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# GENERAL ALGEBRAIC CREEP ANALYSIS OF CONCRETE MEMBERS WITH POINT OR DISTRIBUTED ELASTIC RESTRAINTS

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## Abstract

*In hybrid structures, made of elastically restrained concrete members, the state of stress and strain due to load or imposed deformation is significantly influenced by the delayed deformation of concrete. The structural analysis of such complex structural arrangements can be approached in general form by means of the Reduced Relaxation Function Method (RRFM), or in an approximate form, according to the Trost-Bazant algebraic Age Adjusted Effective Modulus Method (AAEMM). This way of operating, suggested by various Codes, leads to the solution of linear algebraic systems for point elastic restraints and linear differential equations for distributed elastic restraints. In any case, the related computations are tedious, as we have to work in a pseudo-elastic domain with imposed deformation depending on the initial stress. In the present paper, an alternative procedure, based on a theorem proven by the first of the authors, is discussed. This approach allows the AAEMM procedure to be reduced to a convenient combination of linear elastic problems. Some case studies, regarding outstanding hybrid structures, will point out the effectiveness and the feasibility of the proposed approach.*

## 1 INTRODUCTION

In the last three decades, hybrid structures have become very popular and nowadays they represent a reference point in modern structural engineering. Hybrid structures generally consist of two interacting parts, made of different materials, in particular reinforced or prestressed concrete and structural steel. The restraints connecting the two parts can be various, in particular point or distributed. The first type regards structures composed by two different homogeneous materials mutually connected by means of localized elastic restraints. The shear-resisting systems in tall buildings, collaborating with steel frames or trusses, belong to this category , Figure 1a); more examples are represented by reinforced concrete frames containing steel members of large dimension introduced in order to increase structural stiffness, Figure 1b); or by steel frames working as a supports for structures of complex architectural shape, collaborating with shear resisting concrete cores, Figure 1c).

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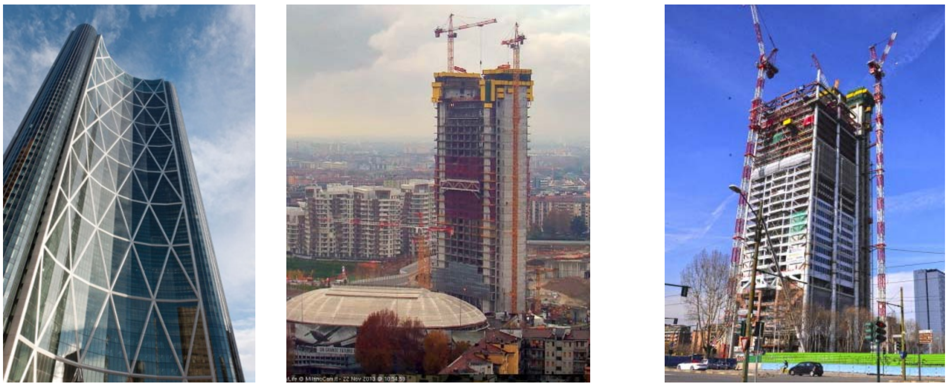

Figure 1: a) Tall building shear restraint core and steel frame b) Concrete frames with large steel elements c) Steel frames collaborating with shear resisting concrete cores

For these structural types, the structural role of each part is clearly defined, as each one is able to counterbalance action effects. Hybrid structures of different nature can be observed when one of the two parts can counterbalance only special sets of actions, in particular the ones deriving from the interaction with the other structural part. In this case, the first part acts as a restraint on the second one. This category includes the outstanding case of cable-stayed bridges, Figure 2, in which the stays act as a system of internal elastic restraints connecting the antenna and the deck.

Figure 2: Cable-stayed Bridge

Other interesting examples are the continuous reinforced or prestressed concrete beams supported by elastic restraints, e.g. the decks of arch bridges, Figure 3.

Figure 3: Deck of r.c. arch bridge

Besides the examples now discussed, other hybrid structures with distributed elastic restraints are of relevant importance. At this regard, we can remember composite steel-concrete columns and beams, Figure 4. For the columns, the restraint between concrete and steel can be assumed rigid, provided by bonding and by connecting devices. For composite beams, the connection between the two parts is provided by special, elastically deformable connectors, so that they can be assimilated to hybrid structures with distributed elastic restraints.

Figure 4: Composite steel-concrete columns and beams

Another example of this type is represented by prestressed concrete members with grouted tendons, Figure 5. In these arrangements, the ordinary reinforcement and the prestressing cables can be regarded as distributed elastic restraints.

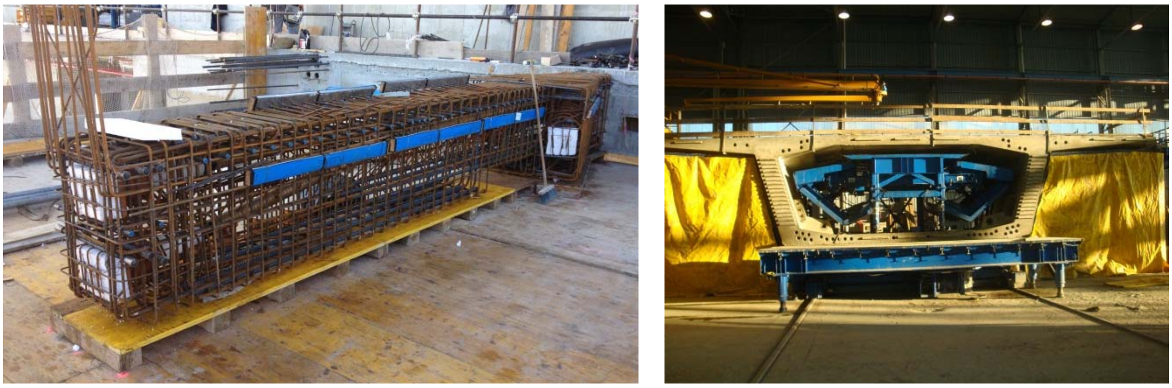

Figure 5: P.c. members

Finally, an interesting example is the slab resting on an elastic soil, Figure 6, in which a set of piles with specified elastic stiffness has been introduced. In this case, the system of the elastic restraints is both distributed and concentrated. In hybrid structures, the non-homogeneous behaviour of the two collaborating parts generates a significant redistribution of stress accompanied by a pronounced variation of strain and displacements, which have to be carefully evaluated.

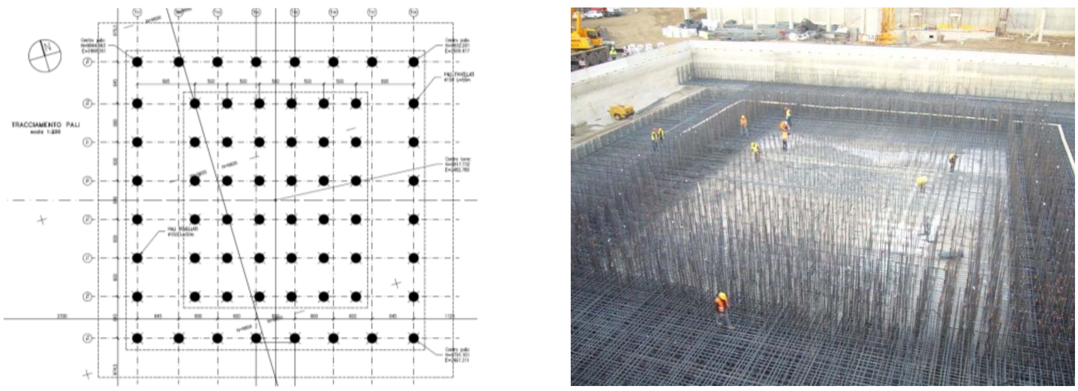

Figure 6: Slab resting on elastic soil and piles

In the following, the solution of the problem related to the evaluation of the state of stress and deformation in hybrid structures subjected to sustained loads will be discussed. The general approach based on the application of FFRM will be briefly outlined, while the approximate approach based on Trost-Bazant approach, applied according to the procedure derived from the Fundamental Theorem stated by the author, will be discussed in detail. Some case studies will illustrate the feasibility of the proposed approach.

