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ABSTRACT

The reduction of gravity data from earth's surface to a reference surface where the proper Geodetic Boundary Value Problem can be solved is a central point in global geopotential modelling, and the analytical downward continuation commonly used, based on a first guess of the gravity field, gives rises to an iterative procedure whose convergence is strongly affected by terrain inclination and roughness. This problem is here formulated as a Dirichlet problem for the earth's surface, whose practical solution is not easy, due to the complexity of the boundary surface, but that is here solved in planar and in spherical approximation. The main result of this paper consists in the proof of a theorem stating that in planar and spherical approximation the first derivative operator is a contraction in the space of the functions continuous up to the boundary Σ (plane or sphere), and therefore the solution of the Dirichlet problem can really be obtained with an iterative procedure that is proved to be convergent. The importance of a good choice of the Taylor point where the first derivative operators are computed is also analysed.
Once that the data have been properly reduced, the corresponding anomalous potential T can be obtained by a convolution integral on the boundary with the simple Kernel $1/2\pi l$, in the planar case, or with the Stokes kernel, in the spherical case.

1 INTRODUCTION

One of the classical tasks of physical geodesy is to derive the anomalous potential $T(\underline{x})$ of the gravity field on the surface approximating the actual surface of the earth (the telluroid), by using gravity anomalies Δg and potential (or orthometric heights) as input data (Heiskanen, Moritz, 1990; Rapp, Pavlis, 1990). This problem is usually translated into a Boundary Value Problem (B.V.P.), namely Molodenskii's problem, and its solution leads, via Bruns's relation

$$\zeta = T/\gamma \quad , \tag{1,1}$$

to the necessary correction between normal and ellipsoidal heights nowadays so important after the advent of spatial positioning methods.
Despite many attempts, most of the practical solutions of the Molodenskii's problem rely on a so-called "change of boundary method" (cf. Sanso', 1993), namely on tricks allowing the reduction of our data to a "simple surface" (e.g. a sphere, a plane) and solving B.V.P.s for such a simple geometrical surfaces (Heck, 1991).

Even in the classical solution by series expansion of this problem, what we do is just a repeated application of a spherical solution, giving rise to a series, the convergence of which is conditioned by the inclination and the roughness of the terrain (Moritz, 1973; Holota, 1991).
After all, by "solution of Molodenskii's problem" it is today understood, with some approximation, a solution in which the free air anomalies are simply reduced by truncated analytical downward continuation (Sideris, 1987), the simplest form of which is (cf. Fig. 1.1)

$$\Delta g_Q = \Delta g_P - h_Q \frac{\partial}{\partial r} \Delta g_P \tag{1.2}$$

$$\Delta g_Q = -\frac{\partial T}{\partial r} - \frac{2}{r} T \Big|_Q \quad . \tag{1.3}$$

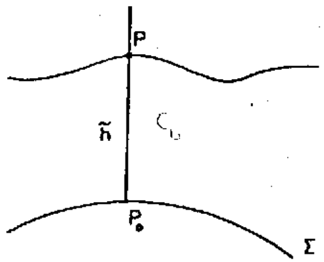

Fig. 1.1. Downward continuation on a spherical boundary.
s= earth's telluroid
Σ=reference sphere

Here, as we see, we are adopting a simple spherical approximation in the boundary operator and a first order continuation only.
Sometimes the problem is even reduced to a pure plane approximation formulation (cf. Fig. 1.2) with

$$\Delta g_Q = \Delta g_P - h_Q \frac{\partial}{\partial z} \Delta g_P \tag{1.4}$$

$$\Delta g_Q = -\frac{\partial T}{\partial z}\Big|_Q \quad . \tag{1.5}$$

Naturally, we could consider higher order expansions, and we shall shortly comment on them at the end of the paper, though we prefer here to concentrate on the first order only because this is by far the most important case.

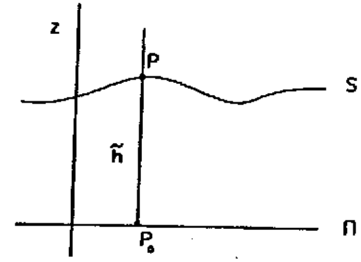

Fig. 1.2. Downward continuation on a reference plane.
s= earth's telluroid
Σ=reference plane

