On the structure of quantum Markov semigroups of weak coupling limit type

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Abstract. We discuss some recent results on the structure of quantum Markov semigroups of weak coupling limit type and their stationary states. In particular, we identify the minimal central projection in the fixed point algebra where they act in a trivial way and show, when they admit a single Bohr frequency, that all invariant states are convex combinations of equilibrium states and arbitrary states supported in the above minimal central projection.

1. Introduction

Quantum Markov Semigroups (QMS) or, in the physical terminology, quantum dynamical semigroups are the fundamental tool for mathematical modelling of open quantum systems interacting with external environments. The increasing interest in phenomena like decoherence [9, 10, 13, 25], coherent quantum computing and control [2], convergence to equilibrium [12, 14, 15, 7, 19] and entropy production [11, 18, 21, 22] motivates investigation on special features of QMSs.

The fundamental papers of Gorini-Kossakowski-Sudharshan [20] and Lindblad [24] in 1976 characterized the structure of generators uniformly continuous QMSs. Yet the emerging structure is still general and needs further specialization if one wants to study really relevant physical models (as, for instance, quantum harmonic oscillators [8, 15, 23] or QMS satisfying detailed balance conditions [5, 11, 8, 16, 17]). The powerful technique of the stochastic limit [3], allows one to deduce, from fundamental physical laws, generators of QMSs with special structures that are rich enough to include several relevant models but not too much to fit (almost) any Markov process (see e.g. [4]).

In [1] we began the study of invariant states of QMS of weak coupling limit type (WCLT) trying to single out special subclasses of invariant states with properties that are rich enough to go beyond the equilibrium situation and possibly allow explicit solutions. Following a suggestion emerging in that work, non-equilibrium states seem to be those which are a function of the system Hamiltonian H_S. In this framework, it would be interesting to answer the following questions:

1. under what conditions invariant densities commute with H_S?
2. under what conditions invariant densities are functions of H_S?
3. if an invariant state is a function of H_S, what function is it?

In this paper, we discuss some recent partial results on the structure of QMS of WCLT and their stationary states. In particular, we show how to identify the minimal central projection in the fixed point algebra where the QMS acts in trivial way. Moreover, we prove in Theorem 12, under some technical assumptions (see (**A**) Section 4), that all invariant states of QMSs with single Bohr frequency, which are a function of the system Hamiltonian, are convex combinations of equilibrium states and arbitrary states supported in the above minimal central projection.

Example 13 shows that the same conclusion does not hold for invariant states in the commutant of the system Hamiltonian.

2. QMSs of WCLT

Let h be a complex Hilbert space and $(\mathcal{T}_t)_{t \geq 0}$ be a uniformly continuous QMS acting on $\mathcal{B}(h)$, the von Neumann algebra of all bounded operators on h with identity operator $\mathbb{1}$. The generator $\mathcal{L}$ has the well-known Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form

$$\mathcal{L}(x) = \mathrm{i}[H, x] - \frac{1}{2} \sum_{\ell \geq 1} (L_\ell^* L_\ell x - 2 L_\ell^* x L_\ell + x L_\ell^* L_\ell)$$

where $H = H^* \in \mathcal{B}(h)$ and $(L_\ell)_{\ell \geq 1}$ is sequence of operators on h such that the series $\sum_{\ell \geq 1} L_\ell^* L_\ell$ is strongly convergent.

Generators of QMSs of WCLT (see [1, 3]) have the following further additional structure determined by a self-adjoint operator H_S, the system Hamiltonian, and an operator $D \in \mathcal{B}(h)$, the system factor of the system - environment interaction operator. Assume that H_S has purely point spectrum and write its spectral decomposition

$$H_S = \sum_{\epsilon_n \in \mathrm{Sp}(H_S)} \epsilon_n P_{\epsilon_n}$$

where $\epsilon_0 < \epsilon_1 < \ldots$ and P_{ϵ_n} is the spectral projection corresponding to the eigenvalue ϵ_n. We call *Bohr frequencies* all differences $\omega = \epsilon_n - \epsilon_m$ with ϵ_n, ϵ_m eigenvalues of H_S and denote by

$$B_+ = \{ \omega = \epsilon_n - \epsilon_m > 0 \mid \epsilon_n, \epsilon_m \in \mathrm{Sp}(H_S) \},$$

the set of all strictly positive Bohr frequencies. We consider a $D \in \mathcal{B}(h)$ and, with each $\omega \in B_+$, we associate the operators

$$
\begin{aligned}
D_\omega &= \sum_{\omega = \epsilon_n - \epsilon_m} P_{\epsilon_m} D P_{\epsilon_n}, & D_\omega^* &= \sum_{\omega = \epsilon_n - \epsilon_m} P_{\epsilon_n} D^* P_{\epsilon_m} \\
H_\omega &= \zeta_{-\omega} D_\omega^* D_\omega + \zeta_{+\omega} D_\omega D_\omega^*, & \zeta_{-\omega}, \zeta_{+\omega} &\in \mathbb{R}
\end{aligned}
$$

where $\beta :]0, +\infty[\to]0, +\infty[$ is a function and constants $\Gamma_{-\omega} \geq \Gamma_{+\omega} \geq 0$ given by

$$\Gamma_{-\omega} = \frac{c_\omega \, e^{\beta(\omega)\omega}}{e^{\beta(\omega)\omega} - 1}, \qquad \Gamma_{+\omega} = \frac{c_\omega}{e^{\beta(\omega)\omega} - 1}, \qquad c_\omega \geq 0.$$

