

**MAXIMAL AND MINIMAL SOLUTIONS
 OF SECOND ORDER ELLIPTIC AND PARABOLIC EQUATIONS
 IN NON-DIVERGENCE FORM WITH MEASURABLE
 COEFFICIENTS**

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ABSTRACT. In this paper we will prove that the supremum and infimum of good solutions of the Dirichlet problem for elliptic and parabolic equations in non divergence form with measurable coefficients, are good solutions to the same problem.

1. INTRODUCTION

Let $D \subset \mathbf{R}^n$ be a bounded domain with smooth boundary ∂D and $I = (0, T)$ be an interval; then Q will be the cylinder $Q = D \times I$ in \mathbf{R}^{n+1} . In this paper L will be either an elliptic operator defined in D or a parabolic operator defined in Q and, to point out that proofs of the main results are basically the same in both cases, we will use a notation that will not distinguish (unless necessary) among them. More precisely let $x = (x_1, \dots, x_n)$ and $t \in \mathbf{R}$; then we will use the same letter z for the variable understanding that $z = x$ in the context of elliptic operators or $z = (x, t)$ in the context of parabolic operators. From now on we will also use N meaning $N = n$ in the elliptic context and $N = n + 1$ in the parabolic one. So e.g. we will write $z \in \mathbf{R}^N$. The operator L will have either the form

$$(1.1) \quad L = \sum_{i,j=1}^n a_{ij}(z) D_{ij}$$

for $z \in D$ or the form

$$(1.2) \quad L = \sum_{i,j=1}^n a_{ij}(z) D_{ij} - \partial_t$$

for $z \in Q$. Here we use the notation $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. In both cases $a_{ij} = a_{ji}$ are measurable functions such that

$$(1.3) \quad \lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(z) \xi_i \xi_j \leq \Lambda |\xi|^2$$

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for all $\xi \in \mathbf{R}^n$ and with λ, Λ positive constants.

With the purpose of unifying even more the two contexts, when it is not necessary to distinguish we will use the same name Ω for the domain of the operators, understanding that $\Omega = D$ in the elliptic context and $\Omega = Q$ in the parabolic one. We will call $\mathcal{L}(\lambda, \Lambda, \Omega)$ the class of either elliptic or parabolic operators defined above.

Finally let the parabolic boundary of Q be $\partial_p Q = \partial_b Q \cup \partial_x Q$, where $\partial_b Q = D \times \{0\}$ and $\partial_x Q = \partial D \times \bar{I}$. Accordingly, we will use the notation $\partial_d \Omega$ to mean ∂D in the elliptic context and to mean $\partial_p Q$ in the parabolic one.

For g continuous on $\partial_d \Omega$ and $f \in L^p$ consider the following Dirichlet problem (D.P):

$$(1.4) \quad \begin{cases} Lu = f & \text{on } \Omega \\ u = g & \text{on } \partial_d \Omega \end{cases}$$

If the coefficients a_{ij} are at least continuous functions in $\bar{\Omega}$ the problem has a unique solution $u \in W_p^2(\Omega')$ if L is elliptic or $u \in W_p^{2,1}(\Omega')$, $p > 1$ if L is parabolic, for any $\Omega' \subset \bar{\Omega}' \subset \Omega$.

For discontinuous a_{ij} it has been shown that Sobolev spaces are not suitable for the solvability of the above D.P., whereas counterexamples showed that either existence or uniqueness may fail, and such solutions may not satisfy basic properties, such as the maximum principle.