Let us first of all remark that, in the plane approximation, $\Delta g = -\frac{\partial T}{\partial z}$ is a harmonic function in the half space z>0, as well as that $r\Delta g = -r\frac{\partial T}{\partial r} - 2T$ is harmonic outside the earth's sphere Σ, when we adopt the spherical approximation.
In both cases therefore we are able to formulate the original problem as a simple Dirichlet problem, i.e. find u harmonic in space, attaining on S given values f(P), as the corresponding potential T can always be achieved by a subsequent integration along a vertical line (parallel to z in plane approximation, in radial direction in spherical approximation).
Yet, even if formulated as a simple Dirichlet problem, the practical solution is far from easy, due to the great complexity of the boundary S, which mirrors the actual topography of the earth.
So we accept, instead of the original Dirichlet problem, the approximate "change of boundary" formulation:

(a) Plane approximation: find u harmonic in z>0 and such that

$$u(Q) = f(P) - h_Q \frac{\partial u(P)}{\partial z} \tag{1.6}$$

or, since P and Q are in biunivocal correspondence and f is the given datum,

$$u(Q) + h_Q \frac{\partial u(P)}{\partial z} = f(P) = G(Q) \quad . \tag{1.7}$$

(b) Spherical approximation: using the radius of the reference sphere as the length unit, find u harmonic in r >1 such that

$$u(Q) + h_Q \frac{\partial u(P)}{\partial r} = f(P) = G(Q) \quad . \tag{1.8}$$

Let us note that in (1.7), (1.8) while Q spans the reference plane, respectively the reference sphere, P spans the telluroid S (of equation $z = h_Q$, respectively $r = 1+h_Q$). Equations (1.7), (1.8) therefore imply the contemporary use of boundary and non-boundary functionals of the harmonic function u; for this reason, we call the corresponding problems pseudo-boundary value problems.
A typical numerical solution of such problems, for instance in the plane case, is given in geodesy by starting with some model u_m to compute $\frac{\partial}{\partial z}u_m(P)$ and then getting a first solution from

$$u_1(Q) = G(Q) - h_Q \frac{\partial}{\partial z} u_m(P) \quad ; \tag{1.9}$$

when needed, a better solution is computed by

$$u_2(Q) = G(Q) - h_Q \frac{\partial}{\partial z} u_1(P) \tag{1.10}$$

and so on.
This procedure seems to be a simple iterative solution of (1.7) and it is the main purpose of this paper to show that this iterative scheme is well founded; it is a striking result that a space where the iteration converges is simply the space of continuous functions $C(\Sigma)$, contrary to most of the main results of potential theory for which the use of Hölder spaces is mandatory (Miranda, 1970; Sansò, Sacerdote, 1991).
To be definite let us specify from now on that we will denote with $C(\Sigma)$ the (Banach) space of harmonic functions continuous up to the boundary endowed with the Sup norm.
We just note, closing this paragraph, that once a solution of (a) or (b) is found, the corresponding anomalous potential T can be obtained by a convolution integral on the boundary, either with a simple kernel $\frac{1}{2\pi}l^{-1} = \frac{1}{2\pi}|\underline{x}-\underline{y}|^{-1}$, for the case (a), or with the Stokes kernel $S(\psi_{\underline{x},\underline{y}})$ for case (b).

2 ON THE IMPORTANCE OF THE CHOICE OF THE TAYLOR POINT P

Before deriving the main result of the paper, we want to illustrate how crucial is the choice of the Taylor point P in (1.7), (1.8).
We do that in spherical approximation by using an elementary case in which S is a sphere concentric with Σ, having equation (see Fig. 2.1)

$$r = 1 + h \qquad\qquad h = \bar{h} \quad constant \tag{2.1}$$

Then, considering the harmonic expansions of the known and of the unknown term

$$G(Q) = \sum_{nm} G_{nm} Y_{nm}(\sigma_Q) \quad , \tag{2.2}$$

$$u(P) = \sum_{nm} u_{nm} \frac{1}{r_P^{n+1}} Y_{nm}(P) \quad , \tag{2.3}$$

the solution of (1,8) becomes

$$\left[1 - \frac{\bar{h}(n+1)}{r_P^{n+2}}\right] u_{nm} = G_{nm} \quad . \tag{2.4}$$

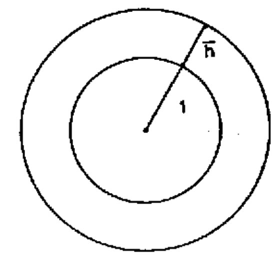

Fig. 2.1. The example of analytical continuation with two spheres.

