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On the connection between Nash equilibria and social optima in electric vehicle charging control games

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Abstract: We consider the problem of optimal charging of heterogeneous plug-in electric vehicles (PEVs). We approach the problem as a multi-agent game in the presence of constraints and formulate an auxiliary minimization program whose solution is shown to be the unique Nash equilibrium of the PEV charging control game, for any finite number of possibly heterogeneous agents. Assuming that the parameters defining the constraints of each vehicle are drawn randomly from a given distribution, we show that, as the number of agents tends to infinity, the value of the game achieved by the Nash equilibrium and the social optimum of the cooperative counterpart of the problem under study coincide for almost any choice of the random heterogeneity parameters. To the best of our knowledge, this result quantifies for the first time the asymptotic behaviour of the *price of anarchy* for this class of games. A numerical investigation to support our result is also provided.

Keywords: Electric vehicles, optimal charging control, optimization, fixed-point theory, mean-field games.

1. INTRODUCTION

The modernization of energy systems and the integration of increasing shares of renewable energy sources has been a major concern worldwide, and has induced an intense research activity towards this direction. However, the expected increase in the share of renewable energy sources will increase the degree of uncertainty in the system, thus posing stability and reliability challenges. To address these contemporary challenges, and to ensure a safe and uninterrupted energy systems operation, novel operational paradigms are needed, exploiting consumers' flexibility and deferrability properties. Electric vehicles, which obtain some or all of their energy from the electricity grid, will play a prominent role in this paradigm shift, since they not only contribute to pollution reduction, but also, by appropriately scheduling their charging status (e.g., charging over low demand/electricity price periods - "valley filling") or shifting their consumption in time, serve as virtual dynamic storage, contributing to the stability of the grid (see Rahman and Shrestha (1993); Denholm and Short (2006); Callaway and Hiskens (2011); Li et al. (2014) and references therein).

Achieving system-wide coordination and control of electric vehicles, and avoiding the severe consequences that a suboptimal design may have, becomes more challenging as the vehicles population size grows, Lemoine et al. (2008). Treating this problem from a social welfare perspective, a centralized solution would be preferable, since it would minimize the global cost. However, centralized computation of the vehicles charging strategies may be challenging both from a communication and a computational point of view, since not all vehicles may be willing to share private information related to their consumption patterns (encapsulated in their individual utility functions and constraints), and even if this was the case the size of the resulting problem might be prohibitive for computations in fleets of realistic size. Decentralized algorithms to calculate social welfare minimizing charging strategies while overcoming the aforementioned challenges can be found in Gan et al. (2013); Deori et al. (2016a), and are based on iterative algorithms and penalty methods.

In a more realistic set-up, vehicles are not concerned with social welfare minimizing paradigms, but act as selfish agents that seek to minimize their local charging energy cost, thus giving rise to multi-agent non-cooperative games. Computation of charging control strategies in such settings is typically approached using tools from aggregative and mean-field game theory. The main concern is the characterization of Nash equilibrium strategies associated with such games, and their computation in a decentralized fashion. The theoretical machinery for the stochastic, continuous-time version of such problems, but in the case where agents are not subject to constraints, is provided in Huang et al. (2007); Lasry and Lions (2007). The deterministic, discrete-time problem variant was investigated in Ma et al. (2013), and was further extended in Parise et al. (2014) to account for the presence of constraints. In Grammatico et al. (2016); Paccagnan et al. (2016) various iterative decentralized PEV charging algorithms are provided, and their convergence properties are analyzed using fixed-point theoretic tools, Berinde (2007).

One challenge associated with the aforementioned references is that there is no common awareness on how the resulting Nash equilibrium solution is related to the associated social welfare optimum, had the PEVs been

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acting in a cooperative manner, and how this is affected by vehicles' heterogeneity. A partial answer to this question was given in Ma et al. (2013) for the case of a homogeneous population of PEVs, that are, however, not subject to constraints. A more general treatment of this problem is proposed in the recent work of Li and Zhang (2016), where the authors show equivalence of Nash equilibria and social optima in terms of value at the limiting case of infinite agent populations. This is achieved by means of a primal-dual analysis, which for the case where the number of agents is finite results in approximate and not exact Nash equilibria.

