Analysing time dependent problems

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Inverse analysis for time dependent problems is discussed in this chapter. When time dependent processes are analysed, further uncertainties come from initial conditions as well as from time dependent boundary conditions and loads, in addition to model parameters. Inverse modelling techniques have been specifically developed for this class of problems, which exploit the availability of a set of measurement and/or monitoring data at given locations at subsequent time instants. Sequential Bayesian data assimilation is introduced, and a brief review of filtering techniques is given. In filtering the problem unknown is the time evolution of the probability density function of the system state, described by means of appropriate time dependent variables and time invariant parameters, conditioned to all previous observations. Particle filtering is chosen to conceptually illustrate the methodology, by means of two simple introductory examples.

1 Introduction

Many engineering systems, and most often geotechnical systems, show time dependent response, due to time dependent loads and boundary conditions, as well as to multiphysics coupling. To assess time dependent, *dynamic*, systems and to predict their evolution in time, a proper model to describe the behaviour of the soil has to be conceived and calibrated, the initial state has to be known, and the time evolution of boundary conditions and of loads has to be described. This adds further uncertainty to the comprehensive *model* we use to describe the physical system. Moreover, small scale laboratory tests can usually give only partial information on the material properties, which have to be upscaled to properly describe the response of the system at the field scale. This is true, in general, for any geotechnical property, but even more for the hydraulic behaviour of soils, which typically shows high non-linearities due to multiphysics coupling, and which is usually strongly affected by scale effects and heterogeneity. Without loss in generality, in this chapter reference is made to the hydraulic behaviour, although the derivations can be equally applied to any other

multiphysics process.

Iterative adjustment of the model parameters, including the soil property values, the initial conditions of the system, and the time dependent boundary conditions, is a powerful tool to infer the future state of a system, given the history of its observed previous response. *Sequential data assimilation* utilises inverse modelling to estimate the state of the dynamic system at each time a measurement or an observation from the system becomes available. In this context, we speak about *measurement* when a variable describing the state of the system is measured directly (e.g. pore water pressure), and about *observation*, when the state variable is inferred by measuring a related quantity, like water content or porosity from electro-magnetic sensors.

When looking at a general time dependent physical system, neither its *"real (true) state"*, $\hat{\mathbf{x}}$, nor the *"real (true) observation"*, $\hat{\mathbf{y}}$, at the current time, $t+1$, are known in reality. The true state is a function of the true history of the state $\hat{\mathbf{x}}|^{0\rightarrow t+1}$, the true and time invariant soil parameters $\hat{\mathbf{p}} = \{\hat{p}_1, \hat{p}_2, ...\}$ and the true history of the boundary conditions $\hat{\mathbf{u}}|^{0\rightarrow t+1}$, while the true observation is a function of the true state of the system at that time

$$\hat{\mathbf{x}}^{t+1}(\hat{\mathbf{x}}|^{0\rightarrow t+1}, \hat{\mathbf{p}}, \hat{\mathbf{u}}|^{0\rightarrow t+1}) \tag{1}$$

$$\hat{\mathbf{y}}^{t+1}(\hat{\mathbf{x}}^{t+1}) \tag{2}$$

with $\hat{\mathbf{x}} \in \mathcal{R}^{N_{\hat{x}}}$, where $N_{\hat{x}}$ is the dimension of the vector describing the system state at a given location, and $\hat{\mathbf{y}} \in \mathcal{R}^{N_{\hat{y}}}$ is the observation vector of dimension $N_{\hat{y}}$.

In order to analyse and infer the response of a time dependent systems, two types of models are required; (a) a model $\mathcal{M}$ of some form describing the transient processes affecting the state, and (b) a model $\mathcal{G}$ relating some observation of the processes to the system state.

(a) The state of a dynamic system is commonly assessed using a discrete-time approach. The *predicted state(s)*, $\mathbf{x}$, may be defined as a first order Markov process, that is, the system state at the current time, $t+1$, is only a function of the state at the previous time step t. Hence

$$\mathbf{x}^{t+1} = \mathcal{M}(\mathbf{x}^t, \mathbf{p}, \mathbf{u}^t) + \epsilon_{\mathbf{x}}^{t+1} \tag{3}$$

where $\mathcal{M}$ is the model operator describing the non-linear physical process as a function of the state $\mathbf{x}^t$ at time t, the time invariant model parameters $\mathbf{p} \in \mathcal{R}^{N_p}$ and the prescribed model boundary conditions $\mathbf{u}^t$ at time t. The Gaussian (white) noise term $\epsilon_{\mathbf{x}} \sim \mathcal{N}(0, \sigma^2_{\epsilon_{\mathbf{x}}})$ (see Chapter 1, [Fen14]) is adding stochastic diffusion and has a mean of zero and a variance of $\sigma^2_{\epsilon_{\mathbf{x}}}$.

The state can be also written as an augmented state variable

$$\mathbf{z}^{t+1} = (\mathbf{x}^{t+1}, \mathbf{p}^{t+1}) \tag{4}$$

[e.g. RHV10, MDS12]. As the parameters in Equations 3 are time invariant, this augmented state variable may be used to describe the estimation of the parameters $\mathbf{p}$ with respect to the state at time $t+1$.

