ELGRAM: an Ellipsoidal Gravity Model Manipulator

FERNANDO SANSÒ - GIOVANNA SONA
DIIAR, Sez. Rilevamento, Politecnico di Milano

Summary. - A gravity potential model manipulator consists of a software package that utilizes a set of coefficients to compute values of gravity potential (and its functionals) at gridded or sparse points on the earth surface, as a finite series of harmonic functions. Different kinds of global model manipulators are available in the I.Ge.S. (International Geoid Service) library (f477/f388 by OSU, Rapp, 1982; GEOCOL in GRAVSOFT package, Tscherning et al., 1991), all of them based on spherical harmonic expansions of the geopotential. In view of the actual requirements of more and more precise potential representation as well as of the foreseeable development of that area, the authors find appropriate to develop a similar manipulator working with ellipsoidal harmonic series of function. The software routines compute values of potential and its functionals at sparse points, using an ellipsoidal system of coordinates and ellipsoidal harmonic function series.

ELGRAM: UN MANIPOLATORE DI MODELLI DI GRAVITÀ ELLISSOIDALI.

Sommario. - Un manipolatore di modelli di potenziale di gravità consiste in un insieme di programmi che utilizza un set di coefficienti e serie finite di funzioni armoniche per il calcolo di valori del potenziale di gravità (e dei suoi funzionali) in punti della superficie terrestre (punti sparsi o organizzati in griglie regolari).

Diversi tipi di manipolatori di modelli globali sono reperibili nella libreria dell'International Geoid Service (I.Ge.S) (f477/f388 di Rapp, OSU, 1982; GEOCOL nel pacchetto softaware GRAVSOFT, di Tscherning et al., 1991) basati tutti sull'espansione del geopotenziale in armoniche sferiche. In vista delle attuali esigenze di rappresentazioni del potenziale sempre più precise gli autori ritengono opportuno sviluppare un manipolatore che utilizza serie di funzioni armoniche ellissoidali. Le routines sviluppate calcolano valori del potenziale e dei suoi funzionali usando un sistema di coordinate ellissoidali e serie di funzioni armoniche ellissoidali.

Keywords: ellipsoidal harmonic expansion, global geopotential model, gravity potential model.
Parole chiave: modello del potenziale di gravità, modello geopotenziale globale, sviluppo in armoniche ellissoidiche.

1. - INTRODUCTION

The study and development of new techniques for the computation of global models (i.e. of the gravitational potential expressed as a truncated series of harmonic functions) is still a topic of great interest, also in view of future dedicated missions like CHAMP, GRACE, GOCE, mainly aimed to the study of gravity field of the earth through measurements of quantities directly or indirectly related to gravity.

Moreover, the development of ultra-high models, up to N_{max} = 1500 (Wenzel, 1999), feasible today also thanks to the increased computational speed and data storage capacity of last generation processors, underline the need of more and more accurate approximations in global models computational techniques.

The commonly used representation of potential as a truncated series of spherical harmonics leads to global models that consist of set of coefficients c_{nm}, s_{nm} (OSU91, EGM96), that can be successively used by a spherical harmonics manipulator to compute values of the potential and its functionals (geoid undulation, gravity anomaly, deflection of the vertical) at any point of given coordinates (φ, λ, h).

The approximations involved both in the computation of the coefficients, from satellite and terrestrial measurements, and in the use of them in the synthesis of potential functionals make desirable the development of different computational techniques. The well known spherical approximation and the ellipsoidal corrections, necessary to resort to such an approximation (Cruz, 1986; Gleason, 1988; Heck, 1991; Rapp and Pavlis, 1990), can be avoided solving the GBVP in ellipsoidal coordinates and writing global geopotential models as series of ellipsoidal harmonics.

Actually, for constructing the OSU91 and the EGM96 global models the two representations have been combined, and the high degree and order coefficients, rigorously computed as ellipsoidal coefficients, are thoroughly transformed into spherical ones, to allow the successive use of spherical harmonic manipulators (Rapp and Pavlis, 1990; Jekeli, 1988; Gleason, 1989).

