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Instability of two-phase flows: A lower bound on the dimension of the global attractor of the Cahn–Hilliard–Navier–Stokes system

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ABSTRACT

We consider a model for the flow of a mixture of two viscous and incompressible fluids in a two or three dimensional channel-like domain. The model consists of the Navier–Stokes equations governing the fluid velocity coupled with a convective Cahn–Hilliard equation for the relative density of atoms of one of the fluids. We prove the instability of certain stationary solutions for such a system endowed with periodic boundary conditions on elongated domains $(0, 2\pi/\alpha_0) \times (0, 2\pi)$ or $(0, 2\pi/\alpha_0) \times (0, 2\pi) \times (0, 2\pi/\beta_0)$ for a special class of periodic body forces, provided that α_0 and β_0 are small enough. As a consequence, we deduce a lower bound for the Hausdorff dimension of the global attractor.

1. Introduction

It is well-known that the long-time behavior of solutions of the two dimensional Navier–Stokes equation can be effectively described in terms of the (finite-dimensional) global attractor of the associated semigroup (see [1] and references therein). Moreover, the related turbulence issues for single-phase flows have been analyzed in many fundamental works (see [2–7] and their references). These aspects are even more challenging when binary fluid mixtures are considered (cf., e.g., [8]). To model their behavior a widely used approach is based on the so-called diffuse-interface method (see, for instance, [9–11]). This method consists in introducing an order parameter, accounting for the presence of two species, whose dynamics interacts with the fluid's velocity. For incompressible fluids a well-known model, known as Cahn–Hilliard fluid, consists of the classical Navier–Stokes equations suitably coupled with a convective Cahn–Hilliard equation, i.e., the so-called "Model H" (see [12,13], cf. also [14–17] and references therein). Of particular interest is the behavior of such mixtures under shear flow (see [18,19]). This is a two-stage evolution of a two-phase mixture: a phase separation stage in which some macroscopic pattern appears, then a shear stage in which these patterns organize themselves into parallel layers (see, e.g., [16] for experimental snapshots). When the two fluids have the same constant density (see [12], cf. also [18,14]), the temperature differences are negligible and the diffusive interface between the two phases has a small but non-zero thickness, the resulting model is a system of equations where an incompressible Navier–Stokes equation for the velocity field $\boldsymbol{u} = (u_1, \ldots, u_{\mathrm{N}})$, N = 2, 3, is coupled with a convective Cahn–Hilliard equation for the order parameter ϕ which represents the relative density of one species of atoms. More precisely, the system reads as follows:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = -\mathcal{K} \mathrm{div}\,(\nabla\phi \otimes \nabla\phi) + \boldsymbol{h}, \tag{1.1}$$

$$\mathrm{div}\, \boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla\phi - \varrho_0 \Delta\mu = 0, \tag{1.3}$$

$$\mu = -\varepsilon\Delta\phi + \frac{1}{\varepsilon} f(\phi), \tag{1.4}$$

in $\Omega \times (0, +\infty)$, where $\nabla\phi \otimes \nabla\phi$ denotes the $N \times N$ matrix whose (i, j)-entry is given by $\partial_i\phi \cdot \partial_j\phi$ for $1 \le i, j \le N$. Here the density has been set equal to one and ν, ϱ_0 and $\mathcal{K}$ are positive constants that correspond to the kinematic viscosity of fluid, mobility of mixture and a capillarity (stress) coefficient, respectively. Moreover, $\boldsymbol{h}$ is an external force, ε is a positive parameter describing the interactions between the two phases and is related to the thickness of the interface separating the two fluids, and f is the derivative of a

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suitable potential F. A typical example of F is the so-called logarithmic potential (see [20])

$$F(s) = \gamma_1 \left(1 - s^2\right) + \gamma_2 \left((1+s)\log(1+s) + (1-s)\log(1-s)\right), \quad \gamma_1, \gamma_2 > 0.$$

However, this potential is very often replaced by a smooth polynomial approximation (typically a fourth-order degree polynomial with positive leading coefficient).

This kind of system was analyzed numerically in a number of papers under different boundary conditions (see, e.g., [21–27]). Taking $\mathcal{K} = \varepsilon$, the singular limit as ε tends to 0 of systems (1.1)–(1.4) endowed with initial and suitable boundary conditions (with $\boldsymbol{h} = \mathbf{0}$) was identified in [28, Appendix A]. The resulting limit is a sharp interface model which combines the classical Navier–Stokes sharp interface model and a Mullins–Sekerka type problem (see [28] and references therein). Qualitative aspects like well-posedness were first analyzed in [29] with $\Omega = \mathbb{R}^2$. Then, a more general analysis was performed in [30] (see also [31,32]) by assuming degenerate mobility, concentration dependent viscosity and considering either smooth or logarithmic potentials. The case of singular potential, viscosity depending on ϕ and constant mobility was carefully analyzed in [33] and, in particular, the convergence to single stationary solutions in absence of external forces was established (see also [34]). It is also worth mentioning that non-Newtonian fluids were considered in [35] (see also [36]) and compressible two-phase fluids were recently studied in [37]. The dynamical system approach was used in [38] to establish the existence of a global attractor for singular potentials (e.g., logarithmic-like). In [39] a more thorough analysis was carried out in the case $N = 2$ for smooth potentials. Besides the existence of a (smooth) global attractor and an exponential attractor, an upper bound on the fractal dimension of the global attractor was obtained. The three dimensional case was examined in [40] by using the trajectory approach (see [41]) for a general class of time-dependent external forces.