Therefore in several papers in the '90's (see e.g. [3],[8], [18]) a different notion of solution (good or weak solution) has been introduced. For sake of completeness we recall it in the following

Definition 1.1. A function $u(z) \in C(\bar{\Omega})$ is a *good solution* to problem (1.4) with $p = n$ in the elliptic case and $p = n + 1$ in the parabolic, if

- there exist a sequence of operators $L^k \in \mathcal{L}(\lambda, \Lambda, \Omega)$ with coefficients $a_{ij}^k \in \mathcal{C}(\bar{\Omega})$ such that $a_{ij}^k \rightarrow a_{ij}$ a.e. in Ω (as $k \rightarrow \infty$, $i, j = 1, \dots, n$). We will say $L^k \rightarrow L$.
- there exists a sequence of smooth functions u^k , solutions of the D.P.'s for the operators L^k (defined respectively either as in (1.1) or as in (1.2) with coefficients a_{ij}^k), i.e.

$$(1.5) \quad \begin{cases} L^k u^k = f & \text{in } \Omega \\ u^k = g & \text{on } \partial_d \Omega \end{cases}$$

and such that $u_k \rightarrow u$ uniformly in $\bar{\Omega}$.

Recall that Krylov-Safonov uniform Hölder estimates hold for the u^k 's independently of the regularity of the coefficients. This implies that the functions u^k are uniformly bounded and equicontinuous in Ω and by Ascoli-Arzelà's theorem, there exists a convergent subsequence. Therefore good solutions always exist.

A result by N. Nadirashvili for elliptic operator (see [13]), states that uniqueness for good solutions may fail if $n \geq 3$. He constructs two sequences of operators with smooth coefficients $a_{ij}^{0,k}$ and $a_{ij}^{1,k}$ which satisfy the ellipticity condition with the same constants and converge to the same a_{ij} a.e., as $k \rightarrow \infty$, in the unit ball $B_1 \subset R^3$, while the corresponding sequences of solutions converge to two different functions

u^0 and u^1 . Because of the probabilistic nature of Nadirashvili's counterexample, it is reasonable to think that it can be slightly modified to provide a counterexample also in the parabolic setting.

In this paper we show that, if uniqueness does not hold, the supremum and the infimum of good solutions are still good solutions. The result is not new for the elliptic case, since it is known to hold for viscosity solutions and Jensen in [7] shows that good solutions in this case coincide with viscosity solutions. Our result, though, besides being new for parabolic equations, provides a unifying proof for both cases and a direct method also for elliptic case.

We hope that this result could be helpful in improving the known results about uniqueness. In fact Nadirashvili's example did not settle completely the matter, since the set of discontinuities of his operators is "very large". Several results for uniqueness (see i.g. [3] and [18] for elliptic equations and [4] for parabolic equations) have been proved that when the set of discontinuities is not too bad, but many problems still remain open such as that of discontinuities along a general line segment or a general hyperplane.

The main result is proved in section 4 and a main role in the proof is played by the parabolic version of Pucci's extremal operators, whose definitions and main properties are recalled in section 2.

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2. DEFINITIONS AND PRELIMINARY RESULTS

A main role in the proofs is played by Pucci's extremal operators (see [14]) that we recall here. We will as well introduce the parabolic version of these operators.

For a symmetric matrix $M \in \mathcal{S}$ (space of real $n \times n$ symmetric matrices), with eigenvalues e_1, \dots, e_n , define

$$\mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$

and

$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i.$$

Let $\mathcal{A}_{\lambda, \Lambda}$ be the set of symmetric matrices with eigenvalues in $[\lambda, \Lambda]$. Define the linear functional L_A on \mathcal{S} by

$$L_A M = \text{tr}(AM)$$

It is well known (see [2]) that

$$\mathcal{M}^-(M) = \inf_{A \in \mathcal{A}} L_A M \quad \text{and} \quad \mathcal{M}^+(M) = \sup_{A \in \mathcal{A}} L_A M .$$

Elliptic Pucci's extremal operators are defined as:

$$L^+ u = \mathcal{M}^+(D^2 u) \quad \text{and} \quad L^- u = \mathcal{M}^-(D^2 u)$$

while parabolic Pucci's extremal operators as:

$$L^+ u = \mathcal{M}^+(D^2 u) - u_t \quad \text{and} \quad L^- u = \mathcal{M}^-(D^2 u) - u_t.$$

We'll also need Hölder and Sobolev spaces and their parabolic counterpart that we'll recall below.

