Isotropic Compliance in the Special Euclidean Group *SE(3)*

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Abstract

In this paper, the isotropic compliance property is examined in the Special Euclidean Group $SE(3)$. The relation between the wrench and the resulting twist is examined, considering an end-effector of 6 D.O.F. serial manipulators. Two properties are introduced. The first one, called *local* isotropic compliance, is verified if the force vector is parallel to the tip point displacement vector, and, at the same time, if the torque vector is parallel to the change of orientation vector. The second one, called *screw* isotropic compliance, is verified if the wrench screw axis is parallel to the twist screw axis. In the latter case, the wrench and twist screw axes are generally not coincident in the Cartesian Space. If they are, the contact point could (*screw-B* isotropic compliance) or could not (*screw-A* isotropic compliance) belong to the coincident axes. Four cases are analyzed and classified. Active stiffness regulation is considered to achieve isotropic compliance in a generic configuration. Two arrangements are taken into account for the control system, which acts either in parallel or as a series with the passive joint stiffness. The control stiffness matrix is then determined for both the arrangements and for all the four kinds of isotropic compliance. One detailed example of application is presented and the obtained results are verified by using multi-body dynamic simulation.

Keywords: isotropic compliance, stiffness matrix, active stiffness regulation, kinetostatics

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1. Definitions and nomenclature

$\mathcal{R}_0$ Base reference frame system $\{O_0, X_0, Y_0, Z_0\}$

P End-effector tip point (or contact point) in $\mathcal{R}_0$

$\hat{\bullet}$ unit vector

$\phi\hat{\boldsymbol{\nu}}$ force vector applied on P

ℓ line of action of $\phi\hat{\boldsymbol{\nu}}$

$\mu\hat{\boldsymbol{u}}$ torque applied to the end-effector

u torque axis

$\hat{\boldsymbol{u}}$ unit vector parallel to the torque axis u

λ $= {}^{\mu}/_{\phi}$ wrench pitch

$\hat{\boldsymbol{j}}$ unit vector parallel to $\hat{\boldsymbol{\nu}}$ and ℓ

$\hat{\boldsymbol{i}}$ unit vector parallel to $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{\mu}}$

$\hat{\boldsymbol{k}}$ $= \hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}$

$\mathcal{R}$ reference system $\{P, x, y, z\}$, where x, y, z are parallel to $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$, respectively

$\boldsymbol{p}$ $= \overrightarrow{PO} = \phi\hat{\boldsymbol{j}} \times \mu\hat{\boldsymbol{u}}/\phi^2 = p\hat{\boldsymbol{i}}$

$\boldsymbol{w}$ $= \begin{Bmatrix} \phi\hat{\boldsymbol{j}} \\ \mu\hat{\boldsymbol{u}} \end{Bmatrix} = \phi \begin{Bmatrix} \hat{\boldsymbol{j}} \\ \lambda\hat{\boldsymbol{j}} + \boldsymbol{p} \times \hat{\boldsymbol{j}} \end{Bmatrix} = \phi \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}_{\boldsymbol{p}} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{j}} \\ \lambda\hat{\boldsymbol{j}} \end{Bmatrix}$, wrench

$\mathbf{S}_{\boldsymbol{p}}$ skew-symmetric tensor defined in such a way that, for a given vector $\boldsymbol{V}$, $\mathbf{S}_{\boldsymbol{p}}\boldsymbol{V} = \boldsymbol{p} \times \boldsymbol{V}$

o screw axis of the wrench, passing through the terminal point O of $\boldsymbol{p} = \overrightarrow{PO}$

$\boldsymbol{\Theta}_{\triangle}$ $= \Theta_{\triangle}\hat{\boldsymbol{h}}$ end-effector change of orientation vector

h screw axis of the twist, or instantaneous rotation axis

$\hat{\boldsymbol{h}}$ unit vector parallel to the change of orientation

$\hat{\boldsymbol{v}}$ unit vector parallel to the displacement vector of point P

$\Delta\boldsymbol{x}_P$ $= \Delta P\hat{\boldsymbol{v}}$ displacement vector of point P

$\boldsymbol{r}$ $= \overrightarrow{PH} = \boldsymbol{\Theta}_\Delta \times \Delta\boldsymbol{x}_P/\Theta_\Delta^2$, minimum distance from P to h

$\Delta\boldsymbol{x}_H$ $= \Delta H\hat{\boldsymbol{h}}$, displacement vector of point H

