Size Estimates of Unknown Boundaries with Robin Type Condition

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Abstract

We deal with the problem of determining an unknown part of the boundary of an electrical conductor that is not accessible from an external observation and where a corrosion process is going on. We obtain estimates from above and below of the size of this damaged region.

1 Introduction

We consider an electrical conductor Ω whose boundary is not fully observable and denote by Γ, the portion of $\partial\Omega$, where it is possible to make on measurements. The aim of this paper is to extract information on an unknown subset E contained in $\partial\Omega \setminus \Gamma$, where a corrosion process is going on, performing boundary measurements on Γ. These problems arise in non-destructive testing of materials, modelling phenomena of surface corrosion in metals (see [Kau-Sa-Vo, Vo-Xu]).
Prescribing a current density g supported on Γ, such that $g = 0$ on $\partial\Omega \setminus \Gamma$, we induce a potential u solution to the problem

$$(1.1) \qquad \begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma u = g, & \text{on } \partial\Omega, \end{cases}$$

where γ denotes the surface impedance of this form

$$(1.2) \qquad \gamma(x) = \gamma_0(x)\chi_\Gamma + k\chi_E \quad \text{for any} \quad x \in \partial\Omega,$$

where k is a constant whose value is not available and $\gamma_0 \equiv 0$ in $\partial\Omega \setminus \Gamma$. The case in which k is replaced by a non constant function, can be treated similarly with minor adjustments. While on the remain portion of the boundary $\partial\Omega \setminus E$, the impedance term γ is fully known.
The goal is to bound the measure of E by comparing the solution u on the boundary with the solution u_0 of the "unperturbed" problem

$$(1.3) \qquad \begin{cases} \Delta u_0 = 0, & \text{in } \Omega, \\ \frac{\partial u_0}{\partial \nu} + \gamma_0 u_0 = g, & \text{on } \partial\Omega, \end{cases}$$

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where $E = \emptyset$, that is, in principle, is completely known. Let us notice that, $\frac{\partial u_0}{\partial \nu}$ vanishes outside Γ.
Specifically, using similar arguments developed in the context of the inverse inclusion problem, see [Al-Mo-Ro03] and the references therein, we deduce information on the size of E by analysing the so called power gap, defined as

$$W - W_0 = \int_{\partial\Omega} gud\sigma - \int_{\partial\Omega} gu_0 d\sigma = \int_\Gamma g(u - u_0)d\sigma.$$

Let us remark here that the quantities W and W_0 can be computed from the boundary data that we measure and are meaningful from a physical viewpoint as they represent the power required to maintain the boundary current g.
The idea of bounding the size of an unknown object D enclosed in a given domain Ω goes back to Friedman [Fr]. The key point is to extract as much as possible information from the available boundary measurements. Precisely, the approach we follow is the one proposed by Alessandrini and Rosset [Al-Ro] and Kang, Sheen and Seo [Ka-Se-Sh] and subsequently refined by some of the authors in [Al-Ro-Se].
The basic argument is to gain information on the hidden boundary by studying the power gap, which is sensitive to the presence of the defect. In particular, since such a power gap contains the information at the accessible boundary, it is possible to carry them up to the inaccessible part of the boundary in a quantitative manner and obtain information on its size. This procedure follows the lines of similar problems studied in [Ka-Se-Sh, Al-Ro] and later developed in [Al-Mo-Ro03, Be-Fr-Ve, Mo-Ro-Ve07, Mo-Ro-Ve12, DC-Li-Mo-Ro-Ve-Wa, DC-Li-Wa, DC-Li-Ve-Wa]. The main novelty of this paper relies on the evaluation of a defect which is located on the boundary. Such a new feature required an original approach in order to relate the power gap and the size of the defect. In order to overcome such a difficulty we find convenient to analyse the problem in an abstract Hilbert setting (Section 3). This argument, due to its general character, can be applied to other practical contexts in inverse problems. The main technical arguments are based on the use of the three spheres inequality and the doubling inequality at the boundary as unique continuation tools which allow to extract information on the unknown defect from the interior and the boundary values of the solution. Another issue that came up dealing with boundary defects, concerns the use of quantitative estimates. With the introduction of a suitable norm (see Remark 2.2) and quantitative estimates of unique continuation (see Proposition 4.2), it has been possible to obtain the desired bounds on the size of the corroded part.
The plan of the paper is the following. In the next Section 2 we define the notation we use, and we state the main theorem. In Section 3 we present an abstract formulation of our problem that will be applied to prove our main result in Section 4.

2 Assumptions and Main Result

For a given vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, we write $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$. Moreover, we denote by $B_r(x)$ and $B'_r(x)$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ of radius r centered at x and x' respectively.