Given $0 < \alpha \leq 1$ the subspace of $C(\Omega)$ of α -Hölder continuous functions is defined as the space of functions u such that the norm

$$(2.1) \quad \|u\|_{C^\alpha(\Omega)} = \sup_{z \in \Omega} |u(z)| + \sup_{z_1, z_2 \in \Omega} \frac{|u(z_1) - u(z_2)|}{d(z_1, z_2)^\alpha}$$

is finite, with d the usual Euclidean distance.

For the parabolic context, we recall a definition of parabolic distance between $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2)$:

$$d(z_1, z_2) = \begin{cases} \max \{ |x_1 - x_2|, |t_1 - t_2|^{1/2} \} & t_1 \leq t_2 \\ \infty & t_1 > t_2 \end{cases}$$

Now given $0 < \alpha \leq 1$, the subspace of $C(\Omega)$ of parabolic α -Hölder continuous functions is defined as the space of functions u that satisfy (2.1) with d the parabolic distance.

As before, throughout the paper, the space $C^\alpha(\Omega)$ will mean either the space of classical or that of parabolic Hölder continuous functions.

Moreover in what follows, let $W(\Omega)$ be the Sobolev space $W_n^2(\Omega)$ in the elliptic context and $W_{n+1}^{2,1}(\Omega)$ for the parabolic one.

The space $W_{loc}(\Omega)$ is the space of functions f defined on Ω such that $f \in W(\Omega')$ for every $\Omega' \subset \bar{\Omega}' \subset \Omega$.

Finally $B_R(z_0) = \{z \in \mathbf{R}^N \mid d(z, z_0) < R\}$ will be the Euclidean ball of radius R in the elliptic context and a parabolic cylinder otherwise.

We will need the following results :

Theorem 2.1. (*ABP/ABPKT-Maximum principle*)(see [1] and [15] for elliptic operators and [9] and [19] for parabolic; a more general version for both cases in [5]). Let L be either the elliptic operator (1.1) or the parabolic operator (1.2), $u \in W_{loc}(\Omega) \cap C(\bar{\Omega})$, then there exists $C = C(\lambda, \Lambda, N, \Omega)$ such that

$$(2.2) \quad \sup_{\Omega} |u| \leq \sup_{\partial_d \Omega} |u| + C \|Lu\|_{L^N(\Omega)}.$$

Theorem 2.2. (*Hölder estimate, see i.e. [16] and [6] for elliptic, [10] and [12] for parabolic*) Let L be either an elliptic or a parabolic operator with smooth coefficients, $u \in W_{loc}(\Omega)$, $B_{R_0}(z) \subset \Omega$, then there exist constants $K(N, \lambda, \Lambda, R_0)$ and $\alpha(N, \lambda, \Lambda)$, such that for $R < R_0$

$$(2.3) \quad \|u\|_{C^\alpha(B_R)} \leq K(\|u\|_\infty + \|Lu\|_N).$$

Moreover, if $u \in W(\Omega)$ and $u = 0$ on $\partial_d \Omega$, there exists $C_2 = C_2(\lambda, \Lambda, N, \Omega)$ such that

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq S_2(\|u\|_\infty + \|Lu\|_N)$$

Theorem 2.3. (see [6] and [12]) Let \mathcal{L} be a finite or countable family of operators L^k with coefficients $C^1(\bar{\Omega})$. Define

$$F_{\mathcal{L}}(u) = \sup_{L \in \mathcal{L}} Lu.$$

Then there exists $\beta = \beta(N, \lambda, \Lambda) \in (0, 1)$ such that, if $f \in C^1(\Omega)$, $g \in C(\partial_d \Omega)$, the problem

$$\begin{cases} F_{\mathcal{L}}(u) = f & \text{in } \Omega \\ u = g & \text{on } \partial_d \Omega \end{cases}$$

has a unique solution $u \in C_{loc}^{2,\beta}(\Omega) \cap C(\overline{\Omega})$.

