TRANSVERSALLY PSEUDOCONVEX SEMIHOLOMORPHIC FOLIATIONS

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ABSTRACT. A *semiholomorphic foliations of type* (n, d) is a differentiable real manifold X of dimension $2n + d$, foliated by complex leaves of complex dimension n. The aim of the present note is to outline some results obtained in studying such spaces along the lines of the classical theory of complex spaces. Complete proofs will appear elsewhere.

1. INTRODUCTION

A *semiholomorphic foliations of type* (n, d) is a differentiable real manifold X of dimension $2n + d$, foliated by complex leaves of complex dimension n. If X is of class C^ω we say that X is a *a real analytic semiholomorphic foliation*. In this case X can be complexified i.e. can be embedded in holomorphic foliations of type (n, d) by a closed real analytic CR embedding (cfr. [15, Theorem 5.1]).

Such spaces were already studied in [10], to some extent, but some of the conclusions reached there are spoiled by the lack of a necessary hypothesis in the statements (the positivity of the transversal bundle mentioned below); in a similar direction moved also the work of Forstneric and Laurent-Thiébaut [7], where the attention was focused on the existence of a basis of Stein neighborhoods for the strongly pseudoconvex compacts of a Levi-flat hypersurface. We should also mention that, more recently, El Kacimi and Slimène studied the vanishing of the CR cohomology for some particular kind of semiholomorphic foliations, in [5, 6].

The aim of this note is to present some results obtained in studying such spaces along the lines of the classical theory of complex spaces, by correcting and further developing what was done in [10]. We are mainly concerned with real analytic semiholomorphic foliations which satisfy some hypothesis of *pseudoconvexity*. Pseudoconvexity we have in mind consists of two conditions: 1-pseudoconvexity (or 1-completeness) along the leaves of X, i.e. the existence of a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is strongly 1-plurisubharmonic along the leaves, and positivity of the bundle N_{tr} transversal to the leaves of X (see 2.4).

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Under these conditions we say that X is 1-*complete*. For 1-complete real analytic semiholomorphic foliation of type (n, d) we can prove that

1) every level set $\{\phi \leq c\}$ has a basis of Stein neighborhoods;
2) an approximation theorem for CR functions on X.

(see Theorem 2.3).

Once proved these results, the methods of complex analysis apply, in order to study the cohomology of the sheaf $\mathcal{CR}$ of germs of CR functions. We show that, for 1-complete real analytic semiholomorphic foliation of type $(n, 1)$, the cohomology groups $H^q(X, \mathcal{CR})$ vanishes for $q \geq 1$ (see Theorem 3.1). This implies a vanishing theorem for the sheaf of sections of a CR bundles, which we use in Section 4 to get a tubular neighborhood theorem and an extension theorem for CR functions from Levi flat hypersurfaces (see Theorems 4.1, 4.2 and Corollary 4.3). In Section 5 we sketch the proof that a 1-complete real analytic semiholomorphic foliation of type (n, d) embeds in $\mathbb{C}^{2n+2d+1}$ as a closed submanifold by a smooth CR map (see Theorem 5.1) and, as an application, that the groups $H_j(X; \mathbb{Z})$ vanish for $j \geq n + d + 1$ (see Theorem 5.2). In the last section a notion of *weakly positivity* is given for the transversal bundle N_{tr} on a compact real analytic semiholomorphic foliation X of type $(n, 1)$ (see (12)) and we prove that, under this condition, X embeds in $\mathbb{CP}^N$ by a real analytic CR map (see Theorem 6.1).

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2. Preliminaries

2.1. **Tangentially q-complete semiholomorphic foliations.** We recall that a *semiholomorphic foliation of type* (n, d) is a (connected) smooth foliation X whose local models are subdomains $U_j = V_j \times B_j$ of $\mathbb{C}^n \times \mathbb{R}^d$ and local change of coordinates $(z_k, t_k) \mapsto (z_j, t_j)$ are of the form

$$\begin{cases} z_j = f_{jk}(z_k, t_k) \\ t_j = g_{jk}(t_k), \end{cases} \tag{1}$$

where f_{jk}, g_{jk} are smooth and f_{jk} is holomorphic with respect to z_k. If we replace $\mathbb{R}^d$ by $\mathbb{C}^d$ and we suppose that f and h are holomorphic we get the notion of *holomorphic foliation of codimension d*.

Local coordinates $z_j^1, \ldots, z_j^n, t_j^1, \ldots, t_j^d$ satisfying (1) are called *distinguished local coordinates*.

A semiholomorphic foliation X is said to be *tangentially q-complete* if X carries a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is q-plurisubharmonic along the leaves i.e. its Levi form has at least $n-q+1$ positive eigenvalues in the tangent directions to the leaves).