Here we want to obtain a lower bound on the Hausdorff dimension of the global attractor in the case of periodic boundary conditions by estimating from below the dimension of a suitable unstable manifold. More precisely, we assume that Ω is either $(0, 2\pi L/\alpha_0) \times (0, 2\pi L)$ or $(0, 2\pi L/\alpha_0) \times (0, 2\pi L) \times (0, 2\pi L/\beta_0)$, where $L > 0$ and $\alpha_0, \beta_0 \in (0, 1]$. The numbers α_0, β_0 are small non-dimensional parameters so that Ω can be thought as an elongated channel in either the direction of x_1 or of x_1 and x_3. Then periodicity conditions are imposed on ϕ, p and $\boldsymbol{u}$, that is (cf., e.g., [42])

$$\boldsymbol{u}, p, \phi \quad \text{are } \Omega\text{-periodic.} \tag{1.5}$$

System (1.1)–(1.5) is also subject to the initial conditions

$$\boldsymbol{u}_{|t=0} = \boldsymbol{u}_0, \qquad \phi_{|t=0} = \phi_0, \quad \text{in } \Omega. \tag{1.6}$$

Note that conditions (1.5) imply the conservation of mass

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle, \quad \forall t \geq 0, \tag{1.7}$$

where $\langle v(t) \rangle = \frac{1}{|\Omega|} \int_\Omega v(x,t)\,\mathrm{d}x$, $|\Omega|$ is the Lebesgue measure of Ω.

As we mentioned, an upper estimate on the Hausdorff and fractal dimensions of the global attractor for system (1.1)–(1.6) in 2D (with no-slip boundary conditions for $\boldsymbol{u}$ and zero-flux conditions for ϕ) was obtained in [39] by means of a volume-contraction argument. It is not difficult to realize that all the results proven in [39] and, in particular, [39, Theorem 4.5] can be easily adapted to the present case of periodic boundary conditions (1.5) (see, for instance, Section 2).

Lower bounds on the dimension of the global attractor are usually based on the observation that the unstable manifold of any equilibrium of the system is always contained in the global attractor (see [43], cf. also [44–50]). This method works very easily for systems possessing a global Lyapunov function. Indeed, in the absence of nongradient external forces (i.e., $\boldsymbol{h} \equiv \mathbf{0}$), system (1.1)–(1.6) is gradient-like (see [39, Section 5]) and the nontransient dynamics essentially reduces to the one associated with the Cahn–Hilliard equation. Thus, a lower bound on the dimension of the attractor can be found by analyzing the dimension of an unstable manifold associated with a constant equilibrium $(\mathbf{0}, M)$. More precisely, in 2D, when (1.1)–(1.5) is a gradient system on a suitable phase space (see below) with a global attractor $\mathcal{A}$, the following bound holds

$$\dim_F \mathcal{A} \geq \dim_H \mathcal{A} \geq \frac{c_f}{\varepsilon^2}|\Omega| - 1, \tag{1.8}$$

for some positive constant c_f, depending only on f, and where $\dim_H$ and $\dim_F$ denote the Hausdorff and the fractal dimension, respectively. However, when $\boldsymbol{h}$ is a nonvanishing nongradient force, the system may exhibit a very complicated (e.g., chaotic) dynamical behavior as it happens with single fluids (see, for instance, [51] and references therein). In this case we need to estimate the dimension $N^+(u_*, \phi_*)$ of the unstable manifold of a stationary solution (u_*, ϕ_*) of (1.1)–(1.5) which is less trivial than before.

The aim of the present paper is to obtain an estimate from below of $N^+(u_*, \phi_*)$ for a special class of periodic external forces $\boldsymbol{h}$ which includes the so-called Kolmogorov problem (see, for instance, [52] and its references). As a consequence, we find a lower bound on the dimension of the global attractor $\mathcal{A}$ for (1.1)–(1.6) (whenever it exists), depending explicitly on the physical parameters ν, ε, α_0 and β_0. The paper is organized as follows. In Section 2, we provide a lower bound for the dimension of the attractor in the case of two dimensional two-phase flows, while Section 3 contains a conditional result concerning the three dimensional case.

2. The two dimensional case

In the two dimensional case, problem (1.1)–(1.6) generates a dynamical system in a suitable phase–space. Then we can state the existence of the global attractor. We can follow [39] closely since no-slip boundary conditions for $\boldsymbol{u}$ and zero-flux conditions for ϕ were considered there.

Let $\Omega = (0, 2\pi L/\alpha_0) \times (0, 2\pi L)$, $L > 0$ being a characteristic macroscopic length and $\alpha_0 > 0$ a given parameter. Suppose that F is a double-well potential of polynomial type (see [39] for more general assumptions).

Let us set (see, e.g., [42])

$$\mathbb{D} = \left\{ \boldsymbol{v} \in (C^\infty_{per}(\bar{\Omega}))^2 : \nabla \cdot \boldsymbol{v} = 0, \ \langle \boldsymbol{v} \rangle = \mathbf{0} \right\}.$$

Then denote by $\mathbb{H}^s_{per}(\Omega)$ the closure of $\mathbb{D}$ with respect to the $(H^s)^2$-norm, for each $s \in \mathbb{R}$, where $H^0 = L^2$. We also indicate by $H^s_{per}(\Omega)$, $s \in \mathbb{R}$, the closure of $C^\infty_{per}(\bar{\Omega})$ with respect to the H^s-norm. From now on, for a generic Hilbert space X, the norms in $L^2_{per}(X)$ and $H^s_{per}(X)$ will be indicated by $\|\cdot\|$ and $\|\cdot\|_{H^s(X)}$, for any $s > 0$, respectively.

Arguing as in [22] (see also [39]), it is possible to prove

Theorem 2.1. *Let $\boldsymbol{h} \in \mathbb{H}^{-1}_{per}(\Omega)$. For any $(\boldsymbol{u}_0, \phi_0) \in \mathbb{H}^0_{per}(\Omega) \times H^1_{per}(\Omega)$ there exists a unique pair $(\boldsymbol{u}, \phi) \in C^0([0, +\infty); \mathbb{H}^0_{per}(\Omega) \times H^1_{per}(\Omega))$ such that*

$$\boldsymbol{u} \in L^2_{loc}([0, +\infty); \mathbb{H}^1_{per}(\Omega)), \qquad \phi \in L^2_{loc}([0, +\infty); H^3_{per}(\Omega)), \tag{2.1}$$

which is a (weak) solution to (1.1)–(1.7). Moreover, the solution depends continuously on the initial data and on the external force in a Lipschitz way.

