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A Fast Computation of the Best k-Digit Rational Approximation to a Real Number

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Abstract. Given a real number α, we aim at computing the best rational approximation with at most k digits and with exactly k digits at the numerator (denominator). Our approach exploits Farey sequences. Our method turns out to be very fast in the sense that, once the development of α in continued fractions is available, the required operations are just a few and their number remains essentially constant for any k (in double precision finite arithmetic). Estimations of error bounds are also provided.

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Keywords. Best rational approximation with k digits, Continued fractions, Farey sequences.

1. Introduction

Given a real number α, the best rational approximation with at most k digits is known in literature since a long time (e.g., [2,6,9]). Here, we focus on the computational cost of such a best rational approximation. Within this context, we present a method which exhibits a very low and essentially constant computational cost for any k. Indeed, our algorithm requires a couple of dozens of floating point operations, once the required coefficients of the continued fraction development of α are known. It is worth noticing that the presented method improves results previously presented in literature, e.g., [1,10]. More precisely, in [10] the reported computational cost increases exponentially with k, whereas in [1] the proposed geometric approach exhibits a computational cost increasing linearly with k.

In addition, we complete our new algorithm with the computation of the best rational approximation with exactly k digits, which comes together with the other one and is proved to be fast as well.

At first we recall some notation. Let α be a positive irrational number and x and x' be two approximations of α; then we say that x is *finer* than x', and x' is *coarser* than x, when $|\alpha - x| < |\alpha - x'|$.

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We write α in its development in infinite continued fraction

$$\alpha = [a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \tag{1}$$

where a_i are positive integers for any $i \geq 1$, while a_0 is a nonnegative integer.

The *convergents* s_n of α are the rational numbers

$$s_n = \frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]. \tag{2}$$

For any non-negative integer n, we have

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_n &= a_n p_{n-1} + p_{n-2}; \\ q_{-2} &= 1, & q_{-1} &= 0, & q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned}$$

The *semiconvergents* $s_{n,m}$ of α are defined by

$$s_{n,m} = \frac{p_{n,m}}{q_{n,m}} = \frac{mp_n + p_{n-1}}{mq_n + q_{n-1}}, \tag{3}$$

where m is an integer with $1 \leq m < a_{n+1}$. In addition, we extend the definition of $s_{n,m}$ to include the two special cases $m = 0$ and $m = a_{n+1}$. Comparing (3) to (2) we see that

$$\begin{aligned} s_{n-1} &= s_{n,0}, & &\text{for } m = 0; \\ s_{n+1} &= s_{n,a_{n+1}}, & &\text{for } m = a_{n+1}. \end{aligned}$$

This means that the very first semiconvergent $s_{n,0}$ coincides with the previous convergent s_{n-1}, and the last semiconvergent $s_{n,a_{n+1}}$ coincides with the subsequent convergent s_{n+1}.

For any nonnegative integer m, the *pseudoconvergent* $\tilde{s}_{n,m}$ of the convergent $s_n = p_n/q_n$ is given by

$$\tilde{s}_{n,m} = \frac{\tilde{p}_{n,m}}{\tilde{q}_{n,m}} = \frac{mp_n + p_{n-1,a_n-1}}{mq_n + q_{n-1,a_n-1}} = \frac{mp_n + (a_n - 1)p_{n-1} + p_{n-2}}{mq_n + (a_n - 1)q_{n-1} + q_{n-2}}.$$

The pseudoconvergents $\tilde{s}_{n,m}$ of s_n are infinite (Farey) fractions, which lie on the same side of s_n with respect to α; their value tends to the value of s_n for increasing m; moreover, they are all coarser than s_n.

We start from a well-known result (see, e.g., [2]): for any positive integer k and any given real number α, the best rational approximation with a numerator (denominator) with at most k digits is either the closest convergent or the closest semiconvergent, whose numerator (denominator) has no more than k digits. One might be lead to the conclusion that this result suffices to solve even the analogous problem of the best approximation with exactly k digits at the numerator (denominator), but we will show that an extra investigation is needed.

In the following, we will consider the numerator only, since results can be easily extended to the case referring to the denominator, working in analogous way.

