INTUITIVE THINKING IN A CONTEXT OF LEARNING

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Traditional studies approach students' intuitions giving a task and observing their reasoning. In Mathematics Education, it would be interesting to provide an understanding of intuitions specifically in learning processes. This is the aim of this paper. To accomplish it, we provide both a theoretical framework to encompass the specificity and the complexity of intuition in learning mathematics, and an example, from classroom activities in a longitudinal study, that shows intuitive thinking as emerging within a socially shared activity, and interrelating with other ways of thinking throughout the students' objectification process. We conclude that intuitions may come from the individual's insight, but it is in the socio-cultural activity that they are part of mathematics learning processes.

Traditionally, from different perspectives (Fischbein, 1987; Gigerenzer & Selten, 2001; Kahneman, Slovic, & Tversky, 1982), intuitions have been regarded as a way of thinking that is in contrast with the mathematical deductive one, identified as mediated, analytical and justification-requiring, in accordance with the rules of formal logic (Fischbein, 1987). The intuitive form of thinking, indeed, establishes a necessity that does not follow the logical necessity criterion.

The *theory of knowledge objectification* (TKO) is specifically interested on *how people think when they learn*, rather than generally *how they think* (Radford, 2008). Looking at intuitions from a teaching/learning point of view requires both a theoretical and a methodological shift. On the one hand, given this change of theoretical focus from *thinking* to *thinking in learning*, instead of looking at cognitive functioning, we need to provide theoretical tools that encompass the *consciousness movement* (Radford, 2012) as reflecting a cultural and historical dimension that transcends it. Therefore, the aim of this study is to frame intuitions within an Activity Theory strand (see Roth & Radford, 2011). On the other hand, an experimental investigation cannot look at the response of the subject exposed to a specific task or problem to scrutinize his cognitive structure, but it is necessary to provide data that show the complexity of the consciousness's movement towards a mathematical generalization.

LITERATURE REVIEW ON INTUITION

Fischbein (1987) underlines that intuitions are neither a source, nor a method, but a type of cognition. He distinguishes between: *perception*, a form of immediate cognition, and *intuition*, which exceeds the given facts. Perceptual knowledge is immediate, while intuitive knowledge is also extrapolative (Fischbein, 1987). Immediacy is the widely acknowledged basic feature of intuition. Furthermore, intuitions refer to self-evident statements that exceed the observable facts. Being apparently self-evident, intuitions appear generally as absolute and unchangeable–they

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possess a coercive character. A certain statement that is accepted as self-evident is also accepted globally as a structured, meaningful, unitary representation. The unity between the particular, the specific, the directly convincing example, and the general principle derived through similarity and proportionality from the particular case, *needs to be established* by the subject in order to have intuitive knowledge. The globality of intuitions – based on tacit, perceptual elaborations – is generally expressed in a selection process which tends to eliminate the discordant clues and to organize the others so as to present a unitary, compact meaning (Fischbein, 1987). In a sense, globalization in intuitive knowledge can be regarded as a form of generalization. This is in line with Radford's (2013) understanding of sensuous cognition as a feature of living material bodies which have responsive sensations. Perception is, according to this view, the substratum of mind and it is culturally shaped. Perception is a sensing form of action and reflection, which can pave the road to culturally-historically forms of mathematical generalization.

Gigerenzer & Selten (2001) consider intuition the most effective form of thinking compared to rational and deductive thinking. Intuition for Gigerenzer is good adaptation, namely both copying prestigious individuals, and conforming to the most common behavior in the population. Fischbein (1987) suggests that intuitions do not disappear from intellectual (mathematical) endeavors, because they are an integral part of any intellectually productive activity. In fact, as a consequence of Goedel's incompleteness theorem, formally, any mathematical system cannot be absolutely closed, it cannot possess in itself all the necessary formal prerequisites for deciding about the validity of all its theorems (see Fischbein, 1987). Psychologically, no productive mathematical reasoning (solving problems, producing theorems and proofs, etc.) is possible by resorting only to formal means (Fischbein, 1987). This is reported by Liljedahl (in press) in a study on famous mathematicians' "AHA!" experiences: the aspects of illumination that sets this occurrence apart from other mathematical experiences are affective in nature. The cognitive components are not absent, but mathematicians comment on attributes such as *a sense of* certainty, *a sense of* significance, *a sense of* simplicity. The "AHA!" would be ascertained through verification, but this "sense of" is the very real aspect of illumination. Following Roth & Radford (2011), within the Vygotsky-Leont'ev strand of cultural-historical activity we stem from in this work, cognition cannot be understood independently of emotions. Emotions are not a static, trait-like feature of the subject, but they constitute a holistic expression of the subject's current state with respect to the object and the subject's *sense of likelihood of* success. Emotions mediate the movement of the activity itself. Hence, we can see intuitive knowledge as the expression of the sensuous-valuational and volitional character of activity.