As a consequence, we can define a strongly continuous (nonlinear) semigroup $S(t)$ acting on $\mathbb{H}^0_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega)$ by setting $(\boldsymbol{u}(t), \phi(t)) = S(t)(\boldsymbol{u}_0, \phi_0)$ for all $t \geq 0$. Then, arguing as in [39, Subsecs. 3.2, 3.3 and Sec. 4] and recalling (1.7), we can also prove that

Theorem 2.2. *Let $\boldsymbol{h} \in \mathbb{H}^0_{\text{per}}(\Omega)$ and $M \in \mathbb{R}$. Set*

$$X_M = \left\{(\boldsymbol{v}, \psi) \in \mathbb{H}^0_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega) : \langle\psi\rangle := M\right\}. \tag{2.2}$$

Then the dynamical system $(X_M, S(t))$ has the global attractor $\mathcal{A}_M$ which is bounded in $\mathbb{H}^2_{\text{per}}(\Omega) \times H^4_{\text{per}}(\Omega)$ and has finite fractal dimension.

From now on, without loss of generality we suppose $L = 1$, $\varepsilon \in (0, 1]$ and we assume that $\boldsymbol{h}(x_1, x_2) = (g(x_2), 0)$ with

$$\int_0^{2\pi} g(x_2)\, dx_2 = 0. \tag{2.3}$$

Then we consider a stationary solution of (1.1)–(1.5) of the form $(\boldsymbol{u}_S, M_0)$, with $p = 0$, such that

$$f'(M_0) < 0, \tag{2.4}$$

and $\boldsymbol{u}_S = (U(x_2), 0)$, where U is the unique 2π-periodic solution of the problem

$$\begin{cases} -\nu U'' = g, \quad \text{in } [0, 2\pi], \\ \int_0^{2\pi} U(x_2)\, dx_2 = 0. \end{cases} \tag{2.5}$$

Observe that, due to (2.5) $_2$, there exists a unique 2π-periodic function θ with zero average such that $\theta'' = U$. Also, it is worth recalling that (2.4) means that M_0 is an unstable state for the Cahn–Hilliard equation (see [53]). We recall that the interval where $f' < 0$ is known as the spinodal region and only in this region the whole process of phase separation sets in at all (see, e.g., [54,55]).

The following bound can be obtained for system (1.1)–(1.6).

Theorem 2.3. *Let $\varepsilon_0 \in (0, 1]$ and choose M_0 in the spinodal region in such a way that*

$$-2f'(M_0) < \varepsilon_0^2, \tag{2.6}$$

that is, M_0 is sufficiently close to the end points of the spinodal interval. Suppose that ν and θ satisfy the inequalities:

$$\nu^2 < (2\pi)^{-1} \left\|\theta'\right\|^2, \tag{2.7}$$

$$\|\theta''\|\,\|\theta\| < -\frac{\pi}{\varepsilon_0} f'(M_0)\sqrt{\varepsilon_0^2 + 2f'(M_0)}. \tag{2.8}$$

Then, there exist positive constants d_0 depending on ν, g and d_1 depending on g, ε_0 such that, if $0 < \alpha_0 \leq \min\{d_0, d_1\}$, then

$$N^+(\boldsymbol{u}_S, M_0) \geq \frac{d_0 d_1}{\alpha_0^2} - 1. \tag{2.9}$$

In particular, for any given $\varepsilon \in [\varepsilon_0, 1]$, the global attractor $\mathcal{A}_{M_0}$ that describes the long-time behavior of (1.1)–(1.6) has a Hausdorff and fractal dimensions that satisfy the following bounds

$$\dim_F \mathcal{A}_{M_0} \geq \dim_H \mathcal{A}_{M_0} \geq \frac{d_0 d_1}{\alpha_0^2} - 1. \tag{2.10}$$

It is well-known (cf., e.g., [43,45]) that, if ν satisfies (2.7), the Hausdorff dimension of the global attractor $\mathcal{A}^{NS}$, of the two dimensional Navier–Stokes equation, satisfies the following bound:

$$\dim_H \mathcal{A}^{NS} \geq \frac{\delta_0}{\alpha_0} - 1, \tag{2.11}$$

for some positive constant δ_0 that depends only on ν and g. The lower bound (2.11) was first given in [44] by keeping the viscosity and the density of volume forces fixed and letting $\alpha_0 \to 0^+$. The authors used a specific volume force g for which simple stationary solutions (also known as Kolmogorov solutions) can be found. Furthermore, the dependence of the dimension of the attractor with respect to the shape factor α_0 of the domain was investigated in details in [50]. There, an upper bound for the dimension of $\mathcal{A}^{NS}$ in 2D is derived and this bound fully agrees with (2.11). In other words, there holds

$$\frac{C_0}{\alpha_0} \lesssim \dim_H \mathcal{A}^{NS} \lesssim \frac{C_1}{\alpha_0}, \tag{2.12}$$

which shows that (2.12) is sharp when the other physical parameters are fixed and the dimension of $\mathcal{A}^{NS}$ depends only on $\alpha_0 \to 0^+$. The constants C_0 and C_1 depend on the viscosity and density of the volume forces but not on the shape ratio α_0. Partial results of the same nature are also obtained for the three dimensional Navier–Stokes equation (see [45,50]).

It was conjectured in [39] that two-phase flows exhibit more complex flow behavior than single-phase fluids. Estimate (2.9) is a first step towards proving that the coupling of the Navier–Stokes equation with a convective Cahn–Hilliard equation may give rise to additional instabilities to the full system (1.1)–(1.6) and, thus, to novel and even more complex flow behavior. However, this does not seem so straightforward. Indeed, upper bounds for the dimension of the attractor $\mathcal{A}_{M_0}$ of the 2D Navier–Stokes–Cahn–Hilliard equation were obtained in [39] under more general assumptions on the potentials F. In particular, assuming that $F \in C^3(\mathbb{R})$ is such that

$$\liminf_{|s|\to\infty} F''(s) > 0; \qquad \left|F'''(s)\right| \leq C_F\left(1 + |s|^m\right), \quad \forall s \in \mathbb{R},$$

for some $C_F > 0$ and $m \geq 0$, we deduced the following upper bound for $\mathcal{A}_{M_0}$, as a function of α_0 only:

$$\dim_H \mathcal{A}_{M_0} \leq \dim_F \mathcal{A}_{M_0} \lesssim \frac{C_2}{\alpha_0^\gamma}. \tag{2.13}$$

Here $\gamma = \gamma(m) > 2$, for $m \geq 0$, depends only on m and the positive constant C_2 depends on ν, ε, $\mathcal{K}$ and $\boldsymbol{h}$. Therefore, there seems to be a discrepancy between (2.10) and (2.13) for the global attractor $\mathcal{A}_{M_0}$ of system (1.1)–(1.6). We believe that our lower bound estimate (2.10) cannot be improved using, for instance, Kolmogorov flows as base solutions. Therefore this poses the question about how sharp are our estimates, both from above and below. To bridge the gap between lower and upper estimates (as in [50] for single fluids), we should possibly consider stationary flows other than Kolmogorov ones and/or to improve the upper bound (2.13). These issues will be the subject of future investigations.

