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Stable determination of an inclusion for a class of anisotropic conductivities

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Abstract

We deal with the inverse problem of determining an inclusion within an electrical conductor from electrostatic boundary measurements. We consider an anisotropic conductivity and provide logarithmic type stability.

Keywords: stability estimates, inclusion determination, anisotropic conductivity

1. Introduction

In this paper we deal with the problem of determining an inclusion D in an electrical conductor Ω. Particularly, we are interested in analysing anisotropic bodies. The region D represents a portion of Ω where the conductivity σ has a jump of discontinuity across the interface. Denoted by $A(x)$ the known matrix conductivity of Ω, and $kA(x)$ the conductivity inside of D, where k is an unknown function. Prescribing a voltage $f \in H^{1/2}(\partial\Omega)$ on the boundary $\partial\Omega$ of the domain, the induced potential $u \in H^1(\Omega)$ is the solution of the boundary value problem

$$\begin{cases} \operatorname{div}((A(x) + (k-1)A(x)\chi_D)\nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where χ_D is the characteristic function of the set D. Our inverse problem is addressed to determine the anomalous region D when the Dirichlet–to–Neumann map Λ_D

$$\begin{array}{cccc} \Lambda_D & : & H^{1/2}(\partial\Omega) & \longrightarrow & H^{-1/2}(\partial\Omega) \\ & & f & \longrightarrow & \frac{\partial u}{\partial \nu}_{|\partial\Omega}, \end{array}$$

is given for any $f \in H^{1/2}(\partial\Omega)$. Here, ν denotes the outer unit normal to $\partial\Omega$, and $\frac{\partial u}{\partial \nu}_{|\partial\Omega}$ corresponds to the current density measured on $\partial\Omega$. Thus, the Dirichlet–to–Neumann map represents the knowledge of infinitely many boundary measurements.

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This problem is considered by Isakov [Is88], where uniqueness for isotropic conductivities is established. Stability is obtained in [Al-DC] for piecewise constant conductivities and is extended in [DC] with conductivities of the form $\sigma(x) = a(x) + b(x)\chi_D$, where a is a $C^{1,\alpha}$ function and b is a C^α unknown function. Concerning the anisotropic case, not many results are available. The uniqueness result is proven in [Kw] with conductivities of the form $\sigma = A + (B - A)\chi_D$, where A, B are matrix valued functions.

In this paper we analyse the stability, given the reasonable *a priori* information on the unknown, the inclusion depends continuously on the Dirichlet–to–Neumann map with modulus of continuity of logarithmic type. Let us stress that this rate of continuity is optimal as it is shown by examples in [DC-Ro].

The argument follows the lines of the isotropic case presented in [Al-DC] (see also [DC2] for the inverse scattering case, [DC-Ve], [DC-Ve2] for the thermal imaging case and [Al-DC-Mo-Ro] for elasticity) and it is based on the use of singular solutions and quantitative estimates of unique continuation. We will carry on an accurate analysis of singular solutions by studying the asymptotic behaviour when the singularity gets close to the interface ∂D. Combining this with a control of boundary smallness propagation, we will get the stability estimates. This is the first analysis to study the stability for more general conductivities of the form $A + (B - A)\chi_D$. It seems reasonable that this argument also works in this case, but a deeper analysis on the asymptotic behaviour of the fundamental solutions (theorem 4.1) is needed.

The paper is organised as follows. In the next section 2, after some notations and definitions, we will state our main result, whose proof is presented in section 3. The proof is based on some auxiliary propositions proven in section 4.

2. Main result

Let us first premise some notations and definitions. For points $x \in \mathbb{R}^n$, we will write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, denoted by $\mathrm{dist}(\cdot, \cdot)$ the standard Euclidean distance, we define

$$B_r(x) = \{y \in \mathbb{R}^n | \mathrm{dist}(x, y) \leqslant r\}, \qquad B'_r(x') = \{y' \in \mathbb{R}^{n-1} | \mathrm{dist}(x', y') \leqslant r\}$$

as the open balls with radius r centered at x and x' respectively. We write $Q_r(x) = B'_r(x') \times (x_n - r, x_n + r)$ for the cylinder in $\mathbb{R}^n$. For simplicity, we use B_r, B'_r, Q_r instead of $B_r(0), B'_r(0')$ and $Q_r(0)$ respectively. We shall also denote half domain, as well as its associated ball and cylinder

$$\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}; \;\; B^+_r = B_r \cap \mathbb{R}^n_+; \;\; Q^+_r = Q_r \cap \mathbb{R}^n_+.$$

Definition 2.1. Let Ω be the bounded domain in $\mathbb{R}^n$. We say a portion S of $\partial\Omega$ is of Lipschitz class with constants $r, L > 0$ if for any point $p \in S$, there exists a rigid transformation $\varphi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ of coordinates under which we have $p = 0$ and

$$\Omega \cap B_r = \{(x', x_n) \in B_r | x_n > \varphi(x')\},$$

where $\varphi(\cdot)$ is a Lipschitz continuous function on B'_r, which satisfies

$$\varphi(0) = 0$$

and

$$\|\varphi\|_{C^{0,1}(B'_r)} \leqslant Lr.$$

We shall say that Ω is of Lipschitz class with constants r, L if $\partial\Omega$ is of Lipschitz class with the same constants.

Definition 2.2. Let Ω be the bounded domain in $\mathbb{R}^n$. Given $\alpha \in (0, 1]$, we say a portion S of $\partial\Omega$ is of $C^{1,\alpha}$ class with constants $r, L > 0$ if for any point $p \in S$, there exists a rigid transformation $\varphi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ of coordinates under which we have $p = 0$ and

$$\Omega \cap B_r = \{(x', x_n) \in B_r | x_n > \varphi(x')\},$$

where $\varphi(\cdot)$ is a $C^{1,\alpha}$ function on B'_r, which satisfies

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,\alpha}(B'_r)} \leqslant Lr$$

where the norm is defined as

$$\|\varphi\|_{C^{1,\alpha}(B'_r)} := \|\varphi\|_{L^\infty(B'_r)} + r\|\nabla\varphi\|_{L^\infty(B'_r)} + r^{1+\alpha}|\nabla\varphi|_{\alpha,B'_r}$$

$$|\nabla\varphi|_{\alpha,B'_r} := \sup_{\substack{x',y' \in B'_r \\ x' \neq y'}} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x' - y'|}.$$

2.1. Assumptions and a priori data

Given constants $r_0, M_0, M_1, \delta_0, \overline{A}, \lambda > 0$ and $0 < \alpha < 1$, we assume the domain $\Omega \subset \mathbb{R}^n$ is bounded

$$|\Omega| \leqslant M_1 r_0^n$$

where $|\cdot|$ denotes the Lebesgue measure.