Table 1. Best approximation to π with at most k digits

k	$lb < \pi < ub$	bra	$a.er.$
1	$\mathbf{3} < \pi < 7/2$	lb	$1.4e-1$
2	$91/29 < \pi < \mathbf{22/7}$	ub	$1.3e-3$
3	$688/219 < \pi < \mathbf{355/113}$	ub	$2.7e-7$
4	$9918/3157 < \pi < \mathbf{355/113}$	ub	$2.7e-7$
5	$99733/31746 < \pi < \mathbf{355/113}$	lb	$1.2e-8$

Indeed three events can happen only:

(i) if both the closest convergent and semiconvergent have a k-digit numerator, then the best approximation with exactly k digits is given by the one closer to α; therefore this case has the same solution as the problem of the best rational approximation with a numerator with at most k digits;

(ii) if the closest convergent has exactly k digits and the closest semiconvergent has less than k digits at the numerator, then obviously the convergent is the best approximation with at most k digits at the numerator and with exactly k digits also (we point out that this is the case when the semiconvergent is $s_{n,0} = s_{n-1}$);

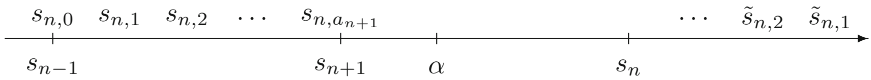
(iii) if the closest convergent has less than k digits and the closest semiconvergent has exactly k digits at the numerator, then the finer one provides the solution with at most k digits, whereas we prove that the best approximation with exactly k digits is given by either the semiconvergent or the pseudoconvergent, depending on which is closer to α.

Let us consider an example referring to π. The following Table 1 reports the best rational approximation (bra) with at most k digits at the numerator together with the corresponding absolute error ($a.er.$) for $k = 1, 2, 3, 4, 5$; lb denotes the lower bound and ub denotes the upper bound; convergents are given in bold face.

It is clear that when the numerator of the convergent has less than k digits (for $k = 4, 5$ in Table 1) then the corresponding closest semiconvergent exhibits exactly k digits, so the best approximation with at most k digits is provided by the finer between the convergent and the semiconvergent (for $k = 4$ it is the convergent and for $k = 5$ it is the semiconvergent). About the best approximation with exactly k digits at the numerator, it may appear that it is provided by the semiconvergent, for both $k = 4$ and $k = 5$. Actually this holds here for π and for very many other numbers, but not always. Indeed something different may happen and an extra investigation has to be done.

Assume n odd; when n is even, the same results hold after some convenient adjustments.

Actually, if s_n is the last convergent with at most k digits but less than k digits at the numerator, we can represent the collocation of convergents, semiconvergents and pseudoconvergents by the following graphical synopsis:



We point out that semiconvergents $s_{n,1}, s_{n,2}, \dots$ monotonically converge to s_{n+1} from the left and pseudoconvergents $\widetilde{s}_{n,1}, \widetilde{s}_{n,2}, \dots$ monotonically converge to s_n from the right.

Now we are looking for the best rational approximation with exactly k digits at the numerator. Let n_k be the greatest integer such that s_{n_k} has less than k digits at the numerator, in the case that s_{n_k+1} has more than k digits; let m_k be the greatest integer such that s_{n_k,m_k} has exactly k digits and M_k be the greatest integer such that $\widetilde{s}_{n_k,M_k}$ has exactly k digits at the numerator. Then, since s_{n_k} has less than k digits at the numerator, we have $s_{n_k,m_k} < \alpha < s_{n_k} < \widetilde{s}_{n_k,M_k}$ and α is such that either $\alpha \in [s_{n_k,m_k}, (s_{n_k,m_k} + \widetilde{s}_{n_k,M_k})/2]$ or $\alpha \in [(s_{n_k,m_k} + \widetilde{s}_{n_k,M_k})/2, \widetilde{s}_{n_k,M_k}]$. We show that if the first event happens, then the best rational approximation with exactly k digits at the numerator is given by the semiconvergent s_{n_k,m_k} (the most frequent case), otherwise it is given by the pseudoconvergent $\widetilde{s}_{n_k,M_k}$. This allows us to build an efficient numerical method whose computation time is very fast.

Our paper is organized in the following way. In Sect. 2 we introduce our numerical method by propositions and proofs about our main theoretical results which allow to build a fast numerical algorithm; furthermore, some details about error estimation are provided. Section 3 is devoted to numerical results discussed by a particular numerical example too. Section 4 contains some concluding remarks.