2.2. **CR-bundles.** Let $\mathsf{G}_{m,s}$ be the group of matrices

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in GL(m;\mathbb{C})$, $B \in GL(m,s;\mathbb{C})$ and $C \in GL(s;\mathbb{R})$. We set $\mathsf{G_{m,0}} = \mathsf{GL(m;}\mathbb{C})$, $\mathsf{G_{0,s}} = \mathsf{GL(s;}\mathbb{R})$. To each matrix $M \in \mathsf{G_{m,s}}$ we associate the linear transformation $\mathbb{C}^m \times \mathbb{R}^s \to \mathbb{C}^m \times \mathbb{R}^s$ given by

$$(z,t) \mapsto (Az + Bt, Ct),$$

$(z,t) \in \mathbb{C}^m \times \mathbb{R}^s$.

Let X be a semiholomorphic foliation of type (n,d). A CR-*bundle of type (m,s)* is a vector bundle $\pi : E \to X$ such that the cocycle of E determined by a trivializing distinguished covering $\{U_j\}_j$ is a smooth CR map $\gamma_{jk} : U_j \cap U_k \to \mathsf{G_{m,k}}$

$$\gamma_{jk} = \begin{pmatrix} A_{jk} & B_{jk} \\ 0 & C_{jk} \end{pmatrix} \tag{2}$$

where $C_{jk} = C_{jk}(t)$ is a matrix with smooth entries and A_{jk}, B_{jk} are matrices with smooth CR entries. Thus E is foliated by complex leaves of dimension $m+n$ and real codimension $d+s$. The *inverse* E^{-1} of E is the CR-bundle E of type (m,s) whose cocycle is

$$\gamma_{jk}^{-1} = \begin{pmatrix} A_{jk}^{-1} & -A_{jk}^{-1}B_{jk}C_{jk}^{-1} \\ 0 & C_{jk}^{-1} \end{pmatrix}. \tag{3}$$

Let X be a semiholomorphic foliation of type (n,d). Then

- the tangent bundle TX of X is a CR-bundle of type (n,d).
- the bundle $T_{\mathcal{F}} = T^{1,0}_{\mathcal{F}}X$ of the holomorphic tangent vectors to the leaves of X is a CR-bundle of type $(n,0)$.
- the transverse bundle $N_{\rm tr}$ (to the leaves of X) is a CR-bundle of type $(0,d)$.

Remark 2.1. *If X is embedded in a complex manifold Z, its transverse TZ/TX is not a $\mathsf{G_{m,s}}$-bundle in general.*

2.3. **Complexification.** A real analytic foliation with complex leaves can be complexified, essentially in a unique way: there exists a holomorphic foliation $\widetilde{X}$ of type (n,d) with a closed real analytic CR embedding $X \hookrightarrow \widetilde{X}$ (cfr. [15, Theorem 5.1]). In particular, X is a Levi flat submanifold of $\widetilde{X}$

In order to construct $\widetilde{X}$ we consider a covering by distinguished domains $\{U_j = V_j \times B_j\}$ and we complexify each B_j in such a way to obtain domains $\widetilde{U}_j$ in $\mathbb{C}^n \times \mathbb{C}^d$. The domains $\widetilde{U}_j$ are patched together by the local change of coordinates

$$\begin{cases} z_j = \widetilde{f}_{jk}(z_k, \tau_k) \\ \tau_j = \widetilde{g}_{jk}(\tau_k) \end{cases} \tag{4}$$

obtained complexifying the (vector) variable t_k by $\tau_k = t_k + i\theta_k$ in f_{jk} and g_{jk} (cfr. (1)).

Let z_j, τ_j be distinguished holomorphic coordinates on $\widetilde{U}_j$ and let $\theta_j = \mathsf{Im}\,\tau_j$. The we have $\theta_j^s = \mathsf{Im}\,\widetilde{g}_{jk}(\tau_k)$, $1 \leq s \leq d$, on $\widetilde{U}_j \cap \widetilde{U}_k$ and consequently, since $\mathsf{Im}\,\widetilde{g}_{jk} = 0$ on X,

$$\theta^r = \sum_{s=1}^{d} \psi_{jk}^{rs} \theta_k^s$$

where $\psi_{jk} = \left(\psi_{jk}^{rs}\right)$ is an invertible $d \times d$ matrix whose entries are real analytic functions on $\widetilde{U}_j \cap \widetilde{U}_k$. Moreover, since $\widetilde{g}_{jk}$ is holomorphic and $\widetilde{g}_{jk|X} = g_{jk}$ is real, we also have $\psi_{jk|X} = \partial g_{jk}/\partial t_k$.

$\{\psi_{jk}\}$ is a cocycle of a CR-bundle of type $(0, d)$ which extends N_{tr} on a neighborhood of X in $\widetilde{X}$.