(b) The observation at $t+1$ can be computed by

$$\mathbf{y}^{t+1} = \mathcal{G}(\mathbf{x}^{t+1}, \mathbf{p}) + \epsilon_{\mathbf{y}}^{t+1} \tag{5}$$

where $\mathcal{G}$ is the measurement function of the system response and $\epsilon_{\mathbf{y}} \sim \mathcal{N}(0, \sigma^2_{\epsilon_{\mathbf{y}}})$ is the Gaussian noise term of the observation.

For most time dependent non-linear soil processes, the inference of the state, as well as of the soil property values, from direct inversion using closed-from analytical frameworks is virtually impossible, as already discussed in the previous chapters [Led14, Cal14]. The local gradient-based search algorithms, previously introduced to iteratively determine the local minimum within a maximum likelihood and weighted least square framework, become more likely to fail in finding the global minimum with increasing non-linearity of the system. Indeed, these algorithms are not designed to handle highly multivariate problems with multiple local optima in parameter space and multiple domains of attraction, and they become less and less effective with increasing domain size or in the presence of discontinuous responses [VSW$^+$08]. More robust global optimisation algorithms have been developed, that use multiple searches from different starting points within the parameter space, to reduce the risk of attraction towards a single local domain. The classical inference methods for non-linear dynamic systems are the *Kalman filters* and its variants, which have been have been successfully applied to many non-linear problems. An alternative sequential Monte Carlo method, the *particle filter*, has been chosen here to introduce the potentials and the limitations of global optimisation methods for time dependent processes. After a brief general introduction to sequential Bayesian data assimilation, a review of sequential inference is given. The Particle Filter is then briefly illustrated, and discussed by means of two introductory examples.

2 Inverse modelling

2.1 Bayesian basics

A background on Bayesian theory is provided in Chapter 1 [Fen14] or in specific monographs [e.g. BT92, Gre05] and thus will be summarised here only briefly. The basic form of Bayes' theorem in a continuous version is

$$P[E_i|A] = \frac{P[A|E_i]P[E_i]}{P[A]} \tag{6}$$

where E_i is the event, i.e. the state to be predicted, A is the occurrence, i.e. the observed data/measurments, $P[E_i]$ is the prior distribution or expectation defining the prior knowledge of the event i ($\int P[E_i] = 1$), $P[A|E_i] \propto P[E_i|A]P[A]$ is the likelihood function ($\int P[A|E] \neq 1$), $P[E_i|A]$ is the posterior distribution estimating

the event E_i given the observed data A ($\int P[E|A] = 1$) and $P[A]$ represents the marginal distribution of A ($\int P[A] = 1$), i.e. the normalisation factor

$$\begin{aligned} P[A] &= P[A|E]P[E] + P[E|A^c]P[A^c] \\ &= P\left[\bigcup_{i=1}^{n}(A \cap E_i)\right] = \sum_{i=1}^{n} P\left[A \cap E_i\right] = \sum_{i=1}^{n} P[A|E_i]P[E_i] \end{aligned} \tag{7}$$

Inserting Equation 7 into Equation 6 the posterior distribution can be written as

$$P[E_i|A] = \frac{P[A|E_i]P[E_i]}{\sum_{i=1}^{n} P[A|E_i]P[E_i]} \tag{8}$$

A simple example will illustrate how a Bayesian scheme can be applied. Let us assume that the volumetric water content θ has to be inferred to describe the state of an unsaturated soil, and that, based on previous experience, laboratory tests, or database, we know that $\theta \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$ with a variance of $\sigma_\theta^2 = 0.0009$. A set of new direct laboratory measurements of θ from soil samples retrieved in the field becomes available, which allows updating our prior knowledge on the mean μ_θ. Given the prior of the mean $\mu_\theta = \Theta \sim \mathcal{N}(\mu_\Theta, \sigma_\Theta^2)$

$$f(\Theta) = \sqrt{2\pi\sigma_\Theta^2}\ \exp\left\{-\frac{\left(\mu_\Theta - \Theta\right)^2}{2\sigma_\Theta^2}\right\} \tag{9}$$

the likelihood of the mean given one measurement θ is proportional to the likelihood of the sample mean $\bar{\theta}$ for a set of N_s independent measurements.

$$p(\theta|\Theta) = \sqrt{2\pi\sigma_\theta^2}\ \exp\left\{-\frac{\left(\theta - \Theta\right)^2}{2\sigma_\theta^2}\right\} \propto \sqrt[N_s]{2\pi\sigma_\theta^2}\ \exp\left\{-\frac{\left(\bar{\theta} - \Theta\right)^2}{2\sigma_\theta^2/N_s}\right\} \tag{10}$$

Therefore, the likelihood is normally distributed with a mean Θ, a variance σ_θ^2/N_s and the shape being controlled by the sample size. The posterior distribution is then obtained via multiplication of the prior and likelihood function

$$p(\Theta|\theta) \propto \exp\left\{-\frac{\left(\bar{\theta} - \Theta\right)^2}{2\sigma_\theta^2/N_s} - \frac{\left(\mu_\Theta - \Theta\right)^2}{2\sigma_\Theta^2}\right\} \tag{11}$$

where $p(\Theta|\theta) \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma})$ with a mean $\tilde{\mu} = \frac{\Theta\sigma_\theta^2/N_s + \sigma_\Theta^2\bar{\theta}}{\sigma_\Theta^2 + \sigma_\theta^2/N_s}$ and variance $\tilde{\sigma} = \frac{\sigma_\Theta^2\sigma_\theta^2/N_s}{\sigma_\Theta^2 + \sigma_\theta^2/N_s}$.