2. Numerical Method

Let us assume that the continued fraction expansion of real number α is available (indeed efficient routines are present in many software libraries). The main feature of the presented method is that it computes just three integers n_k, m_k and M_k, which are defined rigorously in the next subsection, and then by means of these numbers, corresponding convergent, semiconvergent and pseudoconvergent are computed. Exploiting these quantities, the algorithm computes the best rational approximation with at most k digits at the numerator and the best rational approximation with exactly k digits at the numerator, when the two approximations do not coincide; when the two approximations coincide, just one of them is computed. This approach allows to cut computational time significantly and in practice the number of required floating point operations remains very small and constant for any k (in double precision finite arithmetic). This means that our method works better than other methods already presented in literature (e.g., [1,10]).

2.1. Main Results and Algorithm

It was already proved (e.g., [2,6,9]) that for any positive integer k and any given real number α, the best rational approximation with a numerator with at most k digits is either the closest convergent or the closest semiconvergent, whose numerator has no more than k digits. Here we add the following Proposition 1 to complete the computation of the best rational approximation with exactly k digits at the numerator. Before that, some properties are reported, which will be used in the following proofs. In particular, Properties 2–4 derive from known results about Farey sequences (e.g., [5]). Actually, Farey sequences can provide an efficient tool to the rational approximation of real numbers (e.g., [7,8]) and we follow this approach.

Property 1. *For any $i > j$, the convergent s_i is finer than s_j with $p_i > p_j$ and $q_i > q_j$. For any n and $a_{n+1} \geq i > j \geq 0$, the number $s_{n,i}$ is finer than $s_{n,j}$ with $p_{n,i} > p_{n,j}$ and $q_{n,i} > q_{n,j}$.*

Property 2. *If a/b and a'/b' are Farey fractions, any rational x/y with $a/b < x/y < a'/b'$ has a denominator $y \geq b+b'$ ($y = b+b'$ holds only for $x = a+a'$) and numerator $x \geq a + a'$ ($x = a + a'$ holds only for $y = b + b'$).*

Property 3. *The non-zero convergents and semiconvergents of α are exactly the reciprocals of those of $1/\alpha$.*

Property 4. *If a/b and c/d are neighbors in a Farey sequence, with $a/b < c/d$, then their difference $c/d - a/b$ is equal to $1/bd$. This is equivalent to saying that $bc - ad = 1$.*

For the sake of brevity, if w is an integer, we call $d(w)$ its number of digits.

Proposition 1. *For any positive integer k and any given real number α, the best rational approximation with a numerator with exactly k digits is either a convergent or a semiconvergent or a pseudoconvergent.*

Proof. We assume n odd. Then we distinguish two cases:

1. The best approximation with at most k digits has exactly k digits at the numerator; therefore it is either a convergent or a semiconvergent (according to known results, e.g., [2,6,9]).
2. The best approximation with at most k digits at the numerator has less than k digits at the numerator.

   Then, firstly we consider the real number $\beta = 1/\alpha$ and prove results referring to the best rational approximation with exactly k digits at the denominator, since we need to exploit Property 2. Then, by Property 3 we extend results to α.

   Referring to β, by Property 3, there is no convergent s_n with $d(q_n) = k$; consequently we can compute:
   - n_k the greatest integer such that $d(q_{n_k}) < k$,
   - m_k the greatest integer such that $d(q_{n_k,m_k}) = k$,
   - M_k the greatest integer such that $d(\widetilde{q}_{n_k,M_k}) = k$.

   We notice that these three integers exist by Property 1 and by the fact that for any integers m, n we have that $d(m) < k$ and $d(n) < k$ imply $d(m + n) \leq k$. Consider the semiconvergent s_{n_k,m_k} and the pseudoconvergent $\widetilde{s}_{n_k,M_k}$ of β. Indeed, any rational a/b finer than them, as well as α and s_{n_k}, belong to the open interval whose extremes are s_{n_k,m_k} and $\widetilde{s}_{n_k,M_k}$; we point out that both these two numbers are Farey pairs with s_n.