Definition 2.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$*. Given* k, α*, with* $k \in \mathbb{N}$*,* $0 < \alpha \leq 1$*, we say that a portion* S *of* $\partial\Omega$ *is of class* $C^{k,\alpha}$ *with constants* r_0, M*, if for any* $P \in S$*, there exists a rigid transformation of coordinates under which we have* $P = 0$ *and*

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) \,:\, x_n > \psi(x')\},$$

where ψ *is a* $C^{k,\alpha}$ *function on* $B'_{r_0}(0)$ *satisfying*

$$\begin{aligned} &\psi(0) = 0 \\ &\nabla\psi(0) = 0\ ,\ \text{when } k \geq 1 \\ &\|\psi\|_{C^{k,\alpha}(B'_{r_0}(0))} \leq M r_0. \end{aligned}$$

When $k = 0, \alpha = 1$*, we also say that* S *is of Lipsschitz class with constants* r_0, M*.*

Remark 2.1. *We have chosen to normalize all norms in such a way that their terms are dimensionally homogeneous and coincides with the standard definition as the dimensional parameter equals one. For instance, the norm appearing in the previous definition is meant as follows*

$$\|\psi\|_{C^{k,\alpha}(B'_{r_0}(0))} = \sum_{i=0}^{k} r_0^i \|D^i\psi\|_{L^\infty(B'_{r_0}(0))} + r_0^{k+\alpha}|D^k\psi|_{\alpha, B'_{r_0}(0)}\ ,$$

where

$$|D^k\psi|_{\alpha, B'_{r_0}(0)} = \sup_{\substack{x', y' \in B'_{r_0} \\ x' \neq y'}} \frac{|D^k\psi(x') - D^k\psi(y')|}{|x' - y'|^\alpha}$$

Similarly, we shall set

$$\|u\|_{L^2(\Omega)} = r_0^{-\frac{n}{2}} \left(\int_\Omega u^2\right)^{\frac{1}{2}}\ , \tag{2.1}$$

$$\|u\|_{H^1(\Omega)} = r_0^{-\frac{n}{2}} \left(\int_\Omega u^2 + r_0^2 \int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}\ . \tag{2.2}$$

We denote by $< \cdot, \cdot >_{H^{-1/2}, H^{1/2}}$ the duality pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ based on the L^2 scalar product. Given the open and connected portion Γ of $\partial\Omega$, we introduce the trace space $H^{1/2}_{00}(\Gamma)$ as the interpolation space $[H^1_0(\Gamma), L^2(\Gamma)]_{1/2}$ (see [Li-Ma, Chapter 1]). Let us now consider the following space of distributions $H^{-1/2}(\Gamma) = \{\eta \in H^{-1/2}(\partial\Omega) |\, < \eta, \varphi >= 0, \forall \varphi \in H^{1/2}_{00}(\partial\Omega \setminus \overline{\Gamma})\}$.

Assumptions on the domain Ω

Given $r_0, M > 0$ constants, we assume that $\Omega \subset \mathbb{R}^n, n \geq 2$ and

(2.3) Ω is of Lipschitz class with constants r_0, M.

Furthermore, given $L > 0$ we assume that

$$|\partial\Omega| \leq L r_0^{n-1}\ . \tag{2.4}$$

In addition, we assume that the portion of the boundary

(2.5) $$\partial\Omega \setminus \Gamma \text{ is of class } C^{1,1} \text{ with constants } r_0, M \ .$$

Assumptions on the surface impedance γ
Given E an open and connected subset of $\partial\Gamma \setminus \overline{\Gamma}$ and given Γ_0 be an open and connected subset of Γ we assume that

(2.6) $$\gamma \in L^\infty(\partial\Omega) \ .$$

Moreover, for a given constant $c_0, 0 < c_0 \leq 1$ we have that

(2.7) $$\gamma(x) \geq \frac{c_0}{r_0} > 0 \quad \text{on } \Gamma_0 \ .$$

Finally, for a given function $\gamma_0(x) \in L^\infty(\partial\Omega)$ supported on Γ and such that

(2.8) $$\gamma_0(x) \leq c_0^{-1}/r_0$$

we have that

(2.9) $$\gamma(x) = \gamma_0(x)\chi_\Gamma + k\chi_E$$

where $k > 0$ is an unknown constant such that

(2.10) $$0 < \bar{k}_0 < kr_0 < \bar{k}_1$$

for given constants $\bar{k}_0$ and $\bar{k}_1$.
Here and in the following we shall denote with

(2.11) $$\gamma(x) = \frac{\bar{\gamma}(x)}{r_0},$$

(2.12) $$\gamma_0(x) = \frac{\bar{\gamma}_0(x)}{r_0},$$

(2.13) $$k = \frac{\bar{k}}{r_0}.$$

Assumptions on the given data g
Given $g_0 > 0$ we assume that

(2.14) $$\|g\|_{H^{-1/2}(\Gamma)} \leq g_0 \ .$$

Furthermore, given $F > 0$ we assume that

(2.15) $$\frac{\|g\|_{H^{-\frac{1}{2}}(\Gamma)}}{\|g\|_{H^{-1}(\Gamma)}} \leq F \ .$$

This ratio takes into account the oscillation character of the boundary data and it is called frequency. Other choices of norms are possible and we refer to [Al-Mo-Ro03] for a discussion on this topic.