ν $= \Delta H/\Theta_\Delta$ twist pitch

$$\boldsymbol{t} = \begin{Bmatrix} \Theta_\Delta\hat{\boldsymbol{h}} \\ \Delta P\hat{\boldsymbol{v}} \end{Bmatrix} = \Theta_\Delta \begin{Bmatrix} \hat{\boldsymbol{h}} \\ \nu\hat{\boldsymbol{h}} + \boldsymbol{r}\times\hat{\boldsymbol{h}} \end{Bmatrix} = \Theta_\Delta \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{S}\boldsymbol{r} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{h}} \\ \nu\hat{\boldsymbol{h}} \end{Bmatrix}$$

$$\boldsymbol{S}_\Delta = \begin{Bmatrix} \Delta P\hat{\boldsymbol{v}} \\ \Theta_\Delta\hat{\boldsymbol{h}} \end{Bmatrix} = \begin{Bmatrix} \Theta_\Delta\nu\hat{\boldsymbol{h}} + \boldsymbol{r}\times\Theta_\Delta\hat{\boldsymbol{h}} \\ \Theta_\Delta\hat{\boldsymbol{h}} \end{Bmatrix}$$

n number of joints

q_i generic Lagrangian coordinate

$\boldsymbol{q}$ joint coordinates vector

$\mathbf{J}$ Jacobian matrix defined as

$$\boldsymbol{S}_\Delta = \mathbf{J}\Delta\boldsymbol{q} \tag{1}$$

$\boldsymbol{\Delta\tau_c}$ reaction generalized force vector in the joint space due to the control system as a response to a posture modification $\Delta\boldsymbol{q}$ (in the joint space), defined as

$$\boldsymbol{\Delta\tau_c} = -\mathbf{k_c}\boldsymbol{\Delta q} \tag{2}$$

$\mathbf{K_p}$ stiffness matrix in the Cartesian space due to the passive stiffness of the joints

$\mathbf{K_c}$ stiffness matrix in the Cartesian space due to control

$\mathbf{K}$ stiffness matrix in the Cartesian space

$\mathbf{k_p}$ passive stiffness matrix defined in the joint space, $\mathbf{k_p} = \mathrm{diag}(k_{p_1}, k_{p_2}, ..., k_{p_n})$, due to the mechanical compliance of the joints components

$\mathbf{k_c}$ control stiffness matrix defined in the joint space, where the generic element $k_{c_{ij}}$ represents the feedback gain (in terms of torque or force) applied by link $i-1$ to link i, due to feedback acquired from link j

$\mathbf{k}$ overall stiffness matrix defined in the joint space and related to the overall stiffness matrix defined in the Cartesian space by means of [1]

$$\mathbf{k} = \mathbf{J}^{\mathrm{T}}\mathbf{K}\mathbf{J} \tag{3}$$

$\mathbf{C}$ compliance matrix defined in the Cartesian space associating the end-effector displacement $\boldsymbol{S}_{\Delta}$ with wrench $\boldsymbol{w}$ in the Cartesian space by means of the relation

$$\boldsymbol{S}_{\Delta} = \mathbf{C}\boldsymbol{w} \ , \tag{4}$$

and equal to [2]

$$\mathbf{C} = \mathbf{J}\mathbf{k}^{-1}\mathbf{J}^{\mathrm{T}} \tag{5}$$

2. Introduction

Interactions between a manipulator end-effector and the environment occur in many applications, such as assembling or grasping operations. Consequently, considerable attention has been paid to the analysis, modeling, optimization, and control of robotic systems stiffness and compliance [3–18]. Generally, such interactions occur in dynamic conditions, and depend on the compliance of the systems in contact, i.e. both the robot and the environment. A manipulator end-effector, trying to maintain its pose when subjected to an external action, represents a simplified case of such interactions. Assuming that the control system is able to assure local stability in a prescribed pose, a wrench on the end-effector and its consequent twist response can be studied under kinetostatic conditions.

A possible way to characterize such response was presented in Ref. [19], where both the force and the displacement were defined in the Euclidean Space $E(3)$. In this contribution, *isotropic compliance* property was introduced considering 3 D.O.F. serial manipulators. If this property holds in a specific posture, the displacement of the end-effector is parallel to the external action. As described in Refs. [19, 20], to achieve isotropic compliance, the 3×3 compliance matrix of the manipulator, defined in the Cartesian space, has to be equal to a *scalar matrix*. It is worth noting that the well-known condition of *kinematic isotropy* [21–28] generally does not imply *isotropic compliance*. A scalar compliance matrix was obtained, in specific postures, by means of a decoupling control strategy, based on the adoption of a PD controller in each joint. Two arrangements were considered, depending on whether each control torque/force may act either as a series or in parallel with the mechanical passive elastic torque/force, as in some applications of compliant mechanisms and MEMS [29–31].