Proof. Let $\varepsilon \in [\varepsilon_0, 1]$ and consider Eq. (1.1) linearized around $(\boldsymbol{u}_S, M_0)$ with $p = 0$, that is,

$$\partial_t u_1 + U\partial_{x_1} u_1 + u_2 \partial_{x_2} U + \partial_{x_1} p = \nu \Delta u_1, \tag{2.14}$$

$$\partial_t u_2 + U\partial_{x_1} u_2 + \partial_{x_2} p = \nu \Delta u_2, \tag{2.15}$$

$$\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0, \tag{2.16}$$

$$\partial_t \phi - \Delta\left(-\varepsilon\Delta\phi + \frac{1}{\varepsilon} f'(M_0)\phi\right) + U\partial_{x_1}\phi = 0. \tag{2.17}$$

It is worth recalling that Eqs. (2.14)–(2.16) correspond exactly to the linearized equations of the Navier–Stokes equations around $(U(x_2), 0)$ with U as above. We look for solutions of Eqs. (2.14)–(2.17) which satisfy (1.5) and have the form

$$\begin{aligned} u_1(x_1, x_2, t) &= \partial_{x_2}\varphi(x_1, x_2, t), \\ u_2(x_1, x_2, t) &= -\partial_{x_1}\varphi(x_1, x_2, t), \end{aligned} \tag{2.18}$$

where

$$\varphi(x_1, x_2, t) = \tilde{\psi}(x_2)\, e^{\alpha_0 \sigma t + i\alpha_0 x_1}, \tag{2.19}$$

and

$$\phi(x_1, x_2, t) = \psi(x_2)\, e^{\varepsilon\lambda t + i\alpha_0 x_1}. \tag{2.20}$$

Also, we take $p(x_1, x_2, t) = \tilde{p}(x_2)\, e^{\alpha_0 \sigma t + i\alpha_0 x_1}$.

It follows that (2.14)–(2.17) reduce to the uncoupled system of ordinary differential equations ($D^m = d^m/dx_2^m$):

$$\begin{cases} \left(D^4 - 2\alpha_0^2 D^2 + \alpha_0^4\right)\tilde{\psi} = \dfrac{i\alpha_0}{\nu}\left[(U - i\sigma)\left(D^2 - \alpha_0^2\right) - U''\right]\tilde{\psi}, \\ \left(D^4 + \lambda + \alpha_0^4 - b_\varepsilon \alpha_0^2 + U\dfrac{i\alpha_0}{\varepsilon}\right)\psi = \left(2\alpha_0^2 - b_\varepsilon\right) D^2\psi, \end{cases} \tag{2.21}$$

where $b_\varepsilon := -f'(M_0)/\varepsilon^2 > 0$. The first equation of (2.21) is well-known as the Orr–Sommerfeld equation. It is known (cf., e.g., [44,43] and their references) that, if ν is sufficiently small so that (2.7) is satisfied, there exists $d_0 = d_0(\nu, g)$ such that, if $\alpha_0 \in (0, d_0]$, there exists a family $\left(\tilde{\psi}_{\alpha_0}, \sigma_{\alpha_0}\right)$ such that $\|\tilde{\psi}_{\alpha_0}\| = 1$ and

$$\mathrm{Re}\,\sigma_{\alpha_0} > 0, \qquad \frac{d}{d\alpha_0}\mathrm{Re}\,\sigma_{\alpha_0} > 0. \tag{2.22}$$

The corresponding solutions defined by (2.18) and (2.19) are thus unstable.

Let us now focus our attention on Eq. (2.21)$_2$, subject to periodic boundary conditions for ψ, that is, ψ and its higher-order derivatives up to the order three are 2π-periodic. For Eq. (2.21)$_2$ to have a solution, it is necessary that the integral of its left-hand side vanishes, that is,

$$\left(\lambda + \alpha_0^4 - b_\varepsilon\alpha_0^2\right)\int_0^{2\pi}\psi(x_2)\,dx_2 + \frac{i\alpha_0}{\varepsilon}\int_0^{2\pi} U(x_2)\,\psi(x_2)\,dx_2 = 0. \tag{2.23}$$

Thus, it is convenient to require that ψ belongs to an affine space orthogonal to constants, that is,

$$\int_0^{2\pi}\psi(x_2)\,dx_2 = 2\pi i. \tag{2.24}$$

Due to (2.24), from (2.23) we get the following expression for the eigenvalue λ,

$$\lambda = -\alpha_0^4 + b_\varepsilon\alpha_0^2 - \frac{\alpha_0}{2\pi\varepsilon}\int_0^{2\pi} U(x_2)\,\psi(x_2)\,dx_2. \tag{2.25}$$

When $\alpha_0 = 0$, then all non-constant 2π-periodic solutions of (2.21)$_2$ (with (2.25)) satisfy

$$D^4\psi + b_\varepsilon D^2\psi = 0. \tag{2.26}$$

Clearly, (2.26) has infinitely many solutions (other than constant valued solutions) of the form

$$\psi(x_2) = c_1\cos\left(\sqrt{b_\varepsilon}x_2\right) + c_2\sin\left(\sqrt{b_\varepsilon}x_2\right), \quad c_1, c_2 \in \mathbb{R}. \tag{2.27}$$

We are looking for solutions of (2.26) such that $\psi \in H^4_{per}(0, 2\pi)$ and ψ satisfies (2.24). However, on account of (2.6), we have that $\sqrt{b_\varepsilon} < 1$. Thus any nonconstant function given by (2.27) is periodic with fundamental period $\mathbb{T}_\varepsilon = 2\pi/\sqrt{b_\varepsilon}$, which is strictly greater than 2π. Thus, the only solution $\psi \in H^4_{per}(0, 2\pi)$ satisfying (2.24) is $\psi(x_2) = i$.