It's worth noting that the example can be extended to the case of unknown mean and variance, given that the state variable can be described by a normal or log-normal distribution.

The quality of the prior information and the advantage provided by the new measurements are illustrated in Figure 1, where N_s is the number of samples available. Given $f(\Theta) \sim \mathcal{N}(0.42, 0.0009)$ and $p(\theta|\Theta) \sim \mathcal{N}(0.33, 0.18^2/N_s)$ taking only three samples the likelihood is low, and the prior distribution dominates the posterior (Figure

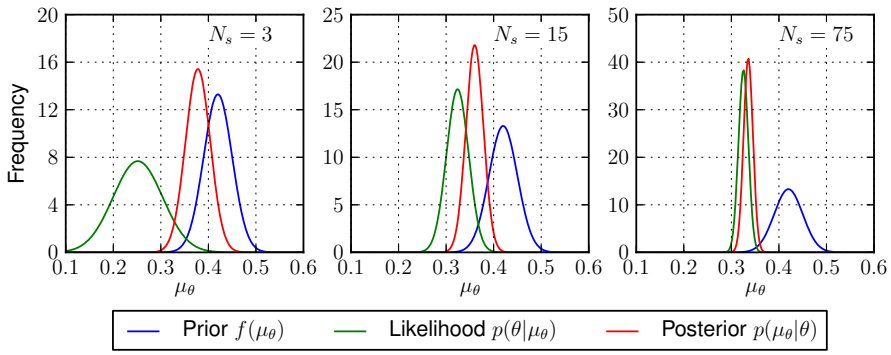

Figure 1: Example of Bayesian inference of normally distributed mean estimate μ_θ of a sample θ.

1(a)). With increasing sample size, the confidence in the observation increases and the prior information becomes less significant for the posterior prediction (Figure 1(b-c)), The difference between the prior and the posterior distributions indicates that the prior information was poor in this case.

Two simplifying assumptions were implicitly introduced in the previous derivation: (i) that the water content does not change in time, and (ii) that it can be sampled by means of direct measurement. In a more realistic scenario, (i) the water content in the field will be time dependent, as it is the result of soil-atmosphere interaction and of the position of the groundwater table, and (ii) non-invasive observations, often based on electromagnetic techniques, are usually preferred to track the physical process. Therefore, the information available usually refers to a sequential distribution of states, and a model, including its own uncertainty (Equation 5), has to be introduced to translate the observations into water content sampling. To deal with information referring to a time dependent sequence of states, the Bayesian scheme can be extended over time.

2.2 Sequential Bayesian Data Assimilation

Given the initial joint *probability density function* (PDF) of the state $p(\mathbf{x}_0|\mathbf{y}_0) \equiv f(\mathbf{x}_0)$, where $\mathbf{y}_0$ indicates no observation, the aim of this assimilation process is to sequentially infer the posterior state $p(\mathbf{x}^{0:t+1}|\mathbf{y}^{1:t+1})$ at present time conditioned by any observation which became available in time. This characterisation of the distribution state of the hidden Markov tracking process is referred to as *filtering* [DJ11].

Given the state $E = \mathbf{z}$ (Equation 4), and the observation $A = \mathbf{y}$ (Equation 5), Equation 8 gives the conditional posterior PDF at time $t + 1$

$$p_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}^{0:t+1}|\mathbf{y}^{1:t+1}) = \frac{p_{\mathbf{Y}|\mathbf{Z}}(\mathbf{y}^{1:t+1}|\mathbf{z}^{0:t+1})p_{\mathbf{Z}}(\mathbf{z}^{0:t+1})}{p_{\mathbf{Y}}(\mathbf{y}^{1:t+1})} \tag{12}$$

where $p_{\mathbf{Y}|\mathbf{Z}}(\mathbf{y}^{1:t+1}|\mathbf{z}^{0:t+1})$ is the likelihood, $p_{\mathbf{Z}}(\mathbf{z}^{0:t+1})$ is the prior and $p_{\mathbf{Y}}(\mathbf{y}^{1:t+1})$ is scaling the numerator to satisfy $\int p_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}^{0:t+1}|\mathbf{y}^{1:t+1}) = 1$. For simplicity the subscripts $\mathbf{Z}$ and $\mathbf{Y}$, indicating the random nature of $\mathbf{z}$ and $\mathbf{y}$, will be omitted from here on.

