{C}_{ilk} + 2\mathrm{R}_{lp}\nabla_i\mathrm{C}_{plk} + 2\mathrm{R}_{lp}\nabla_k\mathrm{C}_{ilp} - 2\mathrm{R}_{lp}\nabla_l\mathrm{C}_{ipk} \\
&\quad + \frac{1}{2}\nabla_p\mathrm{R}\mathrm{C}_{ikp} + 2\mathrm{R}_{lk}\mathrm{B}_{il} - \mathrm{R}\mathrm{B}_{ik}\,.
\end{aligned}$$

The covariant derivative of the evolution of the Cotton tensor is given by

$$\nabla_j(\partial_t - \Delta)\mathrm{C}_{ijk} = \frac{5}{2}\nabla_p\mathrm{R}\mathrm{C}_{ipk} + \nabla_p\mathrm{R}\mathrm{C}_{pki} + \mathrm{R}_{lp}\nabla_p\mathrm{C}_{kli} + \mathrm{R}_{lp}\nabla_p\mathrm{C}_{lki} - \nabla_p\mathrm{R}_{kl}\mathrm{C}_{pli}$$

$$
\begin{aligned}
&-\nabla_p R_{kl} C_{lpi} - R_{kp} B_{pi} + 5\nabla_p R_{il} C_{lkp} - 5R_{ip} B_{pk} + 2RB_{ik} \\
&+2\nabla_s R_{pl} C_{psl} g_{ik} + 2R_{pl} B_{pl} g_{ik} - 2\nabla_i R_{pl} C_{pkl} - 2R_{pl} \nabla_i C_{pkl} \\
&+\frac{1}{2}(|\nabla R|^2 + R\Delta R - \Delta|\mathrm{Ric}|^2) g_{ik} - \frac{1}{2}(\nabla_i R \nabla_k R + R\nabla_i\nabla_k R - \nabla_i\nabla_k |\mathrm{Ric}|^2) \\
&+2\Delta R_{ip} R_{kp} + 2\nabla_l R_{ip} \nabla_l R_{kp} - 2\nabla_l \nabla_k R_{ip} R_{lp} - \nabla_k R_{ip} \nabla_p R \\
&+\nabla_l \nabla_k R R_{il} + \frac{1}{2}\nabla_k R \nabla_i R - \Delta R R_{ik} - \nabla_l R \nabla_l R_{ik}.
\end{aligned}
$$

Finally, the commutator between the covariant derivative and the time derivative can be expressed in terms of the time derivatives of the Christoffel symbols, as follows

$$
\begin{aligned}
[\partial_t, \nabla_j] C_{ijk} &= -\partial_t \Gamma^p_{ij} C_{pjk} - \partial_t \Gamma^p_{jk} C_{ijp} \\
&= \nabla_i R_{jp} C_{pjk} + \nabla_j R_{ip} C_{pjk} - \nabla_p R_{ij} C_{pjk} + \nabla_j R_{kp} C_{ijp} + \nabla_k R_{jp} C_{ijp} - \nabla_p R_{jk} C_{ijp} \\
&= \nabla_i R_{jp} C_{pjk} + \nabla_p R_{ij} C_{jpk} + \nabla_p R_{ij} C_{pkj} + \nabla_p R_{kj} C_{ipj} + \nabla_k R_{jp} C_{ijp} + \nabla_p R_{jk} C_{ipj} \\
&= \nabla_i R_{jp} C_{pjk} - \nabla_p R_{ij} C_{pkj} - \nabla_p R_{ij} C_{kjp} + \nabla_p R_{ij} C_{pkj} + 2\nabla_p R_{kj} C_{ipj} \\
&= \nabla_i R_{jp} C_{pjk} - \nabla_p R_{ij} C_{kjp} + 2\nabla_p R_{kj} C_{ipj}\,.
\end{aligned}
$$

Substituting into (5.3), and making some computations, we obtain the evolution equation