By Property 2, if a/b is in the open interval whose extremes are s_{n_k} and s_{n_k,m_k}, then it is such that $b \geq q_{n_k} + q_{n_k,m_k} \geq (m_k+1)q_{n_k} + q_{n_k-1}$. On the other hand, if a/b is in the open interval whose extremes are s_{n_k} and $\widetilde{s}_{n_k,M_k}$, then $b \geq q_{n_k} + \widetilde{q}_{n_k,M_k} \geq (M_k+1)q_{n_k} + q_{n_k-1,a_{n_k}-1}$. In both cases, $d(b) > k$. Therefore no rational $a/b \in (s_{n_k,m_k}, s_{n_k})$ nor $a/b \in (s_{n_k}, \widetilde{s}_{n_k,M_k})$ can be the best rational approximation with exactly k digits at the denominator. Consequently, the finer between the semiconvergent s_{n_k,m_k} and the pseudoconvergent $\widetilde{s}_{n_k,M_k}$ provides such a best rational approximation we are looking for.

In the case n even, we reach the same result by considering, with some adjustments, the intervals (s_{n_k}, s_{n_k,m_k}) and $(\widetilde{s}_{n_k,M_k}, s_{n_k})$.

By Property 3, the same results hold for the rational number b/a referring to α, so proof is completed. □

Consider $s_{n_k} = \frac{p_{n_k}}{q_{n_k}}$. From Sect. 1 we see that $n_k + 1$ is the smallest integer such that p_{n_k+1} has more than k digits and m_k is the greatest integer such that $m_k p_{n_k} + p_{n_k-1}$ has no more than k digits. Then we can compute m_k by the expression

$$m_k = \left\lfloor \frac{10^k - 1 - p_{n_k-1}}{p_{n_k}} \right\rfloor, \tag{4}$$

so that semiconvergent s_{n_k,m_k} becomes

$$s_{n_k,m_k} = \frac{m_k p_{n_k} + p_{n_k-1}}{m_k q_{n_k} + q_{n_k-1}}. \tag{5}$$

It is worth noticing that in the case $a_{n_k+1} = 1$, we have $m_k = 0$ and then the best approximation to α with at most k digits is the convergent s_{n_k} since it is finer than $s_{n_k,m_k} = s_{n_k-1}$. For example, when α is the golden ratio $\varphi = (1+\sqrt{5})/2 = [1; 1, 1, 1, 1, \ldots]$, the best approximation with at most k digits is for any k, the convergent s_{n_k}; in this case the best approximation with exactly k digits coincides [4].

More, let M_k the greatest integer such that $M_k p_{n_k} + (a_{n_k} - 1)p_{n_k-1} + p_{n_k-2}$ has no more than k digits, then we can compute M_k by the expression

$$M_k = \left\lfloor \frac{10^k - 1 - (a_{n_k} - 1)p_{n_k-1} - p_{n_k-2}}{p_{n_k}} \right\rfloor, \tag{6}$$

so that pseudoconvergent $\widetilde{s}_{n_k,M_k}$ becomes

$$\widetilde{s}_{n_k,M_k} = \frac{M_k p_{n_k} + (a_{n_k} - 1)p_{n_k-1} + p_{n_k-2}}{M_k q_{n_k} + (a_{n_k} - 1)q_{n_k-1} + q_{n_k-2}}, \tag{7}$$

where a_{n_k} belongs to the development in continued fractions (1).

It can be shown that M_k and m_k are such that $M_k = m_k+1$ or $M_k = m_k$ or $M_k = m_k - 1$.

We remark that the development in continued fraction of a real number α is provided by many algorithms, often included in numerical function libraries. Provided that such development is available, we focus on the computation of the best rational approximation with at most k digits and with exactly k digits, which can be achieved by the following algorithm, for any given integer k.

Algorithm

1. Find n_k and then compute m_k and M_k, respectively by (4) and (6).
2. Build s_{n_k} and s_{n_k,m_k} and $\widetilde{s}_{n_k,M_k}$ (using (5) and (7)).
3. If n_k is odd, s_{n_k,m_k} provides the lower bound LB for approximation with at most k digits and lower bound PLB for approximation with exactly k digits; then s_{n_k} provides the upper bound UB for approximation with at most k digits and $\widetilde{s}_{n_k,M_k}$ the upper bound PUB for approximation with exactly k digits. The reverse happens when n_k is even.
4. Call *bra* the best rational approximation with at most k digits at the numerator; if LB is closer to α than UB, then *bra* = LB; otherwise the reverse.
5. Call *pbra* the best rational approximation with exactly k digits at the numerator; if PLB is closer to α than PUB, then *pbra* = PLB; otherwise the reverse.
6. Print results which include *pbra* only if it does not coincide with *bra*.

