On weakly complete surfaces

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Abstract

A weakly complete space is a complex space endowed with a plurisubharmonic exhaustion function. In this paper, we classify the weakly complete surfaces for which such exhaustion function can be chosen to be real analytic: they can be modifications of Stein spaces or proper a non compact complex curve or surfaces of Grauert type i.e. foliated with real analytic Levi flat hypersurfaces whose Levi foliation has dense complex leaves. In the last case, we also show that such Levi flat hypersurfaces are in fact level sets of a global proper pluriharmonic function, up to passing to a holomorphic double covering. *To cite this article: S. Mongodi, Z. Slodkowski, G. Tomassini C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Sur les surfaces faiblement complètes. Un espace complexe est dit faiblement complet s'il est muni d'une fonction d'exhaustion plurisousharmonique. Dans ce papier on classifie les surfaces complexes faiblement complètes qui admettent une fonction d'exhaustion plurisousharmonique et analytique réelle. Elles sont des types suivants : modifications des espaces de Stein, surfaces complexes propres sur des courbes complexes non compactes, ou bien surfaces complexes de type Grauert i.e. feuilletées par des hypersurfaces Levi plates dont les feuilles du feuilletage de Levi sont partout denses. Dans ce dernier cas on montre aussi que, sauf à passer à un double revêtement, les hypersurfaces Levi plates sont en fait les niveaux d'une fonction pluriharmonique globale. *Pour citer cet article : S. Mongodi, Z. Slodkowski, G. Tomassini C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Version française abrégée

Un espace complexe (connexe) X est dit faiblement complet s'il est muni d'une fonction d'exhaustion lisse et plurisousharmonique. Les domains faiblement complets d'un espace de Stein sont des espaces de Stein ; les espaces complexes propres sur un espace de Stein (i.e. munis d'une projection holomorphe et propre sur un espace de Stein), en particulier les modifications des espaces de Stein, sont faiblement complets. Des exemples plus intéressants sont les domaines pseudoconvexes du tore construit par Grauert. Dans le premier cas $\mathcal{O}(X) \neq \mathbb{C}$; dans le deuxième on a $\mathcal{O}(X) = \mathbb{C}$ et il existe une fonction d'exhaustion pluriharmonique dont les niveaux sont des hypersurfaces Levi plates feuilletées par des feuilles qui sont partout denses (lelong des quelles la form de Levi est dégénérée). Dans cette situation on dit que X est un espace de *type Grauert*.

Il est naturel de poser la question si pour un espace faiblement complet les exemples ci-dessus sont les "seuls possibles". En cette généralité le problème semble très difficile même pour les surfaces. Dans ce papier on répond à la question par l'affirmative pour les surfaces faiblement complètes qui admettent une fonction d'exhaustion plurisousharmonique et analytique réelle. À savoir on démontre le théorème suivant :

Théorème principal *Toute surface X faiblement complète, munie d'une fonction d'exhaustion plurisousharmonique et analytique réelle α, est de l'un des types suivants :*

i) une modification d'un espace de Stein ;

ii) une surface propre sur une courbe complex (en général singulière) ;

iii) une surface de type Grauert.

De plus, dans le cas iii), *si le sous ensemble critique* $\mathrm{Crt}(\alpha)$ *de α est de dimension ≤ 2, alors*

iii-a) l'ensemble Z des points de minimum de α est une courbe complexe et il existe une fonction pluriharmonique et propre $\chi : X \setminus Z \to \mathbb{R}$ telle que toute fonction plurisousharmonique sur $X \setminus Z$ soit de la forme $\gamma \circ \chi$;

si $\mathrm{Crt}(\alpha)$ *est de dimension 3, alors*

iii-b) il existe a double revêtement holomorphe $\pi : X^* \to X$ et une fonction pluriharmonique et propre $\chi^* : X^* \to \mathbb{R}$ telle que toute fonction plurisousharmonique sur X^* soit de la forme $\gamma \circ \chi^*$.

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1 Introduction

A *weakly complete space* is a connected complex space X endowed with a smooth plurisubharmonic exhaustion function φ. Interesting examples of weakly complete spaces which are not Stein are the pseudoconvex subdomains of a complex torus constructed by Grauert. In the former, $\mathcal{O}(X) \neq \mathbb{C}$; in the latter, $\mathcal{O}(X) = \mathbb{C}$ and there exists a smooth plurisubharmonic exhaustion function φ whose regular level sets are Levi flat hypersurfaces, foliated by dense complex leaves (along which the Levi form of φ degenerates). In such a situation X is said to be a space of *Grauert type*.