2. – ELLIPSOIDAL HARMONIC EXPANSIONS

Usually the anomalous gravitational potential of the earth is represented in terms of truncated series of spherical harmonic functions:

$$T = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} v_{nm}^{s} \left(\frac{R}{r}\right)^{n+1} Y_{nm}(\theta, \lambda).$$

On the other hand it is possible to expand it in ellipsoidal harmonic functions:

$$T = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} v_{nm}^{e} \frac{Q_{nm}\left(i\frac{u}{E}\right)}{Q_{nm}\left(i\frac{b}{E}\right)} Y_{nm}(\theta_e, \lambda).$$

It is a long time that scientists are studying global models manipulators working with ellipsoidal harmonic functions, but the presence of Legendre functions of second kind $Q_{nm}(i\,(u/E))$ have not allowed up to now an easy realization.

Two difficulties arise in using truncated series of ellipsoidal harmonics, due to the presence of Q_{nm} functions:

– Because of the dependence on both order n and degree m of the Q_{nm}, the computation of the terms containing the ellipsoidal height u requires a much longer computer time, compared with the simple term $(R/r)^{n+1}$. Spherical series can in fact be computed in two separate steps: first the angular part, and then the radial terms, independent on m.

– The computation of the Q_{nm} functions cannot be done using the simple recursive relations that hold identically for Q_{nm} and for P_{nm}, because in this case the recursive relations show an unstable behavior due to their imaginary argument (Sona, 1995).

The first problem is intrinsic in the representation, but with the more and more powerful processors available today it is a minor problem.

The second obstacle can be overcome computing the Q_{nm} as hypergeometric functions, as suggested in Thong and Grafarend, 1989 or in Sona, 1995,

$$Q_{nm}(z) = (-i)^{m} \frac{n!\,(n+m)!\,2^{2n+m+1}}{(2n+1)!} \frac{1}{z^{n+m+1}} (x^2+1)^{m/2}$$
$$F\left(\frac{1}{2}+m, n+m+1; \frac{3}{2}+n; \frac{1}{z^2}\right); \quad (z = iu/b),$$

this however makes furtherly heavier the computation. The values of Q_{nm} have to be evaluated adding several terms of the function F for each n, m and for each u.

Both difficulties are reduced introducing the «limit layer approach»: in most of the computations on the earth surface we can put

$$1 < \frac{u}{b} < 1 + 10^{-3} \quad (h < 6700\text{m})$$

thus getting the following approximate solution of Laplace equation in ellipsoidal coordinates:

$$\overline{Q}_{\text{nm}}(u) = \frac{Q_{\text{nm}}\left(i\frac{u}{E}\right)}{Q_{\text{nm}}\left(i\frac{b}{E}\right)} = \left(\frac{b}{u}\right)^{\text{n+1}}\left(\frac{u}{b}\right)^{e^2\frac{(\text{n+1})(\text{n+2})+\text{m}^2}{2\text{n+1}}} \tag{1}$$

This expression has been proved to be a very good approximation for $\overline{Q}_{\text{nm}}$, up to a very high order and degree ($n < 1.5\ 10^5$) and in the same time does not display any computational irregularity (singularity, instability...) (Sona, 1995).

The «limit layer» approximation is therefore a good solution to simplify and speed up the computation of $\overline{Q}_{\text{nm}}$ making thus sufficiently manageable a computer program working directly in ellipsoidal harmonics.

In order to compute from a set of ellipsoidal coefficients anomalous potential T and its functionals ζ and Δg we started from the usual relations:

$$\zeta = \frac{T}{\gamma}$$

$$\Delta g = \frac{\boldsymbol{\gamma}}{\gamma} \cdot \nabla T + \left(\frac{\partial \gamma}{\partial h}\right)\frac{T}{\gamma}.$$