Remark 2.2. *We first observe that the standard norm in* $H^1(\Omega)$ *and the norm*

$$\|u\|_* = r_0^{-\frac{n}{2}} \left(r_0^2 \int_\Omega |\nabla u|^2 dx + r_0 \int_{\Gamma_0} u^2 d\sigma \right)^{1/2},$$

are equivalent.
Indeed, we notice that, on one hand, by the standard trace estimate we have

$$\|u\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C \|u\|_{H^1(\Omega)} \tag{2.16}$$

where $C > 0$ *is a constant depending on* L *and* M *only. The above inequality leads to*

$$r_0^2 \int_\Omega |\nabla u|^2 dx + r_0 \int_{\Gamma_0} u^2 d\sigma \leq C \left(r_0^2 \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx \right)$$

where $C > 0$ *is a constant depending on* L *and* M *only.*
On the other hand, by the argument in [Al-Mo-Ro01, Example 3.6] we deduce that

$$r_0^2 \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx \leq C \left(r_0^2 \int_\Omega |\nabla u|^2 dx + r_0 \int_{\Gamma_0} u^2 d\sigma \right)$$

where $C > 0$ *is a constant depending on* L *and* M *only.*

Denoting by $< \cdot, \cdot >_{H^{-1/2}, H^{1/2}}$ the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, with a slight abuse of notation, we will write

$$< g, f >_{H^{-1/2}, H^{1/2}} = \int_{\partial\Omega} g f d\sigma$$

for any $g \in H^{-1/2}(\partial\Omega)$ and $f \in H^{1/2}(\partial\Omega)$.

Remark 2.3. *By solution to* (1.1) *we mean a function* $u \in H^1(\Omega)$ *such that*

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \gamma(x) u v d\sigma = \int_{\partial\Omega} g v d\sigma, \qquad \forall\, v \in H^1(\Omega).$$

As a consequence of Remark 2.2, we deduce that the existence and the uniqueness of the weak solution to the problem (1.1) *follow from standard theory on the boundary value problem for the Laplace equation and the sign condition* (2.7).

The inverse problem we are addressing to is to estimate the size of the corroded part E of the boundary from a knowledge of Cauchy data $\{g, u_{|\Gamma}\}$. For this purpose we will compare u with the solution u_0 of the problem when $E = \emptyset$ and $\gamma \equiv \gamma_0$. Precisely $u_0 \in H^1(\Omega)$ is such that

$$\int_\Omega \nabla u_0 \cdot \nabla v dx + \int_{\partial\Omega} \gamma_0(x) u_0 v d\sigma = \int_{\partial\Omega} g v d\sigma, \qquad \forall\, v \in H^1(\Omega).$$

As mentioned before we denote by W, W_0 the power required to maintain the current density g on $\partial\Omega$ when E is present and it is not respectively, namely

$$W = \int_{\partial\Omega} g u d\sigma = \int_\Omega \nabla u \cdot \nabla u dx + \int_{\partial\Omega} \gamma u^2 d\sigma,$$

$$W_0 = \int_{\partial\Omega} g u_0 d\sigma = \int_\Omega \nabla u_0 \cdot \nabla u_0 dx + \int_{\partial\Omega} \gamma_0 u_0^2 d\sigma.$$

From now on we shall refer as the a priori data, the following set of quantities: $M, L, \bar{k}_0, \bar{k}_1, c_0, g_0, F$.
We can now state the main result we want to prove.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class $C^{0,1}$. Let $\gamma, \gamma_0 \in L^\infty(\partial\Omega)$ defined as above. Then there exist positive constants $C_1, C_2, p > 1$ depending on the a priori data only such that*

$$C_1 {r_0}^{n-1} \frac{W - W_0}{W_0} \leq |E| \leq C_2 {r_0}^{n-1} \left(\frac{W - W_0}{W_0} \right)^{1/p} . \tag{2.17}$$

3 Abstract Formulation

To prove Theorem 2.4 we will make use of techniques developed in the context of the inverse conductivity problem [Al-Mo-Ro03]. The difference with respect to other situations, is that we want to determine a defect of the external boundary of the specimen whereas in the other cases the inhomogeneity is fully contained in the domain. To overcome this difficulty we will rephrase the argument in an abstract way disconnecting it from the physical context.
We denote by H a Hilbert space and by H' its dual. Let $a_1(\cdot,\cdot)$ and $a_0(\cdot,\cdot)$ be two bilinear symmetric forms on H and let $F \in H'$. By Lax-Milgram Theorem, there exist u_1 and u_0 in H such that

$$a_j(u_j, v) =< F, v > \qquad \forall\, v \in H, \quad j = 0, 1,$$

where $< \cdot, \cdot >$ denotes the duality pairing between H and H'.