The inclusion D is assumed to stay away from the boundary of the domain, as $\mathrm{dist}(D, \partial\Omega) \geqslant \delta_0$, and also $\Omega \setminus D$ is connected. Both $\partial\Omega$ and ∂D are of $C^{1,\alpha}$ class with constants r_0, M_0. We shall consider the conductivities

$$\sigma_D(x) = A(x) + A(x)(k - 1)\chi_D$$

where $A(x)$ is a known Lipschitz symmetric matrix valued function satisfying $\|A\|_{C^{0,1}(\Omega)} \leqslant \overline{A}$ and ellipticity condition with constant $\lambda > 0$ such that

$$\lambda^{-1}|\xi|^2 \leqslant A(x)\xi \cdot \xi \leqslant \lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

We refer to $n, k, r_0, M_0, M_1, \alpha, \delta_0, \overline{A}, \lambda$ as the *a priori data*.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, $k > 0$, $k \neq 1$ be given, and D_1 and D_2 be two inclusions in Ω. With the assumptions above, for any given $\varepsilon > 0$ if we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} < \epsilon \tag{2.1}$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leqslant \omega(\epsilon)$$

where ω is an increasing function on $[0, +\infty)$ *that satisfies*

$$\omega(t) \leqslant C|\log t|^{-\eta} \qquad \forall t \in (0,1)$$

and $C > 0, 1 \geqslant \eta > 0$ *are constants only depending on the a priori data.*

3. Proof of the main result

The proof of theorem 2.3 is based on some auxiliary propositions, and their proofs are collected in the next section 4. In what follows we define layers of our domains. We denote by $\mathcal{G}$ the connected component of $\Omega\backslash(D_1 \cup D_2)$, whose boundary contains $\partial\Omega$, $\Omega_D = \Omega\backslash\overline{\mathcal{G}}$, $\Omega_r := \{x \in \mathcal{C}\Omega | \mathrm{dist}(x,\Omega) > r\}$, $S_{2r} := \{x \in \mathbb{R}^n | r \leqslant \mathrm{dist}(x,\Omega) \leqslant 2r\}$ and $\mathcal{G}^h := \{x \in \mathcal{G} | \mathrm{dist}(x,\Omega_D) \geqslant h\}$.

We introduce a variation of the Hausdorff distance called the *modified distance*, which can simplify our proof.

Definition 3.1. The modified distance between D_1 and D_2 is defined as

$$d_m(D_1, D_2) := \max\left\{\sup_{x\in\partial\Omega_D\cap\partial D_1} \mathrm{dist}(x, \partial D_2), \sup_{x\in\partial\Omega_D\cap\partial D_2} \mathrm{dist}(x, \partial D_1)\right\}.$$

We remark here that d_m is not a metric, and in general, it does not dominate the Hausdorff distance. However, under our *a priori* assumptions on the inclusion, the following lemma holds.

Lemma 3.2. *Under the assumptions of theorem 2.3, there exists a constant* $c_0 \geqslant 1$ *only depending on* M_0 *and* α *such that*

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leqslant c_0 d_m(D_1, D_2). \tag{3.1}$$

Proof. See [Al-DC], proposition 3.3]. □

Another obstacle comes from the fact that the propagation of smallness arguments are based on an iterated application of the three spheres inequality for solutions of the equation over chains of balls contained in $\mathcal{G}$. Therefore, it is crucial to control from below the radii of these balls. In the following lemma 3.3 we treat the case of points of $\partial\Omega_D$ that are not reachable by such chains of balls. This problem was originally considered by [Al-Si] in the context of cracks detection in electrical conductors.

Let us premise some notations. Given $O = (0,\ldots,0)$ the origin, v a unit vector, $H > 0$ and $\vartheta \in \left(0, \frac{\pi}{2}\right)$, we denote

$$C(O, v, H, \vartheta) = \{x \in \mathbf{R}^n : \ |x - (x\cdot v)v| \leqslant \sin\vartheta |x|, \ 0 \leqslant x\cdot v \leqslant H\}$$

the closed truncated cone with vertex at O, axis along the direction v, height H and aperture 2ϑ. Given R, d, $0 < R < d$ and $Q = -de_n$, where $e_n = (0,\ldots,0,1)$, let us consider the cone $C\left(O, -e_n, \frac{d^2-R^2}{d}, \arcsin\frac{R}{d}\right)$.

From now on, without loss of generality, we assume that

$$d_m(D_1, D_2) = \max_{x\in\partial D_1\cap\partial\Omega_D} \mathrm{dist}(x, \partial D_2)$$

and we write $d_m = d_m(D_1, D_2)$.

Let us define

$$S_{2\rho_0} = \left\{ x \in \mathbf{R}^n \,|\rho_0 < \operatorname{dist}(x, \overline{\Omega}) < 2\rho_0 \right\}.$$

We shall make use of paths connecting points in order that appropriate tubular neighborhoods of such paths still remain within $\mathbf{R}^n \setminus \Omega_D$. Let us pick a point $P \in \partial D_1 \cap \partial\Omega_D$, let ν be the outer unit normal to ∂D_1 at P and let $d > 0$ be such that the segment $[(P + d\nu), P]$ is contained in $\mathbf{R}^n \setminus \Omega_D$. Given $P_0 \in \mathbf{R}^n \setminus \Omega_D$, let γ be a path in $\mathbf{R}^n \setminus \Omega_D$ joining P_0 to $P + d\nu$. We consider the following neighborhood of $\gamma \cup [(P + d\nu), P] \setminus \{P\}$ formed by a tubular neighborhood of γ attached to a cone with vertex at P and axis along ν

$$V(\gamma) = \bigcup_{S \in \gamma} B_R(S) \cup C\left(P, \nu, \frac{d^2 - R^2}{d}, \arcsin\frac{R}{d}\right). \tag{3.2}$$

Note that two significant parameters are associated to such a set, the radius R of the tubular neighborhood of γ, $\cup_{S\in\gamma} B_R(S)$, and the half-aperture $\arcsin\frac{R}{d}$ of the cone $C\left(P, \nu, \frac{d^2-R^2}{d}, \arcsin\frac{R}{d}\right)$. In other terms, $V(\gamma)$ depends on γ and also on the parameters R and d. At each of the following steps, such two parameters shall be appropriately chosen and shall be accurately specified. For the sake of simplicity we convene to maintain the notation $V(\gamma)$ also when different values of R, d are introduced. Also we warn the reader that it will be convenient at various stages to use a reference frame such that $P = O = (0, \ldots, 0)$ and $\nu = -e_n$.