## 2 LONG TERM ANALYSIS OF HYBRID CONCRETE –STEEL STRUCTURES

### 2.1 General Approach

In hybrid structures with point elastic restraints, applying the force method, the two collaborating parts are made independent, pointing out the vector $\underline{X}(t)$ of the mutual reactions at time t, Figure 7a). An analogous procedure can be followed also for distributed restraints, Figure 7b).

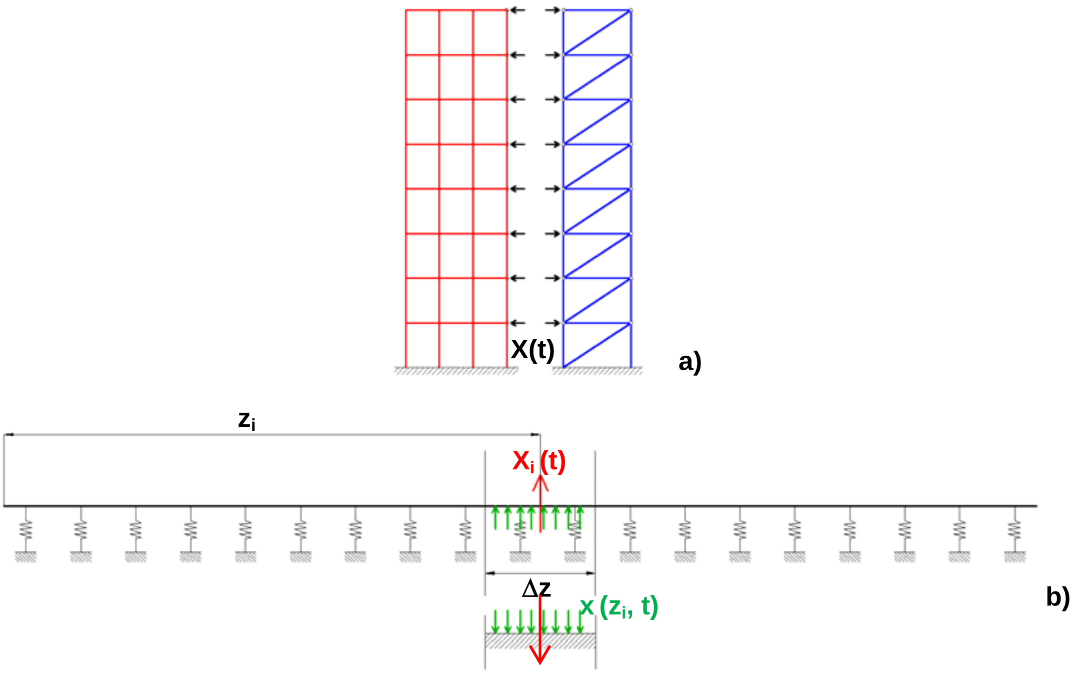


Figure 7: Force method, point and distributed restraints

In this case the mutual reaction is represented by the function x(z, t), so that indicating by $\underline{X}(t)$ the vector of the concentrated reactions acting on a small length $\Delta z$, the i-th component of $\underline{X}(t)$ is defined by the relationship $X_i(t) = x(z_i, t) \cdot \Delta z$. When external loads constant in time are applied, the compatibility equation can be written in the following way:

$$\int_0^t \left[\underline{\underline{F}}_c \cdot E_c(t_0) \cdot J(t, t') + \underline{\underline{F}}_s\right] \cdot d\underline{X}(t') + \underline{\delta}_{c0} \cdot E_c(t_0) \cdot J(t, t_0) = 0 \tag{1}$$

In eq. (1) $\underline{\underline{F}}_c$ , $\underline{\underline{F}}_s$ are respectively the elastic flexibility matrices of the two parts and $\underline{\delta}_{c0}$ is the vector of the elastic displacements produced by the external loads. $\underline{\underline{F}}_c$ , $\underline{\delta}_{c0}$ are calculated referring to the modulus $E_c(t_0)$.

At initial time $t_0$ the structure behaves in the linear elastic field, so eq. (1) becomes:

$$\left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right) \cdot \underline{X}_0 + \underline{\delta}_{c0} = 0 \tag{2}$$

where $\underline{X}_0 = \underline{X}(t_0)$ is the initial value of the unknown vector $\underline{X}(t)$.

From eq. (2) we derive:

$$\underline{X}_0 = -\left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right)^{-1} \cdot \underline{\delta}_{c0} \tag{3}$$

Introducing the matrices:

$$\underline{\underline{D}} = \left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right)^{-1} \cdot \underline{\underline{F}}_c \quad ; \quad \underline{\underline{I}} - \underline{\underline{D}} = \left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right)^{-1} \cdot \underline{\underline{F}}_s \tag{4}$$

with $\underline{\underline{I}}$ unit matrix, eq. (1) assumes the following form:

$$\int_0^t \left[\underline{\underline{D}} \cdot E_c(t_0) \cdot J(t,t') + \underline{\underline{I}} - \underline{\underline{D}}\right] \cdot d\underline{X}(t') = \underline{X}_0 \cdot E_c(t_0) \cdot J(t,t_0) \tag{5}$$

Let us define the linear transformation:

$$\underline{X} = \underline{\underline{K}} \cdot \underline{Y} \tag{6}$$

with $\underline{\underline{K}}$ modal matrix of $\underline{\underline{D}}$. Substituting eq. (6) in eq. (5) we can derive:

$$\int_0^t \left[\underline{\underline{\Omega}} \cdot E_c(t_0) \cdot J(t,t') + \underline{\underline{I}} - \underline{\underline{\Omega}}\right] \cdot d\underline{Y}(t') = \underline{\underline{I}} \cdot \underline{Y}_0 \cdot E_c(t_0) \cdot J(t,t_0) \tag{7}$$

where

$$\underline{\underline{\Omega}} = \underline{\underline{K}}^{-1} \cdot \underline{\underline{D}} \cdot \underline{\underline{K}} \tag{8}$$

is the modal matrix of $\underline{\underline{D}}$. Matrix $\underline{\underline{\Omega}}$ is diagonal and allocates along the principal diagonal the eigenvalues of matrix $\underline{\underline{D}}$.

Consequently, the system of Volterra integral equations (7) is diagonal and the components $Y_i(t)$ of the vector $\underline{Y}$ are uncoupled.