Existing studies on intuition share some limitations. Both Fischbein's and Gigerenzer's understanding of intuition lack a precise explanation of what intuitions in mathematics really are, beyond a definition that casts them in opposition to logical thinking. If, according to Fischbein, globalization, as well as immediacy and self-evidence, does not lead necessarily to an intuitive acceptance, how does the individual establish the unity between the particular, the specific, the directly convincing example, and the general principle derived through similarity and proportionality from the particular case? Furthermore, we believe that it is necessary to look at the emotional and sensuous dimension of the individual not as an element that hinders or enhances thinking, or as a need to care of for a successful cognition, but as a constitutive part of thinking itself. The view of intuitions as an effective form of adaptation (Gigerenzer & Selten, 2001), beyond pure rationality, of the individual to the constraints and challenges of its environment is based on the illusion of an *autonomous self-determined individual* that constructs his knowledge (Radford, 2012). In our view cognition is not only the individual's adaptation but it is a *mediated reflexive activity* (Radford, 2008). The cultural and historical dimension embodied in socially shared activity (Roth & Radford, 2011) is the true substance of the individual's self-determination and cognition.

INTUITIVE THINKING IN A CONTEXT OF LEARNING

We are not disregarding the importance of the previous results, both in psychology and in mathematics education. In fact, we are aware that there exist mathematical concepts that are more intuitive than others (Fischbein, 1987) – or that there are ways of framing mathematical tasks that foster intuitive thinking more than others, but the fact that we recognize concepts that are more intuitive than others is not *absolute*: it is *culturally determined*. And for a certain subject a concept, which is culturally recognized to be intuitive, may be not intuitive – or viceversa, regardless of his incorrect or correct answer to a task. Hence, it is important to consider how the subject relates himself to the concept, and not only the concept itself. Similarly, we are aware that there exist individuals who are more intuitive – and researches studying their behaviour are worth considering, but this approach leads to accounting for learning as adaptation (Gigerenzer & Selten, 2001). Learning is *also* adaptation to a physical/social/cultural environment, but it is *more* than mere adaptation.

In our view, cognition is a mediated reflexive activity (Radford, 2008). The teacher plays a crucial role in learning, since he is the only one who knows *where the activity should lead to*. In this view, intuitive learning is a determinate way to intertwine the subject, with its material and ideal components, a reified cultural and historical activity (the so called mathematical object or mathematical content), and a set of semiotic means (ideal and material) that allow the individual to become part of, re-enact and make sense of such an activity. The intuitive relationship between the individual and the content of knowledge is a reflexive activity mediated at an embodied, perceptual and sensuous level. Thinking is not purely sensorial, nor purely conceptual. Intuition can be seen as the sensuous side of intellectual-emotional activity when the activity is mediated mainly through objects, artefacts, gestures, bodily movements, deictic and generative use of natural language (Andrà & Santi, 2011). Intuitions are a relationship between the subject and a content of knowledge that allows sensibilities to notice, to think, to become in proximity and synchrony with generality. Intuitions are a way of being and becoming of the consciousness in its movement towards the generality of mathematical knowledge, with the feeling you are close to and re-enacting what culturally transcends you. Intuitive thinking allows the students moving towards a more accepted concept, relying on previous knowledge both cultural and historical, and that is being objectified in the reflexive activity. This mode of existence is dialectically entangled with the logical and discursive one. The dialectics between the two modes of existence accounts for the consciousness trajectory towards the recognition of the general that is the essence of mathematics.

AN EXAMPLE FROM CLASSROOM ACTIVITY

Data come from a longitudinal study that observes the development of algebraic thinking in students of a same classroom from grade 2 to grade 6. In particular, this paper focuses on a cycle started in the school year 2010/11. We discuss a teaching/learning sequence involving grade 3 students in the school year 2011/12.

The experimentation has been designed according to activity theory methodology (Roth & Radford, 2011): (a) presentation and discussion of the activity to the whole class, (b) work in small groups of students, with the support of the teacher who goes around and discusses with each group, (c) general discussion and a new cycle begins.

The mathematical content of the activity is part of grade 3 curriculum of Ontario: the search for regularity in number sequences. Data are collected both from videotaping and written material produced by the students.