In this paper we consider the problem of PEV charging control, where each vehicle is subject to possibly different constraints. We represent constraint heterogeneity by assuming that certain parameters in each vehicle constraints are drawn randomly from a given distribution. We show that, for any finite number of possibly heterogeneous agents, the PEV charging control game admits a unique Nash equilibrium, which is the minimizer of an auxiliary minimization program (Proposition 1). We further prove that as the number of agents tends to infinity the value of the game achieved by the Nash equilibrium and the social optimum of the cooperative counterpart of the problem under study coincide for almost any choice of the random heterogeneity parameters (Theorem 3). This result extends Ma et al. (2013) to the case of heterogeneous agents that are subject to constraints, without resorting to approximate Nash equilibria as in Li and Zhang (2016), and following a fundamentally different analysis that does not require primal-dual update steps. At the same time, to the best of our knowledge, this result quantifies for the first time the asymptotic behaviour of the *price of anarchy*, Koutsoupias and Papadimitriou (2016), for this class of games.

The rest of the paper unfolds as follows: Section 2 introduces the non-cooperative PEV charging control game under study, along with its social welfare minimization counterpart. Section 3 shows that, for any finite number of possibly heterogeneous agents, the associated game admits a unique Nash equilibrium, which is the social optimum of an auxiliary minimization program. In Section 4 we show that as the number of agents tends to infinity, the value of the game and the social welfare optimum of the original problem tend to coincide. This is also investigated in the numerical example in Section 5. Finally, Section 6 concludes the paper and provides some directions for future work.

2. ELECTRIC VEHICLE CHARGING CONTROL PROBLEM

Consider the problem of electric vehicle charging where we have m PEVs and each vehicle seeks to determine its charging profile along some discrete time horizon $[0, h-1]$ of arbitrary length $h \in \mathbb{N}$ so as to minimize its own charging cost.

The price of electricity depends on the total demand and we assume that it is given by

$$c^t = p^t \sum_{j \in I} x^{jt}, t \in H, \tag{1}$$

where $I = \{0, 1, \dots, m\}$ and $H = \{0, 1, \dots, h-1\}$ are the agent and time index sets, $x^{it} \in \mathbb{R}$ is the charging rate of vehicle i at time t, and $p^t > 0$ is an electricity price coefficient at time t. Note that the agent index set I is enlarged to include a virtual agent indexed by 0 to represent some additional fixed demand besides the one requested by the PEVs. This choice is to avoid cluttering notation in the subsequent derivations and makes the dependency of the cost c^t in (1) on the PEV demand affine due the presence of x^{0t}. The linear dependency of price with respect to the total demand models the fact that price depends on demand, and, agents/vehicles are price anticipating authorities, anticipating their consumption to have an effect on the electricity price (see Gharesifard et al. (2016) for further elaboration).

Each agent $i \in I$ optimizes its charging profile subject to the following constraints

$$\sum_{t \in H} x^{it} = \gamma^i \tag{2}$$

$$x^{it} \in [\underline{x}^{it}, \overline{x}^{it}], \text{ for all } t \in H \tag{3}$$

where constraint (2) represents a prescribed charging level $\gamma^i \in \mathbb{R}$, $\gamma^i > 0$, to be reached by vehicle i at the end of the considered time horizon H, whereas (3) imposes minimum ($\underline{x}^{it} \in \mathbb{R}$, $\underline{x}^{it} \geq 0$) and maximum ($\overline{x}^{it} \in \mathbb{R}$, $\overline{x}^{it} < \infty$) limits, respectively, on x^{it}. By appropriately choosing $\underline{x}^{0t}$, and setting $\overline{x}^{0t} = \underline{x}^{0t}$ and $\gamma^0 = \sum_{t \in H} \underline{x}^{0t}$, the charging strategy of the virtual agent 0 can match any given non-PEV demand profile.

For all $i \in I$, let $x^i = [x^{i0}, \dots, x^{i(h-1)}]^\top \in \mathbb{R}^{|H|}$, where $|\cdot|$ denotes the cardinality of its argument. Let also $f : \mathbb{R}^{|H|} \times \mathbb{R}^{(m)|H|} \to \mathbb{R}$ be such that, for all $i \in I$, for any $(x^i, x^{-i}) \in \mathbb{R}^{(m+1)|H|}$,

$$f(x^i, x^{-i}) = \sum_{t \in H} x^{it} \Big(p^t \sum_{\substack{j \in I \\ j \neq i}} x^{jt} + p^t x^{it} \Big), \tag{4}$$

where by the notation x^{-i} we imply a vector including the decision variables of all vehicles except vehicle i. Moreover, for all $i \in I$, let

$$X^i = \big\{ x^i \in \mathbb{R}^{|H|} : \sum_{t \in H} x^{it} = \gamma^i \text{ and } x^{it} \in [\underline{x}^{it}, \overline{x}^{it}], \text{ for all } t \in H \big\}, \tag{5}$$

denote the constraint set corresponding to vehicle i.