Denote by $A = D^{-4}$ the isomorphism of $\{\psi \in L^2(0, 2\pi) : \langle\psi, 1\rangle = 0\}$ into $\{\psi \in H^4_{per}(0, 2\pi) : \langle\psi, 1\rangle = 2\pi i\}$, defined by $D^4(A\psi) = \psi$, such that $\langle A\psi, 1\rangle = 2\pi i$. Observe that A also extends to an isomorphism (still denoted by A) which is one-to-one between the space $V := \left\{\psi \in H^2_{per}(0, 2\pi) : \langle\psi, 1\rangle = 2\pi i\right\}$ and $V^* := \left\{\eta \in H^{-2}_{per}(0, 2\pi) : \langle\eta, 1\rangle = 0\right\}$. Consequently, Eq. (2.21)$_2$ can be written as

$$\psi = AB_{\varepsilon,\lambda}\psi, \quad B_{\varepsilon,\lambda} : V \to L^2(0, 2\pi), \tag{2.28}$$

where $B_{\varepsilon,\lambda} = \left[-\lambda - \alpha_0^4 + b_\varepsilon\alpha_0^2 - U\frac{i\alpha_0}{\varepsilon}\right]I + \left(2\alpha_0^2 - b_\varepsilon\right)D^2$ with λ given by (2.25). The mapping $AB_{\varepsilon,\lambda} : V \to V$ is continuous and analytic with respect to α_0 (and ε). Recall that, by assumption (2.6) we have $0 < b_\varepsilon < 1/2$. On account of (2.23), Eq. (2.28) possesses a unique solution (that is, a fixed point) if $0 < \alpha_0 \le \alpha_0^*$ is sufficiently small. More precisely, there exists $\alpha_0^* = \alpha_0^*(\varepsilon_0) > 0$ such that

$$\begin{aligned} \left\|AB_{\varepsilon,\lambda}\psi^1 - AB_{\varepsilon,\lambda}\psi^2\right\|_V &\le \left[\left(2\alpha_0^2 + b_\varepsilon\right) + \varepsilon^{-1}\|U\|\,\alpha_0\right]\left\|\psi^1 - \psi^2\right\|_V \\ &\quad + \frac{\alpha_0}{2\pi\varepsilon}\left\|\langle U, \psi^1\rangle\psi^1 - \langle U, \psi^2\rangle\psi^2\right\| \\ &\le \frac{1}{4}\left\|\psi^1 - \psi^2\right\|_V + b_\varepsilon\left\|\psi^1 - \psi^2\right\|_V \\ &< \frac{3}{4}\left\|\psi^1 - \psi^2\right\|_V, \end{aligned}$$

for all $\alpha_0 \in \left[0, \alpha_0^*\right]$ and every $\psi^1, \psi^2 \in V$, where V is endowed with the $H^2_{per}(0, 2\pi)$-norm. Let us now denote by $\psi = \psi(\alpha_0, \varepsilon) \in V$ the fixed point of (2.28). Since $\psi(\alpha_0, \varepsilon)$ is an analytic function with respect to α_0, by the Cauchy–Riemann integral theorem, we can represent ψ through an analytic power series expansion which holds uniformly with respect to $\varepsilon \in [\varepsilon_0, 1]$ and $\alpha_0 \in \left[0, \alpha_0^*\right]$, that is,

$$\psi(\alpha_0, \varepsilon) = \sum_{k=0}^{\infty}\psi_k(\varepsilon)\alpha_0^k. \tag{2.29}$$

Since $\psi(\alpha_0, \varepsilon) \in V$, from (2.29) it is readily seen that

$$\int_0^{2\pi}\psi_0(x_2)\,dx_2 = 2\pi i, \qquad \int_0^{2\pi}\psi_k(x_2)\,dx_2 = 0, \quad k \ge 1. \tag{2.30}$$

Plugging (2.29) into (2.25), we obtain then

$$\lambda = \lambda(\alpha_0, \varepsilon) = \sum_{k=0}^{\infty}\lambda_k(\varepsilon)\alpha_0^k, \tag{2.31}$$

with the first three eigenvalues $\lambda_k(\varepsilon)$, $k \in \{0, 1, 2\}$, explicitly given by

$$\lambda_0(\varepsilon) = 0, \qquad \lambda_1(\varepsilon) = -\frac{1}{2\pi\varepsilon}\int_0^{2\pi} U(x_2)\,\psi_0(x_2)\,dx_2, \tag{2.32}$$

$$\lambda_2(\varepsilon) = b_\varepsilon - \frac{1}{2\pi\varepsilon}\int_0^{2\pi} U(x_2)\,\psi_1(x_2)\,dx_2. \tag{2.33}$$

Substituting (2.29) and (2.31) into Eq. (2.21)$_2$, we obtain, on account of (2.32), that $\psi_0 \in H^4_{per}(0, 2\pi)$ must solve $\left(D^4 + b_\varepsilon D^2\right)\psi_0 = 0$. Thus ψ_0 must solve $D^2\psi_0 + b_\varepsilon\psi_0 = C$, where the constant C is determined from the first condition of (2.30). We get $C = b_\varepsilon i$. Hence $\psi_0 \in H^2_{per}(0, 2\pi)$ solves

$$D^2\psi_0 + b_\varepsilon\psi_0 - b_\varepsilon i = 0. \tag{2.34}$$

Arguing as for (2.26) (cf. (2.27)), the only solution of (2.34) is $\psi_0(x_2) = i$. Then, recalling (2.5)$_2$, we deduce that $\lambda_1 = 0$.