3. PROPERTIES OF GOOD AND CLASSICAL SOLUTIONS

Let $\mu > 0$, $\Omega_\mu = \{z \in \Omega : \text{dist}(z, \partial_d \Omega) > \mu\} \subset \Omega$.

Lemma 3.1. *Let $L \in \mathcal{L}(\lambda, \Lambda, \Omega)$ and $a_{ij} \in C(\overline{\Omega})$, $u \in W(\Omega) \cap C(\overline{\Omega})$. If $Lu = f$, $u|_{\partial_d \Omega} = g$ and $f \in L^N(\Omega)$, $g \in C(\partial_d \Omega)$, then there exists $\sigma_\mu > 0$, $\sigma_\mu \rightarrow 0$ as $\mu \rightarrow 0$, satisfying:*

$$(3.1) \quad \sup_{z_\mu \in \partial_d \Omega_\mu} \inf_{\xi \in \partial_d \Omega} |g(\xi) - u(z_\mu)| \leq \sigma_\mu$$

The number $\sigma_\mu = \sigma(\lambda, \Lambda, \Omega, \mu, N, \|f\|_{L^N(\Omega)}, g)$.

Proof. Assume for the moment $f = 0$ a.e. Let $\mathcal{L}^* = \{L \in \mathcal{L} : L \text{ has rational constant coefficients in } \Omega\}$ and consider Pucci's maximal and minimal operators:

$$L^+ u = F_{\mathcal{L}^*}(u) \quad \text{and} \quad L^- u = -F_{\mathcal{L}^*}(-u).$$

By Theorem 2.3 there exist $u_\mu^+, u_\mu^- \in C^{2,\beta}(\Omega) \cap C(\overline{\Omega})$ satisfying:

$$(3.2) \quad \begin{cases} L^+(u_\mu^+) = 0 & \text{in } \Omega \\ u_\mu^+ = g & \text{on } \partial_d \Omega \end{cases}$$

and

$$(3.3) \quad \begin{cases} L^-(u_\mu^-) = 0 & \text{in } \Omega \\ u_\mu^- = g & \text{on } \partial_d \Omega \end{cases}$$

As u_μ^+ and u_μ^- are continuous functions, then:

$$\sigma_\mu^- = \sup |g(\xi) - u_\mu^-(z_\mu)| \rightarrow 0 \quad \text{as } \mu \rightarrow 0$$

$$\sigma_\mu^+ = \sup |u_\mu^+(z_\mu) - g(\xi)| \rightarrow 0 \quad \text{as } \mu \rightarrow 0$$

where $z_\mu \in \partial_d \Omega_\mu$ and $\xi \in \partial_d \Omega$

As $Lu_\mu^+ \leq 0$ and analogously $Lu_\mu^- \geq 0$ we have $L(u - u_\mu^+) \geq 0$ and $L(u_\mu^- - u) \geq 0$ in Ω . By the Maximum Principle:

$$\begin{aligned} \sup_\Omega (u - u_\mu^+) &\leq 0 \\ \sup_\Omega (u_\mu^- - u) &\leq 0. \end{aligned}$$

We can write:

$$\begin{aligned} u(z_\mu) - g(\xi) &\leq \sup(u - u_\mu^+) + \sigma_\mu^+ \leq \sigma_\mu^+ \\ g(\xi) - u(z_\mu) &\leq \sigma_\mu^- + \sup(u_\mu^- - u) \leq \sigma_\mu^-, \end{aligned}$$

i.e.

$$|u(z_\mu) - g(\xi)| \leq \max(\sigma_\mu^+, \sigma_\mu^-).$$

The thesis follows, if f vanishes a.e.