Then, each vehicle/agent i, $i \in I$, aims at determining a charging profile x^i that minimizes its pay-off function $f(x^i, x^{-i})$, as this is given by (4), which depends on its own decision vector x^i and on the other agents decision vector x^{-i}, subject to a local constraint $x^i \in X^i$, where X^i is defined in (5).

This non-cooperative behavior naturally gives rise to a gaming setting. We say that for all i, $i \in I$, the tuple (x^i, x^{-i}) is a Nash equilibrium of the game, if each agent i, given the strategies x^{-i} of the other agents, has no interest in changing its own strategy x^i. In other words, unilateral deviations in the agents' local strategies can not lead to an improvement in their pay-offs. This is formally stated in the following definition.

Definition 1. Consider a non-cooperative game where each agent has a pay-off function f and a constraint set X^i, $i \in I$. The set of Nash equilibria N of the game is given by

$$N = \{x \in X : f(x^i, x^{-i}) \leq f(\zeta^i, x^{-i}) \text{ for all } \zeta^i \in X^i, \ i \in I\}, \quad (6)$$

where $x = (x^0, \ldots, x^m)$ and $X = X^0 \times \ldots \times X^m$.

Since each agent has a pay-off function of the same structure, the resulting game is a potential game, Facchinei et al. (2011); Voorneveld (2000).

In Section 3 we show that set of Nash equilibria N coincides with the set of optima of an auxiliary cooperative optimization program involving all agents. This allows to use the decentralized algorithm of Deori et al. (2016b) to determine an element of N, i.e., a Nash equilibrium of the game of interest (see also Section 6 and Deori et al. (2017)). Interestingly, in the limiting case of an infinite population of agents, the auxiliary optimization program tends to the cooperative social welfare optimization problem

$$\mathcal{P}: \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} f(x^i, x^{-i}), \quad (7)$$

so that the m selfish PEVs will eventually choose their own charging profile so as to minimize the total charging cost for the entire fleet and hence achieve a *social optimum*.

We impose the following standing assumption to ensure feasibility of $\mathcal{P}$.

Assumption 1. Fix any $m \geq 1$, and let $\{\underline{x}^{it}, \overline{x}^{it}\}_{t \in H, i \in I}$, $\{\gamma^i > 0\}_{i \in I \setminus \{0\}}$, and $\gamma^0 \geq 0$. The feasibility sets X^i, $i \in I$, are nonempty and compact. Moreover, $p^t > 0$, for all $t \in H$.

Note that γ^0 is allowed to be zero to encode the case where there is no non-PEV demand. The second part of Assumption 1 is only needed for the proof of Theorem 3, but is naturally satisfied in situations of practical relevance. Denote the set of social optima M of $\mathcal{P}$ by

$$M = \arg \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} f(x^i, x^{-i}). \quad (8)$$

Note that (8) involves minimizing a continuous function (as an effect of being convex), over a compact set (due to Assumption 1). As such, the minimum is achieved due to Weierstrass' theorem (Bertsekas and Tsitsiklis, 1989, Proposition A.8, p. 625). Under a similar reasoning all other minimization problems defined in the sequel are well defined. More precisely, problem $\mathcal{P}$ involves a convex, quadratic minimization program.

3. NASH EQUILIBRIA AS SOCIAL OPTIMA OF AN AUXILIARY PROBLEM

We show that the set of Nash equilibria N defined in (6) of the game defined in Section 2 coincides with the set of optimizers of an auxiliary minimization program. To this end, for all $i \in I$, let

$$f_a(x^i) = \sum_{t \in H} p^t (x^{it})^2, \quad (9)$$

and consider the following minimization problem.

$$\mathcal{P}_a: \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} \left[f(x^i, x^{-i}) + f_a(x^i) \right]. \quad (10)$$

We then have the following proposition.