Definition 1 *A (H_S, β)-weak coupling limit type (**WCLT**) Markov generator $\mathcal{L}$ on $\mathcal{B}(h)$ has the GKSL form given by:*

$$\mathcal{L}(x) = \sum_{\omega \in \hat{B}_+} \mathcal{L}_\omega(x), \tag{1}$$

where $\hat{B}_+ = \{ \omega \in B_+ \mid D_\omega \neq 0, \ |\Gamma_{-\omega}| + |\zeta_{-\omega}| + |\zeta_{+\omega}| \neq 0 \}$ and

$$\mathcal{L}_\omega(x) = \mathrm{i}[H_\omega, x] - \Gamma_{-\omega}\left(\frac{1}{2}\{D_\omega^* D_\omega, x\} - D_\omega^* x D_\omega\right) - \Gamma_{+\omega}\left(\frac{1}{2}\{D_\omega D_\omega^*, x\} - D_\omega x D_\omega^*\right).$$

In order the above $\mathcal{L}$ to be the GKSL generator of a norm continuous QMS, throughout the paper we assume that the series

$$\sum_{\omega \in \hat{B}_+} \Gamma_{-\omega} D_\omega^* D_\omega, \quad \sum_{\omega \in \hat{B}_+} \Gamma_{+\omega} D_\omega^* D_\omega, \quad \sum_{\omega \in \hat{B}_+} \zeta_{-\omega} D_\omega^* D_\omega, \quad \sum_{\omega \in \hat{B}_+} \zeta_{+\omega} D_\omega^* D_\omega,$$

are strongly convergent.

Note that, for all Bohr frequency ω, it may happen that either the commutator $\mathrm{i}[H_\omega, \cdot\,]$ or the dissipative part of $\mathcal{L}_\omega$ depending on $\Gamma_{\pm\omega}$ are zero according to the values of the constants $c_\omega, \zeta_{-\omega}, \zeta_{+\omega}$ but they can not be *both* equal to zero.

We now recall a useful property of generators of QMSs of WCLT.

Lemma 2 *The linear map $\mathcal{E}$ on $\mathcal{B}(\mathsf{h})$ defined by*

$$\mathcal{E}(x) = \sum_{n \geq 0} P_{\epsilon_n} x P_{\epsilon_n}$$

is a conditional expectation onto $\{H_S\}'$. Moreover

$$\mathcal{L} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{L}, \qquad \mathcal{L}_* \circ \mathcal{E} = \mathcal{E} \circ \mathcal{L}_*$$

Proof. Hint (see [1, 3]): check that $\mathcal{L} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{L} \circ \mathcal{E}$ and $\mathcal{L}_* \circ \mathcal{E} = \mathcal{E} \circ \mathcal{L}_* \circ \mathcal{E}$ then "dualize"

$$\mathrm{tr}\left(\sigma \mathcal{E}\left(\mathcal{L}(x)\right)\right) = \mathrm{tr}\left(\mathcal{L}_*\left(\mathcal{E}(\sigma)\right)(x)\right) = \mathrm{tr}\left(\mathcal{E}\left(\mathcal{L}_*\left(\mathcal{E}(\sigma)\right)\right)x\right) = \mathrm{tr}\left(\sigma \mathcal{E}\left(\mathcal{L}\left(\mathcal{E}(x)\right)\right)\right)$$

for all trace-class operator σ and all $x \in \mathcal{B}(\mathsf{h})$. □

In this paper we are concerned with the case where

(i) the system Hamiltonian H_S has a lowest energy state, also called ground state, which we will denote by ϵ_0,

(ii) the eigenspace associated with each eigenvalue ϵ_n is finite dimensional.

In order to answer questions raised in the introduction we begin by a simple example inspired by the two-photon creation and annihilation process discussed in [15]. It is reasonable to expect that the interaction operator D plays a key role in the structure of invariant states. This example gives a hint in this direction.

Example 3 Let $\mathsf{h} = \ell^2(\mathbb{N})$ with canonical orthonormal basis $(\epsilon_n)_{n \geq 0}$, and

$$H_S = \sum_{n \geq 0} n\left(|\epsilon_{2n}\rangle\langle\epsilon_{2n}| + |\epsilon_{2n+1}\rangle\langle\epsilon_{2n+1}|\right), \qquad D = \sum_{m \geq 1} |\epsilon_{2m-2}\rangle\langle\epsilon_{2m}|$$

Suppose that there is a single Bohr frequency $\omega = 1$ (or, in an equivalent way all $\zeta_{\pm h\omega}, \Gamma_{\pm h\omega}$ vanish for all $h \geq 2$) for and $P_{\epsilon_n} = |\epsilon_{2n}\rangle\langle\epsilon_{2n}| + |\epsilon_{2n+1}\rangle\langle\epsilon_{2n+1}|$ for all $n \geq 0$ so that $D_\omega = D$. Choosing constants $\Gamma_{-\omega} = 2, \Gamma_{+\omega} = 1$ and leaving $\zeta_{-\omega} = \zeta_{+\omega}$ arbitrary, the QMS $\mathcal{T}$ generated by (1) admits the faithful invariant state

$$\rho = \sum_{n \geq 0} 2^{-n-2} P_{\epsilon_n}.$$

However, it is not the unique invariant state because introducing the even and odd projections

$$p_e = \sum_{m \geq 0} |\epsilon_{2m}\rangle\langle\epsilon_{2m}|, \qquad p_o = \sum_{m \geq 0} |\epsilon_{2m+1}\rangle\langle\epsilon_{2m+1}|,$$

both commuting with H_S and D, one can easily see as in [15] that any invariant state has the form

$$\eta = \sigma p_o + (2\rho)p_e$$

where σ is an arbitrary density matrix supported in p_o and $2\rho = 2\rho p_e$ (the constant 2 is needed for normalization of the state ρ restricted to p_e).

