A MODEL FOR DESCRIBING REASONING IN LOGICAL TASKS

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The paper presents a cognitive model that describes reasoning encompassing formal logic and semiotics, according to two dimensions: formulation and formality. Formulation addresses students' awareness of formal logic rules, referring to both the degree of certainty and Radford's layers of generality. Formality addresses the appropriate use of logic rules. Combining these dimensions, we derive, characterize and discuss five possible behaviours. Evidence is provided by the analysis of written protocols from an experimentation with undergraduate students.

INTRODUCTION AND BACKGROUND

Reasoning plays a crucial role in human activities that are specifically cognitive (such as: learning, development, knowledge processing), but also in those generically creative and social. Amongst reasoning activities, let us consider for example inferences: they allow us to get new information from previous one. Inferences, in fact, help human beings to access knowledge (whether conscious or implicit), and apply it to specific (new) situations. Mathematical formal logic, in any one of its formulations, cannot represent a full formalization of all kinds of reasoning activity. Several researches from the beginning of the 20th century attempt to link reasoning to studies in the field of formal logic (for a review see Casadio, 2006; Toulmin, 1958), but the relationship between reasoning, as an everyday activity, and formal logic in mathematics is still rather complex: according to Dapueto and Ferrari (1988)

> the contexts in which the "daily" reasoning develops and those in which the deductions are built, which (also) mathematical logic deals with, are completely different, with different criteria of acceptability and coherence. (p. 779)

It is possible, however, that some forms of 'daily' reasoning enter also in formal logic tasks. For this reason, we present a cognitive model for describing students' reasoning in typical formal logic tasks, such as syllogisms and if-then statements. Formal logic itself does not only provide a context in which individuals perform tasks, but it can provide tools for analysing such tasks. Indeed, formal logic can be regarded as a tool for studying reasoning processes, when considered in a wider way as the expression of some aspects concerning *language*. Historically, the birth of modern logic and its development are characterized by the change in the role of language in mathematics from being only a communication tool to becoming also a manipulation one. In this paper, formal logic lenses are provided by Reid's model (2002). Since formal logic alone is not enough, we take into account also other Mathematics Education theoretical lenses (Piattelli-Palmarini, 1995; Radford, 2001). All these lenses enter our model as different dimensions and our aim is to study how they are intertwined in framing students' logical reasoning. We do not consider the

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context as a variable in this study, even if we are aware of its importance in mathematics teaching/learning processes.

According to Radford's cultural-semiotic approach that considers cognition as a *reflexive mediated activity*, mathematical concepts are objectified at different layers of generality depending on the semiotic means that mediate activity. Radford (2001) identifies three increasing levels of generality: a *factual generalization,* when the objectification of the general scheme takes the form of a *perceptual/sensorimotor semiosis*; a *contextual generalization* when the general scheme is objectified by more abstract semiotic means that, however, bear the spatial and temporal origin of the situation they come from; a *symbolic generalization* when the general scheme is objectified by symbolic language that does not allow any relation with the spatial-temporal dimension. The learner lives a desubjectification of meaning, namely a rupture with his spatial-temporal and sensorimotor experience.

Reid (2002) describes mathematical reasoning across five dimensions. Among them, "*formulation* refers to the degree of awareness the reasoner has of his own reasoning" (p. 105). Semiotics helps characterizing formulation. According to Radford (2001), a high degree of awareness can be identified with a symbolic generalization, where the cultural logical discourse is objectified by the students at an interpersonal level. On the counterpart, we claim that when there is an unaware inconsistency between the meaning objectified by the individual and the cultural meaning of logical activity, personal opinions and misconceptions play a crucial role in guiding learner's logical reasonings. For instance, several studies show that both children and adults make errors because they infer not only on the basis of the premises, but either introducing other premises or referring to the common sense. The unaware inconsistency can occur both when students use a proper formalism, and when they do not. For this reason, another dimension from Reid's model is taken into account: *formality.* Before talking about it, let us further characterize formulation. Both when there is a good level of awareness and in the opposite case of unaware inconsistency, we claim that the individual tends to feel sure of his resoning. In the first case, such a certainty is provided by the formal logic rules: following them correctly, in fact, leads the reasoner to arrive at a conclusion that is almost always correct and formally grounded. In the second case, according to Piattelli-Palmarini (1995), a student who is leaded by misconceptions and personal opinions tends to be sure of his reasoning, since they seem to be very reliable.