In order to solve system (7) we introduce the diagonal matrix $\underline{\underline{J}}^*(t,t_0)$ of the Varied Creep Functions by means of the following equality:

$$E_c(t_0) \cdot \underline{\underline{J}}^*(t,t_0) = \underline{\underline{\Omega}} \cdot E_c(t_0) \cdot J(t,t_0) + \underline{\underline{I}} - \underline{\underline{\Omega}} \tag{9}$$

from which:

$$\underline{\underline{I}} \cdot E_c(t_0) \cdot J(t,t_0) = \underline{\underline{\Omega}}^{-1} \cdot E_c(t_0) \cdot \underline{\underline{J}}^*(t,t_0) + \underline{\underline{I}} - \underline{\underline{\Omega}}^{-1} \tag{10}$$

and eq. (7) can be written in the following form:

$$\int_0^t E_c(t_0) \cdot \underline{\underline{J}}^*(t,t') \cdot d\underline{Y}(t') = \left[\underline{\underline{\Omega}}^{-1} \cdot E_c(t_0) \cdot \underline{\underline{J}}^*(t,t_0) + \underline{\underline{I}} - \underline{\underline{\Omega}}^{-1}\right] \cdot \underline{Y}_0 \tag{11}$$

Introducing the diagonal matrix $\underline{\underline{R}}^*(t,t_0)$ of the Reduced Relaxation Function [1], solution of the subsequent set of uncoupled Volterra integral equations:

$$\int_0^t \frac{\partial \underline{\underline{R}}^*(\tau,t')}{\partial \tau} \cdot \underline{\underline{J}}^*(t,\tau) \cdot d\tau = \underline{\underline{I}} \tag{12}$$

the solution of eq. (11), according to McHenry principle of superposition [2],assumes the form:

$$\underline{Y}(t) = \underline{\underline{\Omega}}^{-1} \cdot \left[\underline{\underline{I}} + \left(\underline{\underline{\Omega}} - \underline{\underline{I}}\right) \cdot \frac{\underline{\underline{R}}^*(t,t_0)}{E_c(t_0)}\right] \cdot \underline{Y}_0 \tag{13}$$

From eq. (13), remembering eq. (6), we finally obtain:

$$\underline{X} = \underline{\underline{K}} \cdot \underline{\underline{\Omega}}^{-1} \cdot \left[ \underline{\underline{I}} + \left( \underline{\underline{\Omega}} - \underline{\underline{I}} \right) \cdot \frac{\underline{\underline{R}}^{*}(t,t_0)}{E_c(t_0)} \right] \cdot \underline{\underline{K}}^{-1} \cdot \underline{X}_0 \tag{14}$$

Eq. (14) is the general form of the reaction vector $\underline{X}(t)$. This form, in the uncoupled expression (13), shows that the reactions $Y_i(t)$ increase in time, as the components $R^{*}_{ii}(t,t_0)/E_c(t_0)$ of the diagonal matrix $\underline{\underline{R}}^{*}(t,t_0)$ are functions monotonically decreasing in time and the eigenvalues $\Omega_{ii}$ lie at the interior of the interval $0 \le \Omega_{ii} \le 1$. These properties are illustrated in Figure 8 where the functions $E_{c28} \cdot J^{*}_{ii}(t,t_0)$; $R^{*}_{ii}(t,t_0)/E_{c28}$ are reported for various values of $\Omega_{ii}$ .

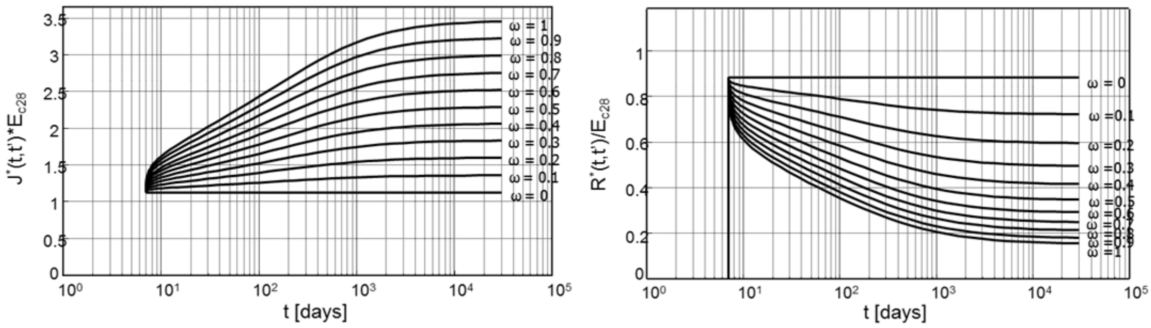


Figure 8: J* and R* functions

The application of the (RRFM) is quite complex as the determination of the eigenvalues of $\underline{\underline{D}}$ requires long and time-consuming computations. For this reason, when the number of unknowns is large, the method becomes too complicate. As an example in Figure 9, the results related to a cable stayed bridge [3], involving seven unknowns are illustrated.

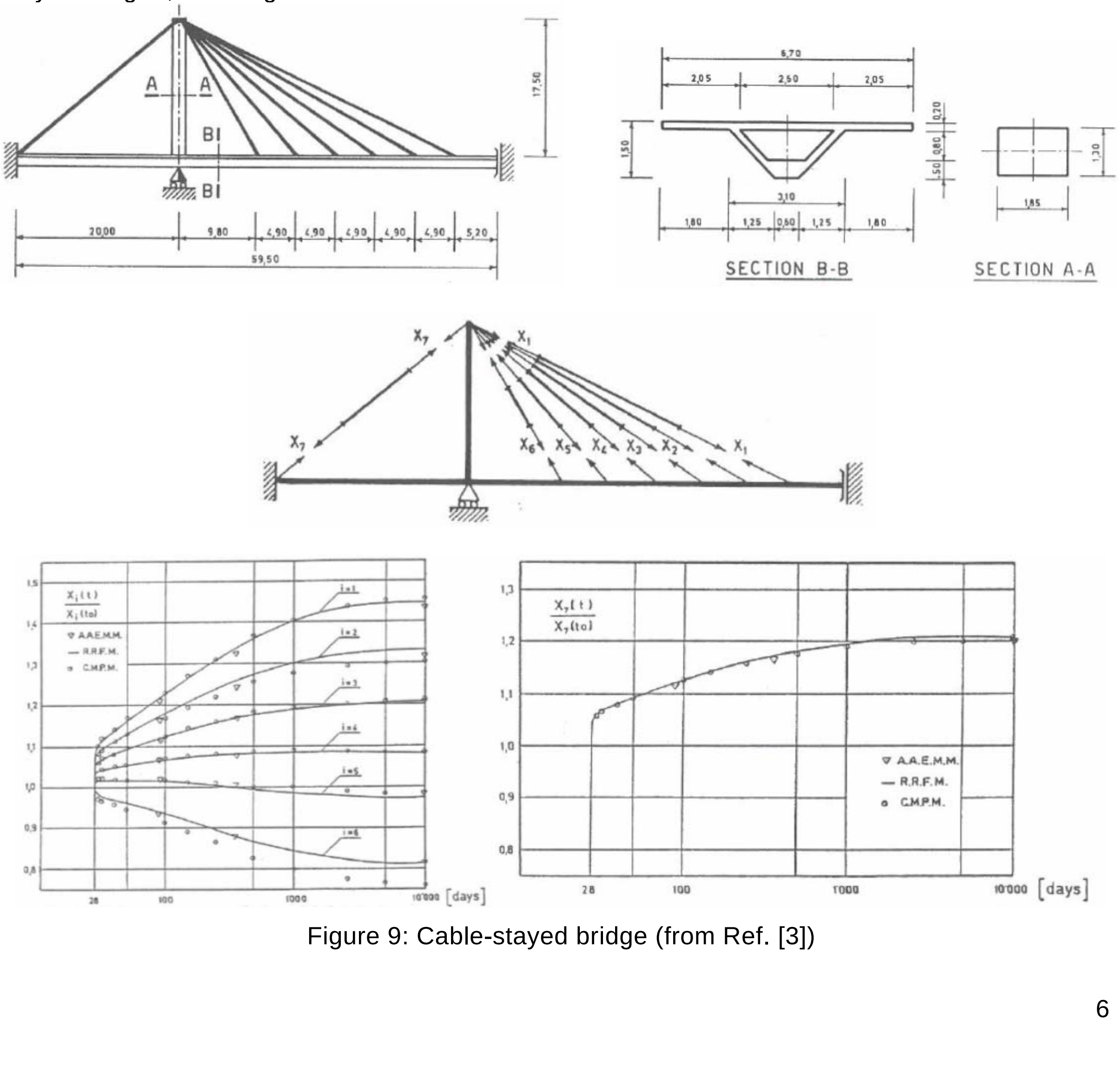


Figure 9: Cable-stayed bridge (from Ref. [3])