Proposition 5.2. *During the Ricci flow of a 3−dimensional Riemannian manifold $(M^3, g(t))$ the Bach tensor satisfies the following evolution equation*

$$
\begin{aligned}
(\partial_t - \Delta) B_{ik} = \; & \big[3\nabla_p R C_{ipk} + \nabla_p R C_{pki} - \nabla_p R \nabla_k R_{ip}\big] \\
+ \; & \big[-2R_{pl} \nabla_p C_{ikl} - 3R_{pk} B_{pi} - 5R_{pi} B_{pk} + 2\Delta R_{ip} R_{kp} \\
& -2\nabla_l \nabla_k R_{pi} R_{pl} + \nabla_l \nabla_k R R_{li} - \Delta R R_{ik}\big] \\
+ \; & \big[-2\nabla_p R_{kl} C_{lpi} - 2\nabla_p R_{kl} C_{ilp} - 4\nabla_p R_{il} C_{lpk} - 2\nabla_i R_{pl} C_{pkl}\big] \\
+ \; & \big[3RB_{ik} + 2\nabla_s R_{pl} C_{psl} g_{ik} + 2R_{pl} B_{pl} g_{ik} \\
& +\frac{1}{2}(|\nabla R|^2 + R\Delta R - \Delta|\mathrm{Ric}|^2) g_{ik} - \frac{1}{2}(R\nabla_i\nabla_k R - \nabla_i\nabla_k|\mathrm{Ric}|^2) \\
& +2\nabla_l R_{ip} \nabla_l R_{kp} - \nabla_l R \nabla_l R_{ik}\big]\,.
\end{aligned}
$$

Hence, if the Bach tensor vanishes identically along the flow, we have

$$
\begin{aligned}
0 = \; & 3\nabla_p R C_{ipk} + \nabla_p R C_{pki} - \nabla_p R \nabla_k R_{ip} - 2R_{pl} \nabla_p C_{ikl} \\
& +2\Delta R_{ip} R_{kp} - 2\nabla_l \nabla_k R_{pi} R_{pl} + \nabla_l \nabla_k R R_{li} - \Delta R R_{ik} \\
& -2\nabla_p R_{kl} C_{lpi} - 2\nabla_p R_{kl} C_{ilp} - 4\nabla_p R_{il} C_{lpk} - 2\nabla_i R_{pl} C_{pkl} \\
& +2\nabla_s R_{pl} C_{psl} g_{ik} + \frac{1}{2}(|\nabla R|^2 + R\Delta R - \Delta|\mathrm{Ric}|^2) g_{ik} \\
& -\frac{1}{2}(R\nabla_i\nabla_k R - \nabla_i\nabla_k|\mathrm{Ric}|^2) + 2\nabla_l R_{ip} \nabla_l R_{kp} - \nabla_l R \nabla_l R_{ik}.
\end{aligned}
$$

Remark 5.3. Note that, from the symmetry property of the Bach tensor, we have that the RHS in the evolution equation of the Bach tensor should be symmetric in the two indices. It is not so difficult to check that this property is verified for the formula in Proposition 5.2. Indeed, each of the terms in between square brackets is symmetric in the two indices.

As a consequence of Proposition 5.2, we get that during the Ricci flow of a 3−dimensional Riemannian manifold the squared norm of the Bach tensor satisfies

$$
\begin{aligned}
(\partial_t - \Delta)|B_{ik}|^2 = \; & -2|\nabla B_{ik}|^2 - 12B_{ik} B_{iq} R_{qk} + 6B_{ik} \nabla_p R - 4B_{ik} R_{pl} \nabla_p C_{ikl} \\
& +4B_{ik} \nabla_p R_{kl} C_{pil} - 8B_{ik} \nabla_p R_{kl} C_{lpi} - 4B_{ik} \nabla_i R_{pl} C_{pkl} + 6R|B_{ik}|^2 \\
& -2B_{ik} \nabla_p R \nabla_k R_{ip} + 4B_{ik} \Delta R_{ip} R_{kp} - 4B_{ik} \nabla_l \nabla_k R_{pi} R_{pl} + 2B_{ik} \nabla_l \nabla_k R R_{li} \\
& -2B_{ik} \Delta R R_{ik} - B_{ik} R \nabla_i \nabla_k R + B_{ik} \nabla_i \nabla_k |\mathrm{Ric}|^2 - 2B_{ik} \nabla_l R \nabla_l R_{ik} \\
& +4B_{ik} \nabla_l R_{ip} \nabla_l R_{kp}.
\end{aligned}
$$

5.2 The Bach Tensor of Three–Dimensional Gradient Ricci Solitons

In what follows, we will use formulas (3.1)–(3.5) to derive an expression of the Bach tensor and of its divergence in the particular case of a gradient Ricci soliton in dimension three.