Let $\widetilde{X}$ be the complexification of X. Then the cocycle of the (holomorphic) transverse bundle $\widetilde{N}_{\mathrm{tr}}$ (to the leaves of $\widetilde{X}$) is

$$\frac{\partial \widetilde{g}_{jk}(\tau_k)}{\partial \tau_k} = \frac{\partial \tau_j}{\partial \tau_k} = \left(\frac{\partial \tau_j^\alpha}{\partial \tau_k^\beta} \right). \tag{5}$$

2.4. **1-complete foliations.** Let X be a semiholomorphic foliation of type (n, d), N_{tr} the transverse bundle to the leaves of X. A metric on the fibres of N_{tr} is an assignement of a distinguished covering $\{U_j\}$ of X and for every j a smooth map λ_j^0 from U_j to the space of symmetric positive $d \times d$ matrices such that

$$\lambda_k^0 = \frac{{}^t\partial g_{jk}}{\partial t_k} \lambda_j^0 \frac{\partial g_{jk}}{\partial t_k}.$$

Denoting ∂ and $\overline{\partial}$ the complex differentiation along the leaves of X, the local tangential forms

$$2\overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{\lambda_j^0 \overline{\partial}\partial \lambda_j^0 - 2\overline{\partial}\lambda_j^0 \wedge \partial \lambda_j^0}{{\lambda_j^0}^3} \tag{6}$$

$$\overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{2\lambda_j^0 \overline{\partial}\partial \lambda_j^0 - 3\overline{\partial}\lambda_j^0 \wedge \partial \lambda_j^0}{{\lambda_j^0}^3} \tag{7}$$

actually give global tangential forms ω, Ω.

The foliation X is said to be:

- *tranversally q-complete* (*strongly tranversally q-complete*) if a metric on the fibres of N_{tr} can be chosen in such a way that the hermitian form associated to ω (Ω) has at least $n - q + 1$ positive eigenvalues.

A semiholomorphic foliation X of type (n,d) is said to be 1-*complete* (*strongly* 1-*complete*) if it is tangentially and transversally 1-complete (tangentially and strongly transversally 1-complete).

Remark 2.2. *Assume $d=1$ and that X is transversally 1-complete. Then, due to the fact that the functions g_{jk} do not depend on z, $\omega_0 = \{\partial\overline{\partial}\log\lambda_j^0\}$ and $\eta = \{\partial\log\lambda_j^0\}$ are global tangential forms on X; moreover, since ω_0 is positive and exact, $\omega_0 = d\eta$, it gives on each leaf a Kähler metric whose Kähler form is exact. In particular no positive dimension compact complex subspace can be present in X.*

Example 2.1. Every domain $D \subset X = \mathbb{C}_z \times \mathbb{R}_u$ which projects over a bounded domain $D_0 \subset \mathbb{C}_z$ is strongly transversally 1-complete. Indeed, it sufficient to take for λ a function $\mu^{-1}\circ\pi$ where μ is a positive superharmonic function on D_0 and π is the natural projection $\mathbb{C}_z \times \mathbb{R}_u \to \mathbb{C}_z$.

Example 2.2. A real hyperplane X in $\mathbb{C}^n$, $n \geq 2$, is not transversally 1-complete. Indeed, assume $n=2$ and let $X \subset \mathbb{C}^2$ be defined by $v=0$, where $z = x+iy$, $w = u+iv$ are holomorphic coordinates. Transverse pseudoconvexity of X amounts to the existence of a positive smooth function $\lambda = \lambda(z,u)$, $(z,u) \in \mathbb{C}^2$, such that

$$\lambda\lambda_{z\overline{z}} - 2|\lambda_z|^2 > 0.$$

Consider the function λ^{-1}. Then

$$\left(\lambda^{-1}\right)_{z\overline{z}} = \frac{2|\lambda_z|^2 - \lambda\lambda_{z\overline{z}}}{\lambda^3} < 0$$

so, for every fixed u, the function λ^{-1} is positive and superharmonic on $\mathbb{C}_z$, hence it is constant with respect to z: contradiction.

Clearly X is an increasing union of strongly transversally 1-complete domains.

We want to prove the following

Theorem 2.1. *Let X be a semiholomorphic foliation of type (n,d). Assume that X is real analytic and strongly transversally 1-complete. Then there exist an open neighborhood U of X in the complexification $\widetilde{X}$ and a non negative smooth function $u : U \to \mathbb{R}$ with the following properties*

i) $X = \{x \in U : u(x) = 0\}$

ii) *u is plurisubharmonic in U and strongly plurisubharmonic on $U \smallsetminus X$.*

If X is tranversally 1-complete then property ii) *is replaced by the following*

iii) *the Levi form of the smooth hypersurfaces* $\{u = \text{const}\}$ *is positive definite.*