However, the external load acting on the end-effector and its consequent displacement are generally not limited to vectors defined in $E(3)$. In fact, a generalized force may be applied to the end-effector, composed of a force and a moment, and a generalized displacement could occur, composed of a displacement and a rotation. The relation of these generalized vectors, that are the applied *wrench* (Poinsot's theorem) and the resultant *twist* (Chasles's theorem), is then represented by a 6×6 compliance matrix. This paper focuses exactly on such relation, considering non-redundant, 6 D.O.F. serial manipulators. The extension of the isotropic compliance property to the Special Euclidean Group $SE(3)$ leads to the definition of two different properties: *local* isotropic compliance and *screw* isotropic compliance. *Local* isotropic compliance holds when the force vector is parallel to the displacement vector and, at the same time, the torque vector is parallel to the change of orientation vector. *Screw* isotropic compliance refers to the condition of parallelism between the wrench and twist screw axes. Although the screw axes are parallel, generally they are not coincident. Furthermore, none of them passes through the end-effector contact point. However, thanks to the active control, it is possible to impose both the above mentioned conditions. For these reasons, two more cases deserve particular attention, namely, *screw-A* isotropic compliance (or *task screw* isotropic compliance), and *screw-B* isotropic compliance (or *tool screw* isotropic compliance). Screw-A isotropic compliance holds when the screw axes of both the wrench and twist are coincident. Screw-B is a particular form of Screw-A isotropic compliance for which the common screw axis passes through the contact point.

Local or screw isotropic compliance can be achieved by means of coupled active stiffness regulation. Two different arrangements are considered for the control system, which may act either as a series or in parallel with the passive stiffness, depending on the mechanism under consideration. The passive stiffness matrix of the serial manipulator is generally diagonal, while the control stiffness matrix can have a more general form, representing the coupling effects of the active regulation. The control stiffness matrix is determined for both the arrangements.

3. Why Struggling with Isotropic Compliance?

A practical application of the concept of *isotropic compliance* is presented in this paragraph. In the situation depicted in Fig. 1, a manipulator end-effector is grasping a cylindrical object while it is *locally* moving in a structured workspace delimited by two parallel planes π_1 and π_2. During its task, the end-effector must avoid collisions between the carried object and the planar boundaries. Possibly, it must avoid collisions with other end-effectors operating in the same space.

Under these conditions, the end-effector should move on a plane parallel to π_1 and π_2 and, hopefully, it should not rotate. Furthermore, we will assume that the interaction load, externally applied in quasi-static conditions, could be represented by a resultant force $\phi\hat{\boldsymbol{j}}$ which is parallel to the boundary surfaces π_1 and π_2 (otherwise such working station would have been very badly designed).

When the *isotropic compliance* property is limited to the Eucliden Space $E(3)$, the application point displacement is parallel to the applied force $\phi\hat{\boldsymbol{j}}$. This result is useful, but there are no guarantees that interference with neighborhood objects is excluded. In fact, with reference to Fig. 1a, the achievement of the property in $E(3)$ does not prevent the manipulator end-effector from accidental collisions with close walls π_1 and π_2 even though the externally applied force was parallel to π_1 and π_2. This circumstance does apply because *isotropic compliance* in $E(3)$ has no control on the final end-effector rotational attitude.

For these reasons, the achievement of such property has been extended to $SE(3)$, because it involves the end-effector rotations. For example, with reference to Fig. 1b, if the pose of such manipulator corresponds to a *local* isotropic compliance posture (see the next sections), the end-effector and the handled tool both translate along the force axis, from the pose $\mathcal{P}_0$ to the pose $\mathcal{P}_1$. In other words, since no moments are applied to the end-effector, no rotations will be contained within the system response motion.

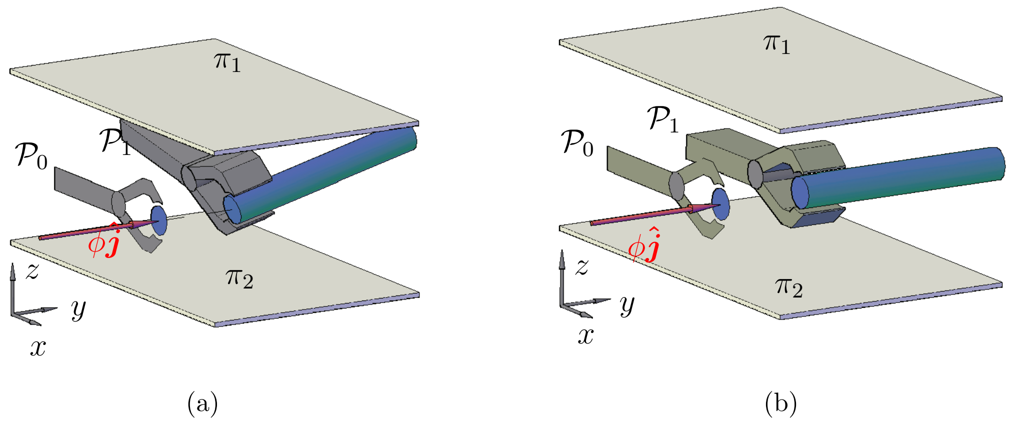


Figure 1: End-effector handling a tool between two parallel surfaces in $E(3)$ isotropic compliance condition (a), and in $SE(3)$ *local* isotropic compliance condition (b)