Lemma 3.3. *Under the above notation, there exist positive constants $\bar{d}$, c_1, where $\frac{\bar{d}}{\rho_0}$ only depends on M_0 and α, and c_1 only depends on M_0, α, M_1, and there exists a point $P \in \partial D_1$ satisfying*

$$c_1 d_m \leqslant \operatorname{dist}(P, D_2),$$

and such that, giving any point $P_0 \in S_{2\rho_0}$, there exists a path $\gamma \subset (\overline{\Omega^{\rho_0}} \cup S_{2\rho_0}) \setminus \overline{\Omega_D}$ joining P_0 to $P + \bar{d}\nu$, where ν is the unit outer normal to D_1 at P, such that, choosing a coordinate system with origin O at P and axis $e_n = -\nu$, the set $V(\gamma)$ introduced in (3.2) satisfies

$$V(\gamma) \subset \mathbf{R}^n \setminus \Omega_D,$$

provided $R = \frac{\bar{d}}{\sqrt{1+L_0^2}}$, where L_0, $0 < L_0 \leqslant M_0$, is a constant only depending on M_0 and α.

Proof. See [Al-DC-Mo-Ro], lemma 4.2]. □

In order to use the information provided by the boundary measurements to evaluate the distance between two inclusions D_1 and D_2, we apply the following identity firstly introduced by Alessandrini in [Al]. Let $u_i \in H^1(\partial\Omega)$, $i = 1, 2$, be solutions to (1.1) with conductivities $\sigma_{D_i} = A(x) + (k-1)A(x)\chi_{D_i}$ respectively, we have

$$\int_\Omega (\sigma_{D_1}\nabla u_1 \cdot \nabla u_2) - \int_\Omega (\sigma_{D_2}\nabla u_1 \cdot \nabla u_2) = \int_{\partial\Omega} u_1[\Lambda_{D_1} - \Lambda_{D_2}]u_2. \tag{3.3}$$

For the operator $\operatorname{div}((A(x) + A(x)(k-1)\chi_{D_i})\nabla\cdot)$, denoted by $\Gamma_{D_i}, i = 1, 2$ the associated fundamental solutions. We apply (3.3) to Γ_{D_1} and Γ_{D_2} with y, z belonging to the complement set of Ω, obtains

$$\int_\Omega (A(x) + (k-1)A(x)\chi_{D_1})\nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z))$$
$$- \int_\Omega (A(x) + (k-1)A(x)\chi_{D_2})\nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z))$$
$$= \int_{\partial\Omega} \Gamma_{D_1}(\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(\cdot, z). \tag{3.4}$$

For $y, z \in \mathcal{G} \cup \mathcal{C}\Omega$, where $\mathcal{C}\Omega$ is the complementary of Ω, we define

$$S_{D_1}(y, z) = (k-1)\int_{D_1} A(x)\nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z)$$
$$S_{D_2}(y, z) = (k-1)\int_{D_2} A(x)\nabla\Gamma_{D_1}(\cdot, y) \cdot \nabla\Gamma_{D_2}(\cdot, z)$$
$$f(y, z) = S_{D_1}(y, z) - S_{D_2}(y, z).$$

Therefore (3.4) can be written as

$$f(y, z) = \int_{\partial\Omega} \Gamma_{D_1}(\cdot, y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(\cdot, z), \quad \forall y, z \in \mathcal{C}\overline{\Omega}.$$

The following two propositions provide quantitative estimates on $f(y, y)$ and $S_{D_1}(y, y)$, when moving y towards O along $\nu(O)$.

Proposition 3.4. *Given $\epsilon > 0$, the domain Ω and inclusions D_1, D_2, and let $y = h\nu(O)$, if we have*

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{L(H^{1/2}, H^{-1/2})} < \epsilon$$

then for every h where $0 < h < cr, 0 < c < 1$, and c depends on M_0, we have

$$|f(y, y)| \leqslant C_0 \frac{\epsilon^{Bh^F}}{h^A} \tag{3.5}$$

here $0 < A < 1$ and $C_0, B, F > 0$ are constants that depend only on the a priori data.

Proposition 3.5. *Given $\epsilon > 0$, the domain Ω and inclusions D_1, D_2, and let $y = h\nu(O)$ defined as above. Then for every $0 < h < h_0/2$*

$$|S_{D_1}(y, y)| \geqslant C_1 h^{2-n} - C_2 d_m^{2-2n} + C_3 \tag{3.6}$$

where $h_0 := \frac{r}{2}\min\left[\frac{1}{2}(8M_0)^{-1/\alpha}, \frac{1}{2}\right]$, and C_1, C_2, C_3 are positive constants depending only on the a priori data.

Now, we have all the ingredients to conclude this section with the proof of theorem 2.3.

Proof of theorem 2.3. We start from the origin of the coordinate system, a point $O \in \partial D_1 \cap \partial\Omega_D$, for which the maximum in definition 3.1 is attainted

$$d_m := d_m(D_1, D_2) = \mathrm{dist}(O, D_2).$$

Then with a transformation of coordinates $y = h\nu(O)$ where $0 < h < h_1, h_1 := \min\{d_m, cr_0, h_0/2\}, 0 < c < 1$, where c depends on M_0. By applying [Al-DC] proposition 3.4 (i), i.e.