Written in ellipsoidal coordinates, the projection of the gradient of T on the normal gravity direction becomes:

$$\frac{\boldsymbol{\gamma}}{\gamma} \cdot \nabla T = \frac{\gamma_{\text{u}}}{\gamma}\frac{\partial T}{\partial s_{\text{u}}} + \frac{\gamma_{\beta}}{\gamma}\frac{\partial T}{\partial s_{\beta}}$$

where γ_u and γ_β can be computed from the closed formula of normal potential U (Heiskanen and Moritz, 1995)

$$\gamma_{\text{u}} = \frac{1}{w}\left[\omega^2 u \cos^2\beta - \frac{kM}{u^2+E^2} + \frac{\omega^2 a^2}{2q(b)}\left(\sin^2\beta - \frac{1}{3}\right)\frac{dq(u)}{du}\right]$$

$$\gamma_{\beta} = \frac{\omega^2 \cos\beta \sin\beta}{w\sqrt{u^2+E^2}}\left[a^2\frac{q(u)}{q(b)} - (u^2+E^2)\right]$$

where

$$w = \sqrt{\frac{u^2 + E^2 \sin^2 \beta}{u^2 + E^2}}$$

and

$$q(u) = \frac{1}{2}\left[\left(1 + 3\frac{u^2}{E^2}\right) \operatorname{artg} \frac{E}{u} - 3\frac{u}{E}\right].$$

The derivatives of the anomalous potential T:

$$\frac{\partial T}{\partial s_u} = \frac{1}{w}\frac{\partial T}{\partial u}$$

$$\frac{\partial T}{\partial s_\beta} = \frac{1}{w\sqrt{u^2+E^2}}\frac{\partial T}{\partial \beta} = \frac{-1}{w\sqrt{u^2+E^2}}\frac{\partial T}{\partial \theta_e}$$

can be derived by writing the anomalous potential T with the expansion

$$T = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \overline{Q}_{nm}(u)\, P_{nm}(\theta_e)\,(c^e_{nm} \cos m\lambda + s^e_{nm} \sin m\lambda)$$

with $\overline{Q}_{n,m}$ given by (1).

Then one gets

$$\frac{\partial T}{\partial u} = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \frac{K_{nm}}{u} \overline{Q}_{nm}(u)\, P_{nm}(\theta_e)\,(c^e_{nm} \cos m\lambda + s^e_{nm} \sin m\lambda)$$

with

$$K_{nm} = e^2 \frac{(n+1)(n+2)+m^2}{2n+1} - (n+1),$$

and

$$\frac{\partial T}{\partial \theta_e} = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \overline{Q}_{nm}(u) \frac{\partial P_{nm}(\theta_e)}{\partial \theta_e}(c^e_{nm} \cos m\lambda + s^e_{nm} \sin m\lambda)$$

Therefore calling

$$S^{(0)} = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \overline{Q}_{nm}(u)\, P_{nm}(\theta_e)\,(c^e_{nm} \cos m\lambda + s^e_{nm} \sin m\lambda)$$

$$S^{(1)} = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \frac{K_{nm}}{u} \overline{Q}_{nm}(u)\, P_{nm}(\theta_e)\,(c^e_{nm} \cos m\lambda + s^e_{nm} \sin m\lambda)$$

$$S^{(2)} = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \overline{Q}_{nm}(u) \frac{\partial P_{nm}(\theta_e)}{\partial \theta_e}(c^e_{nm} \cos m\lambda + s^e_{nm} \sin m\lambda)$$

ζ and Δg can be written as

$$\zeta = \frac{1}{\gamma} S^{(0)} \tag{2}$$

$$\Delta g = \frac{1}{\gamma} [F_1 \cdot S^{(1)} + F_2 \cdot S^{(2)} + F_3 \cdot S^{(0)}]. \tag{3}$$

These relations (similar to those used in spherical manipulators) link values of ζ and Δg to ellipsoidal coefficients c^e_{nm}, s^e_{nm}.

3. – AN ELLIPSOIDAL HARMONIC MANIPULATOR: PROPOSED SOLUTION

The program developed up to now to compute ζ and Δg at points (φ_i, λ_i, h_i) from a set of ellipsoidal coefficients performs the following steps.

– geodetic coordinates (φ_i, λ_i, h_i) are transformed into the ellipsoidal coordinates (β_i, λ_i, u_i) solving the relations

$$(N+h)\cos\varphi = \sqrt{u^2+E^2}\,\cos\beta, \qquad \left(\frac{b^2}{a^2}N+h\right)\sin\varphi = u\cdot\sin\beta$$

for u and $\cos\beta$ (λ is the same in the two coordinate systems).