### 2.2 Trost-Bazant algebraic formulation

In order to simplify the structural analysis, Trost [4] introduced the hypothesis of a linear relationship between the deformation $\varepsilon(t)$ of concrete and the creep coefficient $\varphi(t,t_0)$, according to the following expression:

$$\varepsilon(t) = a + b \cdot \varphi(t, t_0) \tag{15}$$

with a, b arbitrary constants.
As $\varphi(t, t_0) = E_c(t_0) \cdot J(t, t_0) - 1$ , eq. (15) can be written in the equivalent form:

$$\varepsilon(t) = (a - b) + b \cdot E_c(t_0) \cdot J(t, t_0) \tag{16}$$

From eq. (16), remembering the basic theorems of linear viscoelasticity [5], for the state of stress we write:

$$\sigma(t) = (a - b) \cdot R(t, t_0) + b \cdot E_c(t_0) \tag{17}$$

From eq. (15) we derive $\varepsilon(t_0) = a = \sigma(t_0)/E_c(t_0)$, so that eq. (17) can be written:

$$\sigma(t) = \sigma(t_0) \cdot \frac{R(t,t_0)}{E_c(t_0)} + b \cdot [E_c(t_0) - R(t, t_0)] \tag{18}$$

From eq. (18) we so obtain:

$$b = \frac{\sigma(t) - \sigma(t_0) \cdot \frac{R(t,t_0)}{E_c(t_0)}}{E_c(t_0) - R(t,t_0)} \tag{19}$$

and combining eq. (19) and eq. (15) we can finally write:

$$\varepsilon(t) = \sigma(t) \cdot \frac{\varphi(t,t_0)}{E_c(t_0) - R(t,t_0)} + \frac{\sigma(t_0)}{E_c(t_0)} \cdot \left[1 - \frac{R(t,t_0) \cdot \varphi(t,t_0)}{E_c(t_0) - R(t,t_0)}\right] \tag{20}$$

Introducing the function $\chi(t,t_0)$ [6], by means of the following expression:

$$\chi(t, t_0) = \frac{1}{1 - \frac{R(t,t_0)}{E_c(t_0)}} - \frac{1}{\varphi(t,t_0)} \tag{21}$$

eq. (20), after simple algebraic transformations, can be written in its final form:

$$\varepsilon(t) = \frac{\sigma(t)}{E_c(t_0)} \cdot [1 + \chi(t, t_0) \cdot \varphi(t, t_0)] + \frac{\sigma(t_0)}{E_c(t_0)} \cdot \varphi(t, t_0) \cdot [1 - \chi(t, t_0)] \tag{22}$$

Introducing in eq. (22) the equalities: $\varepsilon(t) = \varepsilon$; $\sigma(t) = \sigma$; $\varepsilon(t_0) = \varepsilon_0$; $\sigma(t_0) = \sigma_0$; $E_c(t_0) = E_c$ and neglecting for the various functions to write the dependence on t, $t_0$, the following straightforward relationship can be written:

$$\varepsilon = \frac{\sigma}{E_c} \cdot [1 + \chi \cdot \varphi] + \frac{\sigma_0}{E_c} \cdot \varphi \cdot [1 - \chi] \tag{23}$$

giving at time $t_0$ the elastic form:

$$\varepsilon_0 = \frac{\sigma_0}{E_c} \tag{24}$$

Eqs. (23) – (24) can be immediately extended to force – displacement relationship. In particular for a force F(t), applied at the abscissa $z_i$, the associate displacement s, evaluated for a prescribed direction at the abscissa $z_j$, assumes the form:

$$s = F \cdot \delta_{ji} \cdot (1 + \chi \cdot \varphi) + F_0 \cdot \delta_{ji} \cdot \varphi \cdot (1 - \chi) \tag{25}$$

where $\delta_{ji}$ is the elastic influence coefficient representing the elastic displacement at z = $z_j$ produced by a unit force F = 1 applied at z = $z_i$.

The application of eq. (25) to the problem of Figure 7 drives to the linear algebraic system:

$$\left[\underline{\underline{F}}_c \cdot (1 + \chi \cdot \varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X} + \underline{\underline{F}}_c \cdot \varphi \cdot (1 - \chi) \cdot \underline{X}_0 = -\underline{\delta}_{c0} \cdot (1 + \varphi) \tag{26}$$

and for t = $t_0$:

$$\left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right) \cdot \underline{X}_0 = -\underline{\delta}_{c0} \tag{27}$$

Remembering eq. (4) for the vector $\underline{X}$ we obtain:

$$\underline{X} = \left[\underline{\underline{I}} + \left(\underline{\underline{I}} + \underline{\underline{D}} \cdot \chi \cdot \varphi\right)^{-1} \cdot \varphi \cdot \left(\underline{\underline{I}} - \underline{\underline{D}}\right)\right] \cdot \underline{X}_0 \tag{28}$$

As we can observe, from the results of the example of Figure 9, eq. (28) exhibits a very good level of approximation and can be adopted in structural engineering as suggested by the Codes [7], [8]. Despite its apparent simplicity, nonetheless eq. (28) requires to take into account the initial conditions of the problem, so that when the number of the unknowns is large, e.g. in problems with distributed restraints, the application of eq. (28) becomes tedious and time consuming. In order to make simpler the application of eq. (28), in the following will be discussed the criteria allowing to reach this goal.

### 2.3 Alternate form for the application of Trost-Bazant approach: the Fundamental Theorem

The alternate procedure for the application of eq. (25) is based on a theorem discovered by the first author [9], oriented to define a more convenient form to compute the displacement at first member of eq. (25), without modifying the related approximation level. For this we can observe that the displacement of eq. (25) can be regarded as sum of the following two contribution:

- the displacement:

$$s_1 = F \cdot \delta_{ji} \cdot (1 + \chi \cdot \varphi) \tag{29}$$

produced by the force F at time t, calculated in the elastic stage assuming the varied elastic modulus $E'_c = E_c(t_0)/[1 + \chi(t, t_0) \cdot \varphi(t, t_0)]$;

- the displacement:

$$s_2 = F_0 \cdot \delta_{ji} \cdot \varphi \cdot (1 - \chi) \tag{30}$$

produced by the force F calculated at time $t_0$.

For this second contribution we can assume the following form:

$$s_2 = \mu \cdot s_0 + \lambda \cdot s_{10} \tag{31}$$

where $s_0$ is the elastic displacement $F_0 \cdot \delta_{ji}$ evaluated at t = $t_0$ assuming the elastic modulus $E_c(t_0)$ and $s_{10}$ is the elastic displacement produced by the force $F_0$, calculated assuming the varied elastic modulus $E'_c = E_c(t_0)/[1 + \chi(t, t_0) \cdot \varphi(t, t_0)]$ . $\lambda$, $\mu$ are two functions of t,$t_0$ to be determined.