We now show that $\mathrm{Re}\,\lambda_2(\varepsilon) > 0$ uniformly with respect to $\varepsilon \in [\varepsilon_0, 1]$. To this end, we first must find ψ_1 in (2.33). Recalling the above substitution once more, we have that $\psi_1 \in H^4_{\text{per}}(0, 2\pi)$ solves

$$D^4\psi_1 + b_\varepsilon D^2\psi_1 = \frac{U}{\varepsilon}.$$

Then, recalling that $U = D^2\theta$, we can simplify the latter equation and consider the following

$$D^2\psi_1 + b_\varepsilon \psi_1 = \frac{\theta}{\varepsilon}, \tag{2.35}$$

subject to zero average conditions for both θ and ψ_1 (see (2.30)). We must estimate the last term of (2.33) in terms of the $L^2(0, 2\pi)$-norms of θ and U, respectively. To this end, multiply (2.35) by ψ_1 and integrate over $(0, 2\pi)$. We deduce

$$\varepsilon^2 \|D\psi_1\|^2 - q_0 \|\psi_1\|^2 = -\varepsilon \langle \theta, \psi_1 \rangle, \tag{2.36}$$

where we have set $q_0(M_0) := -f'(M_0) \in \left(0, \varepsilon_0^2/2\right)$. By the Schwarz, Young and Poincaré inequalities (i.e., $\|\psi_1\| \le \|D\psi_1\|$, for $\psi_1 \in H^2_{\text{per}}(0, 2\pi)$ with $\langle \psi_1, 1 \rangle = 0$), we easily see from (2.36) that $\varepsilon^2 \|D\psi_1\|^2 \le 2q_0 \|\psi_1\|^2 + \|\theta\|^2$; hence, since $\varepsilon^2 - \varepsilon_0^2 < \varepsilon^2 - 2q_0 < \varepsilon^2$, for each $\varepsilon \in [\varepsilon_0, 1]$, we have

$$\|\psi_1\| \le \|D\psi_1\| \le \left(\varepsilon^2 - 2q_0\right)^{-1/2} \|\theta\|. \tag{2.37}$$

From (2.33), on account of (2.37), it follows that

$$\operatorname{Re}\lambda_2(\varepsilon) = b_\varepsilon - \frac{1}{2\pi\varepsilon}\langle U, \psi_1 \rangle \ge \frac{q_0}{\varepsilon^2} - \frac{\left(\varepsilon^2 - 2q_0\right)^{-1/2}}{2\pi\varepsilon}\|U\|\,\|\theta\|. \tag{2.38}$$

It is now easy to check that, on account of (2.8), we get

$$\operatorname{Re}\lambda_2(\varepsilon) > \frac{-f'(M_0)}{2\varepsilon^2} \ge \frac{-f'(M_0)}{2}. \tag{2.39}$$

Since $\frac{\partial^2}{\partial\alpha_0^2}\operatorname{Re}\lambda(\alpha_0, \varepsilon)$ is uniformly continuous with respect to α_0 for $\alpha_0 \in \left[0, \alpha_0^*\right]$ and $\varepsilon \in [\varepsilon_0, 1]$, we can find a possibly even smaller number $d_1 = d_1(\varepsilon_0) > 0$ such that

$$\frac{\partial^2}{\partial\alpha_0^2}\operatorname{Re}\lambda(\alpha_0, \varepsilon) \ge \frac{-f'(M_0)}{2}$$

and

$$\begin{aligned} &\operatorname{Re}\lambda(\alpha_0, \varepsilon) \ge \frac{-f'(M_0)\alpha_0^2}{4} > 0, \\ &\frac{\partial}{\partial\alpha_0}\operatorname{Re}\lambda(\alpha_0, \varepsilon) \ge \frac{-f'(M_0)\alpha_0}{2} > 0, \end{aligned} \tag{2.40}$$

for $\alpha_0 \in (0, d_1]$ and $\varepsilon \in [\varepsilon_0, 1]$. Indeed, for $|\alpha_0| \le \alpha_0^*$ and $\varepsilon \in [\varepsilon_0, 1]$, we have

$$\left| \frac{\partial^2}{\partial\alpha_0^2}\lambda(\alpha_0, \varepsilon) - \frac{\partial^2}{\partial\alpha_0^2}\lambda(0, \varepsilon) \right| \le \omega(\alpha_0).$$

Then, we can choose d_1 to be the largest α_0 such that $\omega(\alpha_0) \le -f'(M_0)/2$, for $|\alpha| \le d_1$. Consequently, there exists a family $\left(\psi_{\alpha_0}, \lambda_{\alpha_0}\right)$ satisfying (2.21) $_2$ and (2.40) with $\left\|\psi_{\alpha_0}\right\| = 1$. Hence, the corresponding stationary solutions (2.20) are unstable.

It remains to finalize the proof of our theorem with the following observation. For fixed $\alpha_0, \alpha_0 \le d_0$ and $\alpha_0 \le d_1$ (with d_0 and d_1 introduced above), let n_1 and n_2 be the largest integers such that $\alpha_0 n_1 \le d_0$ and $\alpha_0 n_2 \le d_1$, respectively. To each $l\alpha_0,\ l = 1, 2, \ldots, n_1$ and $m\alpha_0,\ m = 1, 2, \ldots, n_2$, we associate the following pairs of solutions to each of the equations of (2.21), $\{\tilde\psi_{l\alpha_0}, \sigma_{l\alpha_0}\}$ and $\{\psi_{m\alpha_0}, \lambda_{m\alpha_0}\}$ which yield the spatial periodic functions $\boldsymbol{u} = (u_1, u_2)$ and ϕ of the form (2.18) and (2.20) which are solutions to (2.14)–(2.17), respectively. By (2.22) and (2.40), each of the $\sigma_{l\alpha_0}$ or $\lambda_{m\alpha_0}$ are different, while the corresponding $\tilde\psi_{l\alpha_0}$ and $\psi_{m\alpha_0}$ are linearly independent. Therefore, Eq. (2.21) $_1$ (and, hence, (2.14)–(2.16)) possesses at least n_1 unstable modes, whereas Eq. (2.21) $_2$ (that is, (2.17)) possesses at least n_2 unstable solutions. Thus, the number of unstable solutions $(\boldsymbol{u}, \phi)$ to (2.14)–(2.17), subject to periodic boundary conditions (1.5), is at least $n_1 \times n_2$ and the unstable manifold of the shear flow stationary solution $(\boldsymbol{u}_S, M_0)$ has dimension greater or equal than the integer part of $\frac{d_0}{\alpha_0}$ times the integer part of $\frac{d_1}{\alpha_0}$. Consequently, (2.10) follows immediately from (2.9). The proof is now finished. □

Let us now consider the well-known family of Kolmogorov flows (see, e.g., [43]) with external force given by

$$g(x_2) = \Lambda\nu^2 \sin(x_2),$$

for some fixed $\Lambda > \sqrt{2}$. It is easy to check that $(\boldsymbol{u}_S, M_0)$ with $\boldsymbol{u}_S = (U(x_2), 0)$, $U(x_2) = \Lambda\nu\sin(x_2)$ is a stationary solution for (1.1)–(1.5) with $p = 0$. The following result is a consequence of Theorem 2.3.