If f is non-identically zero, we can write $u = u_1 + u_2$, where $u_1 \in W(\Omega)$, $Lu_1 = f$ in Ω and $u_1 = 0$ on the boundary $\partial_d\Omega$, $u_2 \in W(\Omega)$, $Lu_2 = 0$ in Ω and $u_2 = g$ on the boundary. Then u_1 satisfies Hölder estimates of Theorem 1, namely

$$\sup_{z_\mu \in \partial_d\Omega_\mu} |u(z_\mu)| \leq \mu^\alpha K(1+C)\|f\|_{L^N(\Omega)}$$

The thesis follows with $\sigma_\mu = \max(\sigma_\mu^+, \sigma_\mu^-, \mu^\alpha K(1+C)\|f\|_{L^N(\Omega)})$. \square

As a consequence, an equicontinuity result follows:

$\forall z_1, z_2 \in \overline{\Omega}$ such that $d(z_1, z_2) \leq \delta$ and for any $u \in W(\Omega) \cap C(\overline{\Omega})$ with $\|Lu\|_{L^N} \leq F$ and $u = g$ on the boundary; there exists $\omega(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, such that:

$$(3.4) \quad |u(z_1) - u(z_2)| \leq \omega(\delta)$$

Remark 3.2. If the D.P.(1.4) has a strong solution $u \in W(\Omega)$, then u is also the unique good solution to this problem.

Proof. Let $L^k \in \mathcal{L}(\lambda, \Lambda, \Omega)$, $k = 1, 2, \dots$, with coefficients a_{ij}^k that are smooth in $\overline{\Omega}$, $L^k \rightarrow L$ as $k \rightarrow \infty$ and let $\{u^k\}$ be a sequence of solutions to the problems (1.5). Then

$$\begin{cases} L^k(u^k - u) = (L - L^k)u & \text{on } \Omega \\ u^k - u = 0 & \text{on } \partial_d\Omega \end{cases}$$

Since $u^k - u \in W$, by the A.B.P.K.T Theorem 2.1, we have

$$\sup_{\Omega} |u^k - u| \leq K\|(L - L^k)u\|_{L^N(\Omega)} = K \left(\int_{\Omega} \left| \sum_{i,j} (a_{ij} - a_{ij}^k) D_{i,j} u \right|^N dxdt \right)^{1/N}$$

with constant K independent of k . Since the argument in the integral converges to 0 a.e. in Ω as $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} \sup_{\Omega} |u^k - u| = 0.$$

This means that $u(z)$ is the only good solution to the problem (1.4). \square

Remark 3.3. Let $L, L^k \in \mathcal{L}$, $f, f^k \in L^N(\Omega)$, $g \in C(\partial_d\Omega)$, $L^k \rightarrow L$ and $f^k \rightarrow f$ a.e. (meaning the coefficients a_{ij}^k of L^k converge a.e. to a_{ij} of L as in Definition 1.1) in Ω and let $u^k \in C(\overline{\Omega})$ be good solutions to

$$(3.5) \quad \begin{cases} L^k u^k = f^k & \text{on } \Omega \\ u^k = g & \text{on } \partial_d\Omega \end{cases}$$

then, there exists a subsequence of u^k uniformly convergent in $\overline{\Omega}$ to a function $u \in C(\overline{\Omega})$ good solution to (1.4).

Proof. The operators \overline{L}^k can be chosen with coefficients in $C^1(\overline{\Omega})$. As $\{\|f_n\|_{L^N(\Omega)}\}$ is a bounded sequence, by Theorem 2.1, the equicontinuity (3.4) of the u_k 's and Ascoli-Arzelá theorem, a subsequence of $\{u_k\}$ (still named $\{u_k\}$) converges uniformly in $\overline{\Omega}$ to a function $u \in C(\overline{\Omega})$. Since u^k 's are good solutions to the problem (3.5) there exist \overline{L}^k with $\overline{a}_{ij}^k \in C^1(\overline{\Omega})$, satisfying

$$\sum_{i,j} \|a_{ij}^k - \bar{a}_{ij}^k\|_{L(\Omega)} \leq 1/k$$

and functions $v^k \in W(\Omega) \cap C(\bar{\Omega})$, satisfying

$$(3.6) \quad \begin{cases} \bar{L}^k v^k = f^k & \text{on } \Omega \\ v^k = g & \text{on } \partial_d \Omega \end{cases}$$

and $\sup_{\Omega} |v^k - u^k| \leq 1/k$; as $\bar{L}^k \rightarrow L$ and (a subsequence of) $v^k \rightarrow u$ uniformly in $\bar{\Omega}$.