By straightforward computations, we obtain

$$\begin{aligned}
\mathrm{B}_{ik} &= \nabla_j \mathrm{C}_{ijk} \\
&= \frac{\nabla_i\nabla_k \mathrm{R}}{4} - \frac{\Delta \mathrm{R}}{4} g_{ik} - \nabla_j \mathrm{R}_{ik}\nabla_j f + \frac{g_{ik}}{2}\nabla_j \mathrm{R}\nabla_j f \\
&\quad + \left(\mathrm{R}_{ij} - \frac{\mathrm{R}}{2} g_{ij}\right)\nabla_j\nabla_k f - \left(\mathrm{R}_{ik} - \frac{\mathrm{R}}{2} g_{ik}\right)\Delta f \\
&= \frac{1}{4}\nabla_i\nabla_k \mathrm{R} - \frac{1}{4}\Delta \mathrm{R} g_{ik} - \nabla_j \mathrm{R}_{ik}\nabla_j f + \frac{1}{2}\nabla_j \mathrm{R}\nabla_j f g_{ik} - \mathrm{R}_{ij}\mathrm{R}_{jk} + \lambda \mathrm{R}_{ik} \\
&\quad + \frac{1}{2}\mathrm{R}\mathrm{R}_{ik} - \frac{\lambda}{2}\mathrm{R} g_{ik} - 3\lambda \mathrm{R}_{ik} + \mathrm{R}\mathrm{R}_{ik} + \frac{3}{2}\lambda \mathrm{R} g_{ik} - \frac{1}{2}\mathrm{R}^2 g_{ik} \\
&= \frac{1}{2}\nabla_i \mathrm{R}_{lk}\nabla_l f - \frac{1}{2}\mathrm{R}_{lk}\mathrm{R}_{li} + \frac{\lambda}{2}\mathrm{R}_{ik} - \frac{1}{4}\nabla_l \mathrm{R}\nabla_l f g_{ik} - \frac{\lambda}{2}\mathrm{R} g_{ik} \\
&\quad + \frac{1}{2}|\mathrm{Ric}|^2 g_{ik} - \nabla_j \mathrm{R}_{ik}\nabla_j f + \frac{1}{2}\nabla_j \mathrm{R}\nabla_j f g_{ik} - \mathrm{R}_{ij}\mathrm{R}_{jk} + \lambda \mathrm{R}_{ik} + \frac{1}{2}\mathrm{R}\mathrm{R}_{ik} \\
&\quad - \frac{\lambda}{2}\mathrm{R} g_{ik} - 3\lambda \mathrm{R}_{ik} + \mathrm{R}\mathrm{R}_{ik} + \frac{3}{2}\lambda \mathrm{R} g_{ik} - \frac{1}{2}\mathrm{R}^2 g_{ik} \\
&= \frac{1}{2}\nabla_i \mathrm{R}_{lk}\nabla_l f + \frac{1}{4}\nabla_j \mathrm{R}\nabla_j f g_{ik} - \nabla_j \mathrm{R}_{ik}\nabla_j f - \frac{3}{2}\mathrm{R}_{ij}\mathrm{R}_{jk} - \frac{3}{2}\lambda \mathrm{R}_{ik} \\
&\quad + \frac{3}{2}\mathrm{R}\mathrm{R}_{ik} + \frac{\lambda}{2}\mathrm{R} g_{ik} + \frac{1}{2}|\mathrm{Ric}|^2 g_{ik} - \frac{1}{2}\mathrm{R}^2 g_{ik}\,.
\end{aligned}$$

A more compact formulation, employing equations (3.2) and (3.3), is given by

$$\mathrm{B}_{ik} = \frac{1}{2}\nabla_i \mathrm{R}_{lk}\nabla_l f + \frac{1}{4}\Delta \mathrm{R} g_{ik} - \frac{1}{2}\Delta \mathrm{R}_{ik} - \frac{1}{2}\nabla_j \mathrm{R}_{ik}\nabla_j f - \frac{\lambda}{2}\mathrm{R}_{ik} + \frac{1}{2}\mathrm{R}_{ij}\mathrm{R}_{jk}\,.$$