Lemma 3.1. *The following inequalities hold true*

$$J_1 := a_0(u_1 - u_0, u_1 - u_0) - [a_1(u_0, u_0) - a_0(u_0, u_0)] =< F, u_1 - u_0 >, \tag{3.18a}$$

$$J_2 := a_0(u_0 - u_1, u_0 - u_1) - [a_0(u_1, u_1) - a_1(u_1, u_1)] = - < F, u_1 - u_0 >, \tag{3.18b}$$

$$J_3 := a_0(u_1, u_0) - a_1(u_1, u_0) =< F, u_1 - u_0 > . \tag{3.18c}$$

Proof. Let us verify (3.18a).

$$\begin{aligned} & a_0(u_1 - u_0, u_1 - u_0) - [a_1(u_0, u_0) - a_0(u_0, u_0)] \\ = \; & a_1(u_1, u_1) - 2a_1(u_1, u_0) + a_1(u_0, u_0) - a_1(u_0, u_0) + a_0(u_0, u_0) \\ = \; & < F, u_1 > -2 < F, u_0 > + < F, u_0 >=< F, u_1 - u_0 > . \end{aligned}$$

Equalities (3.18b) and (3.18c) can be obtained similarly. □

We define now

$$G(u) := a_1(u, u) - a_0(u, u), \qquad u \in H.$$

Let us observe that G is a functional depending on the defect. We define also

$$\alpha(u, v) := \frac{1}{4}[G(u + v) - G(u - v)], \qquad u, v \in H.$$

Trivially we have

$$a_1(u, v) = a_0(u, v) + \alpha(u, v), \qquad u, v \in H.$$

Lemma 3.2. *If for every $u \in H$ either $\alpha(u,u) \geq 0$, or $\alpha(u,u) \leq 0$, then*

$$|\alpha(u,v)| \leq |\alpha(u,u)|^{1/2}|\alpha(v,v)|^{1/2}, \tag{3.19}$$

for every $u, v \in H$.

Proof. If $\alpha(u,u) = 0$ and $\alpha(v,v) = 0$, then, assuming $\alpha(w,w) \geq 0$, for every $w \in H$, we would have

$$0 \leq \alpha(u+tv, u+tv) = 2t\alpha(u,v), \qquad \forall\, t \in \mathbb{R},$$

which implies $\alpha(u,v) = 0$ and (3.19) is proved.
If $\alpha(u,u) \neq 0$ or $\alpha(v,v) \neq 0$, then assuming for instance $\alpha(v,v) > 0$, we would have

$$0 \leq \alpha(u+tv, u+tv) = t^2\alpha(v,v) + 2t\alpha(u,v) + \alpha(u,u), \qquad \forall\, t \in \mathbb{R},$$

from which

$$(\alpha(u,v))^2 - \alpha(u,u)\alpha(v,v) \leq 0$$

and (3.19) follows.
If $\alpha(w,w) \leq 0$, for every $w \in H$, the thesis follows similarly applying the previous argument to $-\alpha(\cdot,\cdot)$. □

Defining

$$\delta W = < F, u_1 - u_0 >,$$

formula (3.18) can be written as

$$a_1(u_1 - u_0, u_1 - u_0) - \alpha(u_0, u_0) = \delta W, \tag{3.20a}$$

$$a_0(u_1 - u_0, u_1 - u_0) + \alpha(u_1, u_1) = -\delta W, \tag{3.20b}$$

$$\alpha(u_0, u_1) = \delta W. \tag{3.20c}$$

We now prove estimates for a and α that will be useful for our purposes.

Proposition 3.3. *Let $\lambda_0, \lambda_1 \in (0,1]$ be given. Assume that a_0 and a_1 satisfy the following conditions*

$$\lambda_0\|u\|^2 \leq a_0(u,u) \leq \lambda_0^{-1}\|u\|^2, \qquad \forall\, u \in H, \tag{3.21a}$$

$$\lambda_1\|u\|^2 \leq a_1(u,u) \leq \lambda_1^{-1}\|u\|^2, \qquad \forall\, u \in H. \tag{3.21b}$$

If α satisfies the condition

$$0 \leq \alpha(u,u) \leq C_0 a_0(u,u) \quad \forall\, u \in H, \tag{3.22}$$

where C_0 is a positive constant, then

$$|\delta W| \leq \alpha(u_0, u_0) \leq (1 + C_0)|\delta W|. \tag{3.23}$$

Conversely, if α satisfies the condition

$$\alpha(u,u) \leq 0 \quad \forall\, u \in H, \tag{3.24}$$

then

$$C|\delta W| \leq -\alpha(u_0, u_0) \leq |\delta W|, \tag{3.25}$$

where C is a positive constant depending on λ_0, λ_1 only.

Proof. Let us first consider case (3.22). By (3.20b) we have $\delta W \leq 0$ and by (3.20a) we have $-\alpha(u_0, u_0) \leq \delta W$. Thus

$$|\delta W| \leq \alpha(u_0, u_0). \tag{3.26}$$

Let us now obtain the upper bound for $\alpha(u_0, u_0)$. Using Lemma 3.2 we have

$$\begin{aligned}
& \alpha(u_0, u_0) \\
\leq\; & \alpha(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1) + 2|\alpha(u_0 - u_1, u_0 - u_1)|^{1/2}|\alpha(u_1, u_1)|^{1/2} \\
\leq\; & \alpha(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1) + \varepsilon\alpha(u_0 - u_1, u_0 - u_1) + \frac{1}{\varepsilon}\alpha(u_1, u_1) \\
\leq\; & (1+\varepsilon)[C_0 a_0(u_0 - u_1, u_0 - u_1) + \frac{1}{\varepsilon}\alpha(u_1, u_1)] \\
\leq\; & (1 + C_0)|\delta W|,
\end{aligned}$$