The above posterior distribution (Equation 12) joins information of all past states and is commonly referred to as *optimal filtering problem* [e.g. DJ11]. Recursion of Equation 12 satisfying the marginal filtering posterior distribution $p(\mathbf{z}^{t+1}|\mathbf{y}^{1:t+1})$, holding only information of the current state, can be written as

$$p(\mathbf{z}^{t+1}|\mathbf{y}^{1:t+1}) = \frac{g(\mathbf{y}^{t+1}|\mathbf{z}^{t+1})p(\mathbf{z}^{t+1}|\mathbf{y}^{1:t})}{p(\mathbf{y}^{t+1}|\mathbf{y}^{1:t})} \tag{13}$$

where g is the homogeneous likelihood (state and observation densities are time independent) and the prior distribution is estimated as

$$p(\mathbf{z}^{t+1}|\mathbf{y}^{1:t}) = \int_{\mathbf{z}^t} f(\mathbf{z}^{t+1}|\mathbf{z}^t, \mathbf{y}^{1:t})p(\mathbf{z}^t|\mathbf{y}^{1:t})\ \mathrm{d}\mathbf{z}^t = \int_{\mathbf{z}^t} f(\mathbf{z}^{t+1}|\mathbf{z}^t)p(\mathbf{z}^t|\mathbf{y}^{1:t})\ \mathrm{d}\mathbf{z}^t \tag{14}$$

also known as *Chapman-Kolmogorov equation*, which simplifies to the second term due to the first order Markov process [DJ11, MDS12]. The likelihood g is commonly described by a Gaussian with zero mean and a given variance. The normalisation factor may be predicted using the augmented state as intermediate variables [MDS12].

$$p(\mathbf{z}^{t+1}|\mathbf{y}^{1:t}) = \int_{\mathbf{z}^{t+1}} g(\mathbf{y}^{t+1}|\mathbf{z}^{t+1})p(\mathbf{z}^{t+1}|\mathbf{y}^{1:t})\ \mathrm{d}\mathbf{z}^{t+1} \tag{15}$$

Equation 15 is commonly referred to as the *predictive/evolution* step and Equations 12 with 14 are referred to as the *updating/correction* step [e.g. DdG01, CGM07, DJ11].

2.3 Sequential inference of soil water dynamic processes

In geotechnical engineering, the use of inverse models in sequential schemes is still lagging behind. Explicit analysis of transient processes by using advanced constitutive models implemented in numerical frameworks is usually preferred, also due to the difficulties in obtaining a comprehensive body of statistically valuable in situ measurements. The state and the parameters of transient processes have been inferred by using gradient based optimization algorithms, with their application ranging, for example, from deep staged excavations [e.g. RLF08, TK09] to laboratory pulse test [e.g.

GGA+11]. However, in these schemes the different sources of uncertainty are not accounted for, and no or only limited information on the state and parameters are being carried from one measurement time step to the next one. The Bayesian framework presented in Section 2.2 is not exploited in these cases.

Sequential data assimilation is a very commonly applied tool, for instance, in weather forecasting, hydrological modelling and flood protection assessment. For most soil water dynamic processes the non-linearity in the soil response and transient boundary conditions, alongside with the non-Gaussianity of the distributions, makes an analytical solution of the Equations 13-15 untraceable [e.g. CGM07, PCP12]. To overcome this limitation, [Eve94] developed a recursive data-processing algorithm known as the *ensemble Kalman filter* (EnKF), which is an extension to the original *Kalman filter* [Kal60] and the *extended Kalman filter* [Jaz70]. The EnKF is based on a *Markov chain Monte Carlo* (MCMC) method, propagating a large ensemble of model states to approximate the prior state error in time by using the updated states of the previous time step, to predict the current ensemble via forward integration of a stochastic differential equation describing the model dynamics [e.g. BvE98, Eve03, Eve09]. More information, examples and codes can be found on Geir Evensen's EnKF-homepage[1].

The EnKF is one of the most commonly used non-linear filter for state and parameters updating in many fields such as hydrological modelling [e.g. MSGH05, PCP12, SNH12], but it has not often been adopted in the assimilation of geotechnical systems [e.g. HMHV10, CCZ10].

The *particle filter* (PF), which is a *sequential Monte Carlo* (SMC) method, presents one alternative to the EnKF. The PF method is very flexible, easily implementable, strongly parallelisable and, most importantly, it approximates the probability densities directly via a finite number of samples, often referred to as *particles* [DdG01, AMTC02]. A large number of different PF methods was developed in recent years. Some tutorials and state-of-the-art reports provide a good introduction and allow for a more complete overview [e.g. DdG01, DJ11, LW01, AMTC02, CGM07, van09, CR11]. Some useful resources on SMC and PF methods have been compiled by Arnaud Doucet[2].

Most PF frameworks are based on a *sequential importance sampling* (SIS) and *sampling importance resampling* (SIR) algorithm. The SIS is the most basic Monte Carlo method to approximate the prediction and the updating steps (Equations 12-15). It uses a finite set of random samples with associated weights to directly represent the posterior distribution at current time step, and subsequently updates this particles in order to obtain the posterior at the next time step. However, for non-linear systems the sample may tend to degenerate, that is, only a limited number of particles being around the *"real"* state exclusively carry the weights, whilst the remaining majority of samples only carry a negligible weight. To increase the effectiveness of the filter and avoid errors accumulation, the SIR algorithms may be used. SIR introduces a resampling stage at each time step, in which particles with a low weight are eliminated

[1] EnKF sources: `http://enkf.nersc.no/`
[2] SMC and PF sources: `http://www.stats.ox.ac.uk/~doucet/smc_resources.html`

and regenerated in zones in which particles carry a high weight, which renders the approximation of the posterior. Other PF methods include auxiliary particle filters, marginalised particle filters, Markov chain particle filters and may incorporate some particle smoothing algorithm [e.g. AMTC02, CGM07, DJ11]. Sequential smoothing makes use of the estimates of the past states and thus tends to provide a better filter for the current state.