Clearly, all invariant states commute with H_S but only ρ is a function of H_S. It is easy to see that the von Neumann algebra $\{D_\omega, D_\omega^*\}''$ generated by D_ω and D_ω^* is

$$\mathbb{C}p_o \oplus p_e \mathcal{B}(\mathsf{h})p_e$$

and p_o is the orthogonal projection onto $\ker(D_\omega) \cap \ker(D_\omega^*)$. As a consequence $P_{\epsilon_n} \notin \{D, D^*\}''$ for all n.

The above example undoubtedly highlights the role of subspaces of kernels of D_ω and D_ω^* and eigenspaces of H. Operators supported the intersection of these subspaces turn out to be fixed points of completely positive maps $\mathcal{T}_t$ and density matrices supported the intersection of these subspaces turn out to be invariant states.

The set of fixed points $\mathcal{F}(\mathcal{T})$ for completely positive maps $\mathcal{T}_t$, on the other hand, plays an important role in the study of invariant states; next section contains some results on its structure that are very useful in the study of our problems.

3. The set of fixed points $\mathcal{F}(\mathcal{T})$

The set of fixed points of $\mathcal{T}$ is a vector space which is norm-closed, weakly* closed defined as

$$\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = x \text{ for all } t \geq 0\}.$$

Clearly $\mathbb{1} \in \mathcal{F}(\mathcal{T})$ and $a \in \mathcal{F}(\mathcal{T})$ if and only if $a^* \in \mathcal{F}(\mathcal{T})$. The set $\mathcal{F}(\mathcal{T})$ in general is not an algebra (see e.g. [13] Example 2.1), however we have the following

Proposition 4 *If the QMS $\mathcal{T}$ admits a faithful normal invariant state then $\mathcal{F}(\mathcal{T})$ is an atomic von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$.*

Proof. For $x \in \mathcal{F}(\mathcal{T})$, by 2-positivity of maps $\mathcal{T}_t$, we have $\mathcal{T}_t(x^*x) \geq \mathcal{T}_t(x^*)\mathcal{T}_t(x) = x^*x$. Since ρ is an invariant state, we have $(\mathcal{T}_t(x^*x) - x^*x) = 0$. It follows that $\mathcal{T}_t(x^*x) = x^*x$ because ρ is faithful and $x^*x \in \mathcal{F}(\mathcal{T})$.

This shows that $\mathcal{F}(\mathcal{T})$ is a subalgebra of $\mathcal{B}(\mathsf{h})$ which is a von Neumann subalgebra because it is clearly weakly* closed. Recalling that it is the image of a normal conditional expectation by Theorem 1.1 of [19], it follows from a result due to Tomiyama that it is an atomic von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$. $\square$

We recall some properties of $\mathcal{F}(\mathcal{T})$ which we will use throughout the paper, see [13].

Lemma 5   *1) An orthogonal projection $p \in \mathcal{B}(\mathsf{h})$ belongs to $\mathcal{F}(\mathcal{T})$ if and only if it commutes with the operators L_ℓ, L_ℓ^* and H of any GKSL representation of $\mathcal{L}$.*

   2) If $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$, then it coincides with the commutant of the set of operators $\{L_\ell, L_\ell^*, H \mid \ell \geq 1\}$.

Proof. If p commutes with L_l and H then $\mathcal{L}(p) = 0$ and so $\mathcal{T}_t(p) = p$ for all $t \geq 0$. On the other hand, if $\mathcal{T}_t(p) = p$ for all $t \geq 0$, then $\mathcal{L}(p) = 0$. Therefore

$$0 = p^\perp \mathcal{L}(p) p^\perp = \sum_{l \geq 1} (pL_\ell p^\perp)^*(pL_\ell p^\perp) \Rightarrow pL_\ell p^\perp = 0.$$

Analogous computations with $\mathcal{L}(p^\perp) = \mathcal{L}(\mathbb{1} - p) = 0$ give us $p^\perp L_\ell p = 0$. Taking the adjoints it follows that $pL_\ell^* p^\perp = p^\perp L_\ell^* p = 0$, so p commutes with L_ℓ and L_ℓ^*. As a consequence $0 = \mathcal{L}(p) = i[H, p]$ so p commutes with H as well. This proves *1)*.

For *2)*, if $\mathcal{F}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$, then it is contained in its commutant since it is generated by its projections which by *1)* belong to $\{L_\ell, L_\ell^*, H : \ell \geq 1\}'$. Conversely, any x commuting with L_ℓ, L_ℓ^* and H satisfies $\mathcal{L}(x) = 0$ which implies $\mathcal{T}_t(x) = x$ for all $t \geq 0$. $\square$

Corollary 6 *Let ρ be a faithful $\mathcal{T}$-invariant state and p a minimal projection in $\mathcal{F}(\mathcal{T})$. If $\mathrm{tr}(\rho p) \neq 0$ then $\rho_p := \dfrac{p\rho p}{\mathrm{tr}(\rho p)}$ is the unique faithful invariant state of the irreducible QMS $\mathcal{T}^p$, i.e., the restriction of $\mathcal{T}$ to the algebra $p\mathcal{B}(\mathsf{h})p$.*