It is immediate to observe that:

$$s_{10} = s_0 \cdot [1 + \chi(t, t_0) \cdot \varphi(t, t_0)] \tag{32}$$

and introducing eq. (31) in eq. (30) we obtain:

$$F_0 \cdot \delta_{ji} \cdot \left[\mu + \lambda \cdot \left(1 + \chi(t, t_0) \cdot \varphi(t, t_0)\right)\right] = F_0 \cdot \delta_{ji} \cdot \varphi(t, t_0) \cdot [1 - \chi(t, t_0)] \tag{33}$$

Eq. (33) expresses the equality of two first degree polynomials in the variable $\varphi(t,t_0)$. This equality postulates the equality of the related coefficients. From eq. (33) we consequently derive:

$$\mu + \lambda = 0; \quad \lambda \cdot \chi = 1 - \chi \tag{34}$$

or

$$\lambda = \frac{1-\chi}{\chi}; \quad \mu = -\lambda = -\frac{1-\chi}{\chi} \tag{35}$$

Introducing the second of eqs. (35) in eq. (30) and remembering eq. (29) for the displacement we can write:

$$s = s_1 + \mu \cdot (s_0 - s_{10}) \tag{36}$$

Eq. (36) is very suitable for application as it allows the displacements to be obtained by means of two elastic computations. Eq. (36) can be furtherly simplified when the applied load is constant in time. In this case F = $F_0$ so that eq. (36) becomes:

$$s = s_1(1-\mu) + \mu \cdot s_0 \tag{37}$$

eqs. (36) – (37) allow to obtain the solution of the problem, computing the vector $\underline{X}$ by means of a simple elastic analysis. Referring to the general case of eq. (36), the compatibility equation for the vector $\underline{X}$ assumes the form.

$$\begin{aligned}&\left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X} + \mu \cdot \underline{\underline{F}}_c \cdot \underline{X}_0 - \mu \cdot \underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) \cdot \underline{X}_0 + \underline{\delta}_c \cdot (1+\chi\cdot\varphi) + \\ &+\mu \cdot \underline{\delta}_{c0} - \mu \cdot \underline{\delta}_{c0} \cdot (1+\chi\cdot\varphi) = 0\end{aligned} \tag{38}$$

At initial time eq. (38) gives:

$$\left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right) \cdot \underline{X}_0 = -\underline{\delta}_{c0} \tag{39}$$

So that, adding and subtracting at first member of eq. (38) the quantity $\mu \cdot \underline{\underline{F}}_s \cdot \underline{X}_0$, combining eqs. (38), (39) we obtain:

$$\begin{aligned}&\left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X} + \underline{\delta}_c \cdot (1+\chi\cdot\varphi) + \mu \cdot \left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right) \cdot \underline{X}_0 + \mu \cdot \underline{\delta}_{c0} + \\ &-\mu \cdot \left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X}_0 - \mu \cdot \underline{\delta}_{c0} \cdot (1+\chi\cdot\varphi) = 0\end{aligned} \tag{40}$$

Remembering eq. (39) we reach:

$$\left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X} = -\underline{\delta}_c \cdot (1+\chi\cdot\varphi) + \mu \cdot \left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X}_0 + \mu \cdot \underline{\delta}_{c0} \cdot (1+\chi\cdot\varphi) \tag{41}$$

Applying the principle of superposition the vector $\underline{X}$ can thus be obtained by the sum of the solution of the three following linear systems:

$$\begin{aligned}&\left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X}_a = -\underline{\delta}_c \cdot (1+\chi\cdot\varphi) \\ &\left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X}_b = \mu \cdot \left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X}_0 \\ &\left[\underline{\underline{F}}_c \cdot (1+\chi\cdot\varphi) + \underline{\underline{F}}_s\right] \cdot \underline{X}_c = \mu \cdot \underline{\delta}_{c0} \cdot (1+\chi\cdot\varphi)\end{aligned} \tag{42}$$

for which it is immediate to assume:

$$\underline{X}_a = \underline{X}_1 \tag{43}$$

elastic solution assuming the actions at time t and the modulus $E_c' = E_c(t_0)/[1+\chi(t,t_0)\cdot\varphi(t,t_0)]$

$$\underline{X}_b = \mu \cdot \underline{X}_0 \qquad (44)$$

initial elastic solution magnified by the factor $\mu$

$$\underline{X}_c = -\mu \cdot \underline{X}_{10} \qquad (45)$$

elastic solution assuming the loads at time $t_0$ and the elastic modulus $E'_c = E_c(t_0)/[1+\chi(t,t_0)\cdot\varphi(t,t_0)]$ magnified by the factor (-$\mu$).

Summing up eqs. (42), (43), (44) we reach:

$$\underline{X} = \underline{X}_1 + \mu \cdot \left(\underline{X}_0 - \underline{X}_{10}\right) \qquad (46)$$

so that also for the reaction vector $\underline{X}$ a relationship similar to eq. (36) can be derived. For constant actions it results: $\underline{X}_1 = \underline{X}_{10}$, and eq. (46) becomes:

$$\underline{X} = \underline{X}_1(1-\mu) + \mu \cdot \underline{X}_0 \qquad (47)$$

similar to eq. (37).

From eqs. (46), (47) we can thus conclude that the structural analysis according to (AAEMM) can be performed by superposing the results of three elastic analysis, combined by the factor $\mu$. For constant actions the elastic analyses are only two and the same can be said for the case of variable actions, expressing the action A at time t by means of the action $A_0$ at time $t_0$, using the following relationship $A = c(t)\cdot A_0$. In this case $\underline{X}_{10} = \frac{\underline{X}_1}{c(t)}$ and eq. (46) can be written:

$$\underline{X} = \underline{X}_1 \cdot \left(1 - \frac{\mu}{c}\right) + \mu \cdot \underline{X}_0 \qquad (48)$$

which coincides with eq. (47) when A = $A_0$ or c = 1.

The preceding relationship can be extended to the state of stress and strain in the two interacting parts of the hybrid structure. For the restraining elastic part, sustaining only the reaction vector $\underline{X}$ proportional to the displacements, the extension is obvious. Regarding the viscoelastic homogeneous concrete part, the state of stress can be written in the following way:

$$S(t,z) = S_A(t,z) + \underline{S}_{Xe}^T \cdot \left[\underline{X}_1 \cdot \left(1 - \frac{\mu}{c}\right) + \mu \cdot \underline{X}_0\right] \qquad (49)$$

with $S_A(t, z)$ state of stress generated by the applied actions at time t, $\underline{S}_{Xe}$ the column vector of the state of stress generated by the unit vector $\underline{X}$.

Adding and subtracting in eq. (49) the quantity $\mu \cdot S_A(t_0,z)$, from eq. (49) we obtain:

$$S(t,z) = S_A(t,z) + \underline{S}_{Xe}^T \cdot \underline{X}_1 + \mu \cdot \left[S_A(t_0,z) + \underline{S}_{Xe}^T \cdot \underline{X}_0\right] - \mu \cdot \left[S_A(t_0,z) + \underline{S}_{Xe}^T \cdot \underline{X}_{10}\right] \qquad (50)$$

Eq. (50), expressing the first theorem of linear viscoelasticity, allows the following relationship to be immediately written:

$$S(t,z) = S_1(t,z) + \mu \cdot [S_0(t_0,z) - S_{10}(t_0,z)] \qquad (51)$$

similar to eq. (36).

In the same way for the displacements in the viscoelastic part we can write:

$$\begin{aligned} s(t,z) = {} & s_{A1}(t,z) + \mu \cdot [s_{A0}(t_0,z) - s_{A10}(t_0,z)] + \underline{s}_{Xe}^T \cdot (1+\chi\cdot\varphi) \cdot \left[\underline{X}_1 + \mu \cdot \left(\underline{X}_0 - \underline{X}_{10}\right)\right] + \\ & +\mu \cdot \underline{s}_{Xe}^T \cdot \underline{X}_0 - \mu \cdot \underline{s}_{Xe}^T \cdot \underline{X}_0 \cdot (1+\chi\cdot\varphi) \end{aligned} \qquad (52)$$

where $\underline{s}_{Xe}$ is the column vector of the elastic displacements produced by the unit vector $\underline{X}$ .

From eq. (52) we immediately reach:

$$\begin{aligned} s(t,z) = {} & s_{A1}(t,z) + \underline{s}_{Xe}^T \cdot \underline{X}_1 \cdot (1+\chi\cdot\varphi) + \mu \cdot \left[s_{A0}(t_0,z) + \underline{s}_{Xe}^T \cdot \underline{X}_0\right] + \\ & -\mu \cdot \left[s_{A10}(t_0,z) + \underline{s}_{Xe}^T \cdot \underline{X}_{10} \cdot (1+\chi\cdot\varphi)\right] \end{aligned} \qquad (53)$$

so that:

$$s(t,z) = s_1(t,z) + \mu \cdot [s_0(t_0,z) - s_{10}(t_0,z)] \quad (54)$$

similar to eq. (36).