In recent years the PF method became popular and performed well in the assimilation of the state and parameters of different hydrological soil water dynamic processes [e.g. MHGS05, MDS12, KdD05, SF09, QLY$^+$09, RHV10, MMW$^+$11, NTSK11, PDD$^+$12, RVS$^+$12]. A comparison between the EnKF and PF performance using a coupled surface-subsurface flow model has been presented by [PCP12].

In geotechnical engineering the use of the PF method is not common. However, Murakami and co-workers [SMN$^+$12, MSN$^+$13] recently demonstrated that the elastic-plastic Cam Clay model parameters can be successfully inferred using a coupled hydro-mechanical Finite Element program in a PF framework, both on synthetic observation data for soil element loading tests and the construction of a soil embankment, as well as on real observation data related to the construction of the Kobe Airport Island.

3 A simple SIR particle filter implementation

The posterior (Equation 13) is approximated using a discrete set of N_s samples

$$p(\mathbf{x}^{0:t+1}|\mathbf{y}^{1:t+1}) = \sum_{k=1}^{N_s} w_k^{t+1} \delta\left(\hat{\mathbf{x}}^{t+1} - \mathbf{x}_k^{t+1}\right) \tag{16}$$

where w_k^{t+1} are the normalised particle weights

$$w_k^{t+1} = \frac{w_{k^*}^{t+1}}{\sum_{k=1}^{N_s} w_{k^*}^{t+1}} \tag{17}$$

When using the transient prior as importance function, i.e. $q(\mathbf{x}_k^{t+1}|\mathbf{x}_k^t, \mathbf{y}^{t+1}) = p(\mathbf{x}^{t+1}|\mathbf{x}_k^t)$, the updated sequential estimates of the importance weights are

$$w_{k^*}^{t+1} \propto w_{k^*}^t \; \frac{p(\mathbf{y}^{t+1}|\mathbf{x}_k^{t+1}) p(\mathbf{x}^{t+1}|\mathbf{x}_k^t)}{q(\mathbf{x}_k^{t+1}|\mathbf{x}_k^t, \mathbf{y}^{t+1})} = w_{k^*}^t \; p(\mathbf{y}^{t+1}|\mathbf{x}_k^{t+1}) \tag{18}$$

which represent the key part of the SIS filter [e.g. AMTC02, MHGS05, DJ11].

The implementation of a simple SIR filter based on [MHGS05] is schematised in Figure 2. The process can be split into three stages.

Initialisation stage:

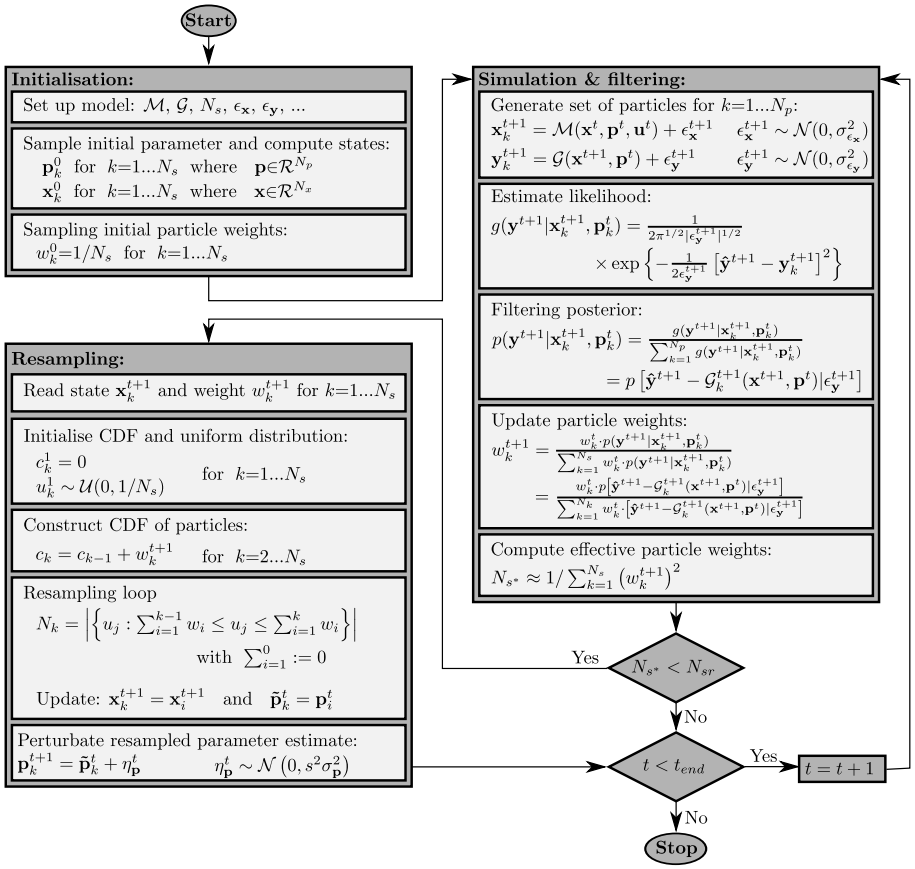

Figure 2: Schematic description of a simple SIR PF.