Proof. Since p commutes with L_ℓ, L_ℓ^* and H we have $\mathcal{T}_t(x) = \mathcal{T}_t(pxp) = p\mathcal{T}_t(x)p$ for all $x \in p\mathcal{B}(\mathsf{h})p$ and $t \geq 0$. Now, ρ_p is a $\mathcal{T}_t^p$-invariant state since

$$\mathrm{tr}(\rho_p \mathcal{T}_t^p(x)) = \frac{\mathrm{tr}(\rho p\mathcal{T}_t(pxp)p)}{\mathrm{tr}(\rho p)} = \frac{\mathrm{tr}(\rho\mathcal{T}_t(x))}{\mathrm{tr}(\rho p)} = \frac{\mathrm{tr}(\rho x)}{\mathrm{tr}(\rho p)} = \mathrm{tr}(\rho_p x).$$

Since p is a minimal projection in $\mathcal{F}(\mathcal{T})$, the QMS $\mathcal{T}^p$ is irreducible. The faithfulness of ρ_p in $p\mathcal{B}(\mathsf{h})p$ follows from the fact that if $pxp \geq 0$ for $x \in \mathcal{B}(\mathsf{h})$ then $0 = \mathrm{tr}(\rho_p x) = \mathrm{tr}(\rho pxp)/\mathrm{tr}(\rho p)$ implies $pxp = 0$ by the faithfulness of ρ. $\square$

Proposition 7 *If $\mathcal{T}$ is a QMS as in Definition 1 with a faithful invariant state then:*

1) *The only eigenvalue of each operator D_ω^* is 0;*

2) *The projection z_0 onto*

$$\bigoplus_{\lambda \in \mathrm{Sp}(H)} \left((\ker(H - \lambda\mathbb{1}) \bigcap_{\{\omega\, :\, \Gamma_{+\omega} > 0\}} (\ker(D_\omega) \cap \ker(D_\omega^*)) \right) \tag{2}$$

belongs to the center $\mathcal{Z}(\mathcal{F}(\mathcal{T}))$ of $\mathcal{F}(\mathcal{T})$;

3) *Every rank one projection in $\mathcal{F}(\mathcal{T})$ is a subprojection of z_0.*

Proof. Suppose $u \neq 0$ is an eigenvector of D_ω^* associated with an eigenvalue $\lambda \neq 0$. Recalling that $D_\omega^* = \sum_{n \geq 0} P_{\epsilon_{n+1}} L^* P_{\epsilon_n}$, we have that $\sum_{n \geq 0} P_{\epsilon_{n+1}} D^* P_{\epsilon_n} u = \lambda \sum_{n \geq 0} P_{\epsilon_n} u$. The existence of a lowest energy level and the orthogonality of $(P_{\epsilon_n})_{n \geq 0}$ implies that $\lambda P_{\epsilon_0} u = 0$, so $P_{\epsilon_0} u = 0$. Proceeding inductively we get that $P_{\epsilon_n} u = 0$ for all $n \geq 0$, thus $u = 0$ which is clearly a contradiction. We conclude that $\lambda = 0$ and *1)* is proved.

The projection z_0 onto the subspace (2) satisfies, by definition $Hz_0 = \lambda z_0$ for some $\lambda \in \mathrm{Sp}(H)$ and $D_\omega z_0 = D_\omega^* z_0 = 0$ for all ω such that $\Gamma_{+\omega} > 0$. Taking the adjoints it follows that $z_0 D_\omega = z_0 D_\omega^* = 0$ for all ω such that $\Gamma_{+\omega} > 0$. Thus z_0 commutes with H and all the operators L_ℓ, L_ℓ^* in a GKSL representation of the generator. By Lemma 5 then $z_0 \in \mathcal{F}(\mathcal{T})$.

To see that z_0 belongs to $\mathcal{Z}(\mathcal{F}(\mathcal{T}))$ notice that z_0 is the sum of spectral projections of

$$(H - \lambda\mathbb{1})^2 + \sum_{\{\omega | \Gamma_{+\omega} > 0\}} \kappa_\omega (D_\omega^* D_\omega + D_\omega D_\omega^*)$$

corresponding to the eigenvalue 0 for $\lambda \in \mathrm{Sp}(H)$ and some strictly positive constants κ_ω because the intersection of kernels in (2) is the kernel of the above positive operator. This proves *2)*.

For *3)*, let $p = |u\rangle\langle u|$, $u \in \mathsf{h}$, $\|u\| = 1$ be a projection in $\mathcal{F}(\mathcal{T})$. By Lemma 5, p commutes with all D_ω and D_ω^* such that $\Gamma_{+\omega} > 0$ which implies that $D_\omega u = \langle u, D_\omega u\rangle u$ and $D_\omega^* u = \langle u, D_\omega^* u\rangle u$. So u is an eigenvector of D_ω and D_ω^*; by *1)* $\langle u, D_\omega u\rangle = \langle u, D_\omega^* u\rangle = 0$ and $u \in \ker(D_\omega) \cap \ker(D_\omega^*)$. Now $\mathcal{L}(p) = i[H, |u\rangle\langle u|] = 0$ which implies that u is an eigenvector of H and so $|u\rangle\langle u|$ is a subprojection of z_0. $\square$

4. The case of a single Bohr frequency

In this section we show, under a technical assumption, that invariant states of QMS of WCLT with a single Bohr frequency ω, which are functions of the system Hamiltonian H_S, are convex combinations of states ρ satisfying the quantum detailed balance condition

$$\rho D_\omega = \frac{\Gamma_{+\omega}}{\Gamma_{-\omega}} D_\omega \rho, \qquad [H, \rho] = 0$$

and arbitrary states supported in z_0. Moreover (see Theorem 12), they are essentially convex combination of the canonical equilibrium state (3) defined here below. In this way, the can be viewed as bona fide equilibrium states for an open quantum system. The notion of detailed balance has several different quantum versions (see e.g. [16, 17] and the references therein). We have no room here for a thorough discussion; suffice it to mention that the above conditions are those that best reflect the quantum detailed condition in the present framework.