Proposition 1. Under Assumption 1, the set of Nash equilibria N, and the set of minimizers of $\mathcal{P}_a$ coincide, i.e.,

$$N = \arg \min_{\{x^i \in X^i\}_{i \in I}} \sum_{i \in I} \left[f(x^i, x^{-i}) + f_a(x^i) \right]. \quad (11)$$

Proof. Problem $\mathcal{P}_a$ is a centralized convex optimization program. By Corollary 1 of Deori et al. (2016b), the set of minimizers of $\mathcal{P}_a$ coincides with the set of fixed points of the mapping $T_a = (T_a^0, \ldots, T_a^m)$ (see also equation (5) in Deori et al. (2016b)), where, for all $i \in I$, for any $c > 0$,

$$T_a^i(x) = \arg \min_{z^i \in X^i} f(z^i, x^{-i}) + f_a(z^i) + \sum_{\substack{k \in I \\ k \neq i}} \left[f(x^k, (z^i, x^{-\{k,i\}})) + f_a(x^k) \right] + 2c\|z^i - x^i\|^2, \quad (12)$$

where $f(x^k, (z^i, x^{-\{k,i\}})) = \sum_{t \in H} x^{kt} \Big(p^t \sum_{\substack{k \in I \\ k \neq i}} x^{kt} + p^t z^{it} \Big)$, for all $k \in I$, $k \neq i$, encodes the fact that the decision vector z^i of agent i appears also in the terms with $k \neq i$. By $x^{-\{k,i\}}$ we mean the elements of x but for the ones corresponding to agents k and i. The interpretation of (12) is that we minimize the objective function in (10) with respect to the decision vector of agent i, where all the other decision vectors are fixed to the values included in vector x.

Therefore, and due to (9), we have that

$$T_a^i(x) = \arg \min_{z^i \in X^i} \Big[\sum_{t \in H} z^{it} \Big(p^t \sum_{\substack{j \in I \\ j \neq i}} x^{jt} + p^t z^{it} \Big) + \sum_{t \in H} z^{it} p^t \sum_{\substack{k \in I \\ k \neq i}} x^{jt} + \sum_{t \in H} p^t (z^{it})^2 \Big] + 2c\|z^i - x^i\|^2, \quad (13)$$

where all terms that have been dropped from the objective function in (12) do not depend on the decision vector z^i. Rearranging the terms, we obtain

$$T_a^i(x) = \arg \min_{z^i \in X^i} 2 \sum_{t \in H} z^{it} \Big(p^t \sum_{\substack{j \in I \\ j \neq i}} x^{jt} + p^t z^{it} \Big) + 2c\|z^i - x^i\|^2 = \arg \min_{z^i \in X^i} f(z^i, x^{-i}) + c\|z^i - x^i\|^2 \quad (14)$$

where in the second equality we used (4) and rescaled the objective by a factor of 2, since this does not affect the resulting minimizer.

By Corollary 1 of Deori et al. (2017) we have that the set of Nash equilibria N coincides with the set of fixed-points of T_a^i as this appears in the second equality in (14). On the same time, by Corollary 1 of Deori et al. (2016b), this set of fixed-points coincides with the set of minimizers of $\mathcal{P}_a$, thus concluding the proof. □

Note that the objective function in (10) is strictly convex due to the presence of the auxiliary term. Therefore, it admits a unique minimizer and, as a result of Proposition 1, the game of Section 2 admits a unique Nash equilibrium. The uniqueness of the Nash equilibrium is due to the equivalence result of (11), which relies on the particular structure of the objective functions in (4); for general convex pay-off functions (11) does not necessarily hold, and as a result N may not be a singleton.

The interpretation of (10) is that the auxiliary term acts like a variance penalty in regularization methods (similar to overfitting prevention in regression algorithms). As shown in the next section, the relative importance of this term becomes negligible as the number of agents increases, since the variance in the agents' decision vectors is reduced automatically as their number tends to infinity.

4. SOCIAL WELFARE ARISING FROM COMPETITIVENESS AND THE PRICE OF ANARCHY IN PEV GAMES

In this subsection we show that in the limiting case of an infinite population of agents, the optimal value of $\mathcal{P}_a$ approaches the one of $\mathcal{P}$. Under Proposition 1, this in turn implies that the Nash equilibrium of the game in Section 2 achieves the social welfare optimum. In other words, even if agents act in a non-cooperative manner, as their number increases, they tend to a social welfare optimizing behavior.

