A chance-constrained approach to the quantized control of a heat ventilation and air conditioning system with prioritized constraints

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***Abstract*— This paper addresses quantized control of a heat ventilation and air conditioning system. The objective is to guarantee comfort, defined in terms of desired temperature and humidity, with a higher priority assigned to the temperature control. The system is described by a linear model with a stochastic input to account for model uncertainty. A chance-constrained control design strategy is proposed where constraints on the temperature and humidity ranges are enforced over some look-ahead time horizon with a predefined (high) probability with respect to the uncertain initial state and the stochastic input. Feasibility of the constraints is guaranteed by minimizing the temperature and humidity variability around the desired set-points, with the variability range on the humidity eventually enlarged when needed to squeeze the one on the temperature. The resulting quantized control is applied in a receding horizon fashion, leading to a closed-loop solution that integrates state filtering to reduce on the fly the uncertainty on the state.**

I. INTRODUCTION

Energy management is an interesting and challenging problem that has recently attracted the attention of many researchers in academy as well as in industry, with focus on microgrid operation, [1]–[5]. Exhaustion of energy resources and heavy environmental impact are two of the main issues that arise from the growing energy consumption. Almost of 40% of the total energy consumption in developed countries is due to buildings [6], a significant part of it being used by Heating, Ventilation and Air Conditioning (HVAC) systems for maintaining comfort conditions for the building occupants. Indeed, indoor air quality seems to have a direct impact on people productivity [7]. Thus a significant part of the research on optimal energy management has been focused on climate control in buildings (see e.g. [8]–[18])

Building energy regulations were established since the 1970s to guarantee a certain energy efficiency level [19]. Efficient management of HVAC units imposes new requirements on their configuration and operation: new components are added, on-off actuators are replaced by multi-staged ones, and more complex control specifications are given. Control specifications for an HVAC system are naturally given in terms of comfort conditions, i.e., desired ranges for temperature and humidity. Due to the limited control authority of HVAC multi-staged actuators, some priority order need to be assigned to the controlled variables and, in particular, temperature regulation has higher priority since temperature is more relevant to comfort than humidity.

HVAC systems have multiple components, and physical modeling based on first principles becomes quite challenging. A possible solution is to determine relevant operating conditions and identify for each of them a linear model that includes a stochastic input to account for actual noises and also model inaccuracy. The resulting model is a stochastic switched linear system, with switches determined by an endogenous signal in that it depends on the values taken by the state variables, [20], [21].

The joint presence of quantization of the control input, prioritization of constraints, stochastic input, and switching dynamics makes the problem hard to tackle with traditional control design methodologies.

In this paper, we focus on a single operating condition and propose a novel chance-constrained approach to solve the quantized control problem with prioritized constraints. Constraints on temperature and humidity are enforced in probability over some look-ahead time horizon, where the probability is induced on the system evolution by the uncertain initial state and the stochastic input. Since constraints on temperature and humidity might cause infeasibility of the chance-constraint optimization program, inspired by [22], [23], feasibility is enforced by optimizing the temperature and humidity ranges. This also allows to handle the prioritization of the temperature constraint over that on the humidity, by eventually enlarging the humidity range with respect to the desired range if this is needed to squeeze the one on the temperature. The resulting quantized control is applied in a receding horizon fashion, thus leading to a closed-loop solution which allows to incorporate state filtering and progressively reduce the uncertainty on the state.

A numerical instance of the problem shows the superiority of the proposed control design methodology against a Model Predictive Control (MPC) approach (see e.g. [24]) where prioritization is accounted for indirectly, via different weights on temperature and humidity in a quadratic average cost.

The rest of the paper is organized as follows. In Section II, we described the model of the HVAC system. In Section III we describe the control problem and the proposed solution, including algorithmic aspects related to the chance constrained optimization program solution via randomized techniques and its receding horizon application integrating state filtering. In Section IV, a numerical example is presented to show the effectiveness of the proposed approach.

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Some concluding remarks are drawn in Section V.

II. HVAC MODELING

HVAC systems are used to provide thermal comfort and suitable air quality inside a building. An HVAC is composed of one or more Roof Top Units (RTUs), which are devices used to regulate and circulate air in different zones of the building. Direct-expansion RTUs use a refrigerant vapor expansion-compression cycle to directly cool the supplied air [25], typically via two- to four-compressor units. A supply fan blows the air across the evaporator, which serves as a cooling coil. In the simplest case, the supply air is directly transported to the conditioned zone. Economiser dampers can be used to mix fresh outside air with air returned from the zone to alter the air properties at the intake of the cooling/heating coils. The considered RTU is equipped with a two-stage compressor, a multi-speed supply fan, and a modulating economiser. It comprises a controller that regulates temperature and humidity of the zone. Measurements of both zone temperature and humidity are made available to the controller and our aim here is to provide a new control algorithm for an RTU operating in a zone of a building.