Proof. (Sketch) For the sake of simplicity we assume $n = d = 1$. Let $\{\lambda_j^0\}$ be a metric on the fibres of N_{tr}. With the notations of 2.3 let $\widetilde{N}_F$ be the tranverse bundle on $\widetilde{X}$ whose cocycle $\psi_{jk} = \widetilde{g}_{jk}$ is defined by (5).Then there exists a metric $\{\lambda_j\}$ on the fibres of $\widetilde{N}_F$ whose restriction to X is $\{\lambda_j^0\}$. Consider on $\widetilde{X}$ the smooth function u locally defined by $\lambda_j\theta_j^2$ (where $\tau_j = t_j + i\theta_j$); u is non negative and positive outside of X. Drop the subscript and compute the Levi form $L(u)$ of u. We have

$$L(u)(\xi,\eta) = A\xi\overline{\xi} + 2\mathsf{Re}\,(B\xi\overline{\eta}) + C\eta\overline{\eta} = \\ \lambda_{j,z\overline{z}}\theta^2\xi\overline{\xi} + 2\mathsf{Re}\left\{(\lambda_{j,z\overline{\tau}}\theta^2 + i\lambda_{j,z}\theta)\xi\overline{\eta}\right\} + \\ (\lambda_{j,\tau\overline{\tau}}\theta^2 + i\lambda_{j,\tau}\theta - i\lambda_{j,\overline{\tau}}\theta + \lambda_j/2)\eta\overline{\eta} \tag{8}$$

and

$$AC - |B|^2 = \theta^2\left(\lambda_{j,z\overline{z}} - 2|\lambda_{j,z}|^2\right) + \theta^3\varrho \tag{9}$$

where ϱ is a smooth function. The coefficient of θ^2 is nothing but that of the form $\lambda_j^2\Omega$ (here we denote ω and Ω, respectively, the forms (6), (7) where λ_j^0 is replaced by λ_j), so if X is strongly transversally pseudoconvex $L(u)$ is positive definite near each point of X and strictly positive away from X. It follows that there exists a neighborhood U of X such that u is plurisubharmonic on U.

Assume now that X is transversally 1-complete. The Levi form $L(u)_{|HT(S)}$ of a smooth hypersurface $S = \{u = \text{const}\}$ is essentially determined by the function

$$\theta^4\big\{\lambda_{j,z\overline{z}}|\lambda_{j,\overline{\tau}}\theta + i\lambda_j|^2 - 2\mathsf{Re}\,(\lambda_{j,z\overline{\tau}}\theta + i\lambda_{j,\overline{z}})(\lambda_{j,\tau}\theta - i\lambda_j)\lambda_{j,\overline{z}} + \\ (\lambda_{j,\tau\overline{\tau}}\theta^2 + i\lambda_{j,\tau}\theta - i\lambda_{j,\overline{\tau}}\theta + \lambda_j/2)|\lambda_{j,z}|^2\big\} = \\ \theta^4(\lambda_j\lambda_{j,z\overline{z}} - (3/2)|\lambda_{j,z}|^2) + \theta^5\varrho \tag{10}$$

where ϱ is a bounded function. The coefficient of θ^4 is nothing but that of the form $\lambda_j^3\omega$ so, if X is transversally pseudoconvex, $L(u)_{|HT(S)}$ is not vanishing near X. It follows that there exists an open neighborhood U of X such that the hypersurfaces $S = \{u = \text{const}\}$ contained in $U \setminus X$ are strongly Levi convex.

Remark 2.3. *In view of* (8)*, the Levi form* $L(u)$ *at a point of* X *is positive in the transversal direction* η*.*

Theorem 2.2. *Let* X *be a semiholomorphic foliation of type* (n,d)*. Assume that* X *is real analytic and strongly 1-complete. Then for every compact subset* $K \subset X$ *there exist an open neighborhood* V *of* K *in* $\widetilde{X}$*, a smooth strongly plurisubharmonic function* $v : V \to \mathbb{R}^+$ *and a constant* $\overline{c}$ *such that* $K \Subset \{v < \overline{c}\} \cap X \Subset V \cap X$*.*

2.5. Stein bases and a density theorem.

Theorem 2.3. *Let X be a semiholomorphic foliation of type (n, d). Assume that X is real analytic and strongly 1-complete and let $\phi : X \to \mathbb{R}^+$ be a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is 1-plurisubharmonic along the leaves. Then*

i) *$\overline{X}_c$ is a Stein compact of $\widetilde{X}$ i.e. it has a Stein basis of neighborhoods in $\widetilde{X}$.*

ii) *every smooth CR function on a neighborhood of $\overline{X}_c$ can be approximated in the C^∞ topology by smooth CR functions on X.*

Proof. (Sketch) In view of Theorem 2.1 we may suppose that $X = \{u = 0\}$ where $u : \widetilde{X} \to [0, +\infty)$ is plurisubharmonic and strongly plurisubharmonic on $\widetilde{X} \smallsetminus X$.