$|\nabla_x \Gamma_{D_i}(x,y)| \leqslant c_1|x-y|^{1-n}, i=1,2$, where $c_1>0$ depending only on k,n,α,M_0; we have

$$|S_{D_2}(y,y)| = (k-1)\int_{D_2} A(x)\nabla\Gamma_{D_1}(\cdot,y)\nabla\Gamma_{D_2}(\cdot,y) \leqslant (k-1)\overline{A}\int_{D_2}(c_1|d_m-h|^{1-n})^2 \leqslant C_4|d_m-h|^{2-2n}|D_2| \tag{3.7}$$

where $|\cdot|$ is the Lebesgue measure and C_4 depends on $k,n,\alpha,M_0,\overline{A}$. From (3.5), we apply the triangular inequality to $f(y,y)$, by its definition, we get

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \leqslant |S_{D_1}(y,y) - S_{D_2}(y,y)| = |f(y,y)| \leqslant C_0\frac{\epsilon^{Bh^F}}{h^A}. \tag{3.8}$$

Therefore, together with (3.6)–(3.8), we obtain

$$C_1h^{2-n} - C_2d_m^{2-2n} + C_3 \leqslant C_4|d_m-h|^{2-2n}|D_2| + C_0\frac{\epsilon^{Bh^F}}{h^A}.$$

Let $C_3 = C_2d_m^{2-2n}$, $C_5 = C_4|D_2|/C_0$ and $C_6 = C_1/C_0$, we have

$$C_5|d_m-h|^{2-2n} \geqslant C_6h^{2-n} - \frac{\epsilon^{Bh^F}}{h^A} = C_6h^{2-n}(1-\epsilon^{Bh^F}h^K)$$

where $0<K=n-2-A$. Now let $h=h(\epsilon)=\min\left\{|\ln\epsilon|^{-\frac{1}{2F}}, d_m\right\}$, for $0<\epsilon\leqslant\epsilon_1, \epsilon_1\in(0,1)$ such that $\exp(-B|\ln\epsilon_1|^{1/2})=1/2$. Thanks to lemma 3.2, if $d_m\leqslant|\ln\epsilon|^{-\frac{1}{2F}}$, the main theorem 2.3 is proved by setting $\eta=\frac{1}{2F}>0$

$$d_{\mathcal{H}}(\partial D_1,\partial D_2) \leqslant c_0d_m \leqslant c_0|\ln\epsilon|^{-\eta} = \omega(\epsilon). \tag{3.9}$$

If $d_m \geqslant |\ln\epsilon|^{-\frac{1}{2F}}$, it is easy to check

$$(d_m-h)^{2-2n} \geqslant \frac{C_5}{2C_6}h^{2-n} \implies d_m \leqslant C_7|\ln\epsilon|^{-\frac{n-2}{4F(n-1)}}$$

here we solve d_m because $h=h(\epsilon)=|\ln\epsilon|^{-\frac{1}{2F}}$, and C_7 depends only on the *a priori* data. Therefore we conclude the proof by setting $\eta=\frac{n-2}{4F(n-1)}$

$$d_{\mathcal{H}}(\partial D_1,\partial D_2) \leqslant c_0d_m \leqslant c_0C_7|\ln\epsilon|^{-\eta} = \omega(\epsilon) \tag{3.10}$$

and for $\epsilon_1\leqslant\epsilon$, we can also conclude the proof because $d_m \leqslant \operatorname{diam}\Omega \leqslant M_1r_0^n$.

$$d_{\mathcal{H}}(\partial D_1,\partial D_2) \leqslant c_0d_m \leqslant c_0M_1r_0^n = \omega(\epsilon) \tag{3.11}$$

Thus, we can conclude the proof theorem 2.3 by (3.9)–(3.11)

$$d_{\mathcal{H}}(\partial D_1,\partial D_2) \leqslant Cd_m = \omega(\epsilon)$$

where C only depends on the *a priori* data. □

4. Proofs of propositions 3.4 and 3.5

Proof of proposition 3.4. Let us consider $f(y,\cdot)$ with a fixed $y \in S_{2r}$, then

$$\operatorname{div}(A(x)\nabla_w f(y,w)) = 0 \ \text{ in } \mathcal{C}\overline{\Omega}_D. \tag{4.1}$$

For $w \in S_{2r}$, by (2.1) and (3.4) and upper bound for Γ_{D_i} (see [Li-St-We]),

$$\begin{aligned} |f(y,w)| &= \left| \int_{\partial\Omega} \Gamma_{D_1}(x,y)[\Lambda_{D_1} - \Lambda_{D_2}]\Gamma_{D_2}(x,w)\mathrm{d}x \right| \\ &\leqslant \int_{\partial\Omega} |x-y|^{2-n} \cdot \|\Lambda_{D_1} - \Lambda_{D_2}\| \cdot |x-w|^{2-n}\mathrm{d}x \\ &= C(r,M_0)\|\Lambda_{D_1} - \Lambda_{D_2}\| = \epsilon, \quad y,w \in S_{2r} \end{aligned} \tag{4.2}$$

we adopt the same notation ϵ to represent the smallness quantity for the above formula, because (2.1) holds for any given $\epsilon > 0$. Now we consider the case when $w \in \mathcal{G}^h$, we have the upper bound of $|S_{D_1}|$

$$\begin{aligned} |S_{D_1}(y,w)| &= \left|(k-1)\int_{D_1} A(x)\nabla\Gamma_{D_1}(x,y)\nabla\Gamma_{D_2}(x,w)\mathrm{d}x\right| \\ &\leqslant |k-1|\overline{A}\int_{D_1} |\nabla\Gamma_{D_1}(x,y)||\nabla\Gamma_{D_2}(x,w)|\mathrm{d}x \\ &\leqslant c_0(k,\overline{A},r,M_0)\int_{D_1} |x-y|^{1-n}|x-w|^{1-n}\mathrm{d}x \leqslant c_0 h^{1-n}. \end{aligned}$$

Similarly, we have $|S_{D_2}(y,w)| \leqslant c_0 h^{1-n}$. By definition, we conclude that

$$|f(y,w)| \leqslant c_0 h^{1-n}, \quad w \in \mathcal{G}^h, y \in S_{2r}. \tag{4.3}$$

Now we apply the three spheres inequality for supremum norms of harmonic function v. In our case, for every $x \in \mathcal{G} \cup S_{2r} \cup \Omega_r$, we apply page 781 [Al-Be-Ro-Ve]. We obtain that for every $1 < \beta_1 < \beta_2$, there exists $\tau \in (0,1)$ and constant $c_1 = c_1(\overline{R}, M_0, \beta_1, \beta_2, n)$ such that

$$\|v\|_{L^\infty(B_{\beta_1\overline{R}}(x))} \leqslant c_1\|v\|^{\tau}_{L^\infty(B_{\overline{R}}(x))} \cdot \|v\|^{1-\tau}_{L^\infty(B_{\beta_2\overline{R}}(x))}.$$

Now we apply the above inequality by choosing $\beta_1 = 3/2, \beta_2 = 2$ and $\overline{R} = r$ for harmonic function $f(y,\cdot)$, we get

$$\|f(y,\cdot)\|_{L^\infty(B_{3r/2}(x))} \leqslant c_1\|f(y,\cdot)\|^{\tau}_{L^\infty(B_r(x))} \cdot \|f(y,\cdot)\|^{1-\tau}_{L^\infty(B_{2r}(x))}. \tag{4.4}$$