A description of the controlled system based on first principles [26], [27] is given by

$$\begin{cases} C_{\mathrm{ZA}}\dot{T}_{\mathrm{ZA}} = P_{\mathrm{ZA}} - P_{\mathrm{RTU}} \\ \dot{w}_{\mathrm{ZA}} = h_{\mathrm{ZA}} - h_{\mathrm{RTU}} \end{cases} \tag{1}$$

where P, h, T, and w represent the heat gains, moisture gains, temperature, and absolute humidity, respectively; and the subscripts $(.)_{\mathrm{ZA}}$ and $(.)_{\mathrm{RTU}}$ refer to zone and RTU. In practice, these equations are not useful for control design purposes since they are quite involved when making the dependence on the control inputs (compressor power, speed of the fan, dampers positions in the economiser) and disturbance inputs (e.g. occupancy and weather conditions) explicit, and physical parameters entering the description are difficult to determine. For instance, the amount of heat P_{RTU} extracted from the supply air by the cooling coil depends in general on the compressor power, mixed air temperature and humidity, and supply airflow. In turn, properties of the mixed air at the intake of the cooling coil depend on the outside air and return air properties and the mixing ratio for the two air streams, as determined by the economiser position; and the airflow is a function of the fan speed and system resistance, which depends on the economiser positions and the pressure changes in the conditioned space due to windows opening/closing.

To the purpose of control design, first-principle equations are then replaced by a simpler approximate description constituted by a set of linear models to describe the behavior of the controlled system around specific operating points, which can be derived via black box identification. If this is done in a laboratory setup, large sources of uncertainty that affect the real operating system like, e.g., the occupancy and location (weather and shading) of the building, and interaction between conditioned zones, are neglected and an additive stochastic input is introduced to account for them. The controlled system (RTU operating in a zone) is hence reduced to a switching stochastic linear system that changes dynamics when commuting between different operating conditions and is subject to a quantized control input given by the multi-stage controls of the RTU.

In formulas, for each operating condition, the controlled system model is given by:

$$\begin{cases} \dot{\xi} = F\xi + G_{\upsilon}\upsilon + G_{\omega}\omega \\ \zeta = H\xi \end{cases} \tag{2}$$

where ξ is the state vector comprising zone temperature and humidity which are made available as output in ζ, the input υ comprises the discrete multi-stage controls of the RTU, and ω is a stochastic input used to capture inaccuracy of the model (2) with respect to (1) and short-term fluctuations in the operating conditions (primarily the loads P_{ZA} and h_{ZA}). Matrices F, G_{υ}, and G_{ω} depend on the operating condition.

In this paper, we focus on control design for a given operating condition.

III. QUANTIZED CONTROL WITH PRIORITIZED CONSTRAINTS

The addressed control problem consists in operating the HVAC system so as to maintain appropriate comfort conditions in the zone, i.e., to keep the zone temperature and humidity within prescribed ranges around given set points. Since the discrete nature of the available control inputs makes it difficult to keep both humidity and temperature within their prescribed ranges, we assign a different priority to the two controlled variables by allowing the humidity specification to be violated if this is needed to satisfy that on the temperature (prioritized constraints).

The control problem is defined over a finite horizon $[0, t_f]$ which is discretized in M time slot of length Δ_t. A discrete-time version of system (2) is then introduced where state and output variables are sampled every Δ_t time units.
If the control input υ is kept constant in each interval $[k\Delta_t, (k+1)\Delta_t)$ for $k = 0, \dots, M-1$, then a discrete-time system equivalent to (2) is given by

$$\begin{cases} x_{k+1} = Ax_k + B_u u_k + B_d d_k \\ y_k = Cx_k \end{cases} \tag{3}$$

where matrices A, B_u, and B_d are given by

$$A = e^{F\Delta_t},\ B_u = \int_0^{\Delta_t} e^{F\tau}G_{\upsilon}d\tau,\ B_d = \int_0^{\Delta_t} e^{F\tau}G_{\omega}d\tau, \tag{4}$$

and we set $x_k = \xi(k\Delta_t)$, $u_k = \upsilon(k\Delta_t)$, $y_k = \zeta(k\Delta_t)$, $d_k = \omega(k\Delta_t)$, assuming that ω is constant over each sample interval, $k = 0, \dots, M-1$. The control input u_k takes value in some discrete set U. The initial condition x_0 may be uncertain and characterized as a random variable with a certain probability distribution.