Let us suppose that good awareness and unaware inconsistency represent two poles. We argue that there is also something in between these poles: in this case, students at the same time have a certain degree of awareness that they cannot use only their personal opinions, and they suspect that they do not have enough theoretical tools for reasoning within the formal logic context. It can be considered as a contextual generalization of logical discourse where the individual's experience and opinions are relative to a particular context (Radford, 2001). Indecision and lack of responses are expected in this kind of behaviour. We claim that this situation is didactically the most interesting, since the learner is in Vygotskij's (1978) Zone of Proximal Development (ZPD), where teaching/learning processes are effective. In our case, ZPD is triggered by a "cognitive conflict" caused by the disagreement between an intuitive model and the mathematical model (Fischbein, 1998). Due to the conflict, the individual is induced to reorganize the previous conceptions for integrating new information coming from the new situation (D'Amore, 1999; Perret-Clermont, 1979). The momentary incorrect conceptions, waiting for a more elaborate cognitive arrangement, are a transitory cognitive moment from a naive conception to a more elaborate one and closer to the (logically) correct conception.

We now consider the aforementioned dimension of *formality* (Reid, 2002): it "refers to the degree to which the expression of the reasoning conforms to the requirements of mathematical style" (p. 105). Combining together *formality* and the two poles of *formulation*, there are four possible distinct cases: (1) good degree of awareness and proper use of formal logical tools; (2) good degree of awareness, but incapability in using logical tools; (3) unaware inconsistency and no use of formal logic; (4) unaware inconsistency, but proper use of formal logical tools. Regarding case (4), is it possible, however, to use formal logical rules when misconceptions/personal opinions enter in reasoning? Some researchers, in fact, claim that in everyday thinking people often use logics that are different from the formal one (Ayalon, 2008). In cognitive psychology, Wason (1966) and Johnson-Laird (1983) point out adults' difficulties in doing also simple inferences. According to Johnson-Laird's theory, common reasoning is not based on formal rules, which are independent from the content, but on construction and manipulation of mental models or representations (Girotto & Legrenzi, 1999). Our study highlights cases of students resorting both to formal logical rules and to misconceptions/personal opinions.

THE COGNITIVE MODEL

Is it possible to describe, and to what extent, reasoning according to the degree of awareness and the accordance with formal logic rules? Is it possible to further characterize it though the means provided by the cultural-semiotic approach, and in terms of certainty? These are the research questions that inform our study. The aim of the paper is the construction of a model for describing students' logical reasoning, according to the theoretical background we presented and discussed in the previous section. The interplay of *formulation* and *formality*, along with the role played by semiotics and the reasoner degree of certainty, allows identifying five behaviours.

We address the first one as R and it is the case of the reasoner that is aware of his reasoning according to formal logic rules (formulation), and also he properly uses them (formality). Moreover, the reasoner is expected sure of his statements. The second behavior is called r: there is the same degree of awareness as in the first one (as well as the same expected certainty), but reasoning has formal imperfections. On the counterpart, when there is an unaware inconsistency between the meaning objectified by the individual and the cultural meaning of logical activity, the reasoner uses misconceptions/personal opinions (formulation) and two behaviours are possible. We refer to *M* when almost none formal logical rule is employed. When we observe some use of formal logical rules, we refer to *m*. According to Piattelli-Palmarini (1995), the reasoner is certain of his statements in both *M* and *m* behaviours. The last behaviour, which corresponds to the intermediate degree of awareness, is called *I*: plausible interpretation. *I* is characterized by containing some dubitative elements (such as conditional verbs). As a consequence, it seems that students perform a reasoning that can be open for being discussed/changed.