Consider now functions $w^k \in W(\Omega) \cap C(\bar{\Omega})$ such that

$$(3.7) \quad \begin{cases} \bar{L}^k w^k = f & \text{on } \Omega \\ w^k = g & \text{on } \partial_d \Omega \end{cases}$$

$$(3.8) \quad \begin{cases} \bar{L}^k (w^k - v^k) = f - f^k & \text{on } \Omega \\ w^k - v^k = 0 & \text{on } \partial_d \Omega; \end{cases}$$

by A.B.P.K.T Theorem 2.1 we have

$$\sup_{\Omega} |w^k - v^k| \leq K \|f - f^k\|_{L^N}$$

As a consequence $w^k \rightarrow u$ uniformly in $\bar{\Omega}$ and u is a good solution to problem (1.4) (with approximating operators \bar{L}^k). \square

From Remark 3.2 and Remark 3.3 if $u \in C(\bar{\Omega})$ is a good solution to problem (1.4) with $f \in C^1$ then u is a good solution to the same problem with $a_{ij}^k \in C^1(\bar{\Omega})$ and functions $u^k \in C^{2,\alpha}(\Omega) \cup C(\bar{\Omega})$, $0 < \alpha < 1$.

4. MAIN RESULT

Lemma 4.1. *Let $L \in \mathcal{L}$, $f \in C^1(\bar{\Omega})$. Let u_1, u_2, \dots, u_S be good solutions to (D.P.) (1.4) with $f \in C^1$. Then, there exists v , good solution to $Lv = f$ in Ω , $v|_{\partial_d \Omega} = g$, such that*

$$(4.1) \quad v(z) \geq \max\{u_1(z), \dots, u_S(z)\}$$

in $\bar{\Omega}$.

Proof. We may assume, by previous observations, that $u_l (l = 1, 2, \dots, S)$ is a good solution to D.P. (1.4) with L_l^k in the definition of good solution with coefficients in $C^1(\bar{\Omega})$; the corresponding $u_l^k \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$, $0 < \alpha < 1$. Moreover, as recalled above, if

$$F^k(\omega) = \max(L_1^k \omega, L_2^k \omega, \dots, L_S^k \omega)$$

the Dirichlet problem

$$(4.2) \quad \begin{cases} F^k(v^k) = f & \text{in } \Omega \\ v^k = g & \text{on } \partial_d \Omega \end{cases}$$

has a unique solution $v^k \in C^{2,\beta}(\Omega) \cup C(\bar{\Omega})$, $0 < \beta < 1$.

Notice that the equation $F^k(v^k) = f$ can also be written as $L^k v^k = f$, where $L^k v^k(z) = L_{l(z)}^k v^k(z)$ and the operators $L^k \in \mathcal{L}$ and have measurable coefficients. Moreover $L^k \rightarrow L$ a.e. in Ω . By A.B.P. or A.B.P.K.T Theorem 2.1 and Hölder estimate Theorem 2.2, the sequence v^k converges uniformly in Ω to v good solution to (1.4).

As $L_{l_1}^k v^k \leq f = L_{l_1}^k u_1^k$ and $u_1^k = v^k = g$ on the boundary, by the Maximum Principle: $v^k \geq u_1^k$. As $k \rightarrow \infty$ (4.1) follows. \square

Here is our main result:

Theorem 4.2. *Let L, f, g be as in Lemma 4.1, and let \mathcal{H} be the family of all good solutions to the Dirichlet Problem: $Lu = f$ in Ω and $u = g$ on the boundary $\partial_d \Omega$. Then*

$$\begin{aligned} U_0(z) &= \sup\{u(z) : u \in \mathcal{H}\} \\ V_0(z) &= \inf\{u(z) : u \in \mathcal{H}\} \end{aligned}$$

for $z \in \Omega$ are good solutions to the same problem.