For our analysis we assume that the price coefficients $\{p^t\}_{t\in H}$ are deterministic known quantities, whereas the consumption level γ^i, $i \in I \setminus \{0\}$ in (2) and the upper and lower limits in (3) are random variables, extracted according to a given probability distribution. Since we will eventually consider an infinite population of agents, we impose the following assumption on the infinite sequence of random vectors $\{\gamma^i,\ \underline{x}^i,\ \overline{x}^i\}_{i\geq 1}$, where $\underline{x}^i = [\underline{x}^{i0}, \dots, \underline{x}^{i(h-1)}]$, $\overline{x}^i = [\overline{x}^{i0}, \dots, \overline{x}^{i(h-1)}]$.

Assumption 2. Let $\{\gamma^i,\ \underline{x}^i,\ \overline{x}^i\}_{i\geq 1}$ be an infinite sequence of random vectors on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ [1]. We assume that

(1) $\{\gamma^i,\ \underline{x}^i,\ \overline{x}^i\}_{i\geq 1}$ are independently and identically distributed (i.i.d.).
(2) $\{\gamma^i\}_{i\geq 1}$ are positive random vairables, while $\{\underline{x}^i,\ \overline{x}^i\}_{i\geq 1}$ are non-negative random vectors.
(3) For any $i \geq 1$, $\mathbb{E}[\gamma^i] < \infty$ and $\mathbb{E}[(\gamma^i)^2] < \infty$, where $\mathbb{E}[\cdot]$ denotes the expectation operator associated with the probability measure $\mathbb{P}$.

Note that we do not impose Assumption 2 for the virtual agent indexed by 0, since its demand is a deterministic quantity. As a consequence of the second part of Assumption 2, $\mathbb{E}[\gamma^i] > 0$, for any $i \geq 1$. Recall that in Assumption 1 we require feasibility of $\mathcal{P}$, $\mathcal{P}_a$ for a finite number of agents, hence the joint probability distribution of $\{\gamma^i,\ \underline{x}^i,\ \overline{x}^i\}_{i\geq 1}$ should be such that feasibility is ensured, namely the lower and upper bounds on charging rate must be compatible with the charging level γ^i to be reached. For the subsequent analysis we employ the following law of large numbers type of argument. Note that we will write that an event holds ($\mathbb{P}$-a.s.) when it holds with probability one with respect to $\mathbb{P}$.

Theorem 2. (Shiryaev (1995), Chapter IV, §3, Theorem 3). Let $\{y^i\}_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[|y^1|] < \infty$. For any given index set I with cardinality $|I| = m$, we then have that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i\in I} y^i = \mathbb{E}[y^1], \quad (\mathbb{P}\text{-a.s.}) \tag{15}$$

Consider any given index set H with $|H| = h$, $h \geq 1$, and let $y^t \in \mathbb{R}$, $y^t \geq 0$, for all $t \in H$. Let also $\bar{y} \in \mathbb{R}$ such that $\sum_{t\in H} y^t = \bar{y}$. Due to norm equivalence we have that $\frac{\|y\|_1}{\sqrt{h}} \leq \|y\|_2 \leq \|y\|_1$, where $y = (y^1, \dots, y^h)$. The latter implies that

$$\frac{\bar{y}^2}{h} \leq \sum_{t\in H} (y^t)^2 \leq \bar{y}^2, \tag{16}$$

which we will exploit in the proof of Theorem 3 below.

Denote by $F^m(x) = \sum_{i\in I} f(x^i, x^{-i})$ the objective function of $\mathcal{P}$ in (7), and let $F_a^m(x) = \sum_{i\in I} f_a(x^i)$. The objective function of $\mathcal{P}_a$ in (10) can be thus written as $F^m(x) + F_a^m(x)$. We introduce the superscript m in our notation to emphasize the fact that the relevant objective functions correspond to a set-up of m agents, since in the sequel we will let m tend to infinity. Notice that, for any $x \in X$,

$$F^m(x) = \sum_{t\in H} p^t \Big(\sum_{i\in I} x^{it}\Big)^2 \geq \underline{p} \sum_{t\in H} \Big(\sum_{i\in I} x^{it}\Big)^2 \geq \underline{p} \frac{\Big(\sum_{i\in I} \gamma^i\Big)^2}{h} > 0, \tag{17}$$