The possibility to operate by means of the combination formulae expressed by eqs. (46), (47) or (47), (48) in order to evaluate all the significant variables of the problem has thus been thoroughly proven. They express the Fundamental Theorem allowing the Trost-Bazant formulation to be turned into a combination of linear elastic solutions maintaining the approximation level of original approach. These equations, which can be conveniently applied for problems with distributed restraints, assume a general character and enable us to easily solve complex problems as it will be illustrated in the following case histories.

## 3 CASE STUDIES

The algebraic procedure based on the theorem discussed in 2 has been applied for the solution of three outstanding cases of hybrid structures. The first case regards the evaluation of the state of stress and deformation in the frame of Figure 10, located on the facade of a tall building. The frame is composed of four r.c. columns pinned to a steel beam. The columns are subjected to sustained axial loads of 1000 t for the internal columns and 500 t for the external ones. The elastic restraint on the column, provided by the beam, reduces the relative column shortening and for this the bending moment in the beam increases in time while the axial load in the internal columns reduces and increases in the lateral ones.

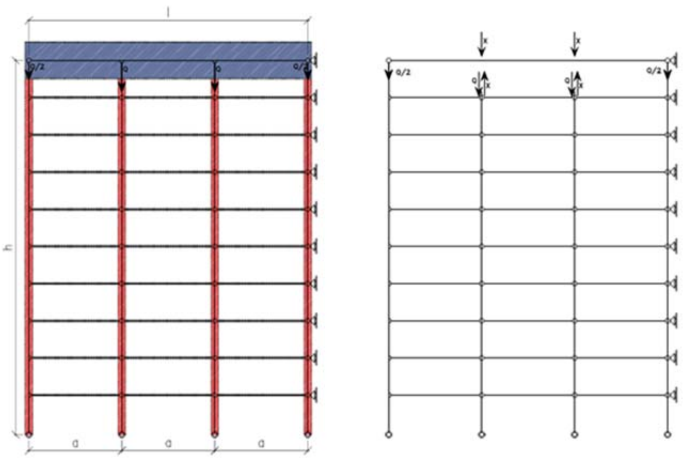

Figure 10: Concrete frame with belt steel beam

Applying eq. (47) to the statically determined scheme of Figure 10, from elementary structural mechanics, neglecting shear deformation in the beam, we derive:

$$X_0 = \frac{Q/4}{1+k_0} \quad (55)$$

$$X_1 = \frac{Q/4}{1+k_1} \quad (56)$$

with

$$k_0 = \frac{a^3}{12h} \cdot \frac{(EA)_c}{(EI)_s} \cdot \left(\frac{3l}{a} - 4\right) \quad (56)$$

$$k_1 = \frac{a^3}{12h} \cdot \frac{(EA)_c}{(EI)_s} \cdot \frac{\left(\frac{3l}{a} - 4\right)}{(1+\chi \cdot \varphi)} \quad (57)$$

and

$$X = X_1 \cdot (1-\mu) + \mu \cdot X_0 \quad (58)$$

In Figure 11 the bending moment in the beam is reported for t = $t_0$ and t = ∞. We observe that the maximum bending moment in the beam nearly redoubles its initial value, while the corresponding vertical displacement increases of about 4 times with respect its initial value. The axial load in the lateral columns increases of about 15% and in central columns we observe a reduction of 10%. A different scenario takes place for reinforced columns, in particular for composite columns.

In this case the reactions $X_0$, X are given by eqs. (55)-(56), assuming for $k_0$, $k_1$ the following values:

$$k_0^* = k_0 \cdot \frac{A_c^*}{A_c} \quad (59)$$

$$k_1^* = k_1 \cdot \frac{A_{c1}^*}{A_c} \quad (60)$$

with

$$A_c^* = A_c \cdot \left[1 + \frac{E_s}{E_c} \cdot \rho_s\right] \quad (61)$$

$$A_{c1}^* = A_c \cdot \left[1 + \frac{E_s}{E_c} \cdot (1 + \chi \cdot \varphi) \cdot \rho_s\right] \quad (62)$$

In the present example, where the geometrical steel ratio $\rho_s \cong 16\%$ of a composite steel-concrete column has been assumed, we observe a dramatic reduction of the bending moment in the beam with a maximum increase of 44%. The reinforcement in the columns markedly reduces the vertical displacements for the internal columns which in this case are about 22% of the corresponding ones of the previous case. The axial loads in this columns also present a variation accompained by a redistribution of the stresses acting on concrete, which are markedly reduced, while on steel we observe a significant increase.

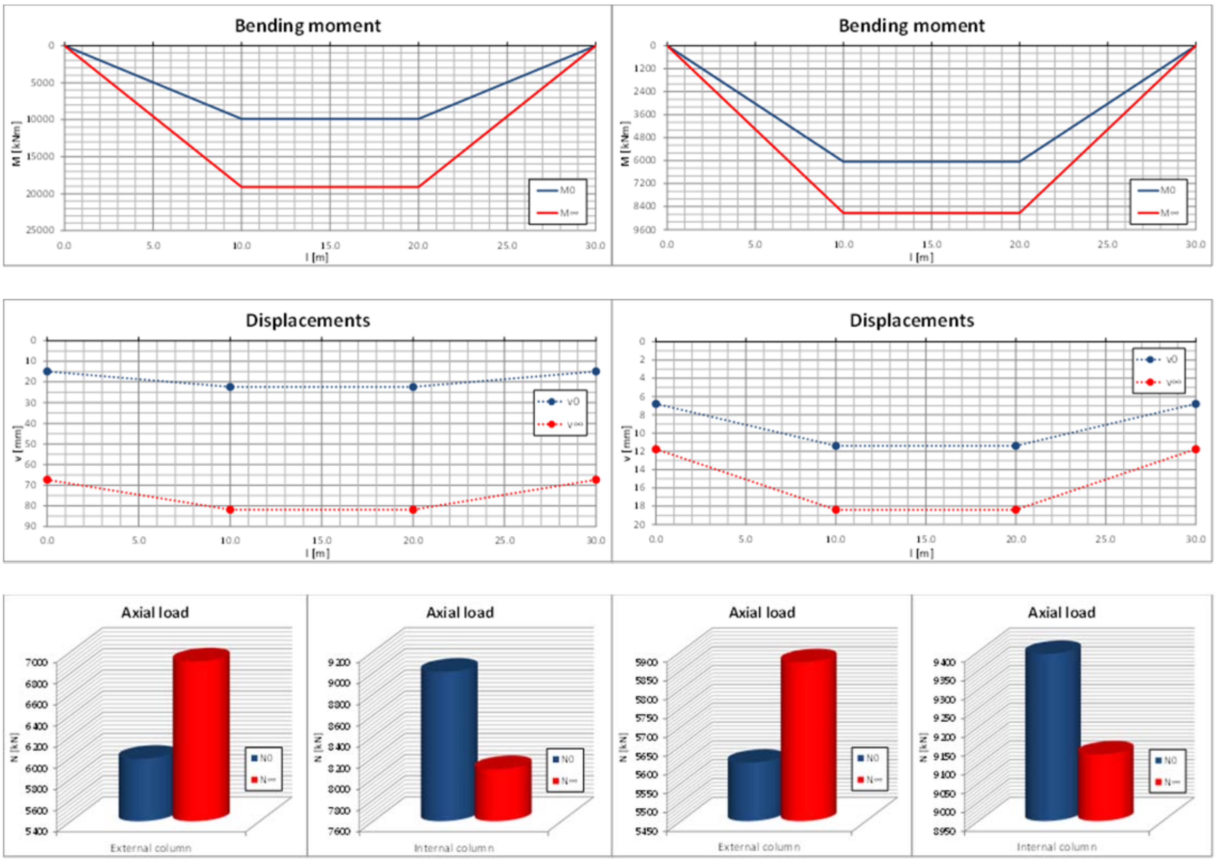


Figure 11: Bending moment, vertical displacement and axial load in the column

The second study regards the r.c. beam resting on an elastic soil represented in Figure 12. Besides the elastic restraint provided by the soil, the beam is restrained by six elastic piles of variable stiffness. The loads, applied in correspondence of the piles are respectively 500 t for the internal piles and 250 t for the external ones. Indicating by $k_W$ the Winkler subgrade coefficient and by $k_0 = k_W \cdot b \cdot l$ a reference stiffness, with b width of the beam, the stiffness of the piles has been assumed $\eta \cdot k_0$ and the cases $\eta = 0; 1; 1.5; 2$ have been studied. Applying the fundamental theorem, the problem is reduced to the well known one regarding the structural analysis of a beam resting on an elastic soil, governed by the following relationships:

$$\left(\underline{\underline{F}}_c + \underline{\underline{F}}_s\right) \cdot \underline{X}_0 = -\underline{\delta}_{c0} \tag{63}$$

$$\left(\underline{\underline{F}}_{c1} + \underline{\underline{F}}_s\right) \cdot \underline{X}_1 = -\underline{\delta}_{c0,1} \tag{64}$$

where $\underline{\underline{F}}_c$ , $\underline{\delta}_{c0}$ are respectively the flexibility matrix and the vector of load effects computed for a beam resting on an elastic soil, assuming the parameter:

$$\alpha = \sqrt[4]{\frac{k_W \cdot b \cdot l}{4EI}} \tag{65}$$

$\underline{\underline{F}}_{c1}$ , $\underline{\delta}_{c0,1}$ are the flexibility matrix and the vector of load effects related to a beam resting on an elastic soil described by the parameter:

$$\alpha' = \alpha \cdot \sqrt[4]{(1 + \chi \cdot \varphi)} \tag{66}$$

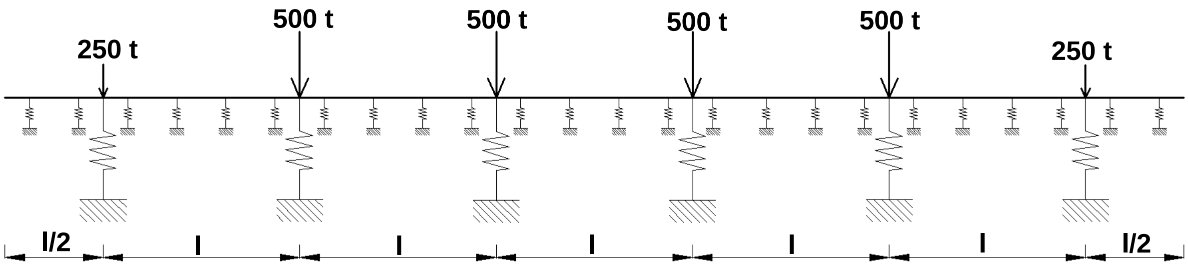


Figure 12: Beam on elastic soil

The results, assuming $\chi$ = 0.8; $\varphi$ = 2.5, b = 4m , E ≅ 33500 MPa, I = 2.11 $m^4$, l = 3.5m, $k_W = 10^4$ $N/mm^3$, are reported in Figure 13. We observe that for $\eta$ = 0, the settlements increase in the central part of the beam and decrease in the lateral part. This produces a generalized reduction of the bending moment in the beam, with a maximum decrease of about 35%. The maximum variation of the settlements is about 11% in the central section and -20% at the edges. The presence of piles strongly mitigates the settlements and the bending moments in the beam. For the maximum value of the piles stiffness, the corresponding bending moment in the beam reduces in time of about 70%, with an increase of about 10% of the mid-span settlement. The obtained results clearly point out the significant contribution of the piles in reducing the settlements and the bending moment in the r.c. beam.

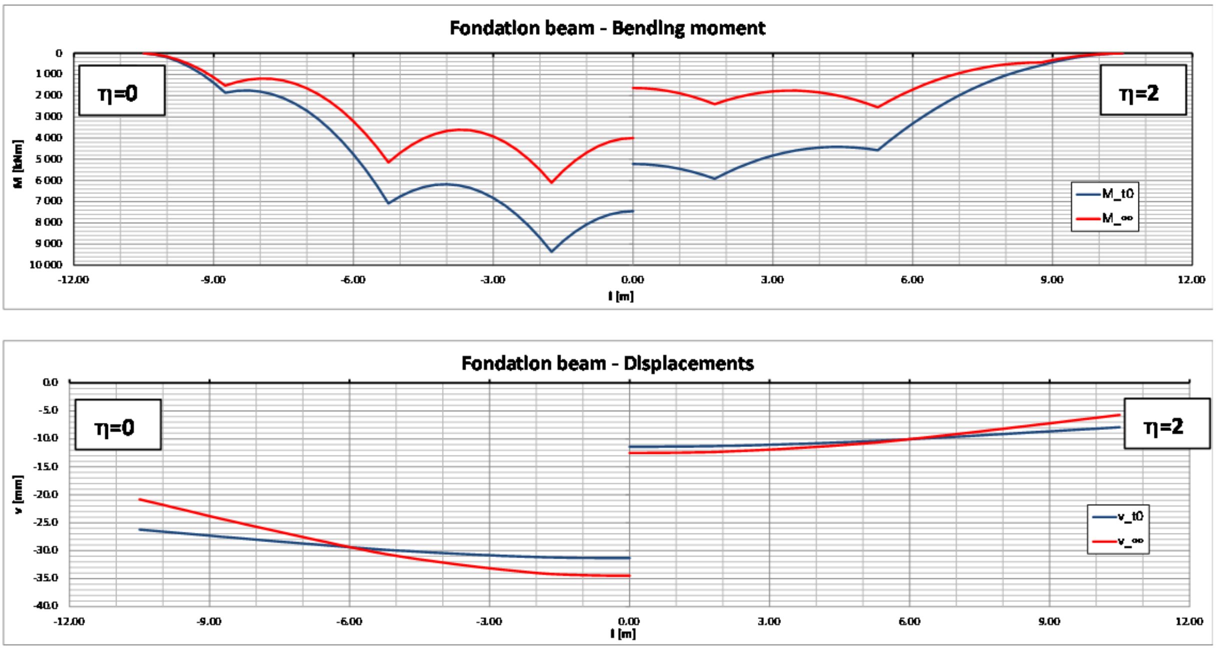


Figure 13: Bending moments and vertical settlements

As a last example we consider the r.c. column of Figure 14, subjected to a bending moment M and to an axial load P applied at the top, where is active an elastic lateral restraint whose deformability is $\delta_s = \frac{c \cdot l^3}{3 \cdot E_c \cdot I_c}$. The slenderness of the column requires to take into account second order effects.