The output y_k is composed of the temperature and humidity variables, which are denoted in the following as y_k^T and y_k^H, respectively: $y_k = [y_k^T\ y_k^H]^\top$. Since the system is linear, we can assume without loss of generality that the desired ranges for y_k^T and y_k^H are both symmetric and centered around zero, namely $[-\bar{T}, \bar{T}]$ and $[-\bar{H}, \bar{H}]$, respectively. Our ideal goal is then to design the control input u_k, $k = 0, \ldots, M-1$, so as to enforce the following constraints:

$$|y_k^T| \leq \bar{T},\ |y_k^H| \leq \bar{H},\ k = 0, 1, \ldots, M.$$

Note, however, that these are constraints posed on variables that depend on the uncertain initial state and stochastic input d_k realizations. To make this dependence explicit, let us introduce some compact notations.

Set

$$\boldsymbol{u} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_M \end{bmatrix}.$$

If we then unroll the dynamics of (3) along the discrete time horizon starting from the initial state x_0, we easily get

$$\boldsymbol{y} = \mathcal{A}x_0 + \mathcal{B}_u \boldsymbol{u} + \mathcal{B}_d \boldsymbol{d},$$

where matrices $\mathcal{A}$, $\mathcal{B}_u$ and $\mathcal{B}_d$ are defined as follows

$$\mathcal{A} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^M \end{bmatrix},$$

$$\mathcal{B}_u = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB_u & 0 & \cdots & 0 & 0 \\ CAB_u & CB_u & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ CA^{M-2}B_u & CA^{M-3}B_u & \cdots & CB_u & 0 \\ CA^{M-1}B_u & CA^{M-2}B_u & \cdots & CAB_u & CB_u \end{bmatrix},$$

$$\mathcal{B}_d = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB_d & 0 & \cdots & 0 & 0 \\ CAB_d & CB_d & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ CA^{M-2}B_d & CA^{M-3}B_d & \cdots & CB_d & 0 \\ CA^{M-1}B_d & CA^{M-2}B_d & \cdots & CAB_d & CB_d \end{bmatrix}.$$

To account for the uncertainty affecting the system evolution, one can opt either for hard constraints or for soft constraints: in the case of hard constraints, they must hold for every and each uncertainty instance, even for very unlikely realizations, while in the case of soft constraints, they are expressed in probability and must hold on a set of uncertainty instances of predefined probability at least $1 - \varepsilon$, with $\varepsilon \in (0, 1)$ set by the user. Since the hard constraint solution may be conservative and hard constraints are indeed not feasible when the stochastic input d_k has unbounded support (d_k enters additively the output and its contribution cannot be canceled exactly), we head for a soft constraint formulation of the form

$$\mathbb{P}_{(x_0,\boldsymbol{d})}\{|\boldsymbol{y}^T| \leq \bar{\boldsymbol{T}},\ |\boldsymbol{y}^H| \leq \bar{\boldsymbol{H}}\} \geq 1 - \varepsilon, \tag{5}$$

with $\mathbb{P}_{(x_0,\boldsymbol{d})}$ denoting the joint probability distribution of the uncertain initial state and the stochastic input. Here

$$\boldsymbol{y}^T = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ y_M^T \end{bmatrix}, \qquad \boldsymbol{y}^H = \begin{bmatrix} y_0^H \\ y_1^H \\ \vdots \\ y_M^H \end{bmatrix},$$

$\bar{\boldsymbol{T}}$ and $\bar{\boldsymbol{H}}$ are column vectors with $M+1$ elements all equal to $\bar{T}$ and $\bar{H}$, respectively, and absolute value and inequalities should be interpreted componentwise.

Still, it might be the case that the probabilistic constraint (5) is unfeasible since it is violated at time $k = 0$ (the air in the zone starts to be controlled at time $k = 0$) or along the control horizon because of the limited actuation capabilities of the control system and the unboundedness of the stochastic input. Inspired by [22], [23], we address this feasibility issue by relaxing the constraint (5) and replacing the threshold values $\bar{T}$ and $\bar{H}$ with optimization variables, say h_k^T and h_k^H, $k = 0, 1, \ldots, M$, representing the bounds on the temperature and the humidity, that are minimized via the introduction of an appropriate cost function.

Interestingly, we can exploit constraint relaxation to account for prioritization of the control specifications. More specifically, we can give more weight to the minimization of the bounds on the temperature with respect to those on the humidity in the cost function, and impose that the bounds on the humidity are not smaller than the desired $\bar{H}$ value. This way, the variability range of the humidity is possibly enlarged with respect to the desired range so as to squeeze that on the temperature.