where the first inequality is obtained by setting $\underline{p} = \min_{t\in H} p^t$. To see the second inequality notice that $\sum_{t\in H}\big(\sum_{i\in I} x^{it}\big) = \sum_{i\in I}\big(\sum_{t\in H} x^{it}\big) = \sum_{i\in I} \gamma^i$. The desired inequality follows then by the left-hand side of (16) with $\sum_{i\in I} x^{it}$, $\sum_{i\in I} \gamma^i$ in place of y^t and $\bar{y}$, respectively. The last inequality is strict, due to the fact that $\underline{p} > 0$ (H is a finite set) as a result of the second part of Assumption 1, and the fact that $\gamma^i > 0$, for all $i \geq 1$, due to the first part of Assumption 2.

We are now in a position to state the following theorem, which is the main result of this section. We show that the value obtained by evaluating F^m at the optimal solution of $\mathcal{P}_a$, which due to Proposition 1 corresponds to the Nash equilibrium for the game of Section 2, tends to the social welfare optimum (optimal value of $\mathcal{P}$) as the number of agents tends to infinity. In other words, the ratio $\frac{F^m(x_a^\star)}{F^m(x^\star)}$ tends to 1 as the number of agents increases; this ratio is the so called *price of anarchy* in the computer science literature, Koutsoupias and Papadimitriou (2016), mostly focused on problems where the decision variables are discrete.

Theorem 3. Consider Assumptions 1 and 2. Let $x^\star \in X$, $x_a^\star \in X$ be any minimizer of $\mathcal{P}$ and $\mathcal{P}_a$, respectively. We then have that

$$\lim_{m\to\infty} \frac{F^m(x_a^\star)}{F^m(x^\star)} = 1, \quad (\mathbb{P}\text{-a.s.}), \tag{18}$$

where $F^m(x^\star) > 0$, i.e., the price of anarchy tends to 1.

Proof. Let $x, x_a \in X$ be feasible solutions, possibly different, of $\mathcal{P}$ and $\mathcal{P}_a$, respectively. By the definition of F^m and F_a^m, and since $F^m(x) > 0$ for any $x \in X$, we have that

$$\frac{F_a^m(x_a)}{F^m(x)} = \frac{\sum_{t\in H} p^t \sum_{i\in I} (x_a^{it})^2}{\sum_{t\in H} p^t \big(\sum_{i\in I} x^{it}\big)^2}. \tag{19}$$

[1] Note that if $\{\gamma^i,\ \underline{x}^i,\ \overline{x}^i\}$, $i \geq 1$, is defined on a given set, by $\mathbb{P}$ we denote the probability measure induced on the infinite cartesian product of these sets. For more details on the mathematical construction of such a measure the reader is referred to Vidyasagar (2003) (Section 2.4.1, p. 29).

Let $\overline{p} = \max_{t\in H} p^t$ and $\underline{p} = \min_{t\in H} p^t > 0$, where the inequality is strict due to Assumption 1. We then have that

$$\frac{F_a^m(x_a)}{F^m(x)} \le \frac{\overline{p}\sum_{t\in H}\sum_{i\in I}(x_a^{it})^2}{\underline{p}\sum_{t\in H}\left(\sum_{i\in I} x^{it}\right)^2}. \tag{20}$$

Since x_a^{it} is feasible for $\mathcal{P}_a$, we have that $\sum_{t\in H} x_a^{it} = \gamma^i$, for all $i \in I$. By the right-hand side of (16) with x^{it}, γ^i in place of y^t and $\bar{y}$, respectively, we obtain that for all $i \in I$,

$$\sum_{t\in H}(x_a^{it})^2 \le (\gamma^i)^2. \tag{21}$$

By the derivation of (17), we obtain that

$$\sum_{t\in H}\left(\sum_{i\in I} x^{it}\right)^2 \ge \frac{\left(\sum_{i\in I}\gamma^i\right)^2}{h}. \tag{22}$$

Employing (21), (22), and by exchanging the summation order in the numerator of (20), we have that

$$\frac{F_a^m(x_a)}{F^m(x)} \le \frac{\overline{p}h\sum_{i\in I}(\gamma^i)^2}{\underline{p}\left(\sum_{i\in I}\gamma^i\right)^2} = \frac{\overline{p}h\frac{\sum_{i\in I}(\gamma^i)^2}{m}}{\underline{p}m\left(\frac{\sum_{i\in I}\gamma^i}{m}\right)^2}. \tag{23}$$