Constructing a chain of balls on an arc connecting x with w for $w \in \mathcal{G}^h$, as in [Al-DC, proposition 3.5, page 212], we have, for any R, $0 < R < r$

$$\|f(y,\cdot)\|_{L^\infty(B_{R/2}(w))} \leqslant c_2 \cdot \epsilon^{\tau^s} \cdot (c_0 h^{1-n})^{1-\tau^s} = c_3(h^{1-n})^A \epsilon^B.$$

Here c_3 depends on τ, s, r, M_0, n, R; and $A = 1 - B, B = \tau^s$. The above inequality concludes the propagation of smallness crossing the layers from outside of the domain Ω to inside. Now we continue to propagate until the smallness reaches the neighborhood of O. To do this, we create the truncated cone $C(O, v, r, \vartheta)$, and apply iteratively the three spheres inequality over the chain of balls $B_{r_1}(w_1), B_{r_2}(w_2), \ldots, B_{r_{k(R)}}(w_{k(R)})$ inside of the cone (see [Al-DC, proposition 3.5]). Denoted by $A_0 = A(1-\tau^{k(R)-1})$, we obtain

$$\|f(y,\cdot)\|_{L^\infty(B_{r_{k(R)}}(w_{k(R)}))} \leqslant \epsilon^{B\tau^{k(R)-1}} \cdot c(h^{1-n})^{A(1-\tau^{k(R)-1})} \leqslant \epsilon^{B\tau^{k(R)}-1} \cdot c(h^{1-n})^{A_0}.$$

For the case where we have the fixed $w \in S_{2r}$, the proof is similar by considering $f(y,w)$ as a function of y. Similarly as (4.1), we have

$$\mathrm{div}(A(x)\nabla_y f(y,w)) = 0 \ \text{ in } \mathcal{C}\Omega_D$$

which allows us to apply the three spheres inequality iteratively over the chain of balls which are on the arc, or contained in a truncated cone. It is easy to conclude (see, [Al-DC, page 214]) that

$$\|f(y,w)\|_{L^\infty(B_{r_{k(h)}}(y_{k(h)}))} \leqslant c(h^{2-2n})^{(A\tau^s+1-\tau^B)(1-\tau^{k(h)-1})} \cdot (\epsilon^{B\tau^{k(h)-1}})^{\tau^{k(h)+B-1}}.$$

Now by choosing $y = w = h\nu(O)$, where $\nu(O)$ is the exterior unit normal vector to $\partial\Omega_D$ in O, we obtain

$$|f(y,y)| \leqslant ch^{A'}(\epsilon^{B\tau^{k(h)-1}})^{\tau^{k(h)+B-1}}$$

where $A' = -(2-2n)B(A\tau^s + 1 - \tau^B) > 0$. We observe that, for $0 < h < cr$, where $0 < c < 1$ depends on M_0 and $k(h) \leqslant c|\log h| = -c\log h$, we write

$$\tau^{k(h)} = \mathrm{e}^{-c\log h \log\tau} = h^{-c\log\tau} = h^{c|\log\tau|} = h^F$$

with $F = c|\log\tau|$. Thus

$$|f(y,y)| \leqslant h^{-A'}\epsilon^{B\tau^{k(h)}} = \mathrm{e}^{-A'\log h}\mathrm{e}^{B\tau^{k(h)}\log\epsilon} = \mathrm{e}^{-A'\log h + Bh^F\log\epsilon} = \frac{\epsilon^{Bh^F}}{h^{A'}}.$$ □

The proof of proposition 3.5 is based on the asymptotic behaviour of the fundamental solutions. In the following theorem we will compare Γ_D with Γ_0, the fundamental solution over the half space. Given a point $x = (x', x_n) \in \mathbb{R}^n$, let us denote $x^* = (x', -x_n)$, and χ_+ be the characteristic function for the half-space $\mathbb{R}^n_+$. The Γ_D and Γ_0 are defined as

$$\begin{aligned} &\mathrm{div}[(A(x) + A(x)(k-1)\chi_D)\nabla\Gamma_D(\cdot,y)] = -\delta(\cdot - y) \\ &\mathrm{div}[(A(0) + A(0)(k-1)\chi_0)\nabla\Gamma_0(\cdot,y)] = -\delta(\cdot - y). \end{aligned} \tag{4.5}$$

When $A(0) = I$, the identity matrix, we have $\Gamma_0 = \Gamma_+$, which is defined as

$$\Gamma_+(x,y) = \begin{cases} \frac{1}{k}\Gamma(x,y) + \frac{(k-1)}{k(k+1)}\Gamma(x,y^*) & \text{for } x_n > 0, y_n > 0 \\ \frac{2}{k+1}\Gamma(x,y) & \text{for } x_n y_n < 0 \\ \Gamma(x,y) - \frac{(k-1)}{k+1}\Gamma(x,y^*) & \text{for } x_n < 0, y_n < 0 \end{cases}$$

where Γ is the fundamental solution for the standard Laplace operator. As for the general case $A(0) \neq I$, we refer to [Ga-Si] by performing a linear change of variable (see (4.80-4.81)) to reduce the general $A(0)$ into a simple case as $A(0) = I$. Now we have the following theorem considering only when $A(0) = I$ and $\Gamma_0 = \Gamma_+$.

Theorem 4.1. *Let Γ_D and Γ_0 be the fundamental solutions for (1.1) and (4.5), respectively. Under a priori assumption of D, the following estimates hold for every $x, y \in \mathbb{R}^n$*

$$|\Gamma_D(x,y) - \Gamma_0(x,y)| \leqslant \frac{C}{r_1^{\alpha}}|x-y|^{\alpha-n+2}$$

$$|\nabla\Gamma_D(x,y) - \nabla\Gamma_0(x,y)| \leqslant \frac{C}{r_1^{\alpha^2}}|x-y|^{\alpha^2-n+1}$$

where $C > 0$ only depends on the a priori data and $r_1 = \frac{r}{2}\min\{\frac{1}{2}(8M_0)^{-1/2}, \frac{1}{2}\}$.