Let U be an open neighborhood of $\overline{X}_c$ in $\widetilde{X}$. With $K = \overline{X}_c$ we apply Theorem 2.2: there exists an open neighborhood $V \Subset U$ of $\overline{X}_c$ in $\widetilde{X}$, a smooth strongly plurisubharmonic function $v : V \to \mathbb{R}^+$ and a constant $\bar{c}$ such that $\overline{X}_c \Subset \{v < \bar{c}\} \cap X \Subset V \cap X$. It follows that for $\varepsilon > 0$, sufficiently small, $W = \{v < \bar{c}\} \cap \{u < \varepsilon\} \Subset V \Subset U$ is a Stein neighborhood of $\overline{X}_c$.

In order to prove ii) consider a smooth CR function f on a neighborhood I of $\overline{X}_c$ in X, and take $c' > c$, such that $\overline{X}_{c'} \subset I$. For every $j \in \mathbb{N}$ define $\overline{B}_j = \overline{X}_{c'+j}$ and choose a Stein neighbourhod U_j of $\overline{B}_j$ such that $\overline{B}_j$ has a fundamental system of open neighborhoods $W_j \Subset U_{j+1} \cap U_j$ which are Runge domains in U_{j+1}. Since U_0 is Stein, the $\mathcal{O}(U_0)$-envelope of $\overline{B}_0$ coincide with its the plurisubharmonic envelope (cfr. [12, Theorem 4.3.4]) hence it is a compact contained in $X \cap U_0$, $\overline{B}_0$ being the zero set of the plurisubharmonic function u. Thus we may assume that $\overline{B}_0$ is $\mathcal{O}(U_0)$-convex. We end the proof applying the approximation theorem of Freeman (cfr. [8, Theorem 1.3]). $\square$

Corollary 2.4. *Let X be as in Theorem 2.3 and $\overline{X}_c$, $c > 0$. Then $\overline{X}_c$ is $\mathcal{CR}(X)$-convex i.e. $\overline{X}_c$ coincides with its $\mathcal{CR}(X)$-envelope.*

Proof. Set $K = \overline{X}_c$. In view of Theorem 2.1, X is the zero set of a plurisubharmonic function defined in an open neighborhood U of X in $\widetilde{X}$. Let $U_\nu \subset U$ be a Stein neighborhood of K and $\mathrm{Psh}(U_\nu)$ the space of the plurisubharmonic in U_ν. Then the envelopes $\hat{K}_{\mathcal{O}(U_\nu)}$ and $\hat{K}_{\mathrm{Psh}(U_\nu)}$ of K coincide (cfr. [12, Theorem 4.3.4]), in our situation we have

$$K \subset \hat{K}_{\mathcal{CR}(X)} \subset \hat{K}_{\mathcal{O}(U_\nu)} = \hat{K}_{\mathrm{Psh}(U_\nu)} \subset U_\nu \cap X.$$

We obtain the thesis letting $U_\nu \subset U$ run in a Stein basis of $\overline{X}_c$. $\square$

3. Cohomology

In this section we sketch the proof of the following

Theorem 3.1. *Let X be a semiholomorphic foliation of type $(n,1)$. Assume that X is real analytic and strongly 1-complete. Then*

$$H^q(X;\mathcal{CR}) = 0$$

for every $q \geq 1$.

Remark 3.1. *The same holds true if X is a closed, orientable, smooth Levi flat hypersurface of a connected Stein manifold D.*

We need some preliminary facts. Let Ω be a domain in $\mathbb{C}^n \times \mathbb{R}^k$ and for every $t \in \mathbb{R}^k$ set $\Omega_t = \{z \in \mathbb{C}^n : (z,t) \in \Omega\}$. Following [2] we say that $\{\Omega_t\}_{t \in \mathbb{R}^k}$ is a *regular family* of domains of holomorphy if the following conditions are fulfilled:

a) Ω_t is a domain of holomorphy for all $t \in \mathbb{R}^k$;
b) for every $t_0 \in \mathbb{R}^k$ there exist a neighborhood $I_0 = \{|t - t_0| < \varepsilon\}$ and a domain $U \subset \mathbb{C}^n$ such that Ω_{t_0} is Runge in U and $\cup_{t \in I_0} \Omega_t \subset I_0 \times U$.

In this situation it can be proved that the sheaf $\mathcal{CR}$ of Ω is cohomologically trivial (cfr. [2, Corollaire pag. 213]).