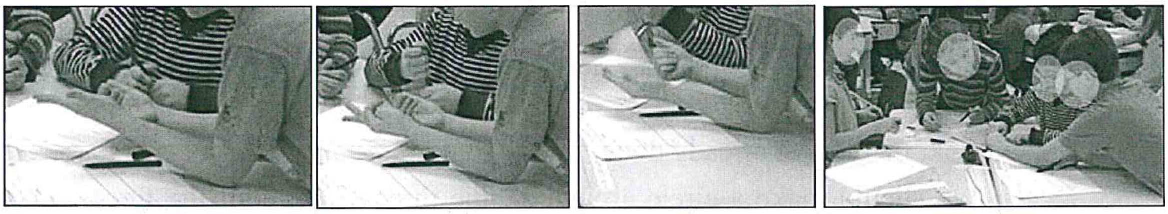
The first task asks the students to find the regularity of the series: 25, 22, 19, 16. The students also know that they should underline the important words on the sheet. In a group of four, Estela proposes "find", "regularity", "this" and "series", and James suggests to reduce to "regularity" and "series". After having agreed about the most important words to underline, Estela proposes to use the number table, and she goes taking one of them from the teacher's desk. In the meanwhile, Mike tells to James that he already knows the answer.

1. James: Seriously. [*he gazes Mike's eyes*]
2. Mike: Yes, I know the answer: one subtracts 3 at any time. You subtract 3 at any time. [*Mike makes no gesture, he stands firmly in front of James*]

Estela comes back with the number table and tries to make sense of the task, addressing James. Alone, Mike counts with his fingers "1,2,3 (*Figure 1-a,b,c*). 1,2,3" (*Figure 1-d*), then he talks to the group:

3. Mike: I know what it is, I know what it is. [*At this point Mike looks at his mates*]
4. Estela: What is it? [*Estela addresses Mike, changing her posture*]
5. Mike: one subtracts 3 any time. [*The students take the number table and Mike counts on it (figure 1-d)*]. 1,2,3. 1,2,3.
6. Mike: 25 minus 3, 1,2,3. 1,2,3. [*James follows Mike's pointing on the table*]
7. Mike: Look, 45
8. James: 25.
9. Mike: then –3, 1,2,3. 1,2,3. [*points on the table, follows numbers in reverse order*]
10. Estela: 25. So 25, 22

11. Mike: 1,2,3. 1,2,3. [*still points on the table*]
12. Estela: 19.
13. Mike: 1,2,3. [*continues to point on the table*]
14. Estela: Oh, yes!



(a) (b) (c) (d)

Figure 1: Mike's intuition on his hand and on the table (Mike is the boy on the right).

We notice that Mike has a starting intuition about the regularity of the series. He tells it to James, but immediately after this, Estela comes back to the group with the table and she catches the group's attention wondering how to deal with the task. Alone, Mike tries his intuition by counting with his hands (*Figure 1-a,b,c*), and then comes back to the group: resorting to the number table, he shows the solution to his mates, repeating several times "1,2,3" and pointing on the number table (*Figure 1-d*). Mike's first intuition can be accounted as an "AHA!" experience, in that he lives a sense of certainty, a sense of significance, about his solution. The nature of Mike's intuition seems to be affective rather than cognitive at this stage. In fact, we can see a sense of proximity with the general rule. In a second moment, he tries the correctness of his solution by counting with his hands, while the other three students wonder for a response. This is a private, more cognitive, moment for Mike. Emotions are still part of his thinking, giving him a sense of likelihood of success about his starting idea. This moment is immediately followed by a public one: Mike shares his intuition. In the rest of the excerpt, Mike repeats again and again the numbers "1,2,3", pointing with his fingers on the number table. Finally, Estela intuits the general rule ("Oh yes!"). Mike's intuition starts to become shared. In Estela's voice we can perceive a sense of disclosing, accompanied with positive emotions. The students' behaviour highlights the need for the intuitive part of mathematical thinking in terms of their space-time and tactile experience, bodily movements, rhythm. This intuitive thinking pivots around the number table as a semiotic means of objectification that allows the synchronic use of gestures and language, through which the students develop their space-time experience. We remark that at this point, in their movement towards mathematical generalization, the students have not yet fully objectified the generality behind the sequence of numbers, they are becoming part and re-enacting what culturally transcends them. Now the group activity goes on: the students have to write their answer on the sheet.

15. Mike: Ok, one does 25 – 3
16. Estela: Yes, yes, yes, yes 25 – 3, so 25 – 3 = 22
17. Maria: Are you sure? Because…

18. Estela: Yes. 22 –, we have already counted on my number table.
19. James: 22 – 3 =
20. Estela: 3 = 19, right? [*echoes James, but addresses Mike*]
21. James: Yes, I believe that it is so.
22. Estela: 19, 19 – 3
23. Mike: I have already done, look.
24. Estela: = 16.
25. Mike: Look, that I have already done.
26. Estela: We should make a circle around all the threes. Circle all the threes, like circle 3, circle 3, circle the third 3. Is there a statement like an idea of a statement which we can write such as: one counts, one subtracts always three, or something similar.
27. Mike: One subtracts three at any number
28. Estela: (*contemporary to writing*) One subtracts, always…