One has to stress that, as both u and β depend on both h and φ, the surfaces h = constant and φ = constant give not u = const or β = const, therefore a geodetic grid with $\Delta\varphi$ = const (typically used in global computations) does not correspond to an ellipsoidal grid with $\Delta\beta$ = const.

– the $\overline{Q}_{nm}$ functions are computed using (1). Given the slow variation of $\overline{Q}_{nm}$ with the ellipsoidal coordinate u and in order to avoid the computation of them for each value of u, it has been verified that a quadratic interpolation between three fixed heights gives $\overline{Q}_{nm}$ with sufficient approximation. Therefore a subroutine computes as a first step the $\overline{Q}_{nm}$ functions at two constant values u_1 = 1000 m, u_2 = 4000 m (the third value is fixed: for $u = b$ $\overline{Q}_{nm} = 1\ \forall\ n, m$) and the coefficients to be used in the interpolation. Another subroutine then interpolates at the point height u the proper values of $\overline{Q}_{n,m}\ \forall\ n, m$.

– the normalized Legendre functions P_{nm} (θ_e) and their derivatives $\partial P_{nm}(\theta_e)/\partial\theta_e$ are computed for the ellipsoidal value of $\theta_e = \pi/2 - \beta$ using recursive relations as in spherical harmonic manipulators.

– the normal gravity $\gamma = \gamma$ (φ, h) and its derivative are approximated by the relations (Heiskanen and Moritz, 1995):

$$\gamma = 9.78032677 \frac{(1+0.00193185135 \sin^2 \varphi)}{\sqrt{1-e^2 \sin^2 \varphi}} - h \cdot 0.3086 \cdot 10^{-5}$$

$$\frac{\partial \gamma}{\partial h} = -\frac{2\gamma_a}{a}(1+f_2 \sin^2 \varphi + f_4 \sin^4 \varphi)\cdot(1+f+m-2f \sin^2 \varphi)$$

– finally, the values of ζ and Δg are computed through (2) and (3).

4. – TEST OF THE NEW SOFTWARE

The target we had in mind in developing the software was to be able to compute geoid undulations to an accuracy level better than 1 cm and gravity anomalies to a level better than 0.1 mGal for models at least complete up to degree and order 360.

To check the results obtained with this new procedure we choose to compare values of ζ and Δg at several points φ_i, λ_i, h_i with those computed by a worldwide used spherical harmonic manipulator: f477 (Rapp, 1982), using the EGM96 global model ($N_{max} = 360$) (Lemoine et al., 1998). For this purpose we needed first to translate the spherical model into the corresponding ellipsoidal one, that means, to transform the set of spherical coefficients v^s_{nm}, into the corresponding set of ellipsoidal coefficients $v^e_{n'm}$.

Exact relations in fact exist between the two sets, that yield the value of $v^e_{n'm}$ as linear combination of v^s_{nm}, with $n = n', n'-2, n'-4, n'-2k$ .....:

$$v^e_{n'm} = S_{nm}\left(\frac{b}{E}\right)\sum_{k=0}^{W} \lambda_{knm} v^s_{n-2k,m} \tag{4}$$

where

$$\lambda_{knm} = \frac{(2n-2k)!n!}{(2n)!k!(n-k)!}\sqrt{\frac{(2n-4k+1)(n-m)!(n+m)!}{(2n+1)(n-2k-m)!(n-2k+m)!}}\left(\frac{E}{R}\right)^2$$

$$\bar{S}_{nm}\left(\frac{b}{E}\right) = \frac{\left(\frac{R}{E}\right)^{n+1}(2n)!}{2^n n!}\sqrt{\frac{\varepsilon_m (2n+1)}{(n-m)!(n+m)!}}\, Q_{nm}\left(i\frac{b}{E}\right) =$$

$$= \left(\left(i\frac{b}{E}\right)^2 - 1\right)\left(i\frac{R}{b}\right)^{n+1}\left(i\frac{E}{b}\right)^m F\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}; n+\frac{3}{2}; -\left(\frac{E}{b}\right)^2\right)$$

and F is the hypergeometric Gauss function (Jekeli, 1988; Petrovskaya, 2000).