Let X be a tangentially 1-complete semiholomorphic foliation of type (n,d), $\phi : X \to [0,+\infty)$ a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is 1-plurisubharmonic along the leaves and $c > 0$ a regular value of ϕ. Since ϕ is a strongly plurisubharmonic along the leaves the "bumps lemma" method applies and allows us to prove the following

Lemma 3.2. *There is $\varepsilon > 0$ such that the homomorphism*

$$H^q(X_{c+\varepsilon};\mathcal{CR}) \to H^q(X_c;\mathcal{CR})$$

is onto for $q \geq 1$. In particular, the homorphism

$$H^q(\overline{X}_c;\mathcal{CR}) \to H^q(X_c;\mathcal{CR})$$

is onto for $q \geq 1$.

Proof of Theorem 3.1(Sketch) We first prove that for every c and $q \geq 1$ the cohomology groups $H^q(X_c;\mathcal{CR})$ vanish and for this that $H^q(\overline{X}_c;\mathcal{CR}) = 0$ for every c and $q \geq 1$ (see Lemma 3.2.)

Let $\widetilde{X}$ be the complexification of X. By Theorem 2.3 there exists a Stein neighborhood $U \Subset \widetilde{X}$ of $\overline{X}_c$ divided by X into two connected components U_+, U_- which are Stein domains. Let $\mathcal{O}_+$ $(\mathcal{O}_-)$ be the sheaf of germs of holomorphic functions in D_+ (D_-) smooth on $U_+ \cup X$ $(U_- \cup X)$ and extend it on U by 0. Then we have on U the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_+ \oplus \mathcal{O}_- \xrightarrow{\mathsf{re}} \mathcal{CR} \longrightarrow 0 \tag{11}$$

where re is definined by $\mathsf{re}(f \oplus g) = f_{|X} - g_{|X}$. Let ξ be a q-cocycle with values in $\mathcal{CR}$ defined in (a neighborhood of) $\overline{X}_c$. We may suppose that ξ is defined on $U \cap X$. Since U is Stein we derive that

$$H^q(U_+ \cup X; \mathcal{O}_+) \oplus H^q(U_- \cup X; \mathcal{O}_-) \to H^q(X; \mathcal{CR})$$

is an isomorphism for $q \geq 1$. In particular, $\xi = \xi_+ - \xi_-$ where ξ_+ and ξ_- are represented by two $\overline{\partial}$-closed $(0,q)$-forms ω_+ and ω_- respectively which are smooth up to X. Now arguing as in [9, Lemma 2.1] (replacing $|z|^2$ by a strictly plurisubharmonic exhaustive function $\phi : U \to \mathbb{R}$) we obtain the following: there exist two bounded pseudoconvex domains $U'_+, \Subset U$, $U'_- \Subset U$ satisfying:

1) $U'_+ \subset U_+$, $U'_- \subset U_-$;
2) $\overline{X}_c \subset \mathrm{b}U'_+ \cap X, \overline{X}_c \subset \mathrm{b}U'_- \cap X$.

By Kohn's theorem (cfr. [13]) $\omega_{+|U'_+} = \overline{\partial}\alpha_+$, $\omega_{-|U'_-} = \overline{\partial}\alpha_-$ with α_+, α_- smoth up to the boundary and this shows that ξ is a q-coboundary. Thus $H^q(\overline{X}_c; \mathcal{CR}) = 0$ for every c and $q \geq 1$.

The vanishing of the groups $H^q(X; \mathcal{CR})$ for $q \geq 2$ now easily follows by a standard procedure. The vanishing of $H^1(X; \mathcal{CR})$ is achieved using the approximation of CR functions. □

Corollary 3.3. *Let X be a semiholomorphic foliation of type $(n,1)$. Assume that X is real analytic and strongly 1-complete. Let $A = \{x_\nu\}$ be a discrete set of distinct points of X and $\{c_\nu\}$ a sequence of complex numbers. Then there exists a smooth CR function $f : X \to \mathbb{C}$ such that $f(x_\nu) = c_\nu$, $\nu = 1, 2, \ldots$. In particular, X is $\mathcal{CR}(X) -$ convex.*

Theorem 3.1 and the Mayer-Vietoris construction of Lemma 3.2 allow us to obtain the following result.

Theorem 3.4. *Let D be a connected Stein domain in $\mathbb{C}^n$ and $X \subset D$ a closed, orientable, smooth Levi flat hypersurface. Let $\phi : D \to \mathbb{R}$ be exhaustive, strictly plurisubharmonic and $X_c = \{\phi < c\} \cap X$, $c \in \mathbb{R}$. Then the image of the CR map $\mathcal{CR}(X) \to \mathcal{CR}(X_c)$ is everywhere dense.*

A vanishing theorem for compact support cohomology $H^*_c(X; \mathcal{CR})$ can be proved in a more general situation. Precisely

Theorem 3.5. *Let X be a 1-complete smooth mixed foliation of type (n,d) with $n \geq 1$. Then*

$$H^j_c(X; \mathcal{CR}) = 0$$

for $j \leq n-1$

Corollary 3.6. *Let X be a 1-complete semiholomorphic foliation of type (n,d) with $n \geq 2$ and K a compact subset such that $X \smallsetminus K$ is connected. Then the homorphism*

$$\mathcal{CR}(X) \to \mathcal{CR}(X \smallsetminus K)$$

is surjective.