Corollary 2.4. *Let $\varepsilon_0 \in (0, 1]$ and M_0 such that (2.6) holds. For a given $\varepsilon \in [\varepsilon_0, 1]$, assume that $\nu > 0$ satisfies*

$$\nu^2 < -\frac{f'(M_0)}{\pi\Lambda^2\varepsilon_0}\sqrt{\varepsilon_0^2 + 2f'(M_0)}. \tag{2.41}$$

Then the dimension $N^+(\boldsymbol{u}_S, M_0)$ of the unstable manifold of $(\boldsymbol{u}_S, M_0)$ is greater or equal than $c(\nu, \Lambda, \varepsilon_0)\alpha_0^{-2} - 1$ and (2.10) follows.

3. The three dimensional case

We now consider system (1.1)–(1.6) when $N = 3$ so that $\boldsymbol{u} = (u_1, u_2, u_3)$. The two-phase flow is supposed to be periodic with period 2π in the x_2-direction, $2\pi/\alpha_0$ in the x_1-direction and $2\pi/\beta_0$ in the x_3-direction. Here $0 < \beta_0 \le \alpha_0 \le 1$ so that the box-domain $\Omega = (0, 2\pi/\alpha_0) \times (0, 2\pi) \times (0, 2\pi/\beta_0)$ is elongated in two directions. The volume body force $\boldsymbol{h}$ has the form

$$\boldsymbol{h}(x_1, x_2, x_3) = (g(x_2), 0, 0),$$

with g satisfying (2.3). Then, system (1.1)–(1.5) has a stationary flow solution $(\boldsymbol{u}_S, M_0)$ such that $f'(M_0) < 0$ and $\boldsymbol{u}_S$ is of the form $(U(x_2), 0, 0)$, with U satisfying (2.5).

Of course in this case we only know that there exists a global weak solution and a trajectory attractor (cf. [40]). Hence, we must assume (see, e.g., [48]), that the unstable manifold of the stationary solution $(\boldsymbol{u}_S, M_0)$ is a functional invariant set bounded in $\mathbb{H}^1_{\text{per}}(\Omega) \times H^2_{\text{per}}(\Omega)$.

Theorem 3.1. *Let $\varepsilon_0 \in (0, 1]$ and M_0 such that (2.6) holds. For a given $\varepsilon \in [\varepsilon_0, 1]$, suppose that $\nu > 0$ and θ satisfy the inequalities:*

$$\nu^2 < (4\pi)^{-1}\left\|\theta'\right\|^2, \tag{3.1}$$

$$\|\theta''\|\,\|\theta\| < -\frac{\sqrt{2}\pi}{2\varepsilon_0}f'(M_0)\sqrt{2\varepsilon_0^2 + 2f'(M_0)}. \tag{3.2}$$

Then, there exist c_0, c_1 depending only on ν, ε_0 and g such that, if

$$0 < \beta_0 \le \alpha_0, \qquad \alpha_0^2 + \beta_0^2 \le c_0^2, \tag{3.3}$$

then

$$N^+(\boldsymbol{u}_S, M_0) \ge \frac{c_1}{(\alpha_0\beta_0)^2}. \tag{3.4}$$

Proof. The linear stability of system (1.1)–(1.6) around the stationary solution $(\boldsymbol{u}_S, M_0)$ is governed by the following system of equations:

$$\partial_t u_1 + U\partial_{x_1} u_1 + u_2 \partial_{x_2} U + \partial_{x_1} p = \nu \Delta u_1, \tag{3.5}$$

$$\partial_t u_2 + U\partial_{x_1} u_2 + \partial_{x_2} p = \nu \Delta u_2, \tag{3.6}$$

$$\partial_t u_3 + U\partial_{x_1} u_3 + \partial_{x_3} p = \nu \Delta u_3, \tag{3.7}$$

$$\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0, \tag{3.8}$$

$$\partial_t \phi - \Delta\left(-\varepsilon\Delta\phi + \frac{1}{\varepsilon} f'(M_0)\phi\right) + U\partial_{x_1}\phi = 0. \tag{3.9}$$

Following [45, Section 3], we seek for solutions to (3.5)–(3.9) of the form

$$\begin{cases} u_1 = u(x_2)\, e^{i\alpha_0 x_1 + i\beta_0 x_3 - i\sigma\alpha_0 t}, \\ u_2 = v(x_2)\, e^{i\alpha_0 x_1 + i\beta_0 x_3 - i\sigma\alpha_0 t}, \\ u_3 = w(x_2)\, e^{i\alpha_0 x_1 + i\beta_0 x_3 - i\sigma\alpha_0 t}, \end{cases} \tag{3.10}$$

and

$$p = \varphi(x_2)\, e^{i\alpha_0 x_1 + i\beta_0 x_3 - i\sigma\alpha_0 t}, \qquad \phi = \psi(x_2)\, e^{i\alpha_0 x_1 + i\beta_0 x_3 + \varepsilon\lambda t}. \tag{3.11}$$