We assume that $\Gamma_{-\omega} \geq \Gamma_{+\omega} > 0$ and

$$Z := \sum_{n \geq 0} \left(\frac{\Gamma_{+\omega}}{\Gamma_{-\omega}} \right)^n \dim(P_{\epsilon_n}) < \infty.$$

Under this condition, one can easily see, as in [1] Theorem 4.1, that the faithful normal state

$$\rho_{\mathrm{e}} = Z^{-1} \sum_{n \geq 0} \left(\frac{\Gamma_{+\omega}}{\Gamma_{-\omega}} \right)^n P_{\epsilon_n} \tag{3}$$

is an invariant state and so, by Lemma 5 part 2, the fixed point algebra $\mathcal{F}(\mathcal{T})$ coincides with the commutant $\{D_\omega, D_\omega^*, H\}'$, i.e., since $H = \zeta_{-\omega} D_\omega^* D_\omega + \zeta_{+\omega} D_\omega D_\omega^*$, in this case, we have $\mathcal{F}(\mathcal{T}) = \{D_\omega, D_\omega^*\}'$.

Inspired by the analogy with classical Markov jump processes, we now introduce a version of "communication classes" allowing us to give an explicit description of $\mathcal{F}(\mathcal{T})$.

Definition 8 *A collection* $(n_0, n_1, \ldots, n_{2k})$, $\epsilon_{n_j} \in Sp(H)$, $k \geq 1$, *is a cycle of length* $2k$, *or a* $2k$-*cycle, rooted in level* n *if*

(i) $n_0 = n_{2k} = n$ *and* $n_j \neq n$ *for all* $1 \leq j \leq 2k - 1$.

(ii) $n_{j+1} = n_j \pm 1$ *for all* $0 \leq j \leq 2k - 1$.

We define the 0-*cycle as* (n) *and two* 1-*cycles as* $(n, n+1, n)$ *and* $(n, n-1, n)$, *all rooted in level* n.

With any $2k$-cycle rooted in level n we associate the following operator acting on $P_{\epsilon_n}\mathsf{h}$:

$$L(n, n_1, \ldots, n_{2k-1}, n) = L(n_{2k-1}, n)L(n_{2k-2}, n_{2k-1}) \cdots L(n_1, n_2)L(n, n_1),$$

$$\text{where} \qquad L(n, n \pm 1) = \begin{cases} P_{\epsilon_{n+1}} D_\omega^* P_{\epsilon_n}, & \text{if } + \\ P_{\epsilon_{n-1}} D_\omega P_{\epsilon_n}, & \text{if } - \end{cases} \quad \text{and } L(n) = P_{\epsilon_n}.$$

Let $\mathcal{A}_n$ be the algebra generated by the above operators associated with all cycles rooted in n. By construction $\mathcal{A}_n$ is canonically identified with a self-adjoint subalgebra of $\mathcal{B}(P_{\epsilon_n}\mathsf{h})$.

The commutant of $\mathcal{A}_n$, in $\mathcal{B}(P_{\epsilon_n}\mathsf{h})$, will not be abelian if z_0 has dimension bigger than 2 and commutes with P_{ϵ_n} because it contains the factor $z_0 \mathcal{B}(P_{\varepsilon_n}\mathsf{h})z_0$. There are several situations, however, in which the commutant of $z_0^\perp \mathcal{A}_n z_0^\perp$ is abelian as, for instance, in Example 3. For this reason, throughout this section we assume that:

$$(\mathbf{A}) \qquad [z_0, P_{\epsilon_n}] = 0 \quad \text{and the commutant in } \mathcal{B}(P_{\epsilon_n}\mathsf{h}) \text{ of } z_0^\perp \mathcal{A}_n z_0^\perp \text{ is abelian for all } n. \tag{4}$$

In this way there exists unique family $\{Q_j^{(n)}\}_j$ of mutually orthogonal minimal projections in the center of $\mathcal{A}_n$ such that its commutant in $\mathcal{B}(P_{\epsilon_n}\mathsf{h})$ is

$$z_0 P_{\epsilon_n} z_0 \oplus \mathbb{C}Q_j^{(n)}.$$

The maximal family of minimal $\mathcal{A}_n$-invariant subspaces of $P_{\epsilon_n}\mathsf{h}$, denoted by $\{S(n,i)\}_i$, is unique and each $Q_j^{(n)}$ is the orthogonal projection onto a certain subspace $S(n, i_n)$. Notice that each vector $u \in S(n, i_n)$ is $\mathcal{A}_n$-cyclic in $S(n, i_n)$ (see e.g. [6] Proposition 2.3.8) otherwise $\mathcal{A}_n u$ would be a $\mathcal{A}_n$-invariant subspace contained in $S(n, i_n)$ for each $u \in S(n, i_n)$, contradicting minimality. We refer to each $S(n, i_n)$ as an n-**level brick**.