Since $r_P = 1 + \bar{h}$, we see that

$$\frac{\bar{h}}{1+\bar{h}} < 1 \tag{2.5}$$

and that

$$\underset{\bar{h}}{Max} \frac{\bar{h}(n+1)}{(1+\bar{h})^{n+2}} < \frac{1}{\left(1+\frac{1}{n}\right)^n} < \frac{1}{2} \qquad (n \geq 1) \quad . \tag{2.6}$$

Accordingly, equation (2.4) is unconditionally solvable and u is a well defined function with the same regularity as G.
On the contrary, we can observe that if we defined a B.V.P. taking Q as the Taylor point, i.e.

$$u(Q) + h_Q \frac{\partial u(Q)}{\partial r} = G(Q) \qquad (h_Q = \bar{h}) \quad , \tag{2.7}$$

we would have obtained the spectral relation

$$\left[1 - (n+1)\bar{h}\right] u_{nm} = G_{nm} \quad ; \tag{2.8}$$

this relation shows that existence and uniqueness of the solution of the problem are no more guaranteed, though, when a solution exists, it happens to be more regular than G.
This counterexample has been useful to us in deciding to treat always problems with the Taylor point P on the external surface.

3 THE MAIN RESULT

We can state the main result of the paper in the form of a theorem:

Theorem 3.1 : the operators

(a) $$h_q \frac{\partial}{\partial z} u(P) \quad ; \quad P \equiv (x, y, h(x,y)) \qquad (3.1)$$

(b) $$h_q \frac{\partial}{\partial r} u(P) \quad ; \quad P \equiv (\varphi, \lambda, 1 + h(\varphi,\lambda)) \qquad (3.2)$$

are contractions, i.e. they can have operator norm smaller than 1 in the space $C(\Sigma)$ of harmonic functions continuous up to the boundary Σ, endowed with norm

$$\|u\| = \sup_{\Sigma} |u(Q)| \qquad (3.3)$$

Remark 3.1: As it is known, $C(\Sigma)$ is as a matter of fact a Banach space, as a consequence of the maximum principle.

Remark 3.2 : Another immediate consequence of the maximum principle is that an operator like $h\frac{\partial}{\partial z}\Big|_h$ is bounded in $C(\Sigma)$, for any Σ with bounded curvature; in fact for small h

$$\frac{\partial}{\partial z} u(P) = \frac{1}{\frac{4}{3}\pi h^3} \int_B \frac{\partial}{\partial z} u(P') dB_{P'} = \frac{1}{\frac{4}{3}\pi h^3} \left\{ \int_{S^+} u(P') dxdy - \int_{S^-} u(P') dxdy \right\}$$

with S^+, S^- the upper and lower hemisphere (cf. Fig. 3.1).

Since $|u(P)| \le \|u\|$ and S^+, S^- project onto a circle of radius h on the x,y plane, we find

$$\left| h \frac{\partial u}{\partial z} \right| \le \frac{3}{2} \|u\| \quad , \qquad (3.4)$$

which is certainly correct but not sufficiently tight for our purposes.

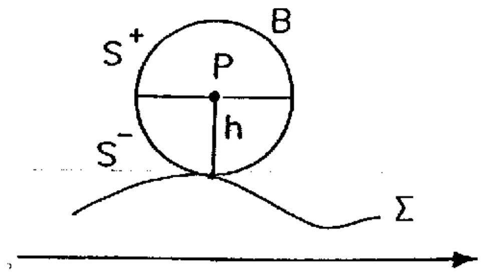

Fig. 3.1. Integration on upper and lower hemisphere

Let us now pass to prove (a). By using the explicit representation

$$u(z, \underline{\xi}) = \frac{1}{2\pi} \int \frac{z}{\left[|\underline{\xi} - \underline{\eta}|^2 + z^2 \right]^{3/2}} u(\underline{\eta}) d_2\eta \qquad (3.5)$$

($\underline{\xi}, \underline{\eta}$ are vectors on the x,y plane), one gets directly, with $\rho = |\underline{\xi} - \underline{\eta}|$ and $\rho = sz$,

$$\left| z \frac{\partial}{\partial z} u(z, \underline{\xi}) \right| = \frac{1}{2\pi} \left| \int \frac{z(\rho^2 - 2z^2)}{\left[\rho^2 + z^2\right]^{5/2}} u(\underline{\eta}) d_2\eta \right| \le \|u\| \int_0^{+\infty} \frac{|s^2 - 2|}{\left[s^2 + 1\right]^{5/2}} s\,ds = \frac{4}{3\sqrt{3}} \|u\| \quad . \qquad (3.6)$$

Since the second term of (3.6) is constant, we see that the operator norm of $z\frac{\partial}{\partial z}$, computed on any surface $z = h(x,y)$, satisfies the inequality

$$\left\| z \frac{\partial}{\partial z} \right\| \le \frac{4}{3\sqrt{3}} < 1 \quad ,$$

so that this operator is a contraction in $C(\Sigma)$.