It is clear that the number of floating point operations is essentially constant for any k and involves the computations of n_k, m_k, M_k, s_{n_k}, s_{n_k,m_k}, $\widetilde{s}_{n_k,M_k}$ only, which means about 25 floating point operations.

Consequently, the increase of total computational cost depends only on the number of required coefficients in the development of α in continued fraction. About this point, we prove the following

Proposition 2. *Let k be a positive integer, α a positive irrational number. The number N of the coefficients a_i in the development of α in continued fraction $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ which have to be computed to solve the k-digit best approximation problem is independent of α and can be linearly bounded, since*

$$N < lk + 3 \tag{8}$$

for any k, with $l = \frac{\ln 10}{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \simeq 4.785$.

Proof. Let $n_k + 1$ the least integer such that $p_{n_k+1} \geq 10^k$. To solve the problem, we need to compute the coefficients a_n and the related convergents s_n, with $n \leq n_k + 1$.

We denote F_n the Fibonacci numbers, with $F_0 = 0, F_1 = 1, \ldots, F_{n+1} = F_n + F_{n-1}$. Since $p_0 \geq F_0$, $p_1 \geq F_1$ and $a_n \geq 1$ for any $n \geq 1$, we obtain inductively

$$p_{n+1} = a_n p_n + p_{n-1} \geq p_n + p_{n-1} \geq F_n + F_{n-1} = F_{n+1}. \tag{9}$$

Denote $\varphi = \frac{1+\sqrt{5}}{2}$; then $\varphi^2 = \varphi + 1$ and consequently $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$; from $\varphi < F_3$ and $\varphi^2 = \varphi + 1 < F_3 + F_2 = F_4$, we obtain inductively

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1} < F_{n+2} + F_{n+1} = F_{n+3}. \tag{10}$$

From the equation $\varphi^x = 10^k$, whose solution is $x = lk$, we have

$$10^k \le \varphi^{\lceil lk \rceil} < F_{\lceil lk \rceil + 2} \le p_{\lceil lk \rceil + 2}. \tag{11}$$

Since $N = n_k + 1$, it follows

$$N \le \lceil lk \rceil + 2 < lk + 3. \tag{12}$$

□

In the case $a_0 \ge 1$, the inequality (12) can be slightly improved in $N \le lk$. However no better improvement can be expected, since for $\alpha = \varphi = (1 + \sqrt{2})/2$, we have $N = \lceil lk \rceil$ for almost all the integers k.

We remark that this increase refers to the development in continued fractions only; therefore, if this development is given, the computational cost remains essentially constant and very small.

2.2. Error Bounds

In [2], using Farey sequences, it is shown that $s_n - s_{n-1} = (-1)^n / q_n q_{n-1}$ and $s_{n,m} - s_{n-1,m} = (-1)^n / q_{n,m} q_{n-1,m}$ as well. Analogously, referring to cases (i) and (ii) in Sect. 1, we can easily derive an estimation of error bound for the best rational approximation with at most k digits at the numerator by the following computation, which exploits Farey sequence Property 4.

$$\begin{aligned} EBM &= |s_{n_k} - s_{n_k,m_k}|/2 \\ &= \frac{1}{2}\left|\frac{p_{n_k}}{q_{n_k}} - \frac{m_k p_{n_k} + p_{n_k-1}}{m_k q_{n_k} + q_{n_k-1}}\right| \\ &= \frac{1}{2}\left|\frac{m_k p_{n_k} q_{n_k} + p_{n_k} q_{n_k-1} - m_k p_{n_k} q_{n_k} - p_{n_k-1} q_{n_k}}{q_{n_k}(m_k q_{n_k} + q_{n_k-1})}\right| \\ &= 1/2 q_{n_k} q_{n_k,m_k} \end{aligned}$$

since the required rational approximation belongs to one of the two half intervals $[(s_{n_k} + s_{n_k,m_k})/2, s_{n_k}], [s_{n_k,m_k}, (s_{n_k} + s_{n_k,m_k})/2]$, assuming n odd; if n_k even, the reverse holds.