We now seek experimental evidence for the relationship between Reid's formulation and formality, their semiotic characterization, and the degree of certainty, according to the five aforementioned behaviours.

METHODOLOGY

A test was administered to 111 undergraduate students (86 in November 2009, 25 in November 2010), with a weak mathematical background, before the beginning of a course in logic. Hence, answers were not influenced by the teaching of logical topics. Students carried out tasks about reasoning. The analysis in this paper refers only to an example of syllogism (task 1), and to one of if-then task (task 4). We choose these tasks because the syllogism is the most classical kind of reasoning scheme and the if-then statement represents one of typical kind of human reasoning. Furthermore, regarding the if-then statements, formal fallacies are very frequent: there is a strong tendency of people to interpret "if-then" statements as "if-and-only-if" statements (Ayalon, 2008; Leron, 2004). In the case of task 4, for example, we predict that a consistent percentage of students would mark the alternative E as the correct one: it would have been true, in fact, if the statement had been an "if-and-only-if" one.

As shown in figure 1, students were requested: (1) to mark one answer, (2) to say how much they feel certain of their response, and (3) to provide a written justification of their answer. On the basis of the written justifications, we make inferences about the reasoning process students activated in their solving activities. According to our theoretical framework, in presenting our results we now provide a detailed analysis of five protocols, respectively classified as *R, r, M, m* or *I.*

Task 1

No ingenuous person is a bad person.
Some bad person is adult.
So??? is not ingenuous.

A) some adult (correct)
B) every adult
C) some ingenuous person
D) some bad person
E) every bad person

How much are you sure of your answer?
Explain your reasoning:

Task 4

Given that *all the dogs that bark do not bite*, one of the following assertions is necessary true:

A) the dogs that bite do not bark (correct)
B) some of the dogs that bark bite
C) all the dogs do not bite
D) all the animals that bite are not dogs
E) if the dogs don't bite, they bark

How much are you sure of your answer?
Explain your reasoning:

Figure 1: tasks 1 (syllogism) and 4 (if-then).

EXPERIMENTAL EVIDENCE

In figure 2 we report two histograms with the relative frequencies of selection of each alternative of tasks 1 and 4. We observe that almost no one omits the answer, and in both cases there is one alternative that has been chosen by a high percentage of respondents (together with the correct answer). In task 4 the choice of alternative E highlights the typical misconception students have, considering the if-then tasks as if-and-only-if ones.

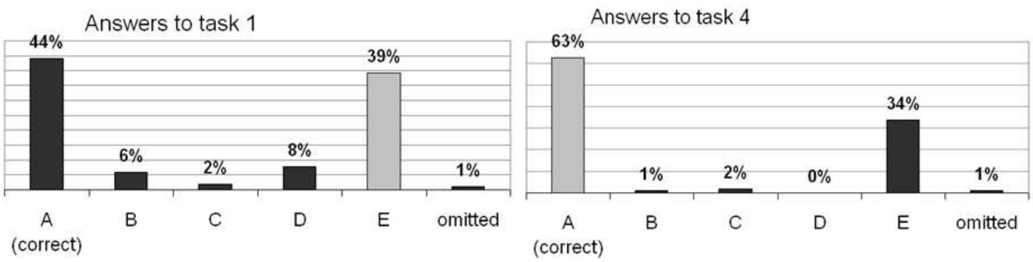

Figure 2: the relative frequencies of selection of each answer.

In figure 3, three answers to task 1 are shown. The English translation is provided in the analysis carried out below.

Let us firstly look at Anna's protocol (figure 3). In providing her justification, she draws a universe set for "human" and inside it she draws three sets for "ingenuous", "adult", and "bad". She matches this graphical representation with the sentence "*Certainly some adult isn't ingenuous because he is a bad person and the bad persons aren't ingenuous*". She answers correctly to task 1. The justification provided by Anna was classified as *R*: her reasoning conforms to the requirements of mathematical style, referring both to *formulation* and *formality*. From a semiotic point of view, she uses a symbolism that is general and desubjectified at an interpersonal level. According to our expectations, Anna is quite sure of her answer, since following a correct logical reasoning provides a good level of certainty.