where in the last line we have chosen $\varepsilon = 1/C_0$. Hence we get

$$\alpha(u_0, u_0) \leq (1 + C_0)|\delta W|.$$

Let us consider now case (3.24). By (3.20a) we get $\delta W \geq 0$ and also

$$|\alpha(u_0, u_0)| \leq \delta W. \tag{3.27}$$

Let us recover an estimate from below for $|\alpha(u_0, u_0)|$. By (3.20c) we get

$$\begin{aligned}
\delta W &= \alpha(u_0, u_1) \leq (-\alpha(u_0, u_0))^{1/2}(-\alpha(u_1, u_1))^{1/2} \\
&\leq \frac{\varepsilon}{2}(-\alpha(u_1, u_1)) + \frac{1}{2\varepsilon}(-\alpha(u_0, u_0)).
\end{aligned} \tag{3.28}$$

Also by (3.20b) we have

$$-\alpha(u_1, u_1) = a_0(u_1 - u_0, u_1 - u_0) + \delta W. \tag{3.29}$$

Moreover by (3.21a) and (3.21b), we have

$$a_0(u_1 - u_0, u_1 - u_0) \leq \lambda_0^{-1}\lambda_1^{-1} a_1(u_1 - u_0, u_1 - u_0).$$

By this last inequality and (3.29) we obtain

$$-\alpha(u_1, u_1) \leq \lambda_0^{-1}\lambda_1^{-1} a_1(u_1 - u_0, u_1 - u_0) + \delta W,$$

that inserting (3.28) and using (3.20a) we have

$$\begin{aligned}
\delta W &\leq \frac{\varepsilon}{2}[A a_1(u_1 - u_0, u_1 - u_0) + \delta W] + \frac{1}{2\varepsilon}(-\alpha(u_0, u_0)) \\
&= \frac{\varepsilon}{2}[A(a_1(u_1 - u_0, u_1 - u_0) - \alpha(u_0, u_0)) + A\alpha(u_0, u_0) + \delta W] \\
&\qquad + \frac{1}{2\varepsilon}(-\alpha(u_0, u_0)) \\
&= \frac{\varepsilon}{2}(1 + A)\delta W + \left(\frac{1}{2\varepsilon} - A\frac{\varepsilon}{2}\right)(-\alpha(u_0, u_0)),
\end{aligned}$$

where we have defined $A = \lambda_0^{-1}\lambda_1^{-1}$. Thus

$$\left(1 - \frac{\varepsilon}{2}(1+A)\right)\delta W \leq \frac{1 - A\varepsilon^2}{2\varepsilon} \leq |\alpha(u_0, u_0)|.$$

If $\varepsilon < \sqrt{A}$ we have

$$\frac{2\varepsilon\left(1 - \frac{\varepsilon}{2}(1+A)\right)}{1 - A\varepsilon^2}\delta W \leq |\alpha(u_0, u_0)|.$$

Finally, choosing $\varepsilon = \frac{1}{1+A}$, we get

$$c\delta W \leq |\alpha(u_0, u_0)|,$$

where c depends on λ_0, λ_1 only. □

Remark 3.4. *In the case* (3.22)*, condition* (3.21) *can be weaken by assuming that $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ are positive semi-definite. Conversely in* (3.24) *case, it is enough to require that $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ are positive semi-definite and such that*

$$a_0(u,u) \leq C_1 a_1(u,u) \qquad \forall\, u \in H,$$

where C_1 is a positive constant.

4 Proof of the Main Result

We want to make use of estimates obtained in the previous section to prove our bounds on the size of E. For this purpose let us define

$$\begin{aligned} a_1(u,v) &= \int_\Omega \nabla u \cdot \nabla v dx + \int_\Gamma \gamma_0 uv d\sigma + k\int_E uv d\sigma, \\ a_0(u,v) &= \int_\Omega \nabla u \cdot \nabla v dx + \int_\Gamma \gamma_0 uv d\sigma, \\ \alpha(u,v) &= k\int_E uv d\sigma, \end{aligned}$$

for $u, v \in H^1(\Omega)$. We have immediately

$$\begin{aligned} &\alpha(u,u) = k\int_E u^2 d\sigma \geq 0, \\ &a_0(u,u) \leq a_1(u,u), \end{aligned} \tag{4.30}$$

for every $u \in H^1(\Omega)$.

Lemma 4.1. *There exists a constant $C > 0$ depending on M and L only such that*

$$\int_{\partial\Omega\setminus\Gamma} u^2 d\sigma \leq C\left(r_0\int_\Omega |\nabla u|^2 + \int_{\Gamma_0} u^2 d\sigma\right) \tag{4.31}$$

for every $u \in H^1(\Omega)$.