Proof. Assume that \mathcal{H} has more than one element (see [13]). Let us show that U_0 exists and is continuous in $\overline{\Omega}$.

By ABPKT Theorem 2.1, U_0 is bounded in Ω ; $\forall z_0, z_1 \in \overline{\Omega}$ let $u^k \in \mathcal{H}$ sequence such that $u^k(z_0) \rightarrow U_0(z_0)$ and by the equicontinuity result, we have

$$u^k(z_0) - u^k(z_1) \leq \omega(d(z_0, z_1)),$$

where $\omega = \omega(\lambda, \Lambda, \Omega, N, g)$.

Since $U_0(z_1) \geq u^k(z_1)$, we have

$$U_0(z_0) - U_0(z_1) \leq U_0(z_0) - u^k(z_0) + u^k(z_0) - u^k(z_1) \leq U_0(z_0) - u^k(z_0) + \omega(d(z_0, z_1))$$

so, as $k \rightarrow \infty$, U_0 is continuous in $\overline{\Omega}$ with the same modulus of continuity ω as u_k . Assume for the moment $f \in C^1(\overline{\Omega})$. For every $\epsilon > 0$, there exist $z_l \in \overline{\Omega}$, $l = 1, 2, \dots, S$, so that $\overline{\Omega}$ is covered by sets $B_\epsilon(z_l)$ (defined in Section 2.) in which $\text{osc } U_0 \leq \epsilon$ and $\text{osc } u \leq \epsilon$ for every $u \in \mathcal{H}$.

Let now $u_1, u_2, \dots, u_S \in \mathcal{H}$ so that

$$(4.3) \quad |u_l(z_l) - U_0(z_l)| < \epsilon.$$

By Lemma 4.1, there exists $v_\epsilon \in \mathcal{H}$, such that $v_\epsilon \geq u_l$ in $\overline{\Omega}$ and $v_\epsilon \leq U_0$ in $\overline{\Omega}$. Moreover $U_0 - \epsilon \leq u_l(x_l) \leq v_\epsilon(x_l)$. Now choose $z \in \overline{\Omega}$ in one of the $B_\epsilon(z_l)$, thus:

$$\begin{aligned} 0 \leq U_0(z) - v_\epsilon(z) &= \\ [U_0(z) - U_0(z_l)] + [U_0(z_l) - u_l(z_l)] + [u_l(z_l) - v_\epsilon(z_l)] + [v_\epsilon(z_l) - v_\epsilon(z)] &= \\ A + B + C + D. \end{aligned}$$

The terms A and D are $\leq \epsilon$ by the oscillation property in $B_\epsilon(z_l)$, term B is $\leq \epsilon$ by (4.3), and term C is ≤ 0 . Then

$$(4.4) \quad 0 \leq U_0(z) - v_\epsilon(z) \leq 3\epsilon.$$

Therefore the family $v_\epsilon \in \mathcal{H}$ converges uniformly to U_0 in $\overline{\Omega}$, as $\epsilon \rightarrow 0$ and U_0 is a good solution to $Lu = f \in C^1(\overline{\Omega})$, $u = g$ on the boundary $\partial_d \Omega$.

For $f \in L^N$, let $f_\nu \in C^1(\overline{\Omega})$, $f_\nu u \rightarrow f$ as $\nu \rightarrow \infty$, \mathcal{H}_ν the family of good solutions to $Lu = f_\nu$ in Ω , $f = g$ on the boundary $\partial_d \Omega$. Then $U_\nu^0 = \sup \mathcal{H}_\nu$ are good solutions to the same problems, since the theorem is proved for $f_\nu \in C^1(\overline{\Omega})$. By Remark 3.3

up to a subsequence $u_n u$ converges uniformly to a function w_0 good solution to $Lu = f$ in Ω , $f = g$.