This finally leads to the following formulation of the control problem:

$$\begin{aligned} &\min_{\boldsymbol{u} \in U^M, \boldsymbol{h} \in \mathbb{R}^{2(M+1)}} \boldsymbol{h}^\top \mathcal{W} \boldsymbol{h} \\ &\text{subject to:} \quad \mathbb{P}_{(x_0,\boldsymbol{d})}\{|\mathcal{A}x_0 + \mathcal{B}_u \boldsymbol{u} + \mathcal{B}_d \boldsymbol{d}| \leq \boldsymbol{h}\} \geq 1 - \varepsilon \\ &\qquad\qquad\quad \boldsymbol{h}^H \geq \bar{\boldsymbol{H}} \end{aligned} \tag{6}$$

where the optimization variables are given by the control input $\boldsymbol{u}$ taking values in the discrete set U^M, and by the bounds on temperature and humidity that are collected in $\boldsymbol{h} = \begin{bmatrix} h_0^\top & h_1^\top & \ldots & h_M^\top \end{bmatrix}^\top$ with $h_k = [h_k^T\ h_k^H]^\top \in \mathbb{R}^2$. Vector $\boldsymbol{h}^H$ appearing in the optimization problem (6) comprises only the bounds on the humidity, i.e., $\boldsymbol{h}^H = \begin{bmatrix} h_0^H & h_1^H & \ldots & h_M^H \end{bmatrix}^\top$. $\mathcal{W}$ is a block diagonal matrix with on the diagonal 2×2 positive definite matrices W_k, each one weighting h_k, $k = 0, 1, \ldots, M$. Matrices W_k can be chosen to be all equal to the diagonal matrix $W = \text{diag}(w_T, w_H)$ with $w_T \gg w_H > 0$ so as to weight more the temperature bounds than the humidity bounds.

Note that (6) is a chance-constrained optimization program in that it involves a bound in probability. Also, it is a mixed integer optimization problem since some of the decision variables are discrete. These two aspects pose some challenges to the solution of (6) that will be addressed in the next subsection.

A. Scenario solution to the chance-constrained optimization

Chance-constrained optimization problems are known to be hard to solve except for few particular cases, [28], [29]. In this work we resort on a randomized technique, known as the scenario approach, [30]–[33], to approximately solve (6). Notably, precise guarantees can be given on the feasibility of the scenario solution for the original chance-constrained problem (6).

We next briefly recall the results on the scenario theory given in the literature that are relevant to our problem. Consider a chance-constrained optimization problem of the form

$$\begin{aligned} \min_{\vartheta\in\mathbb{R}^d} \quad & f(\vartheta) \\ \text{subject to:} \quad & \mathbb{P}_\delta\{\vartheta\in\Theta_\delta\}\geq 1-\varepsilon \end{aligned} \tag{7}$$

where $f(\cdot)$ is a convex function, Θ_δ is a convex set depending on an uncertain parameter δ, which takes values in a set Δ according to a (possibly unknown) probability distribution $\mathbb{P}_\delta$.

The idea of the scenario approach is as simple as follows. Suppose that N samples $\delta^{(1)},\ldots,\delta^{(N)}$ of the uncertain parameter drawn independently according to $\mathbb{P}_\delta$ are available. Then, a randomized solution to (7) can be found by solving the following convex optimization program

$$\begin{aligned} \min_{\vartheta\in\mathbb{R}^d} \quad & f(\vartheta) \\ \text{subject to:} \quad & \vartheta\in\Theta_{\delta^{(i)}} \quad i=1,\ldots,N \end{aligned} \tag{8}$$

and the following Theorem 1 establishes a link between the solution to (8) and its feasibility for (7).

Theorem 1 (Scenario Guarantees): Let problem (8) be feasible for every multi-sample extraction $\delta^{(1)},\ldots,\delta^{(N)}$. Choose a *confidence parameter* $\beta\in(0,1)$. If N is selected so as to satisfy

$$\sum_{i=0}^{d}\binom{N}{i}\varepsilon^i(1-\varepsilon)^{N-i}\leq\beta, \tag{9}$$

where d is the number of decision variables, then, with probability at least $1-\beta$, the solution $\vartheta_N^\star$ to (8) is feasible for (7). ■

Note that the explicit bound

$$N\geq\frac{d+1+\ln(1/\beta)+\sqrt{2(d+1)\ln(1/\beta)}}{\varepsilon} \tag{10}$$

derived in [34] from (9) shows that the dependence on the confidence parameter β is logarithmic so that β can be set very small, say $\beta=10^{-6}$, to get a result that holds deterministically (with probability $\simeq 1$) without having a too large sample size N.