Definition 9 *Let $S(n, i_n)$ and $S(m, i_m)$ respectively be n-level and m-level bricks, $n \neq m$.*

  1) We write $S(n \pm 1, j_{n\pm1}) \rightsquigarrow S(n, i_n)$, and say that $S(n, i_n)$ is accesible from the $(n \pm 1)$-level brick $S(n \pm 1, i_{n\pm1})$, if there exist $u_n \in S(n, i_n)$ and $v_{n\pm1} \in S(n \pm 1, j_{n\pm1})$ such that $\langle L(n, n \pm 1)u_n, v_{n\pm1}\rangle \neq 0$.

  2) $S(m, i_m) \rightsquigarrow S(n, i_n)$, or $S(n, i_n)$ is accesible from the m-level brick $S(m, i_m)$, if

$$S(n + l, i_{n+l}) \rightsquigarrow S(n + l + 1, i_{n+1+l}) \quad for\ l = 0, 1, \ldots, m - n - 1, \quad when\ n < m,\ or$$
$$S(n - l, i_{n+l}) \rightsquigarrow S(n - l - 1, i_{n+1+l}) \quad for\ l = 0, 1, \ldots, n - m - 1, \quad when\ m < n.$$

  3) The bricks $S(n, i_n)$ and $S(m, i_m)$ are said to communicate if $S(m, i_m) \rightsquigarrow S(n, i_n)$ and $S(n, i_n) \rightsquigarrow S(m, i_m)$. This relation is denoted by $S(n, i_n) \leftrightsquigarrow S(m, i_m)$.

Remark. The minimality of $\{S(n, i)\}_i$ as $\mathcal{A}_n$ invariant subspace, for every level n implies that a brick of a given level may communicate with at most one brick of a different level. Indeed, if $S(n, i) \leftrightsquigarrow S(n \pm 1, j)$ and $S(n, i) \leftrightsquigarrow S(n \pm 1, j')$, where $j \neq j'$, then both $S(n \pm 1, j) \leftrightsquigarrow S(n \pm 1, j')$ contradicting the fact that $S(n \pm 1, j)$ is not accessible from $S(n \pm 1, j')$. As a consequence the range of the restriction of $L(n, n \pm 1)$ to $S(n, i)$ is either contained in $\{0\}$ or, if $S(n, i) \leftrightsquigarrow S(n \pm 1, j)$, contained in $S(n \pm 1, j)$. Thus it is always true that $Q_{i_{n\pm1}}^{(n\pm1)}L(n, n \pm 1) = L(n, n \pm 1)Q_{i_n}^{(n)}$.

The communication relation is an equivalence relation. It is reflexive since any non-zero $u \in S(n, i_n)$ satisfies $\langle L(n)u, u\rangle \neq 0$. It is symmetric since $\langle L(n, n \pm 1)u_n, v_{n\pm1}\rangle = \langle u_n, L(n \pm 1, n)v_{n\pm1}\rangle$ implies

$$S(n, i_n) \rightsquigarrow S(n \pm 1, i_{n\pm1}) \Leftrightarrow S(n \pm 1, i_{n\pm1}) \rightsquigarrow S(n, i_i).$$

Transitivity follows from the definition.

Definition 10 *An equivalence class $\mathcal{C}$ induced by $\leftrightsquigarrow$ is called a communication class. The set of all classes will be denoted by $\mathscr{C}$. For each $\mathcal{C} \in \mathscr{C}$ let $Q_{\mathcal{C}}$ be the orthogonal projection onto $\bigoplus_{S(n,i_n)\in\mathcal{C}} S(n, i_n)$. Moreover denote $Q_f = z_0$.*

For a communication class $\mathcal{C}$ we denote by $M_{min}(\mathcal{C})$ (resp. $M_{max}(\mathcal{C})$) the lowest (resp. highest) level n containing a brick in $\mathcal{C}$ with the understanding $M_{max} + \infty$ if there are infinitely many levels with a brick in the class. To simplify the notation we will omit the dependence on the class when no confusion is possible. Therefore any class can be described as $\mathcal{C} = \{S(n, i_n) : M_{min} \leq n \leq M_{max}\}$ and the corresponding projection is $Q_{\mathcal{C}} = \sum_{n=M_{min}}^{M_{max}} Q_{i_n}^{(n)}$.

Theorem 11 *Projections $Q_{\mathcal{C}}$, for $\mathcal{C} \in \mathscr{C}$, are minimal in $\mathcal{F}(\mathcal{T})$. Moreover $Q_f + \sum_{\mathcal{C}\in\mathscr{C}} Q_{\mathcal{C}} = \mathbb{1}$.*
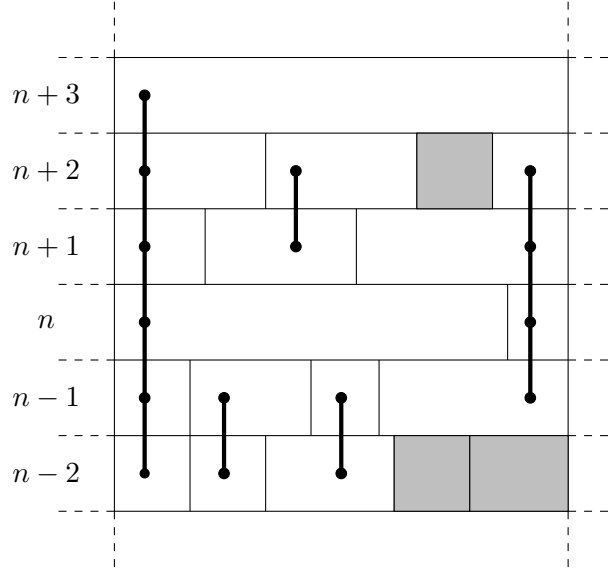
Figure 1. Bricks (rectangles) and communication classes (unions of rectangles joined by solid lines), shaded regions represent subspaces of z_0.