In order to show that $w_0 = \sup \mathcal{H}$ let $u \in \mathcal{H}$ and $u_\nu \in \mathcal{H}_\nu$ with same approximating sequence L^k in the definition of good solution (Definition 1.1). Now $u_\nu^0 \geq u_\nu$ implies $w_0 \geq u$ and since $w_0 \in \mathcal{H}$ we have proved the theorem. \square

The following result has been proved by Krylov in [8] for generalized Green's functions in the elliptic case.

Corollary 1. Let $L \in \mathcal{L}(\lambda, \Lambda, \Omega)$, $f \in L^N(\Omega)$, $g \in \mathcal{C}(\partial_d \Omega)$. Let \mathcal{H} , U_0 and V_0 be as in Theorem 4.2 and assume $U_0 \neq V_0$. Then for every $\theta \in (0, 1)$, we have that $tU_0(z) + (1 - \theta)V_0(z) \in \mathcal{H}$.

Proof. We'll prove the result for $f \in \mathcal{C}^1(\overline{\Omega})$ as the result for general f follows as in the proof of Theorem 4.2.

Being U_0 and V_0 good solutions by previous theorem, let respectively $L_k^{(1)}, u_k^{(1)}$ and $L_k^{(2)}, u_k^{(2)}$ be approximating operators and functions that appear in the definition of good solution.

Let $u_k^+, u_k^- \in \mathcal{C}^{2,\alpha}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be the solutions to the Dirichlet problems:

$$\sup \left(L_k^{(1)} u_k^+, L_k^{(2)} u_k^+ \right) = f \quad \text{in } \Omega \quad u_k^+|_{\partial_d \Omega} = g$$

$$\inf \left(L_k^{(1)} u_k^-, L_k^{(2)} u_k^- \right) = f \quad \text{in } \Omega \quad u_k^-|_{\partial_d \Omega} = g$$

There exist $L_k^+, L_k^- \in \mathcal{L}(\lambda, \Lambda, \Omega)$ such that

$$L_k^+ u_k^+ = f, \quad L_k^- u_k^- = f \quad \text{in } \Omega.$$

As $L_k^{(1)} u_k^+ \leq f$, $L_k^{(2)} u_k^+ \leq f$, $L_k^{(1)} u_k^- \geq f$, $L_k^{(2)} u_k^- \geq f$ and on each point of Ω L_k^+ , L_k^- equal either $L_k^{(1)}$ or $L_k^{(2)}$ we get that

$$L_k^+ u_k^- \geq f, \quad L_k^- u_k^+ \leq f$$

Let $t \in (0, 1)$, $w_n = tu_k^+ + (1 - t)u_k^- \in \mathcal{C}^{2,\alpha}(\Omega) \cap \mathcal{C}(\overline{\Omega})$; the w_k 's satisfy

$$L_k^+ w_k = t f + (1 - t)L_k^+ u_k^- \geq f, \quad L_k^- w_k = t f + (1 - t)L_k^- u_k^+ \leq f.$$

Now let's define $L_k = \theta_k L_k^+ + (1 - \theta_k)L_k^-$, where

$$\theta_k = \begin{cases} 1 & \text{if } L_k^+ w_k = L_k^- w_k = f \\ \frac{f - L_k^- w_k}{L_k^+ w_k - L_k^- w_k} & \text{otherwise} \end{cases}$$

θ_k is a measurable function, $\theta_k \in [0, 1]$, $L_k \in \mathcal{L}(\lambda, \Lambda, \Omega)$ and $L_k w_k = f$ in Ω . Observing that $L_k^+, L_k^-, L_k \rightarrow L$ as $k \rightarrow \infty$, Remarks 3.2 and 3.3 imply that there exist subsequences still named $\{u_k^+\}, \{u_k^-\}$ that converge uniformly to elements of \mathcal{H} . By the maximum principle $u_k^+ \geq u_k^{(1)}$ and $u_k^- \leq u_k^{(2)}$ and therefore $u_k^+ \rightarrow U_0$ and $u_k^- \rightarrow V_0$ and $w_k \rightarrow tU_0 + (1 - t)V_0$ uniformly in Ω . This proves that the latest is in \mathcal{H} . \square

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