In this first stage the process and the observation model, $\mathcal{M}$ and $\mathcal{G}$, (Equations 3 and 5) as well as the stochastic model, e.g. the number of particles (samples) N_s and the error functions ϵ, are set up. The initial state $\mathbf{x}^0$ is computed based on a set of parameters $\mathbf{p}^0$ representing the prior knowledge, and an initial set of uniform weights w_k^0 is assigned to each particle k.

Simulation stage:

In the simulation stage, the filtering of the state at $t + 1$ is performed. By means of the state and the observation models, $\mathbf{x}^{t+1}$ and $\mathbf{y}^{t+1}$ are computed for each particle. Subsequently the homogeneous likelihood function g is estimated to compute the filtering posterior (Equations 13 and 16). Utilising Equations 17 and 18 the weights are assigned to each the particle. Given that the effective particle size N_{s^*} is smaller than a minimum effective particle size N_{sr}, representing a resampling threshold below

which degeneration of the samples occurs, the resampling stage is entered.

Resampling stage:

Different schemes for state and parameters resampling have been proposed. Using one of the systematic schemes [e.g. AMTC02, DJ11], the particles are resampled by relating a *cumulative distribution function* (CDF) for the particle, c, to a uniform CDF, u. After the update of the particle states and parameters, the resampled parameter estimate $\tilde{\mathbf{p}}_k^t$ is perturbed to obtain

$$\mathbf{p}_k^{t+1} = \tilde{\mathbf{p}}_k^t + \eta_{\mathbf{p}}^t \tag{19}$$

with $\eta_{\mathbf{p}}^t \sim \mathcal{N}(0, s^2\sigma_{\mathbf{p}}^2)$ being a Gaussian noise term, as suggested by [LW01] and [MHGS05]. The variance of the parameter particles, $\sigma_{\mathbf{p}}^2$, is multiplied by a small tuning parameter s, which determines the exploration radius around each particle, and for which values between 0.005 and 0.025 have been commonly used [MDS12].

4 Examples

Two introductory examples will be discussed in this section to demonstrate the working principle and the efficiency of the simple SIR PF implementation. In the first benchmark example synthetic observations are used, while the second example refers to a typical field case where direct measurements are available.

4.1 Example 1: an analytical benchmark

The first example has been used as benchmark as well as for illustrative purpose by several authors [e.g. KdD05, MHGS05]. The non-linear state model and the observation function are both one-dimensional and described by the following analytical functions

$$x^{t+1} = \frac{1}{2}x^t + a\frac{x^t}{1+(x^t)^2} + b\cos(1.2t) + \epsilon_x \tag{20}$$

$$y^{t+1} = \frac{(x^{t+1})^2}{20} + \epsilon_y \tag{21}$$

where $a = 25$ and $b = 8$ are the *parameters*, $\epsilon_x \sim \mathcal{N}(0, \sigma_{\epsilon_x}^2)$ and $\epsilon_y \sim \mathcal{N}(0, \sigma_{\epsilon_y}^2)$ are the random noise terms for the state and the observation respectively, with $\sigma_{\epsilon_x}^2 = 10$ and $\sigma_{\epsilon_y}^2 = 1$. The initial state is taken as $x^0 = 10$ and $N_s = 500$ particles are used. The initial parameter estimates are $a^0 = 30$ and $b^0 = 4$.

Figure 3 shows the state, the observation and the inferred parametric response with time, using a sampling time interval of $\Delta t = 1$. The results of the simulation show

that the sequential assimilation technique succeeds in predicting quickly and accurately the state, with some negative peaks not being detected until the end of the simulation. During the time lapse analysed, also the parameters converge close to the real values ($m_{a^{100}} = 25.1$ and $m_{b^{100}} = 7.869$). The remaining variation of the parameters depends partly on the tuning parameter s, which ensures that the filter were able to react to any significant variation in the observation.

In this benchmark case, resampling was required in most of the time steps to avoid degeneracy. Figure 4 exemplary illustrates the resampling scheme for the last time step $t = 100$. The reduction of the variance of the filtering posterior from (b) to (d) due to resampling allows for a more effective use of the particles. A more detailed description of the filtering and resampling process can be found for instance in [NTSK11], [MDS12] and [SMN$^+$12].