About the best approximation with exactly k digits at the numerator (when it does not coincide with the best approximation with at most k digits, i.e., case (iii) in Sect. 1) we already pointed out that, assuming n_k odd, such best approximation belongs to the interval $[s_{n_k,m_k}, \widetilde{s}_{n_k,M_k}]$; consequently we have that the required estimation of error bound can be computed in the following way

$$\begin{aligned} EBE &= |s_{n_k,m_k} - \widetilde{s}_{n_k,M_k}|/2 \\ &= |(m_k + 1 + M_k)/q_{n_k,m_k}\widetilde{q}_{n_k,M_k}|/2 \\ &\le (m_k + 1)/q_{n_k,m_k}\widetilde{q}_{n_k,M_k} \end{aligned}$$

exploiting the relation between M_k and m_k (from (4) and (6)).

Consequently, for $k \to \infty$, both $EBM \to 0$ and $EBE \to 0$.

Table 2. Best approximation with at most k digits

k	LB	UB	bra	$AS.ER.$
1	2	9/4	**LB**	0.111111111136
2	19/9	93/44	**LB**	$2.4747e-11$
3	19/9	986/467	**LB**	$2.4747e-11$
4	19/9	9992/4733	**LB**	$2.4747e-11$
5	19/9	99995/47366	**LB**	$2.4747e-11$

Table 3. Best approximation with exactly k digits

k	PLB	PUB	$pbra$	$AS.ER.$	EBE
3	990/469	986/467	PLB	$2.3691e-4$	$2.3742e-5$
4	9996/4735	9992/4733	PLB	$2.3466e-5$	$2.3471e-5$
5	99999/47368	99995/47366	PLB	$2.3457e-6$	$2.3457e-6$

3. Numerical Test

We implemented two versions of the Algorithm presented in the previous section. One of them runs in MATLAB, where we used the extended accuracy available by the Symbolic Toolbox, and the other one runs in MATHEMATICA. Using a 64-bit Intel Core i5 PC, both versions exhibit a constant computational time for each $k \leq 16$, which is about 0.01 *sec.* for the MATHEMATICA version and 0.04 *sec.* in the MATLAB version. Using these programs, we already presented many numerical tests in [4] and [3]. Here we show a new peculiar numerical example to enlighten the features of our method.

Consider $\alpha = 2.11111111115 - \sqrt{2} \times 10^{-11}$ which, when rounded, becomes $\alpha \simeq 2.11111111113585803167 3$; the first coefficients in its development in continued fractions are $[2; 8, 1, 498877383, 1, 13, 6, 2, \dots]$.

For $k = 1, 2, 3, 4, 5$, Table 2 reports the best approximations of α by rationals with at most k-digit numerators, whereas Table 3 shows the best approximations of α by rationals with exactly k-digit numerators, only when it does not coincide with the previous one. It is clear that in Table 3 pseudoconvergents appear.

In Tables we use the same notation already used for the algorithm and we call $AS.ER$ the absolute error of the best approximation. Moreover, in both columns *bra* and *pbra* we write convergents in bold face and pseudoconvergents in italics.

We point out that in Table 3 lower bounds in column PLB are provided by pseudoconvergents, whereas upper bounds in column PUB are provided by semiconvergents; for each k, all of them are the closest ones to α with exactly k digits at the numerator. For reader's convenience, we report the absolute errors referring to semiconvergents (to be compared with values in column AS.ER.):

$$\begin{aligned} &\text{for} \quad k = 3, \; |\alpha - 986/467| = 2.3793e-5; \\ &\text{for} \quad k = 4, \; |\alpha - 9992/4733| = 2.3476e-5; \\ &\text{for} \quad k = 5, \; |\alpha - 99995/47366| = 2.3458e-6. \end{aligned}$$

It is clear that for this case the behavior of approximation with exactly k digits at the numerator is completely different from the usual one such as the case reported in Table 1.

At last, we notice that if the real number α is rational then k is such that $k \leq d(\alpha)$ and for $k = d(\alpha)$, the provided best approximation with exactly k digits is the number α itself.

4. Conclusion

We present a fast method for computing the best rational approximation with at most k digits and with exactly k digits at the numerator, for any given real number α. Here fast method means that the computational time remains very small and essentially constant for any k, once the development in continued fractions of number α is available. This improves known results already presented in literature. This method is based on some known theoretical results along with some new results, which are proved using an approach based on Farey sequences. Error bounds are theoretically estimated and numerical examples are reported to enlighten the features of the presented method.

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