A question naturally arises: are these two phenomena the only possible for a weakly complete space? Even if this problem has never been explicitly addressed, we find some partial results throughout the literature. See for instance [4], [2].

The general problem is already hard for complex surfaces thus, it seems worthwile to analyse the special case of surfaces admitting a *real analytic* plurisubharmonic exhaustion function. Observe that Brunella (cfr.[1]) constructed an example of weakly complete surface which does not admit a real analytic plurisubharmonic exhaustion function.

Under these hypothesis we obtain the following

Main Theorem *Let X be a weakly complete surface, with a real analytic plurisubharmonic exhaustion function α. Then one of the following three cases occurs:*

i) X is a modification of a Stein space of dimension 2,

ii) X is proper over a (possibly singular) open complex curve,

iii) X is a Grauert type surface.

Moreover, in case iii), *either the critical set* $\mathrm{Crt}(\alpha)$ *of α has dimension ≤ 2 and then*

iii-a) the absolute minimum set Z of α is a compact complex curve $Z \subset X$ and there exists a proper pluriharmonic function $\chi : X \setminus Z \to \mathbb{R}$ such that every plurisubharmonic function on $X \setminus Z$ is of the form $\gamma \circ \chi$,

or it is of dimension 3 and then

iii-b) there exist a double holomorphic covering map $\pi : X^* \to X$ and a proper pluriharmonic function $\chi^* : X^* \to \mathbb{R}$ such that every plurisubharmonic function on X^* is of the form $\gamma \circ \chi^*$.

The method we adopted to tackle the problem consists of a careful analysis of the structure of the level sets of α and their behaviour with respect to the *minimal singular set*. This set was introduced in [5] for any weakly complete space; let us recall its definition. Given any plurisubharmonic exhaustion function φ

in X, let Σ^1_φ be the minimal closed set such that φ is strictly plurisubharmonic on $X \setminus \Sigma^1_\varphi$, and set $\Sigma^1 = \Sigma^1(X) = \bigcap_\varphi \Sigma^1_\varphi$. A plurisubharmonic exhaustion function φ is called *minimal* if $\Sigma^1 = \Sigma^1_\varphi$. The following crucial properties were proved in [5] when X is a complex manifold:

a) there exist minimal functions φ (cfr. [5, Lemma 3.1]);
b) if φ is minimal the nonempty level sets $\Sigma^1_c = \{\varphi = c\} \cap \Sigma^1$ have the local maximum property (cfr. [5, Theorem 3.6]);
c) if $\dim_{\mathbb{C}} X = 2$ and c is a regular value of φ, then the (nonempty) level sets Σ^1_c are compact sets foliated by Riemann surfaces (cfr. [5, Lemma 4.1]).

If we restrict ourselves to the real analytic category, while property a) still holds, the proofs for b) and c) given in [5] do not apply anymore. However, if X is a weakly complete surface, it is possible to show that b) and c) hold for any plurisubharmonic exhaustion function and not just for the minimal ones: this is an important point for the proof.

2 Levi flat levels

We link Σ^1 to the Levi flatness of the levels of α by studying the complex foliation induced by the degeneracy of the Levi form. This allows us to the following

Theorem 2.1 *Let X be a complex surface with a real analytic plurisubharmonic exhaustion function $\alpha : X \to \mathbb{R}$ and Y a connected component of the regular level set $\{\alpha = c\}$ such that $Y \cap \Sigma^1 \neq \varnothing$. Assume that Y does not contain compact complex curves. Then Y is Levi flat, the leaves of the Levi foliation are dense in Y and $Y \subseteqq \Sigma^1$.*

If some regular Levi flat level Y of α contains a compact complex curve C we have the folllowing

Theorem 2.2 *X is proper over a (possibly singular) complex curve.*

— We first prove that there is exist a neighborhood V of C and holomorphic function $G : V \to \mathbb{C}$ such that $C = \{G = 0\}$.
— The family $\mathfrak{F}_0 = \{G = \zeta, |\zeta| < \epsilon\}$, for some $\epsilon > 0$, consists of compact complex curves so, by [3, III.5.B], $\mathfrak{F}_0$ extends to a family $\mathfrak{F}$ globally defined on X by a holomorphic map $\Phi : X \to R$, where R is an open Riemann surface. In particular, X admits a non constant holomorphic function.
— In view of [4], X is holomorphically convex hence proper over a Stein space.