The scenario solution $\boldsymbol{h}_N^\star$, $\boldsymbol{u}_N^\star$ to problem (6) is then obtained via the optimization program

$$\begin{aligned} \min_{\boldsymbol{u}\in U^M,\ \boldsymbol{h}\in\mathbb{R}^{2(M+1)}} \quad & \sum_{k=0}^{M} h_k W h_k^\top \\ \text{subject to:} \quad & |\mathcal{A}x_0^{(i)}+\mathcal{B}_u\boldsymbol{u}+\mathcal{B}_d\boldsymbol{d}^{(i)}|\leq\boldsymbol{h}, i=1,\ldots,N \\ & \boldsymbol{h}^H\geq\bar{\boldsymbol{H}} \end{aligned} \tag{11}$$

where $(x_0^{(i)},\boldsymbol{d}^{(i)})$, $i=1,\ldots,N$ are independently extracted from $\mathbb{P}_{(x_0,\boldsymbol{d})}$.

Unfortunately, the optimization variables $\boldsymbol{u}$ in problem (6) are discrete so that Theorem 1 does not apply directly to the scenario solution $\boldsymbol{h}_N^\star$, $\boldsymbol{u}_N^\star$. However, we can easily generalize Theorem 1 to our setting by considering $|U|^M$ instances of problem (6) (here, $|U|$ denotes the cardinality of the discrete set U), one for each possible value $\boldsymbol{u}^j$, $j=1,\ldots,|U|^M$, of $\boldsymbol{u}\in U^M$:

$$\begin{aligned} \min_{\boldsymbol{h}\in\mathbb{R}^{2(M+1)}} \quad & \sum_{k=0}^{M} h_k W h_k^\top \end{aligned} \tag{12}$$

subject to:

$$\mathbb{P}_{(x_0,\boldsymbol{d})}\{|\mathcal{A}x_0+\mathcal{B}_u\boldsymbol{u}^j+\mathcal{B}_d\boldsymbol{d}|\leq\boldsymbol{h}\wedge\boldsymbol{h}^H\geq\bar{\boldsymbol{H}}\}\geq 1-\varepsilon.$$

Let us denote problem (12) as P_C^j. A scenario solution to P_C^j can be computed with the guarantees provided by Theorem 1 since the assumptions of the theorem are now satisfied. In particular, the solution $\boldsymbol{h}_{N,j}^\star$ to the scenario version

$$\min_{\boldsymbol{h}\in\mathbb{R}^{2(M+1)}} \sum_{k=0}^{M} h_k W h_k^\top$$

subject to:

$$|\mathcal{A}x_0^{(i)}+\mathcal{B}_u\boldsymbol{u}^j+\mathcal{B}_d\boldsymbol{d}^{(i)}|\leq\boldsymbol{h},\ i=1,\ldots,N,$$
$$\boldsymbol{h}^H\geq\bar{\boldsymbol{H}}$$

of problem P_C^j in (12) satisfies

$$\mathbb{P}_{(x_0,\boldsymbol{d})}\{|\mathcal{A}x_0+\mathcal{B}_u\boldsymbol{u}^j+\mathcal{B}_d\boldsymbol{d}|\leq\boldsymbol{h}_{N,j}^\star\}\geq 1-\varepsilon \tag{13}$$

with probability at least $1-\beta$ if N is chosen according to (9) with $d=2(M+1)$. Now, the solution $\boldsymbol{h}_N^\star$, $\boldsymbol{u}_N^\star$ to (11) can be obtained as $\boldsymbol{h}_N^\star=\boldsymbol{h}_{N,j_N^\star}^\star$ and $\boldsymbol{u}_N^\star=\boldsymbol{u}^{j_N^\star}$ where

$$j_N^\star=\arg\min_{j\in\{1,2,\ldots,|U|^M\}}\sum_{k=0}^{M} h_{N,j\,k}^\star W h_{N,j\,k}^{\star\,\top}$$

and, hence, it satisfies

$$\mathbb{P}_{(x_0,\boldsymbol{d})}\{|\mathcal{A}x_0+\mathcal{B}_u\boldsymbol{u}_N^\star+\mathcal{B}_d\boldsymbol{d}|\leq\boldsymbol{h}_N^\star\}\geq 1-\varepsilon,$$

with probability at least $1-|U|^M\beta$, since conditions (13), $j=1,\ldots,|U|^M$, hold jointly with such a probability.
This finally leads to the following statement.

Proposition 1: Choose a *confidence parameter* β. If N is selected so as to satisfy

$$\sum_{i=0}^{2(M+1)}\binom{N}{i}\varepsilon^i(1-\varepsilon)^{N-i}\leq\frac{\beta}{|U|^M}, \tag{14}$$

then, with probability at least $1-\beta$, the solutions $\boldsymbol{h}_N^\star$ and $\boldsymbol{u}_N^\star$ to (11) are feasible for the original chance-constrained mixed integer program (6). ■

Since N satisfying (14) depends logarithmically on $\frac{\beta}{|U|^M}$, then, N scales linearly with the time horizon length M (see (10) where d should be set equal to $2(M+1)$ and β should be replaced by $\frac{\beta}{|U|^M}$).