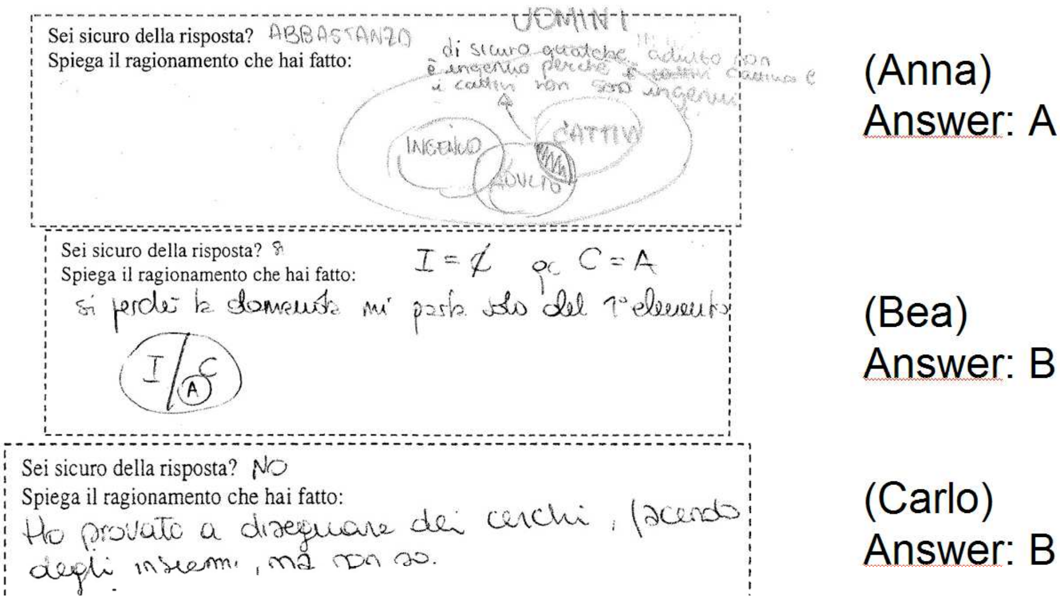

Figure 3: three answers given to task 1.

On the counterpart, Bea answers wrongly to task 1. Looking at figure 3, she makes a partition of the universe in I (ingenuous persons) and C (bad persons). Moreover, she puts the set A (adults) inside the set C, revealing also an incorrect interpretation of the existential quantifier "some" ('qc' in the protocol is an Italian abbreviation for 'some'). She writes also: "*Yes, because the question speaks only about the first element*". The justification is classified as *m*, because Bea fails in interpreting and representing the statements according to *formulation*, but we observe the use of formal logic rules according to *formality*. Bea's mistakes involve both the interpretation of the question ('only the first element' has been interpreted as 'only the first sentence', and the fact the only 'ingenuous' is stated in the conclusion may have lead her to think that only the first sentence is involved), and the representation through Venn diagrams: she represents a partition in C and I, with A included in C. In Bea's case, there is some semiotic manipulation, but misconceptions have a central role in guiding her reasoning. According to our expectations, she asserts to be sure of her (incorrect) answer: when leaded by misconceptions, students feel to be sure.

Let us now look at Carlo's protocol (figure 3). In his justification he uses some dubitative elements. He writes: "*I tried to draw some circles, making some sets, but I don't know*". Mentioned circles and sets are not present in the protocol. We classified it as *I*. We infer that maybe there is a tension between at least two representations in Carlo's mind, hence he is in doubt about his answer. Moreover, Carlo says to be not sure of it, according to our expectations.

In figure 4, two answers to task 4 (an example of if-then task) are shown.

(Daniela) Answer: A (Elvira) Answer: E

Figure 4: two answers given to task 4.