After the substitution of (3.10)–(3.11) into (3.5)–(3.9), elimination of φ, subsequent simplifications and setting

$$\mathcal{L} := D^2 - (\alpha_0^2 + \beta_0^2) - i\nu^{-2}\alpha_0 (U - \sigma),$$

we find that $\boldsymbol{u} = (u_1, u_2, u_3)$ and ϕ, determined by (3.10)–(3.11), are solutions of the following differential equations:

$$i(\alpha_0^2 + \beta_0^2)\mathcal{L}w + \beta_0[\mathcal{L}D + i\nu^{-1}\alpha_0 U']v = 0, \tag{3.12}$$

$$i\alpha_0 u + Dv + i\beta_0 w = 0, \tag{3.13}$$

$$\begin{aligned}&[D^4 - 2(\alpha_0^2 + \beta_0^2)D^2 + (\alpha_0^2 + \beta_0^2)^2]v \\ &\quad = i\nu^{-1}[\alpha_0(U - i\sigma)(D^2 - (\alpha_0^2 + \beta_0^2)) - U'']v,\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\left[D^4 + \lambda + (\alpha_0^2 + \beta_0^2)^2 - b_\varepsilon(\alpha_0^2 + \beta_0^2) + U\frac{i\alpha_0}{\varepsilon}\right]\psi \\ &\quad = [2(\alpha_0^2 + \beta_0^2) - b_\varepsilon]D^2\psi.\end{aligned} \tag{3.15}$$

Note that, if v and ϕ are solutions of (3.14) and (3.15), respectively, then we can also find w and u by solving (3.12) and (3.13), respectively.

Thus, we focus our attention on Eqs. (3.14) and (3.15). We observe that they are similar to Eqs. (2.21) in the sense that we can easily recover both equations of (2.21) by taking $\beta_0 = 0$ in (3.14)–(3.15). For $\beta_0 \neq 0$, we notice that (3.14)–(3.15) still have the same form as Eqs. (2.21), provided we replace in (2.21) α_0, ν and ε by

$$\tilde{\alpha}_0 = (\alpha_0^2 + \beta_0^2)^{1/2}, \qquad \tilde{\nu} = \frac{\tilde{\alpha}_0}{\alpha_0}\nu, \qquad \tilde{\varepsilon} = \frac{\tilde{\alpha}_0}{\alpha_0}\varepsilon. \tag{3.16}$$

Obviously, for $0 < \beta_0 \le \alpha_0$, we have $2^{-1/2}\nu^{-1} \le (\tilde{\nu})^{-1} \le \nu^{-1}$ and analogously, $2^{-1/2}\varepsilon^{-1} \le (\tilde{\varepsilon})^{-1} \le \varepsilon^{-1}$. It was shown in Section 2 (see (2.22) and (2.40)) that if $\tilde{\nu}$ satisfies (2.7) and (2.8) (compare with (3.1) and (3.2)), then Eqs. (3.14)–(3.15) have solutions (v, ψ) corresponding to pairs $(v_{\tilde{\alpha}_0}, \sigma_{\tilde{\alpha}_0})$, $(\psi_{\tilde{\alpha}_0}, \lambda_{\tilde{\alpha}_0})$ such that

$$\begin{cases} \operatorname{Re}\sigma(\tilde{\alpha}_0, \tilde{\varepsilon}) > 0, & \dfrac{\partial}{\partial\tilde{\alpha}_0}\operatorname{Re}\sigma(\tilde{\alpha}_0, \tilde{\varepsilon}) > 0, \\ \operatorname{Re}\lambda(\tilde{\alpha}_0, \tilde{\varepsilon}) > 0, & \dfrac{\partial}{\partial\tilde{\alpha}_0}\operatorname{Re}\lambda(\tilde{\alpha}_0, \tilde{\varepsilon}) > 0, \end{cases} \tag{3.17}$$

for $0 < \tilde{\alpha}_0 \le c_0$, where c_0 depends on ν, ε_0 and g.

On account of (3.3), for fixed α_0, β_0 such that $0 < \beta_0 \le \alpha_0$, we have

$$k^2\alpha_0^2 + l^2\beta_0^2 \le c_0^2, \tag{3.18}$$

for certain pairs $(k, l) \in \mathbb{N}_+^2$. The number n_+ of pairs $(k, l) \neq (0, 0)$ such that $l \le k$ and (3.18) holds true is proportional to the area of some sector of the ellipse $k^2\alpha_0^2 + l^2\beta_0^2 = c_0^2$, that is, of the order of $(\alpha_0\beta_0)^{-1}$, for α_0 and β_0 sufficiently small. To each of these pairs (k, l), we can associate a solution (v, ϕ) to Eqs. (3.14) and (3.15), respectively, producing at least n_+ unstable solutions to either (3.14) or (3.15) and, therefore, to systems (3.5)–(3.9). Due to (3.17), these unstable modes are linearly independent. Thus, the number of unstable solutions $(\boldsymbol{u}, \phi)$ to (3.5)–(3.9) which satisfy (1.5) is at least $n_+ \times n_+$. Consequently, the unstable manifold of the shear flow solution $(\boldsymbol{u}_S, M_0)$ has dimension greater or equal than $(n_+)^2$ and (3.4) follows. The proof of theorem is now finished. □

Assume that, in the case $N = 3$, system (1.1)–(1.6) has the global attractor $\mathcal{A}_{M_0}$ bounded in $\mathbb{H}^1_{\text{per}}(\Omega) \times H^2_{\text{per}}(\Omega)$. Then, the fractal and Hausdorff dimensions of $\mathcal{A}_{M_0}$ satisfy the inequality

$$\dim_F \mathcal{A}_{M_0} \ge \dim_H \mathcal{A}_{M_0} \ge \frac{c_1}{(\alpha_0\beta_0)^2} - 1.$$

We recall that a lower bound on the dimension of the smooth global attractor $\mathcal{A}^{NS}$ (if it exists) for the 3D Navier–Stokes equations was obtained in [45, Theorem 3.1] (see also [50]). Indeed, if ν, α_0, β_0 satisfy (3.1) and $0 < \beta_0 \le \alpha_0$, $\alpha_0^2 + \beta_0^2 \le c_2^2$, ($c_2$ depends only on ν and g), then the following inequalities hold

$$\dim_F \mathcal{A}^{NS} \ge \dim_H \mathcal{A}^{NS} \ge \frac{c_3}{\alpha_0\beta_0} - 1,$$

for some positive constant c_3 depending only on ν and g.

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