As a consequence we have the following fundamental corollary

Corollary 2.3 *Let X be a complex surface, α a real analytic plurisubharmonic exhaustion function, Y a compact connected component of the regular level set $\{\alpha = c\}$. Assume that Y is Levi flat and contains a non closed leaf. Then there exist a pluriharmonic function $\chi : V \to \mathbb{R}$ on a connected neighbourhood of Y and $\epsilon_0 > 0$ such that*

a) $V \cap \{\alpha = c\} = Y,\ Y = \{\chi = 0\}$

b) *the set*
$$H = \{p \in V : 0 < \chi(p) < \epsilon_0\} \ ,$$
is relatively compact in V, does not contain a critical point of α or χ, and does not contain any compact complex curve;

c) *if $0 < \epsilon < \epsilon_0$, the Levi flat hypersurface $\{\chi = \epsilon\}$ is foliated by dense complex leaves and α is constant on it;*

d) $\partial\bar\partial\alpha \wedge \partial\bar\partial\alpha = 0$ *and* $\partial\alpha \wedge \bar\partial\alpha \wedge \partial\bar\partial\alpha = 0$ *on the whole of X;*

e) *there is a real analytic function $\mu : H \to \mathbb{R}$ such that*
$$\partial\bar\partial\alpha = \mu\partial\alpha \wedge \bar\partial\alpha \quad \text{and} \quad d\mu \wedge d\alpha = 0\ .$$

3 Proof of the Main Theorem

The above results allows to prove the following theorem, which is the first part of the Main Theorem:

Theorem 3.1 *Let X be a weakly complete surface and $\alpha : X \to \mathbb{R}$ a real analytic plurisubharmonic exhaustion function. Then three cases can occur:*

1) X is a modification of a Stein space;

2) X is proper over a (possibly singular) complex curve;

3) the connected components of the regular levels of α are foliated with dense complex curves, i.e. X is of Grauert type.

Proof. (Sketch.) Let us suppose that there exists a sequence of real numbers $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $c_n \to +\infty$ and $\{\alpha = c_n\} \cap \Sigma^1 = \emptyset$. Let φ be minimal for X. For every n there is $\epsilon_n > 0$ small enough such that $\{\alpha + \epsilon_n\varphi = c_n\} \cap \Sigma^1 = \emptyset$. Since the function $\alpha + \epsilon_n\varphi$ is minimal as well, $X_n = \{\alpha + \epsilon_n\varphi < c_n\}$ is a relatively compact strictly pseudoconvex domain (which we can suppose smoothly bounded up to some small perturbation of c_n), hence it is a modification of a Stein surface. Moreover, X_n is Runge in X_{n+1}, therefore X itself is holomorphically convex and, possessing a plurisubharmonic function which is strictly plurisubharmonic at some point, it has to be a modification of a Stein space as well: this is the case 1).

Assume that there is $c_0 \in \mathbb{R}$ such that $\{\alpha = c\} \cap \Sigma^1 \neq \varnothing$ for every $c > c_0$. Suppose that there exists $c_1 > c_0$, regular value such that there is a connected componet Y of $\{\alpha = c_1\}$ which does not contain compact complex curves and such that $Y \cap \Sigma^1 \neq \emptyset$; we apply Theorem 2.1, obtaining that Y is Levi flat and by Corollary 2.3, part d), we get that $\partial\alpha \wedge \bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0$ on the whole of X, hence every regular level of α is Levi flat and every such level has a neighbourhood where it is given as the zero of a pluriharmonic function. Therefore, no regular level can contain a compact complex curve, otherwise, by Theorem 2.2, every level would, so all the regular levels are Levi flat and containing no complact complex curves, hence by Corollary 2.3 part c), their connected components are foliated with dense complex leaves. This is case 3)

If every regular level $\{\alpha = c\}$, for $c > c_0$, contains a compact complex curve, then X contains uncountably many compact complex curves; by [3, Proposition 9 and 7] there exist a neighbourhood V of one of these curves and $f : V \to \mathbb{C}$ a holomorphic function which induces on V a foliation in compact curves. By Theorem 2.2 we conclude that X is proper over a non-compact (possibly singular) complex curve, which is case 2). $\square$

The proof of parts iii-a), iii-b) of the Main Theorem is rather difficult and requires a very careful analysis of the geometric structure of the critical set $\mathrm{Crt}(\alpha)$ of α.

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