Proof. By a standard trace inequality (see [Ada, Chap. 7]) we get

$$r_0 \int_{\partial\Omega\setminus\Gamma} u^2 d\sigma \le C \left(\int_\Omega |u|^2 dx + r_0^2 \int_\Omega |\nabla u|^2 dx \right) . \tag{4.32}$$

Moreover, by the equivalence between the norm in $H^1(\Omega)$ and the norm $\|\cdot\|_*$ introduced in Remark 2.2, the thesis follows. □

The main tools of unique continuation needed in the proof of our main result are contained in [Si, Lemma 4.5,Theorem 4.6, Corollary 4.7] and for a detailed proof of them we refer to [Si]. However, for the reader's convenience and for making the paper as much self-contained as possible we give below a sketch of the proof of our main ingredient of unique continuation.

Proposition 4.2. *(A_p property on the boundary) Let u_0 be a solution to the problem* (1.3)*, then there exist constants $p>1, A>0, \bar{r}>0$ depending on the a priori data only such that for every $x_0 \in \Gamma_{1,2\bar{r}}$ the following holds*

$$\left(\frac{1}{|\Delta_r(x_0)|} \int_{\Delta_r(x_0)} |u_0|^2 \right) \left(\frac{1}{|\Delta_r(x_0)|} \int_{\Delta_r(x_0)} |u_0|^{\frac{-2}{p-1}} \right)^{p-1} \le A \tag{4.33}$$

where $\Gamma_{1,2\bar{r}} = \{x \in \partial\Omega \ : \ dist(x,\Gamma_1) < 2\bar{r}\}$, $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}$ and $\Delta_r(x_0) = \Gamma_{1,2\bar{r}} \cap B_r(x_0)$ with $0<r<\bar{r}$.

Proof. We recall that, as main tool of unique continuation, the so called *surface* doubling inequality has been achieved in [Si] and it reads as follows. There exists a constant $K_1>0$ depending on the a priori data only, such that for any $x_0 \in \Gamma_{1,\frac{\bar{r}}{2}}$ and for every $r \in (0,\bar{r})$ the following holds

$$\int_{\Delta_{2r}(x_0)} u_0^2 \le K_1 \int_{\Delta_r(x_0)} u_0^2 \ . \tag{4.34}$$

The proof of the latter relies on two main ingredients. The first one is the well-known stability estimate for the Cauchy problem (see for instance [Tr]), which reads as follows

$$\begin{aligned} \int_{B_{\frac{r}{2}}(x_0)\cap\Omega} |u_0|^2 \le \quad & Cr \left(\int_{\Delta_r(x_0)} u_0^2 + r^2 \int_{\Delta_r(x_0)} |\nabla_t u_0|^2 \right)^{1-\delta} \\ & \cdot \left(\int_{\Delta_r(x_0)} u_0^2 + r^2 \int_{\Delta_r(x_0)} |\nabla_t u_0|^2 + \int_{B_r(x_0)\cap\Omega} |\nabla u_0|^2 \right)^{\delta} \end{aligned} \tag{4.35}$$

where ∇_t denotes the tangential gradient on $\Delta_r(x_0)$ (more precisely we have $\nabla_t u_0 = \nabla u_0 - (\nabla u_0 \cdot \nu)$) and $C>0,\ 0<\delta<1$ are constants depending on the a priori data only.
The second main ingredient is the following *volume* doubling inequality (see Lemma 4.5 [Si])

$$\int_{B_{\beta r}(x_0)\cap\Omega} |u_0|^2 \le C\beta^K \int_{B_r(x_0)\cap\Omega} |u_0|^2 \tag{4.36}$$

for every r, β such that $\beta > 1$ and $0 < \beta r < 2\bar{r}$ where C, K are positive constants depending on the a priori data only. The inequality (4.36) has been achieved in [Si] by combining the techniques introduced in [Ad-Es] which apply to homogeneous Neumann boundary conditions with a suitable change of variable which fits the problem under the assumption required in [Ad-Es].
The control on the vanishing rate of the solution on the boundary provided by inequality (4.34) allows the author to obtain in [Si, Corollary 4.7] the following reverse Hölder inequality

$$(4.37) \qquad \left(r^{-2}\int_{\Delta_r(x_0)} u_0^2\right)^{\frac{1}{4}} \leq \left(Cr^{-2}\int_{\Delta_r(x_0)} u_0^2\right)^{\frac{1}{2}}$$

which in turn, combined with the powerful theory of Muckenhoupt weights (see [Co-Fe]) leads to the desired integrability property for $|u_0|^{-1}$ in (4.33).

□

Proof of Theorem 2.4. By Lemma 4.1, there exists a positive constant C_1, depending on M, L, c_0, such that

$$0 \leq \alpha(u,u) \leq C_1 a_0(u,u), \quad \forall u \in H^1(\Omega).$$

By the above inequality and by Proposition 3.3, we have

$$(4.38) \qquad |\delta W| \leq k\int_E u_0^2 d\sigma \leq (1+C_1)|\delta W|\ ,$$

where $\delta W = \int_{\partial\Omega} g u_0$. The left hand side inequality and standard bounds on Neumann problem solution lead to the following inequality

$$|\delta W| \leq |E| k \|u_0\|^2_{L^\infty(E)}.$$

Moreover, by an uniform boundness type estimate (see [Gi-Tr, Chapter 8]) we have that

$$|\delta W| \leq C\bar{k} r_0^{-1} |E| \|u_0\|^2_{H^1(\Omega)},$$

where C depends on the a priori data only. By Remark 2.2 we also have that

$$|\delta W| \leq C\bar{k}|E| r_0^{-n-1}\left(r_0\int_{\Gamma_0} u_0^2 + r_0^2\int_\Omega |\nabla u_0|^2\right).$$