Proof. Let 0 be the origin of the coordinate system, $\partial D \cap B_r = \{(x', x_n) \in B_r | x_n = \varphi(x')\}$ where the rigid transformation $\varphi \in C^{1,\alpha}(\mathbb{R}^{n-1})$ satisfies the definition 2.2 in which $\varphi(0) = |\nabla\varphi(0)| = 0$ holds. Let $\tau \in C^{\infty}(\mathbb{R})$ be such that $\tau(t) \in [0,1]$, and $\tau(t) = 1$, for $|t| < 1$, $\tau(t) = 0$, for $|t| > 2$ and $|\frac{d\tau}{dt}| \leqslant 2$ for $1 \leqslant |t| \leqslant 2$. We define the change of variables $\xi = \Phi(x)$

$$\begin{cases} \xi' = x' \\ \xi_n = x_n - \varphi(x')\tau(\frac{|x'|}{r_1})\tau(\frac{x_n}{r_1}). \end{cases}$$

It is easy to see $\Phi(\cdot)$ is a $C^{1,\alpha}$ type diffeomorphism from $\mathbb{R}^n$ into itself. Also, $\Phi(\cdot)$ satisfies the following properties where $c \geqslant 1$ depends only on α, M_0

$$\begin{aligned} &\Phi(Q_{2r_1}) = Q_{2r_1}, \qquad \Phi(Q_{r_1} \cap D) = Q_{r_1}^+ \\ &c^{-1}|x_1 - x_2| \leqslant |\Phi(x_1) - \Phi(x_2)| \leqslant c|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n, \\ &|\Phi(x) - x| \leqslant \frac{c}{r^{\alpha}}|x|^{1+\alpha}, \qquad |D\Phi(x) - I| \leqslant \frac{c}{r^{\alpha}}|x|^{\alpha}, \quad \forall x \in \mathbb{R}^n. \end{aligned} \tag{4.6}$$

From the fundamental solution for the half space $\Gamma_0(x,y)$, we proceed a change of variables on (4.5) by choosing $\xi = \Phi(x), \eta = \Phi(y)$. Thus, we obtain another fundamental solution $\tilde{\Gamma}(\xi,\eta)$ such that

$$\operatorname{div}[(1 + (k-1)\chi^+)B\nabla\tilde{\Gamma}(\xi,\eta)] = -\delta(\xi - \eta) \tag{4.7}$$

where $\tilde{\Gamma}(\xi,\eta) = \Gamma_D(\Phi^{-1}(\xi), \Phi^{-1}(\eta))$, $J(\xi) = (D\Phi)(\Phi^{-1}(\xi))$ and $B(\xi) = \frac{JJ^T}{\det J}A(\Phi^{-1}(\xi))$. If we define the residual, as

$$\tilde{R}(\xi,\eta) = \tilde{\Gamma}(\xi,\eta) - \Gamma_0(\xi,\eta).$$

Then we can solve $\tilde{R}$ by convolution with respect to Γ_0 and Γ_+, which gives

$$\operatorname{div}((1 + (k-1)\chi^+)\nabla\tilde{R}(\xi,\eta)) = \operatorname{div}((1 + (k-1)\chi^+)(I - B)\nabla\tilde{\Gamma}(\xi,\eta))$$

$$-\tilde{R}(\xi,\eta) = \int_{B_{\tilde{L}}} (1 + (k-1)\chi^+)(B - I)\nabla\Gamma_0(\cdot,\xi)\nabla\tilde{\Gamma}(\cdot,\eta) + \tilde{C}$$

here $\overline{\Omega} \subset B_{\tilde{L}}$, and $\tilde{L} > 0$ depends only on the *a prior* data. Also the boundary term is bounded by a constant $\tilde{C}$. Now let us estimate the residual $\tilde{R}$ by considering $\xi \in Q_{r_1/2}^+$ and $\eta = e_n\eta_n$. We also separate the domain of integral into two parts $-\tilde{R} = \tilde{R}_1 + \tilde{R}_2$ by using a cylinder Q_{r_1}.

$$\tilde{R}_1(\xi,\eta) = \int_{B_{\tilde{L}}\setminus Q_{r_1}} (1+(k-1)\chi^+)(B-I)\nabla\Gamma_0(\cdot,\xi)\nabla\tilde{\Gamma}(\cdot,\eta)$$
$$\tilde{R}_2(\xi,\eta) = \int_{Q_{r_1}} (1+(k-1)\chi^+)(B-I)\nabla\Gamma_0(\cdot,\xi)\nabla\tilde{\Gamma}(\cdot,\eta).$$

Thanks to the upper bound of the known matrix, and the ellipticity condition. We can combine Schwartz inequality with the Caccioppoli inequality, we get

$$|\tilde{R}_1(\xi,\eta)| \leqslant \frac{C_1}{r_1^2}\|\Gamma_0(\cdot,\xi)\|_{L^2(\Omega\setminus Q_{3r_1/4})}\cdot\|\tilde{\Gamma}(\cdot,\eta)\|_{L^2(\Omega\setminus Q_{3r_1/4})} \tag{4.8}$$

where $C_1 = C_1(\overline{A},\alpha,M_0,\lambda)$ is a positive constant. Moreover, we use (3.3) in [Ga-Si], the standard behavior of the Green functions at hand to obtain

$$|\tilde{R}_1(\xi,\eta)| \leqslant C_1 r_1^{2-n}. \tag{4.9}$$

Now we combine (4.8) and (4.9), we have

$$|\tilde{R}(\xi,\eta)| = |\tilde{R}_1+\tilde{R}_2| \leqslant |\tilde{R}_1|+|\tilde{R}_2| \leqslant \frac{C_2}{r_1^\alpha}(I_1+I_2) \leqslant \frac{C_4}{r_1^\alpha}h^{\alpha-n+2} \tag{4.10}$$

where C_4 depends only on $\lambda, M_0, \alpha, \overline{A}, n, k$. Now our final step is to bound $R(x,y) = \Gamma_D(x,y) - \Gamma_0(x,y)$ under the original coordinate system. Arguing like in [Al-DC, proposition 3.4], we obtain

$$|R(x,y)| = |\Gamma_D(x,y)-\Gamma_0(x,y)| \leqslant |\tilde{R}(\xi,\eta)| + |\Gamma_0(\Phi(x)-x,y)|$$
$$\leqslant \frac{C_4}{r_1^\alpha}h^{\alpha-n+2} + \frac{C_5}{r_1^{2\alpha}}h^{\alpha-n+2} \leqslant \frac{C_6}{r_1^\alpha}h^{\alpha-n+2}$$