Moreover, as we know that $\nabla_k \mathrm{B}_{ik} = \mathrm{C}_{lji}\mathrm{R}_{lj}$, we have

$$\begin{aligned}
\nabla_k \mathrm{B}_{ik} &= \frac{1}{4}\mathrm{R}\nabla_i \mathrm{R} - \frac{1}{4}\mathrm{R}_{ij}\nabla_j \mathrm{R} + |\mathrm{Ric}|^2\nabla_i f - \frac{1}{2}\mathrm{R}^2\nabla_i f - \mathrm{R}_{il}\nabla_j f \mathrm{R}_{lj} + \frac{1}{2}\mathrm{R}\mathrm{R}_{ij}\nabla_j f \\
&= \frac{1}{2}\mathrm{R}\nabla_i \mathrm{R} - \frac{3}{4}\mathrm{R}_{il}\nabla_l \mathrm{R} + |\mathrm{Ric}|^2\nabla_i f - \frac{1}{2}\mathrm{R}^2\nabla_i f\,.
\end{aligned}$$

Therefore, if the divergence of the Bach tensor vanishes, we conclude

$$\frac{1}{2}\mathrm{R}\nabla_i \mathrm{R} - \frac{3}{4}\mathrm{R}_{ik}\nabla_k \mathrm{R} + |\mathrm{Ric}|^2\nabla_i f - \frac{1}{2}\mathrm{R}^2\nabla_i f = 0\,.$$

Taking the scalar product with ∇f in both sides of this equation, we obtain

$$0 = \frac{1}{2}\mathrm{R}\langle\nabla \mathrm{R}, \nabla f\rangle - \frac{3}{8}|\nabla \mathrm{R}|^2 + |\mathrm{Ric}|^2|\nabla f|^2 - \frac{1}{2}\mathrm{R}^2|\nabla f|^2$$

and, from formulas (3.5) and (3.6), we compute

$$\begin{aligned}
|\mathrm{C}_{ijk}|^2 &= (\mathrm{R}_{ij}\nabla_k f - \mathrm{R}_{ik}\nabla_j f)\left(\frac{\nabla_k \mathrm{R}}{4} g_{ij} - \frac{\nabla_j \mathrm{R}}{4} g_{ik} + \left(\mathrm{R}_{ij} - \frac{\mathrm{R}}{2} g_{ij}\right)\nabla_k f - \left(\mathrm{R}_{ik} - \frac{\mathrm{R}}{2} g_{ik}\right)\nabla_j f\right) \\
&= \frac{\mathrm{R}}{4}\nabla_k \mathrm{R}\nabla_k f - \frac{1}{4}\mathrm{R}_{kj}\nabla_j \mathrm{R}\nabla_k f + |\mathrm{Ric}|^2|\nabla f|^2 - \frac{\mathrm{R}^2}{2}|\nabla f|^2 - \mathrm{R}_{ij}\nabla_j f \mathrm{R}_{ik}\nabla_k f + \frac{\mathrm{R}}{2}\mathrm{R}_{kj}\nabla_k f\nabla_j f \\
&\quad - \frac{1}{4}\mathrm{R}_{jk}\nabla_j f\nabla_k \mathrm{R} + \frac{\mathrm{R}}{4}\nabla_j \mathrm{R}\nabla_j f - \mathrm{R}_{ik}\nabla_k f \mathrm{R}_{ij}\nabla_j f + \frac{\mathrm{R}}{2}\mathrm{R}_{jk}\nabla_j f\nabla_k f + |\mathrm{Ric}|^2|\nabla f|^2 - \frac{\mathrm{R}^2}{2}|\nabla f|^2 \\
&= 2|\mathrm{Ric}|^2|\nabla f|^2 - \mathrm{R}^2|\nabla f|^2 + \mathrm{R}\nabla_k \mathrm{R}\nabla_k f - \frac{3}{4}|\nabla \mathrm{R}|^2\,,
\end{aligned}$$

where we repeatedly used equation (3.4).

Therefore, we obtain

$$\nabla_k \mathrm{B}_{ik}\nabla_i f = \frac{1}{2}|\mathrm{C}_{ijk}|^2\,,$$

so, if the divergence of the Bach tensor vanishes then the Cotton tensor vanishes as well (this was already obtained in [2]). As a consequence, getting back to Section 3, the soliton is locally a warped product of a constant curvature surface on a interval of $\mathbb{R}$.

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