It follows from relation (4) that, as each spherical coefficient v^s_{nm} is involved in the computation of all $v^e_{n'm}$ of the same order m and $n' \geq n$, $n' = n, n+2, n+4, \ldots$ .., $n+2k$, a global model, described by spherical coefficients up to order and degree

360, yields through (4) ellipsoidal coefficients that are significantly different from zero also for $n' > 360$.

Therefore, for the purpose to make the new computation comparable with the spherical one, a fictitious ellipsoidal model EGM96ell has been built up to degree $N^e_{max} = 400$ (of course, coefficients with $m > 360$ are missing in this model).

The same values of the fundamental constants a, e^2, ω^2, kM as adopted by f477 have been used to compare values of height anomaly ζ and gravity anomaly Δg computed at the same test points.

Here follows the results of comparisons performed along a quarter of the reference meridian $\lambda = 0°$ and along a quarter of the parallel $\varphi = 45°$, with a spacing of 0.5^0, at various heights.

The following plots display the satisfactory values obtained for both the absolute and relative differences of height anomalies:

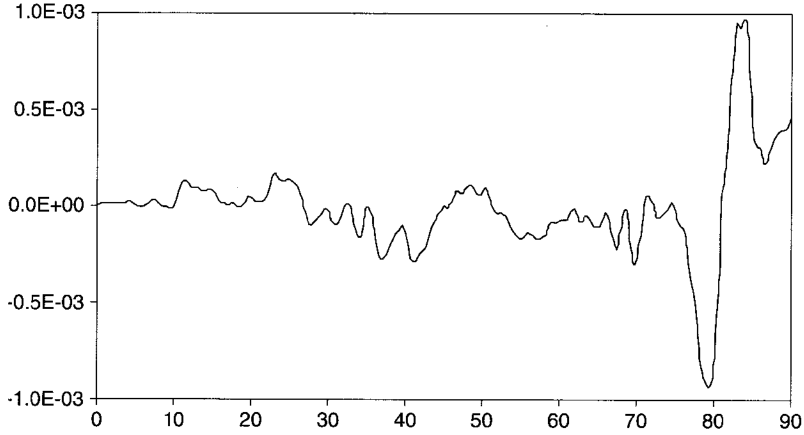

Fig. 1 – Height anomalies absolute differences (m) along a quarter of parallel $\varphi = 45°$.

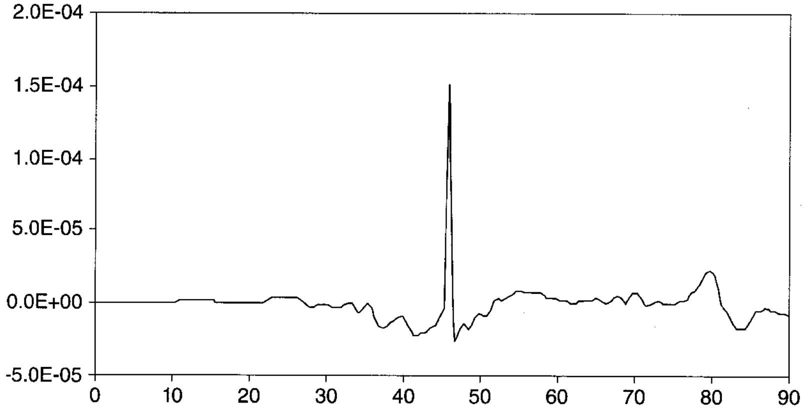

Fig. 2 – Height anomalies relative differences along a quarter of parallel $\varphi = 45°$.