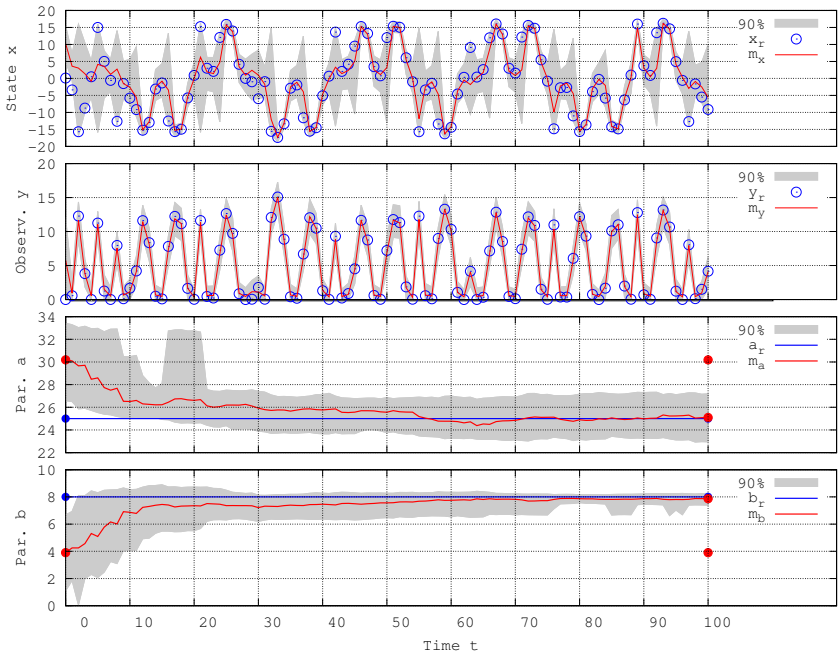

Figure 3: Sequential assimilation of state and parameters using a SIR PF with 500 particles showing the state x and observation y response as well as the evolution of the parameters a and b with time t. Subscripts r indicate the synthetic *"real"* values, m the sample mean and 90% the 90% bounds, respectively.

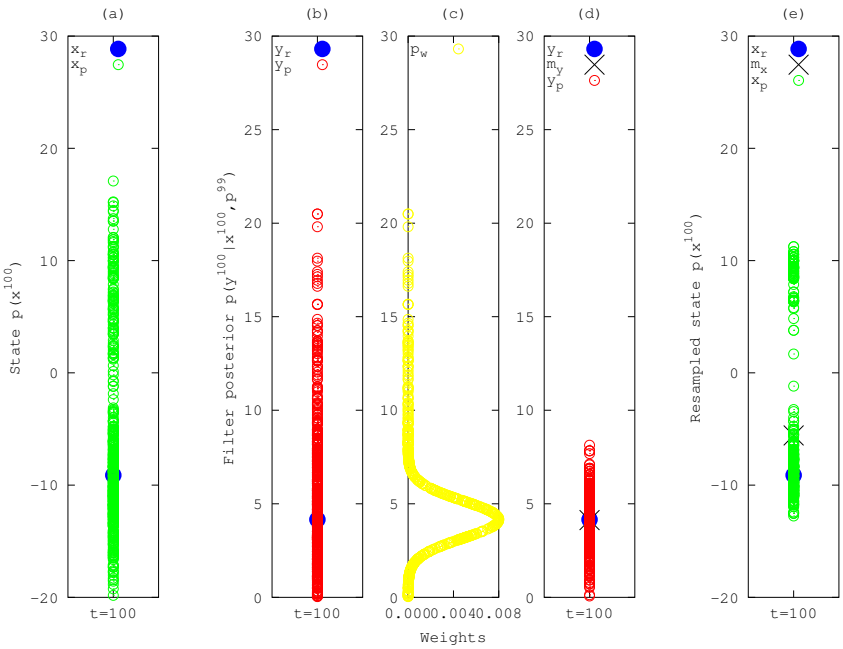

Figure 4: Particle resampling at time step $t = 100$ showing (a) the state, (b) the filter posterior, (c) the weights assigned to each particle and (d-e) the resampled posterior and state. Subscripts r indicate the synthetic *"real"* values, subscripts p the particles and m the sample mean, respectively.

4.2 Example 2: response of pore water pressure below a dike

In the assessment of dikes, different failure mechanisms have to be analysed. Most of them are likely to be initiated by the transient pore groundwater response to the time dependent external forcing conditions. The worst conditions are not necessarily associated to the steady state pore water distribution in equilibrium with the maximum expected water height. Therefore, proper assessment of the potential failure mechanisms requires the analysis of the fully coupled time dependent hydro-mechanical response of the water defense structure, including the dike body and the subsoil.

Explicit coupled numerical finite elements analyses can be performed to this aim, but the computational effort needed to include uncertainty in the model is still high. A valuable alternative consists in relying on simplified analytical solutions of the coupled hydro-mechanical consolidation process, and perform an inverse analysis able to sequentially assimilate the parameters of the simplified model by comparison with observation in time. Figure 5 gives a simplified illustration of the hydro-mechanical

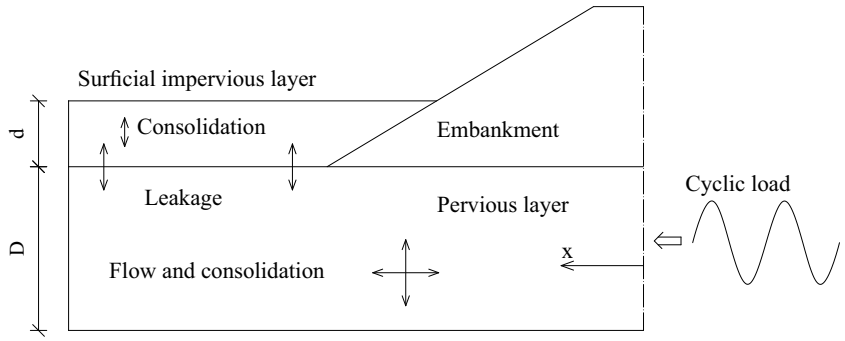

Figure 5: Simplified description of the hydro-mechanical response of a dike subsoil subjected to cyclic hydraulic boundary conditions

Table 1: Parameters of the simplified consolidation model.