The students are in the process of agreeing about the written answer to report on the sheet. Firstly, they write down the computations, accounting of what they have done on the number table. Again, the number table is a SMO the students resort to in order to have a sense of likelihood about what they are expressing in written words. Maria's doubt ("are you sure?", 17) comes from her emotional sense of likelihood, given that she is still struggling to intuit the general rule. The reference to the number table is made explicit in 18 by Estela, who replies to Maria. Then, the students discuss about the statement to be written, and a new word arises in the discourse: "always" (26). This can be taken as a movement towards mathematical generalization, but also allows us to infer that intuitive thinking does not inform only an initial moment, nor it should be disregarded as if there is a moment in the students' trajectory in which they think "truly mathematically". In fact, this intuitive thinking both supports and triggers the need for a formal and discursive objectification of the general rule. The students, even if it isn't required in the task, try to express the general rule: you *always* subtract three, at *any* number. There is also a redundancy in the use of both "always" and "any" in their shared written solution.

In the two sequences already shown, it is evident how intuitive thinking, in our socio-cultural approach, sometimes can be an autonomous stroke of genius of a student, but it is in the communitarian self (Radford, 2012) –the shared activity between the members of the group– that it becomes objectified, as it is testified in the students' evolving use of language: already in 2, Mike uses the expression "at any time", and he repeats it in 5. But it is Estela who, in 26, suggests to use "always". Mike echoes her words, saying "at any number" (27), and recognizing the emerging generality. The communitarian self resorts to the territory of artifactual thought: artifacts, in fact, constitute what we are, feel, think, etcetera. The number table is a good example of a semiotic means belonging to the territory of the artifactual thought: it is not just a representation of the first 50 natural numbers, but it culturally determines the way the students make sense of the sequence. James, for example, in 6-13, follows with his fingers Mike's pointing on the table, and without saying a word he objectifies the rule intuited by Mike. Both in 8 and in 19, we have a clue to infer that James can recognize such a rule, since he contributes to the discussion, correcting Mike in 8 and suggesting how to go on in 19. And the number table belongs to the culture, both in general as part of the mathematical knowledge, and in particular as part of the classroom culture. Also the list of computations (15-24) is a semiotic means of objectification, which (differently from the number table that is given) is created and shared within the group, and allows a further leap towards the general rule: the students, in fact, firstly underline the threes on their sheets (26), then they write a closing statement where they underline "One subtracts, always" (28). The consciousness' movement of the students is constantly culturally and historically determined both by their previous knowledge and the semiotic means of objectification to which they are exposed. The interplay between intuitive (e.g. –3) and more abstract (e.g. "always") forms of thinking, and the communitarian consciousness' movement, is expressed throughout the dialogue, especially when the students invite each other to pay attention ("look").

DISCUSSION AND CONCLUSION

We have argued that our understanding of intuition doesn't allow us to cast them in the individual's cognitive and psychological behaviour, nor in the structure of the mathematical object: we must look at the dialectical relationship—culturally mediated and transcended—between the individual consciousness and the content of knowledge, between the particular (–3) and the general ("One subtracts, always"), that gives subjective activity objective reality and bestows objective reality with the subject's determinations. Intuitive thinking can be seen as the sensuous side of intellectual activity when the activity is mediated mainly through objects, artefacts, gestures, bodily movements, deictic and generative use of natural language. In intuitive thinking the mathematical content belongs to the student's space-time experience in terms of emotions (Mike's sense of certainty in 1), feelings, perception, movement, rhythm (Mike's counting 1-2-3 on his hand and then on the number table, sharing his intuition), manipulation of objects (the number table itself), which account for the sense of proximity with, and enactment of, the general ("any", "always") that transcends the individual. This act of mediated recognition determines both the content and the subject. It is a process of being and becoming that could not take place without the intuitive part of this double-sided activity that we call thinking.

From an educational standpoint, this student-content relationship we termed as intuition is a way of thinking, or rather, a mode of existence that is always present in the consciousness' trajectory towards the objectification of mathematical knowledge both in learning-teaching processes and in the cultural and historical development of mathematics. It is necessary to further scrutinize the nature of this special type of mediated reflexive activity that we bound to intuitive thinking both to better understand this phenomenon and to design suitable instruction in the classroom.

Finally, this work can be taken as an attempt to show that intuition may belong to a private, individual sphere, but it is in the *communitarian self* that it becomes part of the mathematical activity we call learning. In that, we are also addressing an issue on the political, as insightfully pointed out by Pais and Valero (2012). We remark that our understanding of politics follows Milani's (1967) words: "through teaching I have learned that the problem of the others is the same as mine: coming out alone is avarice, coming out together is politics".

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