Proof. Recall that, under our assumptions $\mathcal{F}(\mathcal{T}) = \{D_\omega, D_\omega^*\}'$ and we already know from Proposition 7 part 2 that $Q_f \in \mathcal{F}(\mathcal{T})$.

We first check we that D_ω and D_ω^* commute with any $Q_\mathcal{C}$. Consider a communication class $\mathcal{C} = \{S(n, i_n) : M_{min} \leq n \leq M_{max}\}$. By the above remark we have

$$Q_\mathcal{C} D_\omega = \sum_{n=M_{min}}^{M_{max}} Q_{i_n}^{(n)} P_{\epsilon_n} D_\omega P_{\epsilon_{n+1}} = \sum_{n=M_{min}}^{M_{max}} P_{\epsilon_n} D_\omega P_{\epsilon_{n+1}} Q_{i_{n\pm 1}}^{(n\pm 1)} = D_\omega Q_\mathcal{C}$$

$$Q_\mathcal{C} D_\omega^* = \sum_{n=M_{min}}^{M_{max}} Q_{i_{n\pm 1}}^{(n\pm 1)} P_{\epsilon_{n+1}} D_\omega^* P_{\epsilon_n} = \sum_{n=M_{min}}^{M_{max}} P_{\epsilon_{n+1}} D_\omega^* P_{\epsilon_n} Q_{i_n}^{(n)} = D_\omega^* Q_\mathcal{C}.$$

This proves that $Q_\mathcal{C} \in \mathcal{F}(\mathcal{T})$.

We finally check that each projection $Q_\mathcal{C}$ is minimal in $\mathcal{F}(\mathcal{T})$. Let q be a projection in $\mathcal{F}(\mathcal{T})$ such that $q \leq Q_\mathcal{C}$. Note that, by $P_{\epsilon_n} q P_{\epsilon_n} = |qP_{\epsilon_n}|^2$, we have $P_{\epsilon_n} q P_{\epsilon_n} = 0$ if and only if $qP_{\epsilon_n} \neq 0$ or $P_{\epsilon_n} q \neq 0$. Let $N(q)$ be the set of indexes n for which $qP_{\epsilon_n} \neq 0$.

From $D_\omega D_\omega^* q = q D_\omega D_\omega^*$ and $D_\omega^* D_\omega q = q D_\omega^* D_\omega$, left and right multiplying by P_{ϵ_n} we, find

$$P_{\epsilon_n} q P_{\epsilon_n} D P_{\epsilon_{n+1}} D^* P_{\epsilon_n} = P_{\epsilon_n} D P_{\epsilon_{n+1}} D^* P_{\epsilon_n} q P_{\epsilon_n}, \quad P_{\epsilon_n} q P_{\epsilon_n} D^* P_{\epsilon_{n-1}} D P_{\epsilon_n} = P_{\epsilon_n} D^* P_{\epsilon_{n-1}} D P_{\epsilon_n} q P_{\epsilon_n}$$

for all n. In a similar way, starting from commutations with non-commutative monomials in D_ω and D_ω^*, one can see that $P_{\epsilon_n} q P_{\epsilon_n}$ commutes with generators of $\mathcal{A}_n$ for all n and so the self-adjoint operator $P_{\epsilon_n} q P_{\epsilon_n}$ belongs to the commutant of $\mathcal{A}_n$ in $\mathcal{B}(P_{\epsilon_n}\mathsf{h})$. As a consequence, all its spectral projections belong to the same commutant. The range of these spectral projections, however, is an $\mathcal{A}_n$-invariant subspace of $P_{\epsilon_n}\mathsf{h}$ contained in the range of some $Q_i^{(n)}$ because of the inequality $P_{\epsilon_n} q P_{\epsilon_n} \leq P_{\epsilon_n} Q_\mathcal{C} P_{\epsilon_n}$. It follows that, for all $n \in N(q)$, $P_{\epsilon_n} q P_{\epsilon_n} = P_{\epsilon_n} Q_\mathcal{C} P_{\epsilon_n} = Q_{i_n}^{(n)}$ for some i_n by minimality of $P_{\epsilon_n} Q_\mathcal{C} P_{\epsilon_n}$.

Moreover, if

$$Q_\mathcal{C} = \sum_{n=M_{min}}^{M_{max}} Q_{i_n}^{(n)},$$

then, for all $n \in N(q)$, we can not have $P_{\epsilon_n} q P_{\epsilon_n} \neq 0$ and $P_{\epsilon_{n+1}} q P_{\epsilon_{n+1}} = 0$, otherwise, since $Q_{\mathcal{C}} \leq z_0^{\perp}$, starting from $D_{\omega} q = q D_{\omega}$ and $D_{\omega}^* q = q D_{\omega}^*$ we would find a contradiction with

$$P_{\epsilon_{n+1}} q P_{\epsilon_{n+1}} D_{\omega}^* P_{\epsilon_n} = P_{\epsilon_{n+1}} D_{\omega}^* P_{\epsilon_n} q P_{\epsilon_n}, \qquad P_{\epsilon_n} q P_{\epsilon_n} D_{\omega} P_{\epsilon_{n+1}} = P_{\epsilon_n} D_{\omega} P_{\epsilon_{n+1}} q P_{\epsilon_{n+1}}.$$

It follows that

$$Q_{\mathcal{C}} = \sum_{n=M_{min}}^{M_{max}} Q_{i_n}^{(n)} = \sum_{n=M_{min}}^{M_{max}} P_{\epsilon_n} q P_{\epsilon_n}.$$