B. Receding horizon implementation with state filtering

Due to the presence of the stochastic input $\boldsymbol{d}$, a closed-loop solution would be more desirable for control purposes rather than an open-loop one. We here adopt a receding-horizon approach in which, at each time step ℓ, a new instance of problem (6) is solved over the shifted time window $[\ell, M+\ell]$, the first control action $u_\ell^\star$ is applied and then the procedure is repeated at $\ell+1$. Note that when solving problem (6) over the shifted time window $[\ell, M+\ell]$, the initial state x_0 becomes x_ℓ and the stochastic input realization $\boldsymbol{d}$ contains shifted samples of d_k, $k=\ell,\ell+1,\dots,\ell+M-1$. Some knowledge is acquired on the probability distributions of the shifted initial state x_ℓ and of the shifted stochastic input realization $\boldsymbol{d}$, if d_k is a correlated process, so that the a-posteriori probability distributions of state and stochastic input given the past output observations can be used to get feedback in the receding horizon implementation.

Filtering techniques can be adopted to determine the a-posteriori distribution of the current state and future stochastic input realizations given the past output observations. Interestingly, given the adopted scenario solution to the chance-constrained optimization program, we actually need N samples extracted from such a-posteriori distributions so that particle filtering techniques can be adopted, [35], [36]. Particle filtering are indeed of general applicability. If the stochastic input d_k is modeled as a colored Gaussian process obtained by filtering a white Gaussian process e_k with a linear system and the initial state x_0 is Gaussian, the state of the system can be enlarged to include the input d_k (see the numerical example in Section IV), and Kalman filtering can then be applied to get covariance and mean of the a-posteriori enlarged state distribution, see e.g. [37].

IV. NUMERICAL EXAMPLE

In this section we present some simulation results on a numerical instance of the problem to show the effectiveness of the proposed approach.

The considered RTU is equipped with a two-stage compressor, a multi-speed supply fan, and a modulating economiser. For the sake of simplicity, the economiser is considered fixed. The controlled system operating in some nominal condition and the adopted continuous time linear system model (2) has order $n=4$. Its state $\xi=[\xi_1\ \xi_2\ \xi_3\ \xi_4]^\top$ comprises the temperature (ξ_1) and the humidity (ξ_3), which are provided as output of the system $\zeta=[\zeta_1\ \ \zeta_2]^\top$. The control input $\upsilon=[\upsilon_1\ \upsilon_2]^\top$ includes the speed of the supply fan and the compressor, both taking three possible values: OFF, LOW, HIGH, which are coded here as -50, 0, 50. The disturbance ω is modeled as a Gaussian process. The matrices of system (2) are as follows

$$F = 10^{-4}\begin{bmatrix} -28 & -5.6 & 0 & 0 \\ 0 & -8.3 & 0 & 0 \\ 0 & 0 & -17 & 1 \\ 0 & 0 & 0 & -2.8 \end{bmatrix}$$

$$G_\upsilon = G_\omega = 10^{-4}\begin{bmatrix} -0.8 & -1.7 \\ 0 & 5.8 \\ -1.7 & 0.08 \\ 0 & 2.3 \end{bmatrix} \qquad \tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The discretization time interval Δ_t is set equal to 5 minutes. Matrices of the discrete time model (3) can be derived via (4). Note that $B_u = B_d = B$ in this example.

The stochastic input d_k acting on (3) is modeled as the following filtered Gaussian process

$$d_{k+1} = A_d d_k + e_k, \tag{15}$$

where $A_d = \mathrm{diag}(a,a)$ with $a=0.9835$, and e_k is a white Gaussian process $\mathcal{N}(0_2, \sigma_e^2 I_2)$, 0_2 being the zero vector with two components, I_2 the identity matrix of order two, and $\sigma_e^2 = 0.6147$. Filter (15) is assumed to be initialized with the stationary distribution: $d_0 \sim \mathcal{N}(0_2, \sigma_d^2 I_2)$, where $\sigma_d^2 = \sigma_e^2/(1-a^2)$. Figure 1 plots a realization of the stochastic input for reference.

The desired bounds on temperature and (relative) humidity are $\bar{T} = 0.5$°C and $\bar{H} = 5\%$. The weighting matrix W entering the definition of the chance-constrained problem (6) is set equal to $W=\mathrm{diag}(10,1)$ so as to give priority to the temperature control. The discrete control horizon length is set equal to $M=3$. A receding horizon implementation using Kalman filtering is adopted.