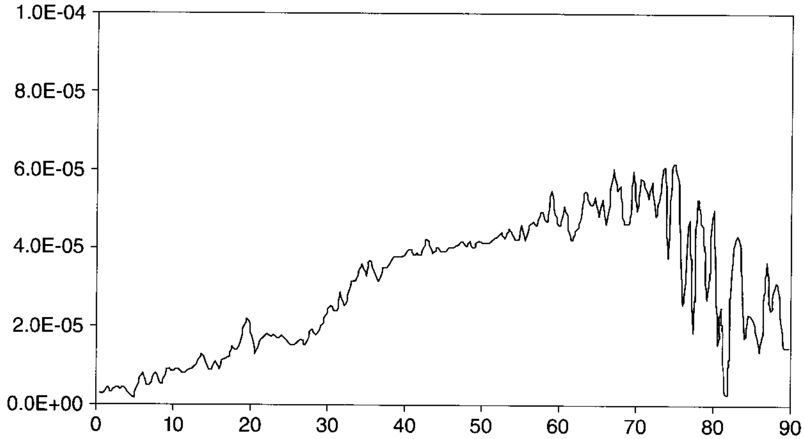

Fig. 3 – Height anomalies absolute differences (*m*) along a quarter of reference meridian λ = 0°.

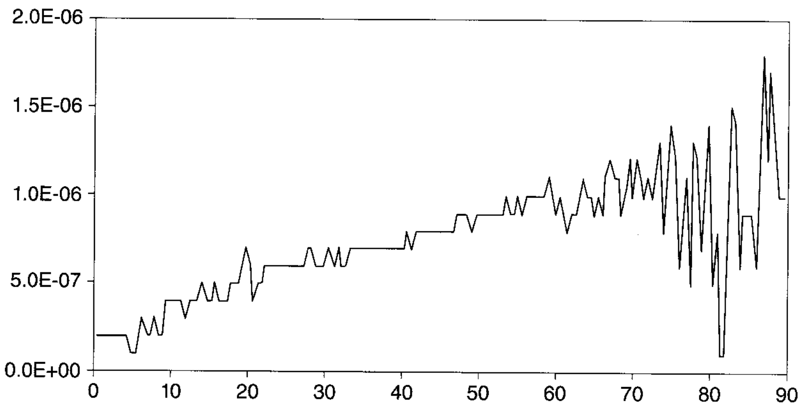

Fig. 4 – Height anomalies relative differences along a quarter of reference meridian λ = 0°.

The spike of relative errors at $\varphi = 45°$, $\lambda = 0°$ in Fig. 2 is explained by the very small value of ζ at that point.

On the contrary, when we came to Δg we didn't get small errors, and this is easy to understand remembering the different approximations used in the two manipulators leading to Δg values which are not comparable.

In order to find an appropriate check for something similar to Δg it was decided to compute at the same test points $|\nabla T|$ in spherical and ellipsoidal expansion, as this quantity is coordinate independent:

$$|\nabla T|^2 = \left(\frac{\partial T}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial T}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial T}{\partial \lambda}\right)^2$$

$$|\nabla T|^2 = \left(\frac{1}{w}\frac{\partial T}{\partial u}\right)^2 + \left(\frac{1}{w\sqrt{u^2+E^2}}\frac{\partial T}{\partial \beta}\right)^2 + \left(\frac{1}{\sqrt{u^2+E^2}\,\cos\beta}\frac{\partial T}{\partial \lambda}\right)^2$$

All the terms needed to compute $|\nabla T|$ are already present in f477 as part of computation of ζ, Δg, ξ, η, therefore only few changes in the source code have been added.

The horizontal derivatives in ellipsoidal coordinates had to be added too to the new software, which is in any way the basic step to include in it the computation of vertical deflections.

As it can be deduced from the following plots, the comparisons of $|\nabla T|$ give very satisfactory results.

It is then possible to conclude that the new ellipsoidal harmonic manipulator is really able to perform the synthesis of ζ and Δg in the topographic layer given an ellipsoidal geopotential model, that means, a set of coefficients complete up to degree and order N_{max}.

One small remark is that in this way the accuracy of the horizontal derivatives has been simultaneously checked, what opens the way to the inclusion of the computation of (ξ, η) into the new software.