Moreover, by the lower bound in (2.3) we deduce that

$$\begin{aligned} |\delta W| &\leq C\bar{k}|E| r_0^{1-n}\max\{{c_0}^{-1},1\}\left(\int_{\Gamma_0}\gamma_0 u_0^2 + \int_\Omega |\nabla u_0|^2\right) \\ &\leq C\bar{k}|E| r_0^{1-n}\max\{{c_0}^{-1},1\}\left(\int_{\partial\Omega}\gamma u_0^2 + \int_\Omega |\nabla u_0|^2\right). \end{aligned}$$

Finally, by the weak formulation for u_0 (see Remark 2.3) we have that

$$|\delta W| \leq C\bar{k}|E| r_0^{1-n}\max\{{c_0}^{-1},1\}\left(\int_{\partial\Omega} g u_0 d\sigma\right).$$

Let us consider now the upper bound for E. First we have to cover properly the unknown part of the boundary (we refer the reader to [Be-Fr-Ve] where a similar construction has been carried on). Let r be such that

$$r = \frac{1}{4} \min\left\{ \frac{r_0}{8\sqrt{n}}, \frac{r_0}{2\sqrt{n}M} \right\}. \tag{4.39}$$

and define

$$\Gamma_1^r = \{x \in \Omega \ : \ \mathrm{dist}(x, \Gamma_1) < r\} \tag{4.40}$$

where $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma}$.
Let $\{Q_j\}_{j=1}^J$ be a family of closed mutually internally disjoint cubes of size $2r$ such that

$$\Gamma_1^r \cap Q_j \neq \emptyset, \ \ j = 1, \dots, J, \tag{4.41}$$

$$\Gamma_1^r \subset \cup_{j=1}^J Q_j. \tag{4.42}$$

Let $x_j \in \Gamma_1^r \cap Q_j, \ j = 1, \dots, J$. We have that

$$\Gamma_1^r \subset \cup_{j=1}^J B_{4\sqrt{n}r}(x_j).$$

Indeed for $x \in \Gamma_1^r$, there exists $\overline{x} \in \Gamma_1$ such that $\mathrm{dist}(x, \overline{x}) < 2\sqrt{n}r$. Let j be such that $\overline{x} \in Q_j$, since $|\overline{x} - x_j| \leq 2\sqrt{n}r$, we have

$$|x - x_j| \leq |x - \overline{x}| + |\overline{x} - x_j| \leq 4\sqrt{n}r,$$

which implies $x \in B_{4\sqrt{n}r}(x_j)$. Using the construction argument in [Be-Fr-Ve, Proposition 5.2] we can infer that there exists a constant $C > 0$ depending on M and L only such that

$$\begin{aligned} &\cup_{j=1}^J Q_j \subset \{x \in \mathbb{R}^n \ : \mathrm{dist}(x, \Gamma_1) \leq 4\sqrt{n}r\} \ \text{ and} \\ &J \leq C \ , \end{aligned} \tag{4.43}$$

where $C > 0$ is a constant depending on M and L only. By the Hölder inequality, (3.23) and (4.32) we have

$$\begin{aligned} |E| &= \int_E |u_0|^{-\frac{2}{p}} |u_0|^{\frac{2}{p}} \leq \left(\int_E |u_0|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_E |u_0|^2 \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\Gamma_1} |u_0|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}} ((1 + C_0)|\delta W|)^{\frac{1}{p}} , \end{aligned} \tag{4.44}$$

where C depends on and M, L, c_0 only. Now

$$\begin{aligned} \int_{\Gamma_1} |u_0|^{-\frac{2}{p-1}} &\leq \int_{\Gamma_1 \cap (\cup_{j=1}^J B_{4\sqrt{n}r}(x_j))} |u_0|^{-\frac{2}{p-1}} \leq \sum_{j=1}^J \int_{\Delta_j} |u_0|^{-\frac{2}{p-1}} \\ &\leq \sum_{j=1}^J \frac{L r_0^{n-1}}{|\Delta_j|} \int_{\Delta_j} |u_0|^{-\frac{2}{p-1}}, \end{aligned} \tag{4.45}$$

where $\Delta_j = B_{4\sqrt{n}r}(x_j) \cap \Gamma_1$. By Proposition 4.2 , we have that

$$\frac{1}{|\Delta_j|}\int_{\Delta_j} |u_0|^{-\frac{2}{p-1}} \leq \left(\frac{A}{\frac{1}{|\Delta_j|}\int_{\Delta_j}|u_0|^2}\right)^{\frac{1}{p-1}}, \tag{4.46}$$

where A is a constant depending on M, L, c_0, F only. Let us assume that the index $\bar{j}, 1 \leq \bar{j} \leq J$ is such that

$$\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}} |u_0|^2 = \min_{1\leq j\leq J} \frac{1}{|\Delta_j|}\int_{\Delta_j} |u_0|^2. \tag{4.47}$$