Since the dynamic of the stochastic input d_k is known, we can augment the state of the system including d_k. The augmented state is defined as $\bar{x}_k = [x_k^\top\ d_k^\top]^\top$.
From (3) and (15), we get the following equivalent system

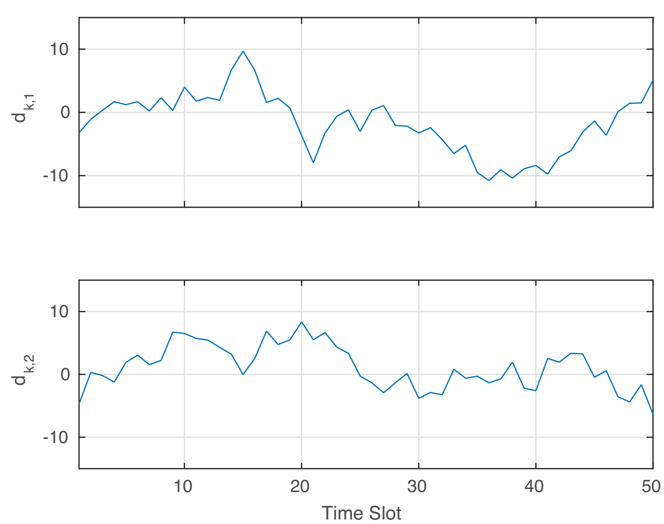

Fig. 1. A stochastic input realization.

where the original state x_k is replaced by the augmented state $\bar{x}_k$ and the stochastic input dynamic is included:

$$\begin{cases} \bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}u_k + \bar{S}e_k \\ y_k = \bar{C}\bar{x}_k \end{cases} \quad (16)$$

e_k being the white Gaussian noise feeding (15). Matrices $\bar{A}$, $\bar{B}$, $\bar{S}$ and $\bar{C}$ are given by:

$$\bar{A} = \begin{bmatrix} A & B \\ 0 & A_d \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{S} = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}, \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \quad (17)$$

with 0 representing a zero matrix of appropriate dimensions. A reduced order filter can then be derived to obtain an estimate of the components, say $\tilde{x}_k$ of the state $\bar{x}_k$ that are not available as measurements. If the initial state is Gaussian and the gain of the filter is set equal to the Kalman gain, the obtained estimate of $\tilde{x}_k$ is the mean of its a-posteriori Gaussian distribution given the output observations up to time k, and the variance of such a distribution can be obtained via the Riccati equations used to set the Kalman gain, see [38] for details of the implementation. The realizations adopted at time k for the scenario solution to the chance-constrained optimization are then obtained by sampling from this a-posterior distribution. In particular, the realizations of $\boldsymbol{d}$ are generated by initializing system (15) with the extracted samples of the d_k component of $\bar{x}_k^{(i)}$.

In the following results, the initial state $x_0 = [x_{1,0}\, x_{2,0}\, x_{3,0}\, x_{4,0}]^\top$ of the system has a Gaussian distribution with all independent components with zero mean and standard deviation 0.17 for $x_{1,0}$ and 1.7 for the others, and $\varepsilon = 0.1$. In the scenario implementation, we set $\beta = 10^{-6}$. For comparative purposes, we consider the average cost function

$$J(\boldsymbol{u}) = \mathbb{E}_{(x_0,\boldsymbol{d})}[\boldsymbol{y}^\top \mathcal{W} \boldsymbol{y}]$$

and the (standard) MPC controller obtained by minimizing $J(\boldsymbol{u})$ and implementing the solution in a receding horizon fashion, integrating state filtering via Kalman filtering as in the proposed chance-constrained approach.

For comparative purposes, the two control strategies are applied using the same initial state and the same realization for the stochastic input, over a time horizon of 250 minutes ($L = 50$ sampled times). Figures 2 and 3 represent the output and control input (u^C for the compressor and u^F for the fan) obtained by applying the proposed approach and the standard MPC approach, respectively. Note that in the case of the approach proposed in this paper, humidity stay close to the boundary of the desired $[-\bar{H}, \bar{H}]$ range to keep the temperature closer to 0 and hence inside $[-\bar{T}, \bar{T}]$, whereas in the standard MPC approach the humidity is better centered in the range $[-\bar{H}, \bar{H}]$ but the temperature has higher fluctuations which brings it outside $[-\bar{T}, \bar{T}]$.

We applied $N_v = 1981$ times the two control laws to different realizations of the initial state x_0 and stochastic input $\{d_k,\ k = 0, 1, \dots, L-1\}$ and computed the expected average temperature violation as

$$\frac{1}{N_v}\sum_{i=1}^{N_v}\left[\frac{1}{L}\sum_{k=0}^{L}\left(1 - \mathbf{I}_{[-\bar{T},\bar{T}]}(y_k^{T^{(i)}})\right)\right]$$

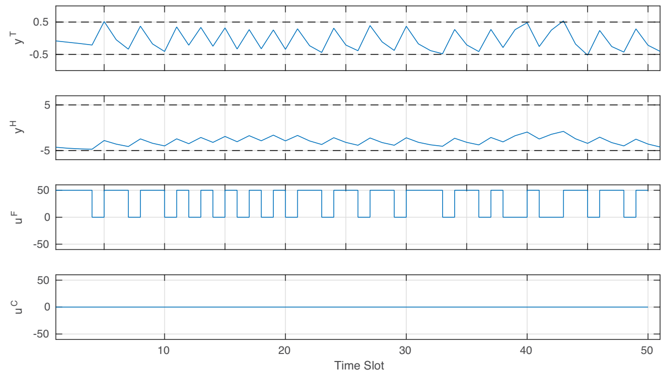

Fig. 2. Proposed chance-constrained approach.