We can now consider (b). Following a similar line, we start from the Poisson representation

$$u(r, \sigma) = \frac{(r^2 - 1)}{4\pi} \int \frac{u(\omega)}{\left[r^2 + 1 - 2r\cos\psi_{\sigma\omega} \right]^{3/2}} d\omega \qquad (3.7)$$

to obtain

$$\left| h \frac{\partial u}{\partial r} \right| \le \frac{h}{4\pi} \int_\sigma \frac{\left| 5r - r^3 - (r^2 + 3)\cos\psi \right|}{\left[r^2 + 1 - 2r\cos\psi \right]^{5/2}} |u(\omega)| d\omega \quad . \qquad (3.8)$$

Using in (3.8) the spherical surface element

$$d\omega = \sin\psi d\psi d\alpha \quad ,$$

we see that

$$\left|h\frac{\partial u}{\partial r}\right| \leq \|u\|\frac{h}{2}\int_0^{\pi}\frac{\left|5r - r^3 - (r^2+3)\cos\psi\right|}{\left[r^2+1-2r\cos\psi\right]^{3/2}}\operatorname{sen}\psi d\psi \quad ; \tag{3.9}$$

with the substitutions

$$\cos\psi = 1 - h^2\tau \quad , \quad h = r - 1 \quad ,$$
$$a = h^2 + 2h + 4 \quad , \quad b = h + 4 \quad ,$$

we recognise that

$$\left\|h\frac{\partial u}{\partial r}\right\| < \frac{\|u\|}{2}\int_0^{+\infty}\frac{|a\tau - b|}{\left[1+2r\tau\right]^{3/2}}d\tau = K(h)\|u\| = \frac{1}{2r^2}\left\{\frac{rb}{3} - \frac{a}{3} + \frac{2a\sqrt{a}}{3(2rb+a)^{1/2}}\right\}\|u\| \tag{3.10}$$

With some further algebra, we find for $K(h)$ the exact expression

$$K(h) = \frac{1}{2(h^2+2h+1)}\left\{h + \frac{2(h^2+2h+4)^{3/2}}{3\sqrt{3}(h+2)}\right\} \tag{3.11}$$

and it is not difficult to prove that $K(h) < 1$ for every $h \geq 0$ and in particular

$$K(0) = \frac{4}{3\sqrt{3}} \quad , \tag{3.12}$$

coinciding with the planar estimate, as it had to be.

So, also for the spherical case, the pseudo-boundary operator (3.2) is a contraction in $C(\Sigma)$. Therefore, Theorem 3.1 is completely proved and by standard theorems on contractions we can draw the following conclusions:

1) there is one and only one solution in $C(\Sigma)$ of pseudo-B.V.P.'s (1.7), (1.8);

2) the solution can be attained as limit of a simple iterative scheme

$$u_{n+1} = G - Bu_n \quad , \qquad u_0 = G \quad , \tag{3.13}$$

with B equal to $h\frac{\partial}{\partial z}$ or to $h\frac{\partial}{\partial r}$ depending on the case considered.

<u>Remark</u> 3.3: We conclude the paragraph by observing that once the iterative scheme (3.13) is known to be convergent we can go back to the original problem and write it directly in terms of the anomalous potential:

(a) In the plane case, we had $u = -\frac{\partial T}{\partial z}$, so

$$-\frac{\partial T_{n+1}(\underline{\xi},0)}{\partial z} = G + h(\underline{\xi})\frac{\partial^2 T_n(\underline{\xi},h(\underline{\xi}))}{\partial z^2} \tag{3.14}$$

which convoluted with the plane Stokes operator

$$(S\cdot u)(\underline{\xi}) = \frac{1}{2\pi}\int\frac{1}{|\underline{\xi}-\underline{\eta}|}u(\underline{\eta})d_2\eta \tag{3.15}$$

gives

$$T_{n+1}(\xi,0) = S\{G\} + S\left\{h\frac{\partial^2 T_n}{\partial z^2}\right\} \quad ; \tag{3.16}$$

To continue the iteration, it is necessary to know the vertical derivatives of T at height h; this is achieved by noting that the kernel $(2\pi\,|\underline{\xi} - \underline{\eta}|\,)^{-1}$ is continued harmonically in space by $(2\pi\,[\rho^2 + z^2]\,)^{-1}$, with $\rho = |\underline{\xi} - \underline{\eta}|$, and that, whatever is u,

$$h\frac{\partial^2}{\partial z^2}(Su)(\underline{\xi},z)\bigg|_{z=h} = \frac{h}{2\pi}\int\frac{2h^2-\rho^2}{\left[\rho^2+h^2\right]^{5/2}}u(\underline{\eta},0)d_2\eta \quad . \tag{3.17}$$