where C_6 depends only on $M_0, \alpha, n, \lambda, k, \overline{A}$. Now we prove the gradient of the residual $\nabla R(x,y)$ is also bounded from above. We start to estimate the first derivative for $\tilde{R}(\xi,\eta)$ by considering a cylinder $Q \subset B^+_{r_1/4}(\xi)$. Let us fix $\xi \in B^+_{r_1/4}$ and $\eta_n \in (-r_1/4, 0)$ and the cylinder is defined as

$$Q = B'_{h/8}(\xi') \times \left(\xi_n, \xi_n + \frac{h}{8}\right).$$

Since $h = |\xi-\eta| = |\xi-(0,\eta_n)| \leqslant \frac{r_1}{2}$, we can reduce the cylinder as $Q \subset Q_{\frac{h}{4}(\xi)}$; moreover, $\xi \in \partial Q$. Then by applying [Li-Vo, theorem 1.1], we have the estimate for the semi-norm $|\nabla\tilde{\Gamma}(\xi,\eta)|_{\alpha,Q}$ by choosing $\beta = \frac{1}{2}\min\left\{\alpha, \frac{\alpha}{(\alpha+1)n}\right\}$

$$|\nabla\tilde{\Gamma}(\xi,\eta)|_{\alpha,Q} \leqslant |\nabla\tilde{\Gamma}(\xi,\eta)|_{\beta,Q_{\frac{h}{4}(\xi)}} \leqslant C_L h^{-\beta-\frac{n}{2}-1}\|\nabla\tilde{\Gamma}(\xi,\eta)\|_{L^2(Q_{h/2}(\xi))}$$

where C_L depends only on the *a priori* data. Again, we use [Al-DC] proposition 3.4 (i) to obtain

$$|\nabla\tilde{\Gamma}(\xi,\eta)|_{\alpha,Q} \leqslant C_L h^{-\beta-\frac{n}{2}-1} c_1\left(\frac{h}{2}\right)^{2-n} = C_7 h^{-\beta-\frac{3n}{2}+1} \leqslant C_7 h^{\alpha-n+1} \tag{4.11}$$

where C_7 depends only on the *a priori* data. By analogous argument, we could also have

$$|\nabla\Gamma_0(\xi,\eta)|_{\alpha,Q} \leqslant C_7 h^{\alpha-n+1}. \tag{4.12}$$

Let's notice the interpolation inequality

$$\|\nabla\tilde{R}(\xi,\eta)\|_{L^\infty(Q)} \leqslant C_I\|\tilde{R}(\xi,\eta)\|^{1-\delta}_{L^\infty(Q)} \cdot |\nabla\tilde{R}(\xi,\eta)|^{\delta}_{\alpha,Q}$$

where $\delta = \frac{1}{1+\alpha}$, and C_I depends only on $M_0, \alpha, \overline{A}, n$, see [Al-Si, proposition 8.3]. Since the first term of the right-hand side is already bounded in (4.10), we just need to estimate the second term $|\nabla\tilde{R}|^{\delta}_{\alpha,Q}$. By (4.11) and (4.12) and triangular inequality, we get

$$|\nabla\tilde{R}(\xi,\eta)|_{\alpha,Q} \leqslant |\nabla\tilde{\Gamma}(\xi,\eta)|_{\alpha,Q} + |\nabla\Gamma_0(\xi,\eta)|_{\alpha,Q} \leqslant C_7 h^{\alpha-n+1}.$$

Now if we plug into the interpolation inequality with (4.10), we obtain

$$\begin{aligned}\|\nabla\tilde{R}(\xi,\eta)\|_{L^\infty(Q)} &\leqslant C_I\frac{C_4^{1-\delta}}{r_1^{\alpha(1-\delta)}}h^{(\alpha-n+2)(1-\delta)}C_7h^{(\alpha-n+1)\delta}\\ &= \frac{C_8}{r_1^{\tau}}h^{\tau-n+1}\end{aligned}$$

where $\tau = \frac{\alpha^2}{1+\alpha}$. Thus we use the definition of L^∞ norm to obtain

$$|\nabla\tilde{R}(\xi,\eta)| \leqslant \frac{C_8}{r_1^{\tau}}h^{\tau-n+1} \tag{4.13}$$

where C_8 depends only on the *a priori* data. Now we consider the original coordinate system

$$\begin{aligned}\nabla R(x,y) &= \nabla\Gamma_D(x,y) - \nabla\Gamma_0(x,y)\\ &= \nabla\Gamma_D(x,y) - \nabla\Gamma_0(\xi,\eta) + \nabla\Gamma_0(\xi,\eta) - \nabla\Gamma_0(x,y)\\ &= \nabla\Gamma_D(\Phi^{-1}(\xi),\Phi^{-1}(\eta)) - \nabla\Gamma_0(\xi,\eta) + \nabla\Gamma_0(\Phi(x),\Phi(y)) - \nabla\Gamma_0(x,y)\\ &= \nabla\tilde{R}(\xi,\eta) + \nabla\Gamma_0(\Phi(x),y) - \nabla\Gamma_0(x,y).\end{aligned}$$

Concerning the absolute value of the second term on the right-hand side, we apply the the properties of diffeomorphism $\Phi(\cdot)$, as well as results from (4.6), (4.10) and (4.12), we obtain

$$\begin{aligned}|\nabla\Gamma_0(\Phi(x),y) - \nabla\Gamma_0(x,y)| &= |D\Phi(x)^T\nabla\Gamma_0(\cdot,y)_{|_{\Phi(x)}} - \nabla\Gamma_0(x,y)|\\ &\leqslant |(D\Phi(x)^T - I)\nabla\Gamma_0(\cdot,y)_{|_{\Phi(x)}}| + |\nabla\Gamma_0(\cdot,y)_{|_{\Phi(x)}} - \nabla\Gamma_0(x,y)|\\ &\leqslant |D\Phi(x)^T - I|\cdot\|\nabla\Gamma_0(\cdot,y)\|_{L^\infty(Q_{r_1})}\cdot|x-\Phi(x)|\\ &+ |\nabla\Gamma_0(\cdot,y)|_{\alpha,Q}\cdot|\Phi(x)-x|^{\alpha}\\ &\leqslant \frac{C_9}{r_1^{\alpha^2}}h^{\alpha^2-n+1}.\end{aligned}$$