4. Applications

In this section we state some applications of Theorem 3.1

4.1. **A vanishing theorem for CR-bundles.** Let X be a real analytic semiholomorphic foliation of type $(n, 1)$, $\pi : E \to X$ a real analytic CR-bundle of type $(n, 0)$. Then it is not difficult to show that

i) if X is 1-complete E is 1-complete;
ii) if X is transversally 1-complete (strongly transversally 1-complete) E is transversally 1-complete (strongly transversally 1-complete).

Theorem 4.1. *Let X be a real analytic semiholomorphic foliation of type $(n, 1)$, $\pi : E \to X$ a real analytic CR bundle of type $(m, 0)$ and $\mathcal{E}_{CR}$ the sheaf of germs of smooth CR-sections of E. Assume that X is strongly 1-complete. Then*

$$H^q(X, \mathcal{E}_{CR}) = 0$$

for $q \geq 1$.

The proof consists in connecting the mentioned cohomology with the cohomology of the sheaf of CR functions on the total space of the bundle, which vanishes by Theorem 3.1.

4.2. **CR tubular neighborhood theorem and extension of CR functions.** Let M be a real analytic, closed, Levi flat hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$. In view of [15, Theorem 5.1], there exist a neighborhood $U \subseteq \mathbb{C}^{n+1}$ of M and a unique holomorphic foliation $\widetilde{\mathcal{F}}$ on U extending the foliation $\mathcal{F}$. A natural problem is the following: given a smooth CR function $f : M \to \mathbb{C}$ extend it on a neighborhood $W \subset U$ by a smooth function $\widetilde{f}$ holomorphic along the leaves of $\widetilde{\mathcal{F}}$. In the sequel we will answer this question.

The key point for the proof is the following "CR tubular neighborhood theorem":

Theorem 4.2. *Assume that M is strongly transversally 1-complete. Then there exist an open neighborhood $W \subset U$ of M and a smooth map $q : W \to M$ with the properties:*

i) *q is a morphism $\widetilde{\mathcal{F}}_{|W} \to \mathcal{F}$;*
ii) *$q_{|M} = \mathsf{id}_M$.*

Along the lines of the corresponding result for complex submanifolds, the main idea of the proof is to use Theorem 4.1 to deduce the exactness of the appropriate short sequence of bundles (cfr. [3]).

Corollary 4.3. *Let M be a real analytic Levi flat hypersurface in $\mathbb{C}^{n+1}$, $n \geq 1$, $\mathcal{F}$ the Levi foliation on M, $\widetilde{\mathcal{F}}$ the holomorphic foliation extending $\mathcal{F}$ on a neighborhood of M. Then every smooth CR function $f : M \to \mathbb{C}$ extends to a smooth function $\widetilde{f}$ on a neighborhood of M holomorphic along the leaves of $\widetilde{\mathcal{F}}$.*

Proof. Take $\widetilde{f} = f \circ q$. $\square$

5. An embedding theorem

We have the following

Theorem 5.1. *Let X be a real analytic semiholomorphic foliation of type (n, d). Assume that X is strongly 1-complete. Then X embeds in $\mathbb{C}^{2n+2d+1}$ as a closed submanifold by a CR map.*

For the proof we argue as in [12, Theorem 5.3.9] using in a crucial way Theorem 2.3 and Corollary 2.4.

Remark 5.1. *The example 2.2 shows that the converse is not true, namely a real analytic semiholomorphic foliation embedded in $\mathbb{C}^N$ is not necessarily transversally 1-complete.*

As an application, we get the following

Theorem 5.2. *Let X be a real analytic semiholomorphic foliation of type (n, d). Assume that X is strongly 1-complete. Then*

$$H_j(X; \mathbb{Z}) = 0$$

for $j \geq n + d + 1$ and $H_{n+d}(X; \mathbb{Z})$ has no torsion.

Embed X in $\mathbb{C}^N$, $N = 2n + 2d + 1$ by a CR map f and choose $a \in \mathbb{C}^N \setminus X$ in such a way that $\psi = |f - a|^2$ is a Morse function [1]. Then, in order to get the result, we apply Morse theorem proving that the Hessian form $H(\psi)(p)$ of ψ at a point $p \in X$ has at most $n + d$ negative eigenvalues.

Remark 5.2. *In particular, the statement holds for smooth foliations of type (n, d) embedded in some $\mathbb{C}^N$.*

Corollary 5.3. *Let $X \subset \mathbb{CP}^N$ be a closed, oriented, semiholomorphic foliation of type (n, d), V a nonsingular algebraic hypersurface which does not contain X. Then the homomorphism*

$$H^j_c(X \smallsetminus V; \mathbb{Z}) \to H^j(X; \mathbb{Z})$$

induced by $V \cap X \to X$ is bijective for $j < n - 1$ and injective for $j = n - 1$. Moreover, the quotient group

$$H^{n-1}(V \cap X; \mathbb{Z})/H^{n-1}(X; \mathbb{Z})$$

has no torsion.