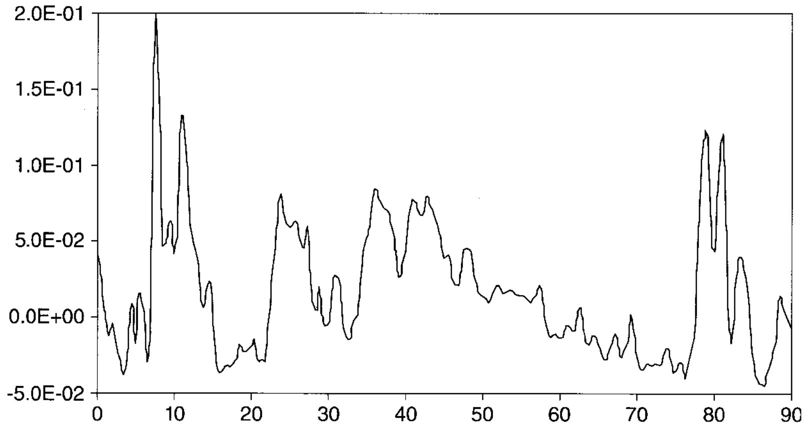

Fig. 5 – Absolute differences of $|\nabla T|$ (mGal) along a quarter of parallel $\varphi = 45°$.

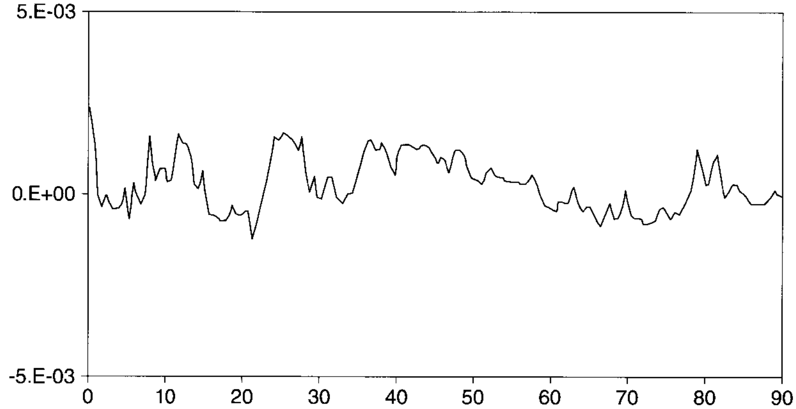

Fig. 6 – Relative differences of $|\nabla T|$ along a quarter of parallel $\varphi = 45°$.

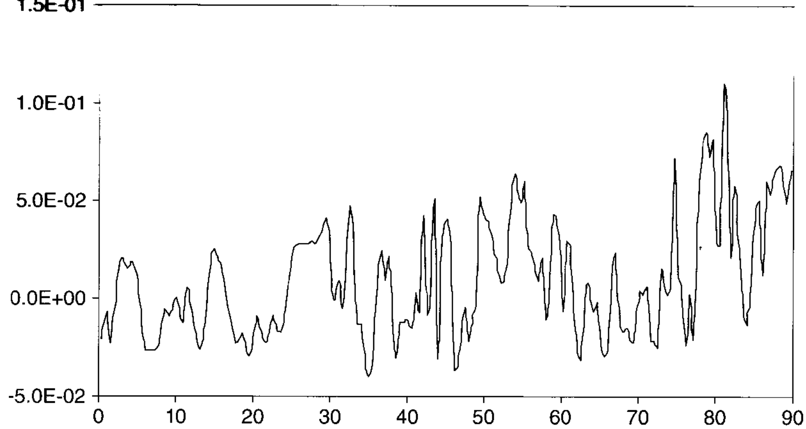

Fig. 7 – Absolute differences of $|\nabla T|$ (mGal) along a quarter of reference meridian $\lambda = 0°$.

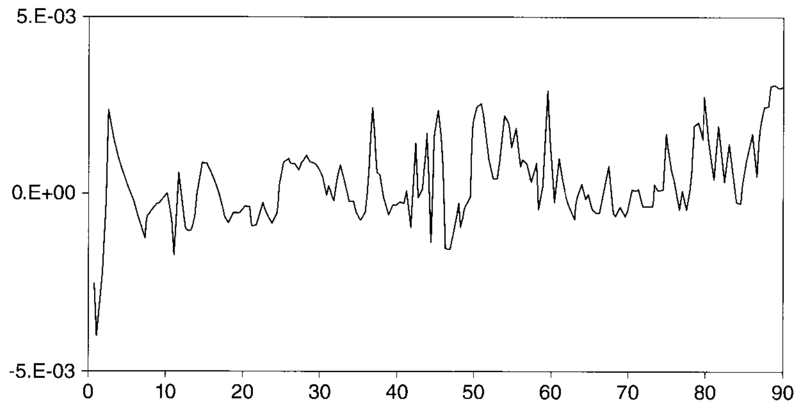

Fig. 8 – Relative differences of $|\nabla T|$ along a quarter of reference meridian $\varphi = 0°$.