Applying Theorem 2 twice, once with γ^i and once with $(\gamma^i)^2$ in place of y^i, we have that $\mathbb{P}$-a.s.

$$\lim_{m\to\infty}\frac{\sum_{i\in I}\gamma^i}{m} = \lim_{m\to\infty}\frac{\sum_{i\in I\setminus\{0\}}\gamma^i}{m} + \frac{\gamma^0}{m} = \mathbb{E}[\gamma^1] \tag{24}$$

$$\lim_{m\to\infty}\frac{\sum_{i\in I}(\gamma^i)^2}{m} = \lim_{m\to\infty}\frac{\sum_{i\in I\setminus\{0\}}(\gamma^i)^2}{m} + \frac{(\gamma^0)^2}{m} = \mathbb{E}[(\gamma^1)^2]$$

However, since $\mathbb{E}[\gamma^1] > 0$ and $\frac{\mathbb{E}[(\gamma^1)^2]}{\left(\mathbb{E}[\gamma^1]\right)^2} < \infty$ due to the third part of Assumption 2,

$$\lim_{m\to\infty}\frac{\overline{p}h\frac{\sum_{i\in I}(\gamma^i)^2}{m}}{\underline{p}m\left(\frac{\sum_{i\in I}\gamma^i}{m}\right)^2} = 0. \quad (\mathbb{P}\text{-a.s.}) \tag{25}$$

Therefore, since (23) holds for any $\{\gamma^i\}_{i\in I}$, we have that

$$\lim_{m\to\infty}\frac{F_a^m(x_a)}{F^m(x)} = 0, \quad (\mathbb{P}\text{-a.s.}) \tag{26}$$

Let now $x^\star, x_a^\star \in X$ denote an optimal solution of $\mathcal{P}$ and $\mathcal{P}_a$, respectively. By optimality of $x_a^\star$ we thus have that

$$F^m(x_a^\star) + F_a^m(x_a^\star) \le F^m(x^\star) + F_a^m(x^\star). \tag{27}$$

Rearranging the terms in (27), and since $F^m(x^\star) > 0$ (see discussion above Theorem 3), we obtain

$$\frac{F^m(x_a^\star) - F^m(x^\star)}{F^m(x^\star)} \le \frac{F_a^m(x^\star) - F_a^m(x_a^\star)}{F^m(x^\star)} \le \frac{F_a^m(x^\star)}{F^m(x^\star)}, \tag{28}$$

where the last inequality is due to the fact that $F_a^m(x_a^\star) \ge 0$. Since (26) holds for any feasible solutions $x, x_a \in X$, it will also hold for $x = x_a = x^\star$. Therefore, (26) and (28) lead to

$$\lim_{m\to\infty}\frac{F^m(x_a^\star) - F^m(x^\star)}{F^m(x^\star)} = 0, \quad (\mathbb{P}\text{-a.s.}) \tag{29}$$

which in turn implies (18), thus concluding the proof. □

From the proof of Theorem 3 it can be observed that (26) holds even if F, F_a, are evaluated at a possibly different feasible solution of $\mathcal{P}$ and $\mathcal{P}_a$, respectively. This implies that the auxiliary term included in $\mathcal{P}_a$ tends to be negligible compared to the objective function of $\mathcal{P}$ as the number of agents increases.

Informally speaking, the price of anarchy quantifies the gap between the social optimum and the value of the non-cooperative game; Theorem 3 implies that this gap tends to zero as the number of agents increases, i.e., even if agents act in a non-cooperative manner, as their number increases they tend to a social welfare optimizing behavior. Theorem 3 extends the results of Ma et al. (2013) that show asymptotic agreement between Nash equilibria and social optima for the case of homogeneous agents in the absence of constraints, to the more general set-up where agents are subject to heterogenous constraints.

Note that the aggregate quantity $\frac{1}{m}\sum_{i\in\mathcal{I}} x^{it}$ exhibits the same behaviour with the corresponding objective functions in Theorem 3, since the latter are strictly convex with respect to the agents' aggregate.