A warning is necessary here. Although (3.16) with (3.17) supply a closed iteration scheme that could be started from $T_0 = S\{G\}$, what we have proved up to now is not sufficient to claim its convergence; in fact the convergence in $C(\Sigma)$ of $-\frac{\partial T_n}{\partial z}$ does not imply a similar convergence for T_n.
However it is easy to see that if both $G(\xi)$, $h(\xi)$ have compact support then, the same being true for $-\frac{\partial T_n}{\partial z}$, we can in fact guarantee the uniform convergence of (3.16). This hypothesis is in fact not too restrictive and is usually accepted.

(b) Also for the spherical case, the iteration can be expressed in terms of T recalling that

$$u = r\Delta g = -r\frac{\partial T}{\partial r} - 2T \quad . \tag{3.18}$$

By using this relation, the iterative scheme becomes

$$-\frac{\partial T_{n+1}}{\partial r} - 2T_{n+1} = G + 3h\frac{\partial T_n}{\partial r}\bigg|_{r=1+h} + h(1+h)\frac{\partial^2 T_n}{\partial r^2}\bigg|_{r=1+h} \tag{3.19}$$

i.e. if we use the Stokes operator S (Heiskanen and Moritz, 1990)

$$T_{n+1}=S\{G\}+S\left\{3h\frac{\partial T_n}{\partial r}\bigg|_{r=1+h}\right\}+S\left\{h(1+h)\frac{\partial^2 T_n}{\partial r^2}\bigg|_{r=1+h}\right\} \quad . \tag{3.20}$$

As we see, in order to continue the iteration we need to compute $-\frac{\partial T_n}{\partial r}$, $-\frac{\partial^2 T_n}{\partial r^2}$ at height h, which can be done with the help of the relations

$$\frac{\partial}{\partial r}S\{u\}=\Pi u-\frac{2}{r}S\{u\} \quad , \tag{3.21}$$

$$\frac{\partial^2}{\partial r^2}S\{u\}=\left(\frac{\partial}{\partial r}\Pi\right)u+\frac{6}{r^2}S\{u\}-\frac{2}{r}\Pi u \quad , \tag{3.22}$$

where Π is the Poisson operator with kernel given in (3.7) and $\frac{\partial}{\partial r}\Pi$ is the radial derivative of Π, i.e. the operator with kernel

$$\frac{\partial}{\partial r}\Pi \approx \frac{1}{4\pi}\frac{5r-r^3-(r^2+3)\cos\psi}{\left[r^2+1-2r\cos\psi\right]^{5/2}} \qquad ;$$

relations (3.21), (3.22) hold throughout the space.
In the spherical case, the convergence of $\{T_n\}$ in $C(\Sigma)$ is guaranteed without further restrictions.

4 DISCUSSION

The idea of the paper is that practical implementations of solutions of Molodenskii's problem are in fact solutions of approximate pseudo-boundary value problems where the boundary is changed to a simple surface and a suitable pseudo-boundary operator provides the necessary downward continuation.
The case of first order downward continuation is completely analysed here; one could however ask what would be the characteristics of a problem with higher order terms.
For instance with second order downward continuation between two spheres, like the example discussed in §2, one would get the solution, analogous to (2.4),

$$\left[1-\frac{\bar{h}(n+1)}{r_P^{n+2}}+\frac{1}{2}\frac{\bar{h}^2(n+1)(n+2)}{r_P^{n+3}}\right]u_{nm}=G_{nm} \quad , \tag{4.1}$$

which is indeed unconditionally solvable.
It is interesting here to notice that, in this case, even taking the Taylor point P on the internal sphere ($r_P=1$), we would obtain conversion coefficients

$$\left[1-(n+1)\bar{h}+\frac{1}{2}(n+1)(n+2)\bar{h}^2\right]>\frac{1}{2}+\frac{1}{2}\left[1-(n+1)\bar{h}\right]^2$$

which are unconditionally positive too, showing that u always exists and is even significantly more regular than G.
So this case seems to be worthwhile of further investigation.
Another point of interest would be to generalise this analysis to an ellipsoidal geometry, because the use of the ellipsoid as reference surface is nowadays necessary to produce high degree, highly accurate, global solutions.

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