By combining (4.44), (4.45) and (4.47) we have that

$$|E| \leq \left(JLr_0^{n-1}\left(\frac{A}{\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}}|u_0|^2}\right)^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}} ((1+C_0)|\delta W|)^{\frac{1}{p}}\,. \tag{4.48}$$

By the a priori bound $|\partial\Omega| \leq Lr_0^{n-1}$, we easily get that

$$\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}} |u_0|^2 \geq \frac{1}{Lr_0^{n-1}}\int_{\Delta_{\bar{j}}} |u_0|^2. \tag{4.49}$$

By (4.34) and by a standard trace inequality we can infer that

$$\int_{\Delta_{\bar{j}}(x_j)} |u_0|^2 \geq Cr_0^{-1}\int_{B_{2\sqrt{n}r}(x_j)\cap\Omega} |u_0|^2, \tag{4.50}$$

where $C > 0$ is a constant depending on $\bar{k}_0, \bar{k}_1, M, L, F$ only. Let $\bar{x} \in B_{2\sqrt{n}r}(x_j)\cap\Omega$ be such that $B_{\frac{\sqrt{n}}{4}r}(\bar{x}) \subset B_{2\sqrt{n}r}(x_j) \cap \Omega$. Hence we get

$$\int_{\Delta_{\bar{j}}(x_j)} |u_0|^2 \geq Cr_0^{-1}\int_{B_{\frac{\sqrt{n}}{4}r}(\bar{x})} |u_0|^2. \tag{4.51}$$

Now, using the arguments developed in [Mo-Ro, Proposition 3.1] (see also [Al-Si-Ve, Lemma 5.3]), relying on a standard propagation of smallness, we get that

$$\int_{B_{\frac{\sqrt{n}}{4}r}(\bar{x})} |u_0|^2 \geq C\int_{\Omega} |u_0|^2, \tag{4.52}$$

where $C > 0$ is a constant depending on $M, L, \bar{k}_0, \bar{k}_1, F$ only. Hence combining (4.49), (4.51) and (4.52) it easily follows that

$$\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}} |u_0|^2 \geq Cr_0^{-n}\int_{\Omega} |u_0|^2, \tag{4.53}$$

where $C > 0$ is a constant depending on $M, L, \bar{k}_0, \bar{k}_1, F$ only. By the estimate (4.51) and the Caccioppoli inequality we get that

$$\int_{\Delta_{\bar{j}}} |u_0|^2 \geq Cr_0\int_{B_{\frac{\sqrt{n}}{8}r}(\bar{x})} |\nabla u_0|^2, \tag{4.54}$$

where $C > 0$ is a constant depending on the $M, L, \bar{k}_0, \bar{k}_1, F$ only. Repeating again the propagation of smallness techniques described in [Mo-Ro, Proposition 3.1] but for the gradient instead we get that

$$\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}}|u_0|^2 \geq Cr_0^{2-n}\int_{\Omega}|\nabla u_0|^2, \tag{4.55}$$

where $C > 0$ is a constant depending on $M, L, \bar{k}_0, \bar{k}_1, F$ only. We can then infer that

$$\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}}|u_0|^2 \geq C\left(r_0^{-n}\int_{\Omega}|u_0|^2 + r_0^{2-n}\int_{\Omega}|\nabla u_0|^2\right), \tag{4.56}$$

where $C > 0$ is a constant depending on $M, L, \bar{k}_0, \bar{k}_1, F$ only. By the equivalence between the standard $H^1(\Omega)$ norm and the norm introduced in Remark 2.2 we find that

$$\frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}}|u_0|^2 \geq Cr_0^{-n}\left(r_0\int_{\Gamma_0}|u_0|^2 + r_0^2\int_{\Omega}|\nabla u_0|^2\right) \tag{4.57}$$

where $C > 0$ is a constant depending on $M, L, \bar{k}_0, \bar{k}_1, F$ only. Now by the a priori bound $\gamma_0(x) \leq c_0^{-1}/r_0$ on Γ we get that

$$\begin{aligned} \frac{1}{|\Delta_{\bar{j}}|}\int_{\Delta_{\bar{j}}}|u_0|^2 &\geq Cr_0^{2-n}\min\{1, c_0\}\left(\int_{\Gamma_0}\gamma_0|u_0|^2 + \int_{\Omega}|\nabla u_0|^2\right) \\ &\geq Cr_0^{2-n}\min\{1, c_0\}\int_{\partial\Omega}gu_0. \end{aligned} \tag{4.58}$$

Combining (4.43), (4.48) and (4.58) and recalling that $\int_{\partial\Omega}gu_0 = W_0$ we obtain that

$$|E| \leq Cr_0^{n-1}\left(\frac{W - W_0}{W_0}\right)^{\frac{1}{p}} \tag{4.59}$$

where $C > 0$ is a constant depending on $M, L, \bar{k}_0, \bar{k}_1, F, c_0$ only. □

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