6. The compact case

Let X be a compact real analytic semiholomorphic foliation of type $(n, 1)$. With the notations of 2.3 let $\{\psi_{\alpha\beta}\}$ be the cocycle of the CR-bundle of type $(0, 1)$ on the complexification $\widetilde{X}$ which extends N_{tr}. The local smooth functions h_α on $\widetilde{X}$ satisfying $h_\beta = \psi^2_{\alpha\beta} h_\alpha$ define a metric on the fibres of $N_{\rm tr}$. We say that $N_{\rm tr}$ is *weakly positive* if a smooth metric $\{h_\alpha\}$ can be chosen in such a way that the function ϕ on $\widetilde{X}$ locally defined by $h_\alpha\psi^2_{\alpha\beta}$ satisfies

$$i\partial\bar\partial\log\phi = i\partial\bar\partial\log h_\alpha + 2i\frac{\partial\tau_\alpha\wedge\bar\partial\bar\tau_\alpha}{(\tau_\alpha-\bar\tau_a)^2}\geq 0 \tag{12}$$

near X, on the complement of X. Clearly, $\phi\geq 0$ near X and $\phi>0$ away from X. By hypothesis, $i\partial\bar\partial\log\phi\geq 0$ away from X and

$$\partial\bar\partial\phi = \phi\,\partial\bar\partial\log\phi + \frac{\partial\phi\wedge\bar\partial\phi}{\phi};$$

moreover, locally on X

$$2\partial\bar\partial\phi = h\partial\tau\wedge\bar\partial\bar\tau.$$

$h=h_\alpha$, $\tau=\tau_\alpha$. Then, since $h>0$ and $i\partial\tau\wedge\bar\partial\bar\tau$ is a positive $(1,1)$-form, ϕ is plurisubharmonic on a neighborhood of X and its Levi form has one positive eigenvalue in the transversal direction τ. In particular, for $c>0$ small enough, the sublevels $\widetilde{X}_c=\{\phi<c\}$ are weakly complete manifolds and give a fundamental system of neighborhoods of X.

Remark 6.1. *Observe that the form (12) is non negative near X in the complement of X, if X admits a space of parameters.*

A CR-bundle $L\to X$ of type $(1,0)$ is said to *positive along the leaves* if there is a smooth metric $\{h_\alpha\}$ (on the fibres of) L such that

$$\sum_{1\leq\alpha\leq n}\frac{\partial^2\log h_\alpha}{\partial z_r\partial\bar z_s}\xi^r\bar\xi^s>0. \tag{13}$$

where (z,t) are distinguished coordinates.

Under the condition (12) we have the following

Theorem 6.1. *Suppose that there exists on X an analytic CR-bundle L of type $(1,0)$ and positive along the leaves. Then X embeds in $\mathbb{CP}^N$, for some N, by a real analytic CR map.*

Proof. Extend L on a neighborhood of X to a holomorphic line bundle $\widetilde{L}$ and the metric $\{h_\alpha\}$ to a smooth metric $\{\widetilde{h}_\alpha\}$ preserving the condition (13) near X. For every positive $C\in\mathbb{R}$ consider the new metric $\{\widetilde{h}_{\alpha,C}=e^{C\phi}\widetilde{h}_\alpha\}$ (where ϕ is locally defined by $h_\alpha\psi^2_{\alpha\beta}$) and set $\zeta_1=\zeta_1,\ldots,\zeta_n=z_n,\zeta_{n+1}=\tau$. At a point of X we have

$$\sum_{1\leq r,s\leq n+1}\frac{\partial^2\log\widetilde{h}_{\alpha,C}}{\partial\zeta_r\partial\bar\zeta_s}\eta^r\bar\eta^s=\mathcal{L}_1+\mathcal{L}_2.$$

where $\mathcal{L}_2$ is positive near X for $|\eta_{n+1}|$ small enough, say $|\eta_{n+1}|<\varepsilon$ and $\mathcal{L}_1$ is nonegative and positive for $|\eta_{n+1}|>0$. It follows that for large enough C the hermitian form $\mathcal{L}_1 + \mathcal{L}_2$ is positive on a neigbourhood of X say $\{\phi < c\}$ for c small enough. In this situation a theorem of Hironaka [14, Theorem 4] applies to embed $\{\phi < c\}$ in some $\mathbb{CP}^N$ by a locally closed holomorphic embedding. In particular, X itself embeds in $\mathbb{CP}^N$ by a CR-embedding. □

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