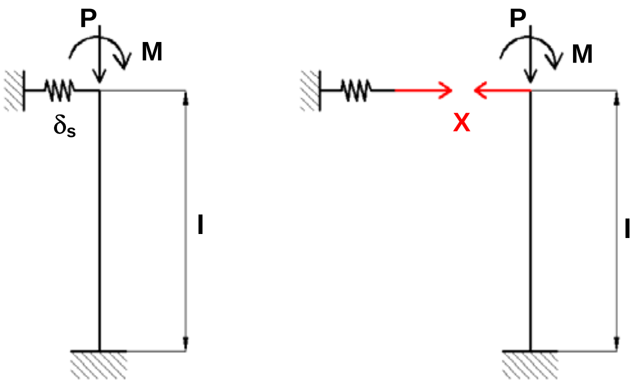


Figure 14: Slender concrete column elastically restrained

The problem can be solved by generalizing Timoshenko linear elastic approach[10], obtaining the following solutions:

$$X_0 = \frac{3}{2} \cdot \frac{M}{l} \cdot \left[ \frac{2\alpha l \cdot (1 - \cos\alpha l)}{3\sin(\alpha l) + \alpha l \cdot \cos(\alpha l) \cdot (c \cdot \alpha^2 l^2 - 3)} \right] \tag{67}$$

$$\alpha = \sqrt{\frac{P}{EI}} \tag{68}$$

$$X_1 = \frac{3}{2} \cdot \frac{M}{l} \cdot \left[ \frac{2\alpha_1 l \cdot (1 - \cos\alpha_1 l)}{3\sin(\alpha_1 l) + \alpha_1 l \cdot \cos(\alpha_1 l) \cdot \left( \frac{c \cdot \alpha_1{}^2 \cdot l^2}{1 + \chi \cdot \varphi} - 3 \right)} \right] \tag{69}$$

$$\alpha_1 = \alpha \cdot \sqrt{(1 + \chi \cdot \varphi)} \tag{70}$$

$$X = X_1 \cdot (1 - \mu) + \mu \cdot X_0 \tag{71}$$

Assuming $\chi$ = 0.8; $\varphi$ = 2.5, in Figure 15, for t = $t_0$ and for t $\rightarrow \infty$, the variation of the adimensional reaction Xl/M of the elastic restraint with the parameter $\alpha l = \sqrt{\frac{P}{E \cdot I}} \cdot l$ is reported. At initial time, for $l = \pi/2$ , the unrestrained column reaches instability, so its stiffness vanishes. For this, the reaction of the restraint is the same, irrespectively of its rigidity. It is noteworthy to observe that for $\alpha l < \pi/2$ the reaction of the elastic restraint reduces when reducing the rigidity, while for $\alpha l > \pi/2$ the reaction of the elastic restraint increases. This means that for $\alpha l < \pi/2$ the behaviour of the column is governed by the restraint rigidity, while for $\alpha l > \pi/2$ second order effects are prevailing.

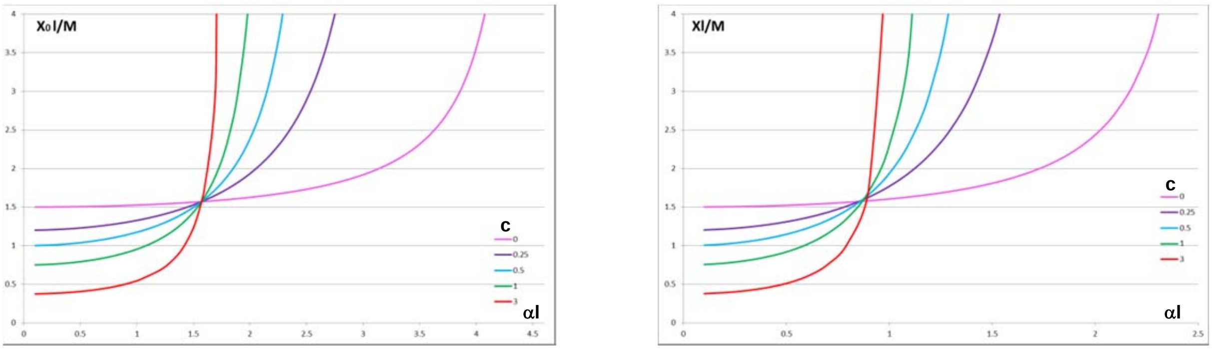


Figure 15: Adimensional restraint reaction

The effects of the delayed deformation due to creep are different if the contribution of reinforcement is accounted for. The corresponding solutions are expressed by the following relationships:

$$X_0 = \frac{3}{2} \cdot \frac{M}{l} \cdot \left[\frac{2\alpha^* l \cdot (1-\cos\alpha^* l)}{3\sin(\alpha^* l)+\alpha^* l \cdot \cos(\alpha^* l) \cdot (c \cdot \alpha^{*2} l^2 \cdot \beta^2 - 3)}\right] \quad (72)$$

$$\alpha^* = \frac{\alpha}{\beta} \quad ; \quad \beta = \sqrt{1 + \frac{E_s \cdot I_s}{E_c \cdot I_c}} \quad (73)$$

$$X_1 = \frac{3}{2} \cdot \frac{M}{l} \cdot \left[\frac{2\alpha_1^* l \cdot (1-\cos\alpha_1^* l)}{3\sin(\alpha_1^* l)+\alpha_1^* l \cdot \cos(\alpha_1^* l) \cdot \left(c \cdot {\alpha_1^*}^2 l^2 \cdot \frac{\beta_1^2}{1+\chi\varphi} - 3\right)}\right] \quad (74)$$

$$\alpha_1^* = \frac{\alpha_1}{\beta_1} \quad ; \quad \beta_1 = \sqrt{\beta^2 + \frac{E_s}{E_c} \cdot \chi \cdot \varphi \cdot \frac{I_s}{I_c}} \quad (75)$$

Assuming $\frac{I_s}{I_c} = 0.15$; $\frac{E_s}{E_c} = 5.5$, Figure 16 shows that the initial instability point is reached for $\alpha l \cong 2$ and the various curves of parameter c exhibit lower values for $\alpha$l constant as a consequence of the higher rigidity of the column due to the distributed internal elastic restraint provided by the steel reinforcement.

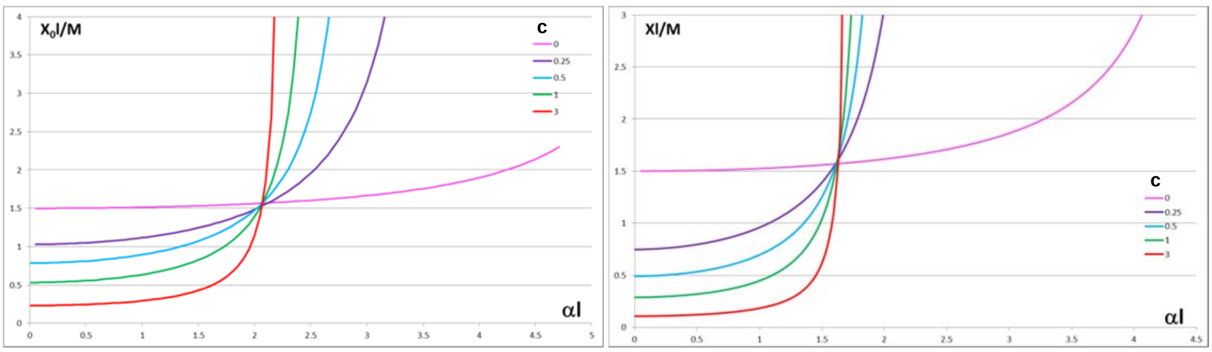


Figure 16: Effect of column reinforcement on the adimensional restraint reaction

## 4 CONCLUSIONS

The structural analysis of hybrid structures subjected to sustained loads can be approached in an exact way according to RRFM, by arriving for point elastic restraint to diagonal systems of Volterra integral equations and to integro-differential equations in space and time for distributed elastic restraints. A consistent simplification of the mathematical procedure can be achieved by adopting the algebraic approach AAEMM, reducing the problem to the solution of linear algebraic systems in the pseudo elastic domain with imposed deformation depending on the initial state of stress. This procedure guarantees a good level of approximation; nevertheless, it requires a lot of computations, so it becomes unpractical when a large number of unknowns is involved. An alternative procedure, stated by the author and yielding the same results of AAEMM, allows the problem to be solved by means of the superposition of two elastic solutions which can be easily obtained by recurring to standard methods of structural analysis. Some interesting case histories point out the feasibility of the proposed approach and the need of a reliable analysis in order to correctly evaluate the evolution of the state of stress and deformation in hybrid structures subjected to long-term action.

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