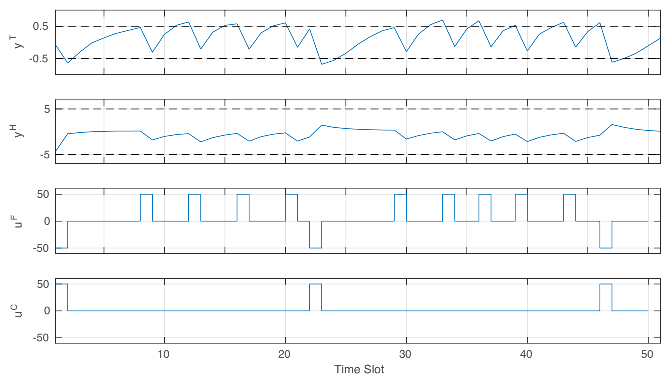

Fig. 3. Standard average approach

where $\mathbf{I}_{[\ell_1,\ell_2]}(\cdot)$ is the indicator function of the interval $[\ell_1, \ell_2]$. Results are shown in Figure 4 plotting the expected average temperature violations as a function of the thresholds $\bar{T}$ in the range $[0.3, 0.6]$ for the two approaches. Note that the curve obtained with the standard MPC approach is above that obtained with the proposed approach for all threshold values $\bar{T} \in [0.3, 0.6]$.

V. CONCLUSION

In this paper we introduced a novel control design strategy that is suitable for a heat, ventilation and air conditioning system. Due to the multi-stage nature of its actuators, the system is characterized by limited control capabilities, which calls for a prioritization of control goals, when aiming at the regulation of multiple variable at some desired set-point. The main features of the proposed approach can be summarized as follows:

- *quantized nature of the control input is accounted for*: the control design problem is formulated as an optimization problem over a finite-horizon with respect to the discrete control inputs;
- *probabilistic prioritized constraints on the state are incorporated in the design*: constraints on variables affected by possibly unbounded stochastic inputs are introduced, their feasibility is guaranteed by adding optimizations variables that relax the constraints if needed, and the constraint on the variable with lower priority is

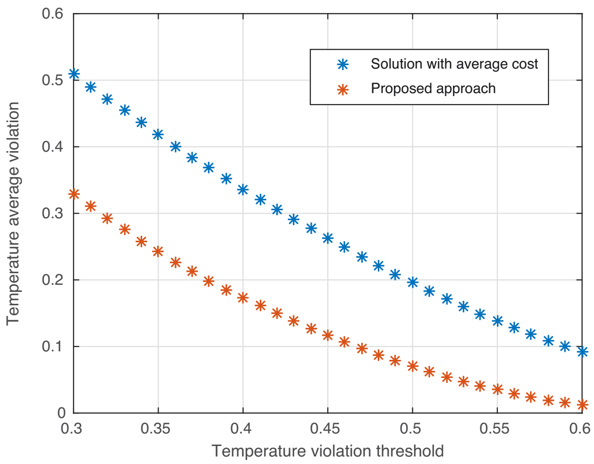

Fig. 4. Average violation for the temperature.

made loose so as to favor the one on the variable with higher priority;

- *receding horizon implementation with state filtering*: the finite-horizon solution is recomputed based on the updated information on the uncertainty obtained via state filtering, and only the first control sample is applied. This allows to obtain a closed-loop controller that can better counteract uncertainty.

The effectiveness of the proposed approach was shown on a numerical case study. Real experiments should be run to better assess its performance and practical impact. Note that in this paper, we focus on control of the HVAC system in a given operating condition. Further work is needed to account for changes in the operating condition. A possible solution is to adopt an adaptive switching mechanism based on state estimation, [39], [40], [41]. This solution has been explored in [38] with reference to a particular setting where the same model of the stochastic input is adopted for the systems associated to different operating conditions. Results are preliminary but appear promising. Alternatively, one could adopt a Linear Parametric Varying (LPV) model (see e.g. [42]) for the HVAC system that varies with continuity in a family of linear systems. This would call for a gain scheduling control solution, which should, however, be able to cope with probabilistic prioritized constrained and quantized control inputs.

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