5. NUMERICAL EXAMPLE

To illustrate the result of Theorem 3, we performed a numerical investigation parametric with respect to the number of agents m. We considered a time horizon $h = 12$, and price coefficients $(p^0, \ldots, p^{h-1}) = (0.1, 1, 1.9, 2.8, 3.7, 4.6, 5.5, 6.4, 7.3, 8.2, 9.1, 10)$. For simplicity we assumed that the probability mass is concentrated to the lower and upper limits $\underline{x}^{it} = 0$ and $\overline{x}^{it} = 1$ for all $i \in I \setminus \{0\}$, $t \in H$ (assuming normalized charging rates) that are effectively being treated as deterministic, whereas the charging levels γ^i, $i \in I\setminus\{0\}$, were extracted in an i.i.d. fashion from a uniform distribution with support $[0, 12]$. We consider a zero non-PEV demand and we set the virtual agent 0 accordingly. For any m, we performed 100 multi-extractions of $\{\gamma^i\}_{i\ge1}$, and calculated the average (across these extractions) of the ratio $\frac{F^m(x_a^\star)-F^m(x^\star)}{F^m(x^\star)}$, where $x^\star, x_a^\star$ are minimizers of $\mathcal{P}$ and $\mathcal{P}_a$, respectively. Note that $x^\star, x_a^\star$ depend on the extracted $\{\gamma^i\}_{i\ge1}$; however, we suppress this dependence in the notation for simplicity. Figure 1 shows that this ratio tends to zero as the number of agents m increases, as this is expected from (18) (see also (29) in the proof of Theorem 3).

6. CONCLUDING REMARKS

In this paper we considered Plug-in Electric Vehicles (PEVs) charging control problems as a multi-agent game. Each vehicle/agent was subject to possibly different constraints, where constraint heterogeneity was represented by assuming that the parameters defining each vehicle constraints are drawn randomly from a given distribution. We formulated an auxiliary minimization program and showed that, for any finite number of possibly heterogeneous agents, its solution is the unique Nash equilibrium of the PEV charging control game. Moreover, we showed that, as the number of agents tends to infinity, the value of the game achieved by the Nash equilibrium and the optimum of the cooperative counterpart of the problem under study coincide for almost any choice of the random heterogeneity parameters, thus quantifying the price of anarchy for this class of games. This result is particularly interesting and somehow counterintuitive: agents are selfishly minimizing their own cost but they end up minimizing the overall cost

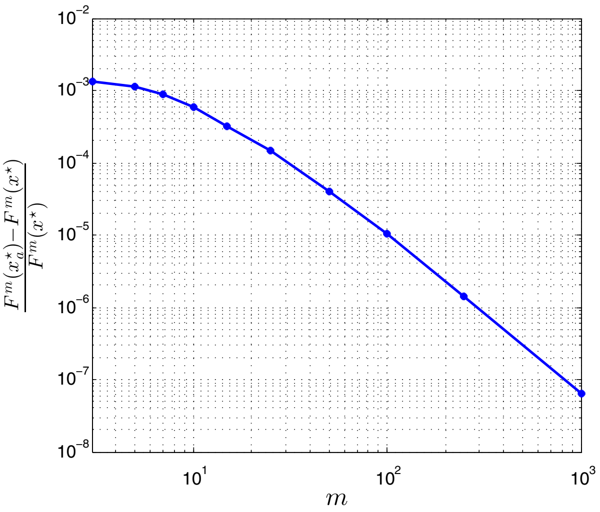

Fig. 1. Relative error $\frac{F^m(x_a^\star)-F^m(x^\star)}{F_m(x^\star)}$, averaged across 100 multi-extractions of $\{\gamma^i\}_{i\geq 1}$ from a uniform distribution and $\underline{x}^{it}=0$, $\overline{x}^{it}=1$ for all $i \in I \setminus \{0\}$, $t \in H$, as a function of the number of agents m. The error tends to zero as m increases.

and achieving the social welfare. This is due to the fact that each agent has a negligible effect on the price if the number of agents is large.

Current work concentrates towards extending the developed methodology to a more general class of non-cooperative games, with more involved pay-off functions. Moreover, Deori et al. (2017) shows that in the case where the agents' heterogeneity parameters follow a discrete probability distribution, agents can be abstracted in homogeneous groups, while the effect of heterogeneity averages out as their number tends to infinity. It should be also noted that the established equivalence between Nash equilibria and social minimizers of an auxiliary problem, opens the road for the use of the regularized Jacobi algorithm, constructed in our earlier work for decentralized optimization Deori et al. (2016b), for decentralized computation of Nash equilibria Deori et al. (2017).

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