5. – CONCLUSIONS AND FURTHER DEVELOPMENTS

As we hoped the check of the ellipsoidal gravity manipulator ELGRAM has got a very positive score; at present this software is able to compute at gridded (in ellipsoidal coordinates) or sparse points the two basic functionals of the anomalous potential ζ, Δg. To this, soon, the computation of the vertical deflections (ξ, η) will be added. By the way we already know that its approximation is good too.

Before we start spreading the software by the I.Ge.S. web, some additional work has still to be done; namely we have to push the testing for models up to order and degree 1440 and the routines have to be carefully formulated with the purpose of saving computer time, in particular FFT have to be introduced along parallels.

REFERENCES

J.Y. CRUZ (1986), *Ellipsoidal Corrections to Potential Coefficients obtained from Gravity Anomaly Data on the Ellipsoid.* Report n. 371, Dept.of Geodetic Science and Surveying, Ohio State University.

D.M. GLEASON (1988), *Comparing Ellipsoidal Corrections to the Transformation between the Geopotential Spherical and Ellipsoidal Spectrums.* Man. Geod. n. 13-2, pp. 114-129.

D.M. GLEASON (1989), *Some Notes on the Evaluation of Ellipsoidal and Spherical Harmonic Coefficients from Surface Data.* Man.Geod. n. 14, pp. 110-116.

B. HECK (1965), *On the linearized Boundary Value Problems of Physical Geodesy.* Report n. 407, Dept. of Geodetic Science and Surveying, Ohio State University.

W.A. HEISKANEN, H. MORITZ (1990), *Physical Geodesy.* Reprint, Institute of Physical Geodesy, Technical University, Graz, Austria.

C. JEKELI (1988), *The Exact Transformation between Ellipsoidal and Spherical Harmonic Expansions.* Man. Geod. n. 13, pp. 106-113.

F.G. LEMOINE, S.C. KENYON, J.K. FACTOR, R.G. TRIMMER, N.K. PAVLIS, D.S. CHINN, C.M. COX, S.M. KLOSKO, S.B. LUTHCKE, M.H. TORRENCE, Y.M. WANG, R.G. WILLIAMSON, E.C. PAVLIS, R.H. RAPP, T.R. OLSON (1998), *The Development of the Joint NASA GSFC and the NIMA Geopotential Model EGM96.* NASA/TP-1998-206861, Goddard Space Flight Center, Greenbelt.

M. PETROVSKAYA, A.N. VERSHKOV (2000), *Simplified relations between the ellipsoidal and spherical harmonic coefficients of the external earth's potential.* Bollettino di Geodesia e Scienze Affini, vol. LIX, n. 1, pp. 57-72.

H.R. RAPP (1982), *A Fortran Program for the computation of gravimetric quantities from high degree spherical harmonic expansions.* Report n. 334, Dept. of Geodetic Science and Surveying, The Ohio State University, Columbus, 1982.

H.R. RAPP, N.K. PAVLIS (1990), *The Development and Analysis of Geopotential Coefficient Models to Spherical Harmonic Degree 360.* Journal of Geophysical Research, Vol. 95, n. B13, pp. 21, 885-21, 911.

G. SONA (1995), *Numerical Problems in the Computation of Ellipsoidal Harmonics.* Journal of Geodesy, Vol. 70, pp. 117-126.

N.C. THONG, E.W. GRAFAREND (1989), *A Spheroidal Harmonic Model of the Terrestrial Gravitational Field.* Man. Geod. n. 14., pp. 285-304.

C.C. TSCHERNING, P. KNUDSEN, R. FORSBERG (1991), *Description of the GRAVSOFT package.* Geophysical Institute, University of Copenhagen, Technical Report.

G. WENZEL (1999), *Global Models of the Gravity Field of High and Ultra-High Resolution.* Lecture Notes for the International Geoid School, I.Ge.S.