However,

$$Q_{\mathcal{C}} \geq q = \sum_{n \in N(q)} P_{\epsilon_n} q P_{\epsilon_n} + \sum_{n,m \in N(q), n \neq m} P_{\epsilon_n} q P_{\epsilon_n} = Q_{\mathcal{C}} + \sum_{n,m \in N(q), n \neq m} P_{\epsilon_n} q P_{\epsilon_n}$$

so that the double sum in the right-hand side is a negative self-adjoint operator with zero diagonal part and, as such, it must be zero. This proves that $Q_{\mathcal{C}} = q = \sum_{n \in N(q)} P_{\epsilon_n} q P_{\epsilon_n}$ and shows that $Q_{\mathcal{C}}$ is minimal. □

Theorem 12 *Let $\mathcal{L}$ be the generator of a QMS of WCLT as in Definition 1 with a single Bohr frequency. Suppose that the abelianess assumption ($\boldsymbol{A}$) holds and let ρ be an invariant state. The following statements hold:*

(i) For every communication class $\mathcal{C} \in \mathscr{C}$ such that $tr(\rho Q_{\mathcal{C}}) > 0$ we have

$$\frac{Q_{\mathcal{C}} \rho Q_{\mathcal{C}}}{tr(\rho Q_{\mathcal{C}})} = \frac{Q_{\mathcal{C}} \rho_e Q_{\mathcal{C}}}{tr(\rho_e Q_{\mathcal{C}})}$$

where ρ_e is the invariant state (3),

(ii) Every invariant state ρ which is a function of H_S, i.e. $\rho = \sum_{n \geq 0} \rho_n P_{\varepsilon_n}$ with $\rho_n \geq 0$, $\sum \rho_n = 1$ has the form

$$\rho = Q_f \rho Q_f + \sum_{\mathcal{C} \lambda_{\mathcal{C}} \in \mathscr{C}} \lambda_{\mathcal{C}} Q_{\mathcal{C}} \rho_e Q_{\mathcal{C}}$$

where

$$\lambda_{\mathcal{C}} = \frac{tr(\rho Q_{\mathcal{C}})}{tr(\rho_e Q_{\mathcal{C}})}.$$

Proof. (i) Follows immediately from Corollary 6 and uniqueness of invariant states of irreducible QMSs.

(ii) Since projections $Q_f, Q_{\mathcal{C}}$ commute with projections P_{ε_n}, the density matrix ρ commutes with projections $Q_f, Q_{\mathcal{C}}$. It follows then from (i)

$$\rho = Q_f \rho + \sum_{\mathcal{C} \in \mathscr{C}} Q_{\mathcal{C}} \rho = Q_f \rho Q_f + \sum_{\mathcal{C} \in \mathscr{C}} Q_{\mathcal{C}} \rho Q_{\mathcal{C}} = Q_f \rho Q_f + \sum_{\mathcal{C} \in \mathscr{C}} \lambda_{\mathcal{C}} Q_{\mathcal{C}} \rho_e Q_{\mathcal{C}}.$$

□

The above result is no longer true if we consider invariant states commuting with H_S as shows the following example.

Example 13 *Let $\mathsf{h} = \ell^2(\mathbb{N})$ with canonical orthonormal basis $(\epsilon_n)_{n \geq 0}$, and*

$$H_S = \sum_{n \geq 0} n \left(|\epsilon_{2n}\rangle\langle\epsilon_{2n}| + |\epsilon_{2n+1}\rangle\langle\epsilon_{2n+1}| \right), \qquad D = \sum_{m \geq 1} \left(|\epsilon_{2m-2}\rangle\langle\epsilon_{2m}| + |\epsilon_{2m-1}\rangle\langle\epsilon_{2m+1}| \right)$$

Suppose that there is a single Bohr frequency $\omega = 1$ (or, in an equivalent way all $\zeta_{\pm h\omega}, \Gamma_{\pm h\omega}$ vanish for all $h \geq 2$) for and $P_{\epsilon_n} = |\epsilon_{2n}\rangle\langle\epsilon_{2n}| + |\epsilon_{2n+1}\rangle\langle\epsilon_{2n+1}|$ for all $n \geq 0$ so that $D_\omega = D$. Choose constants $\Gamma_{-\omega} = 2, \Gamma_{+\omega} = 1$ and leave $\zeta_{-\omega} = \zeta_{+\omega}$ arbitrary. The QMS $\mathcal{T}$ generated by (1) admits the following faithful invariant states

$$\eta = \sum_{n \geq 0} 2^{-n-1} \left(\lambda|\epsilon_{2n}\rangle\langle\epsilon_{2n}| + z|\epsilon_{2n}\rangle\langle\epsilon_{2n+1}| + \overline{z}|\epsilon_{2n+1}\rangle\langle\epsilon_{2n}| + (1-\lambda)|\epsilon_{2n+1}\rangle\langle\epsilon_{2n+1}| \right)$$

where $0 \leq \lambda \leq 1$ and $|z|^2 \leq \lambda(1-\lambda)$. Indeed, straightforward computations show that $DD^* = \mathbb{1}$, $D^*D = \mathbb{1} - P_{\epsilon_0}$, $D\eta D^* = 2^{-1}\eta$, $D^*\eta D = 2\eta(\mathbb{1} - P_{\epsilon_0})$ and η commutes with H so that $\mathcal{L}_*(\eta) = 0$. Clearly, η is a function of H_S if and only if $\lambda = 1/2$ and $z = 0$,

Acknowledgements

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