Now with the above estimates and (4.13), it can be concluded

$$|\nabla R(x,y)| \leqslant |\nabla\tilde{R}(\xi,\eta)| + |\nabla\Gamma_0(\Phi(x),y) - \nabla\Gamma_0(x,y)| \leqslant \frac{C}{r_1^{\alpha^2}}h^{\alpha^2-n+1}$$

where C depends only on the *a priori* data and r_1. □

Proof of proposition 3.5. We write the upper bound of S_{D_1} as

$$\begin{aligned}|S_{D_1}(y,y)| &= \Big|(k-1)\int_{D_1} A(x)\nabla\Gamma_{D_1}(x,y)\nabla\Gamma_{D_2}(x,y)\mathrm{d}x\Big| \\ &\geqslant C\Big|\Big(\int_{D_1\cap B_r(O)\cap D_2} + \int_{D_1\cap B_\rho(O)\cap \mathcal{C}D_2}\Big)\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \\ &- C\Big|\int_{D_1\cap B_r(O)\cap\mathcal{C}B_\rho(O)\cap\mathcal{C}D_2}\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \\ &- C\Big|\int_{D_1\backslash B_r(O)}\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \end{aligned} \tag{4.14}$$

where C depends on $k, \overline{A}$ only, $r=|x-y|, 0<r<r_0$, $0<\rho<\min\{d_m, r\}$. To explain the formula, notice we separate the integrand $\int_{D_1\cap B_r(O)}\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}$ into two parts, because we do not have any information on x. So, either it can be $x\in D_1\cap B_r(O)\cap D_2$ or $x\in D_1\cap B_r(O)\cap\mathcal{C}D_2$. Then we separate the integrand again with respect to an even smaller ball $B_\rho(O)$.

If $x\in D_1\cap B_r(O)\cap D_2$, we use [A] lemma 3.1, which gives

$$\nabla\Gamma_{D_1}(x,y)\cdot\nabla\Gamma_{D_2}(x,y)\geqslant C_A|x-y|^{2-2n}=C_A r^{2-2n}>0 \tag{4.15}$$

where C_A depends on the *a priori* data. If $x\in D_1\cap B_r(O)\cap\mathcal{C}D_2$, we consider in a smaller ball $B_\rho(O)$. In this case, we actually have $x\in D_1\cap B_\rho(O)\cap\mathcal{C}D_2$. By definition of d_m, $B_\rho(O)\cap D_2=\emptyset$, for $x,y\in B_\rho(O)$, we have

$$\begin{cases}\Delta\Big(\Gamma_{D_2}(x,y)-\Gamma(x,y)\Big)=0 \ \text{ in } B_\rho(O)\\ \Big(\Gamma_{D_2}(x,y)-\Gamma(x,y)\Big)|_{\partial B_\rho(O)}\leqslant C_K\rho^{2-n}\end{cases}$$

by the maximum principle, the value on interior is smaller than boundary

$$\Big|\Gamma_{D_2}(x,y)-\Gamma(x,y)\Big|\leqslant C_K\rho^{2-n}\ \ \forall x,y\in B_\rho(O).$$

And by interior gradient bound, we have

$$\Big|\nabla\Gamma_{D_2}(x,y)-\nabla\Gamma(x,y)\Big|\leqslant C_{K_0}\rho^{1-n}\ \ \forall x\in B_{\rho/2}(O);\forall y\in B_\rho(O).$$

Applying [A] lemma 3.1 in $B_{\rho/2}(O)$, we have (notice $|x-y|=r>\rho$)

$$\nabla\Gamma_{D_1}(x,y)\cdot\nabla\Gamma_{D_2}(x,y)\geqslant C_A|x-y|^{2-2n}-C_K\rho^{2-2n}=C_A r^{2-2n}-C_K\rho^{2-2n}>0. \tag{4.16}$$

Now we can bound the first term of (4.14) thanks to (4.15) and (4.16)

$$\begin{aligned}&\Big|\Big(\int_{D_1\cap B_r(O)\cap D_2}+\int_{D_1\cap B_\rho(O)\cap\mathcal{C}D_2}\Big)\nabla\Gamma_{D_1}\nabla\Gamma_{D_2}\Big| \\ \geqslant&\Big|\Big(\int_{D_1\cap B_r(O)\cap D_2}+\int_{D_1\cap B_\rho(O)\cap\mathcal{C}D_2}\Big)(C_A r^{2-2n}-C_K\rho^{2-2n})\Big| \\ \geqslant&\Big|\Big(\int_{[D_1\cap B_r(O)\cap D_2]\cup[D_1\cap B_\rho(O)\cap\mathcal{C}D_2]}\Big)c_1 r^{2-2n}\Big|\geqslant c_1 h^{2-n}.\end{aligned} \tag{4.17}$$

For the upper bounds of the second and third term, we can apply the natural bound of $\nabla\Gamma_{D_i}, i = 1, 2$. When $x \in D_1 \cap B_r(O) \cap \mathcal{C}B_\rho(O) \cap \mathcal{C}D_2$, we have

$$\begin{aligned}
&\left| \int_{D_1\cap B_r(O)\cap \mathcal{C}B_\rho(O)\cap \mathcal{C}D_2} \nabla\Gamma_{D_1}\nabla\Gamma_{D_2} \right| \\
&\leqslant \left| \int_{D_1\cap B_r(O)\cap \mathcal{C}B_\rho(O)\cap \mathcal{C}D_2} c_1|x-y|^{1-n}\cdot c_1|x-y|^{1-n} \right| \\
&\leqslant \left| \int_{D_1\cap B_r(O)\cap \mathcal{C}B_\rho(O)\cap \mathcal{C}D_2} c_1 r^{1-n}\cdot c_1 r^{1-n} \right| \\
&\leqslant c_2 d_m^{2-2n}
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\left| \int_{D_1\setminus B_r(O)} \nabla\Gamma_{D_1}\nabla\Gamma_{D_2} \right| &\leqslant \left| \int_{D_1\setminus B_r(O)} c_1|x-y|^{1-n}\cdot c_1|x-y|^{1-n}\mathrm{d}x \right| \\
&= \left| \int_{D_1\setminus B_r(O)} c_1^2 r^{2-2n}\mathrm{d}x \right| \\
&= c_3.
\end{aligned} \tag{4.19}$$

Now we can plug (4.17)–(4.19) into (4.14), we obtain the lower bound for $S_{D_1}(y, y)$

$$|S_{D_1}| \geqslant c_1 h^{2-n} - c_2 d_m^{2-2n} - c_3$$

where $c_i, i = 1, 2, 3$ depends only on the *a prior* data. □

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