Daniela answers correctly to task 4 and she represents a set for "dog barks" and a set for "dog bites". There is no intersection between the two sets. This is accompanied by written explanation in a quite formalized daily language: "*the biting dog doesn't belong to the barking dogs' set, because all those which bark don't bite.*" Daniela's justification is classified as *r*: according to *formulation*, in fact, her reasoning is correct, but there is some mistake in her representations. The *barking dogs' set* is not explicitly drawn into the *not biting dogs' one*. She asserts to be quite sure of her (correct) answer.

Elvira provides an incorrect answer: E. In her explanation, Elvira writes: "*alternative E states the same thing of the assertion in the question*". This is a typical misconception about the "if-then" statements: 'if-then' is, in fact, regarded as an if-and-only-if statement. This leads us to classify it as *M.* According to our expectations, Elvira declares to be very sure of the answer.

DISCUSSION AND CONCLUDING REMARKS

In this paper we carried out an analysis of reasoning that pivoted around formal logic. We provided a model for reasoning that, along with formal logic, considers the role played by semiotics and the degree of certainty. The interplay of *formulation* and *formality* allows identifying five reasoning behaviours. It would be interesting to investigate why the use of (proper) logical formalisms does not ensure overcoming the hindrance of misconceptions and personal opinions, as in the *m* behaviour. Data prove the existence of the aforementioned reasoning behaviours and the degree of certainty predicted for each of them. This has been also confirmed by a first quantitative analysis that has not been reported in this paper: both when students are aware of their reasoning, and when misconceptions/personal interpretations play a central role, the majority of them declare to be sure of the answer. Even if we observed few students performing *I*, the majority declares to be unsure of the answer.

In order to corroborate and strengthen our model, it is necessary to both go beyond Radford's layers of generality and analyse students' use and integration of different semiotic registers, and go beyond the mere use of syllogisms and if-then tasks, investigating on the connection between argumentative and proving processes, in order to contribute to the international debate regarding the relationship between argumentation and proof.

References

Ayalon, M., and Even, R. (2008). Deductive reasoning: in the eye of the beholder. *Educational Studies in Mathematics, 69*, 235-247.

Casadio, C. (2006). *Logica e Psicologia del pensiero*. [Logic and Psychology of thinking]. Roma: Carocci Editore.

D'Amore, B. (1999). *Elementi di Didattica della Matematica*. [Hints of Mathematics Education]. Bologna: Pitagora Editrice.

Dapueto, C., and Ferrari, P.L. (1988). Educazione logica ed educazione matematica nella scuola elementare. [Logic education and mathematics education in primary schools]. *L'insegnamento della Matematica e delle Scienze integrate, 11,* 773-810.

Fischbein, E. (1998). Conoscenza intuitiva e conoscenza logica nella attività matematica. [Intuitive knowledge and cognitive knowledge in mathematical activity]. *La Matematica e la sua Didattica, 4*, 365-410.

Girotto, V., and Legrenzi, P. (1999). *Psicologia del pensiero*. [Psychology of thinking]. Bologna: Il Mulino.

Johnson-Laird, P.N. (1983). *Mental models*. Cambridge: Cambridge University Press.

Leron, U. (2004). Mathematical thinking and human nature: Consonance and conflict. Retrieved from: http://edu.technion.ac.il/Faculty/uril/Papers/Leron_ ESM_%20Human_ Nature.pdf

Piattelli-Palmarini, M. (1995). *L'illusione di sapere.* [The illusion of knowing]. Milano: Mondadori.

Perret-Clermont, A.N. (1979). *La construction de l'intelligence dans l'interaction sociale*. Berne: Peter Lang, coll. Exploration.

Radford, L. (2001). Factual, Contextual and Symbolic Generalizations in Algebra. In: M. van den Hueuvel-Panhuizen (ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (pp. 81-88). Utrecht: PME.

Reid, D.A. (2002). Describing young children's deductive reasoning. In: A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (pp. 105-112). Norwich: PME.

Toulmin, S. (1958). *The uses of arguments*. Cambridge: Cambridge University Press.

Vygotskij, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Wason, P.C. (1966). Reasoning. In: B. Foss (Ed.), *New Horizons in Psychology* (pp. 135-151). Harmondsworth: Penguin.