Variable		Unit	$\hat{\mathbf{p}}$	$\mathbf{p}^0$
Thickness of clay layer	d_c	[m]	2.0	–
Thickness of sand layer	D_s	[m]	7.0	6.25
Sat. hydr. conductivity of clay layer	K_c	[m s^{-1}]	0.00001	–
Sat. hydr. conductivity of sand layer	K_s	[m s^{-1}]	0.0005	0.005
Compressibility of clay layer	α_c	[m kN^{-2}]	0.005	–
Compressibility of sand layer	α_s	[m kN^{-2}]	0.0000001	–

processes taking place in the typical foundation subsoil of a dike, with a pervious aquifer underlying an impervious surficial layer, subjected to a cyclic variation of the pore water pressure at the boundary representing the river bed. The effective simple analytical solution proposed by Baudin & Barends [BB88] for this problem has been used in this second example.

The adopted analytical solution gives the pore pressure distribution in the two layers at any given distance from the river bed, and is a function of the variables listed in Table 1. The response of the system depends on the thickness, on the compressibility and on the hydraulic conductivity of the two layers, and on the period of the forcing boundary condition. Table 1 summarises the synthetic soil property values $\hat{\mathbf{p}}$ representing the *real* state of the system $\hat{\mathbf{x}}$ (Equation 1) assumed to be represented by the analytical solution. Prior investigation using Monte Carlo simulations had shown that the response of the hydraulic head $H(x)$ in the sand layer is most sensitive to variations in the saturated hydraulic conductivity of the sand layer K_s, while the thickness of the sand layer D_s has a less dominating role. For the sake of simplicity, in this example the random model parameters are limited to this two variables, for which the initial guess is $\mathbf{p}^0 = \{D_s, K_s\} = \{6.25, 0.005\}$, and the remaining four parameters are assumed to be known.

We assumed - as this is the case in the field test to which this example refers to - that a piezometer is installed in the sand layer at a distance $x = 18.6$m from the river bed, where a direct measurement of the pore water pressure is taken at each hour. These pore pressure measurements are used for sequential data assimilation, using the SIR PF previously described.

The period of the forcing function is $T = 10$d ($86\,400$s), and the measurements are taken at each hour ($3\,600$s). The total time was set to 200h ($720\,000$s), the number of particles to $N_s = 400$ and s to 0.005. The maximum hydraulic head at the river was normalised to $h_0(x = 0) = 1.0$m.

Figure 6 shows the parameter estimate with time. The hydraulic conductivity converges very rapidly to the synthetic *"real"* value, confirming the high potential of the adopted algorithm in sequential data assimilation. On the contrary, the convergence for the thickness of the sand layer is much slower, and the uncertainty does not decrease monotonically. Indeed, the two variables were chosen with the purpose of assessing the performance of the algorithm in identifying parameters which have different relative weights on the prediction. The hydraulic conductivity of the pervious layer, which dominates the response of the system, could be rapidly inferred. As for the thickness of the same layer, a reasonable convergence could be achieved, in spite of its minor role in the response of the synthetic *true* system.

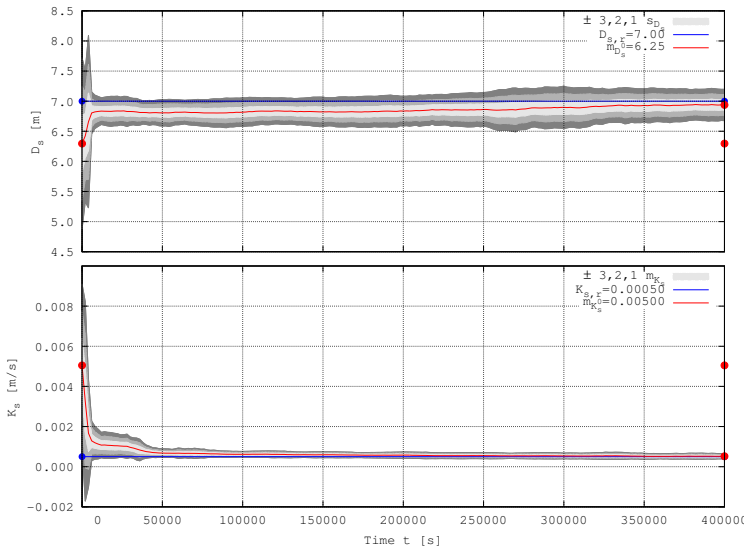

Figure 6: Estimation of the thickness of sand layer, D_s, and the saturated hydraulic conductivity of the sand layer, K_s, using a SIR PF. The subscripts r indicate the synthetic *real* soil property values, the superscript 0 the initial state, and m and s the sample mean and standard deviation, respectively.

5 Final remarks

In this last chapter, a basic introduction to the inverse analysis of time dependent problems was given. The provided overview is far from being a complete review, and to this aim the reader is referred to the references for further reading.

The two basic aims of this contribution were: (i) to combine the theoretical and numerical developments on random fields, presented in the first part of this book, with the general concepts on inverse analysis illustrated in the previous two chapters; and (ii) to open a window on sequential data assimilation, which can be fruitfully exploited in the practice, when time dependent problems have to be analysed. The examples discussed at the end of this chapter are meant just as an introduction to the powerful approaches which can be adopted in these cases. Nonetheless, they suggest that if information from measurement and monitoring in *space* of a time dependent system is accompanied by a thorough analysis of the observed behaviour in *time*, identification of the variables and parameters dominating the response of the models can be effectively accomplished by means of rather simple dedicated algorithms.

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