4. Isotropic compliance in *SE(3)*: problem formulation

To define the isotropic compliance property in *SE(3)*, we first introduce the manipulator base frame $R_0 \equiv \{O_0, x_0, y_0, z_0\}$, and we consider a generic point P defined by the position vector $\boldsymbol{v_P} = \overrightarrow{O_0 P}$. The external load conditions can be reduced to a force, whose line of action passes through P, and to a torque. With reference to Fig. 2, we define $\hat{\boldsymbol{j}}$ as the unit vector corresponding to the force line of action, ϕ the force magnitude, $\hat{\boldsymbol{u}}$ the unit vector corresponding to the axis torque u, and μ as the torque magnitude. Furthermore, we define $\hat{\boldsymbol{k}}$ as the unit vector belonging to the plane defined by $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{u}}$, whose direction is perpendicular to $\hat{\boldsymbol{j}}$, and whose sense is the same as the component of $\hat{\boldsymbol{u}}$ perpendicular to $\hat{\boldsymbol{j}}$. From the definition of $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, it follows that $\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}}$. Finally, we introduce the reference frame $R_P \equiv \{P, x, y, z\}$, having its origin in P and axes x, y, and z, defined by the unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$, respectively.

The torque component $\mu\hat{\boldsymbol{u}}$ of the wrench, $\boldsymbol{w} = \left\{\phi\hat{\boldsymbol{j}}; \mu\hat{\boldsymbol{u}}\right\}$, is equal to the sum of a component parallel to the force line of action, $\lambda\phi\hat{\boldsymbol{j}}$, where λ is the wrench pitch, and a component perpendicular to such line, that is $\phi \boldsymbol{p} \times \hat{\boldsymbol{j}}$. In the reference frame $R_P \equiv \{P, x, y, z\}$, $\boldsymbol{p} = p\hat{\boldsymbol{\imath}}$, and then $\phi \boldsymbol{p} \times \hat{\boldsymbol{j}} = \phi p \hat{\boldsymbol{k}}$. We define o as the screw axis of the wrench, and O the point defined by the position vector $\boldsymbol{p} = \overrightarrow{PO}$. Also, we define π as the plane xy, and σ as the plane normal to the unit vector $\hat{\boldsymbol{u}}$.

As it is well known [32], the wrench $\boldsymbol{w}$ is energetically conjugated to a generalized displacement $\boldsymbol{S}_\Delta$, that depends on the twist $\boldsymbol{t} = \left\{\Theta_\Delta \hat{\boldsymbol{h}}; \Delta P \hat{\boldsymbol{v}}\right\}$, where $\Delta P \hat{\boldsymbol{v}} = \Theta_\Delta \nu \hat{\boldsymbol{h}} + \Theta_\Delta \boldsymbol{r} \times \hat{\boldsymbol{h}}$. We define h as the screw axis of the twist, $\hat{\boldsymbol{h}}$ as the unit vector corresponding to h, and with $\boldsymbol{r}$ the vector representing the position of h with respect to P. The point H is defined by the position vector $\boldsymbol{r} = \overrightarrow{PH}$. Also, we define χ as the plane normal to the unit vector $\hat{\boldsymbol{h}}$.

Generally, there are no particular relations between the directions of the unit vectors $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{v}}$, that represent the directions of the applied force and of the displacement, respectively, nor between the unit vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{h}}$, the unit vectors corresponding to the axis torque and to the screw axis of the twist, respectively. In Ref. [19], the condition of parallelism between $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{v}}$ was called *isotropic compliance in E(3)*.

In the present investigation, we define *local isotropic compliance*, in *SE(3)*, the condition of parallelism between the unit vectors $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{v}}$, and between the unit vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{h}}$.

Also, we define *screw isotropic compliance*, in *SE(3)*, the condition of parallelism between the unit vectors $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{h}}$, that is the condition of parallelism between the screw axes of the wrench and of the twist.

So far, nothing has been assumed about the position of point P. Thus, in general, the formulation is valid for every point defined by the position vector $\overrightarrow{O_0P}$ in the the manipulator base frame $R_0 \equiv \{O_0, x_0, y_0, z_0\}$. Nevertheless, the generalized displacement of the manipulator end-effector due to the external load is of particular interest in many practical applications. For this reason, we consider as point P the origin of the end-effector frame (or *tool* frame) $R_n \equiv \{O_n, x_n, y_n, z_n\}$, defined according to the Denavit-Hartenberg convention; namely, we assume $P \equiv O_n$.

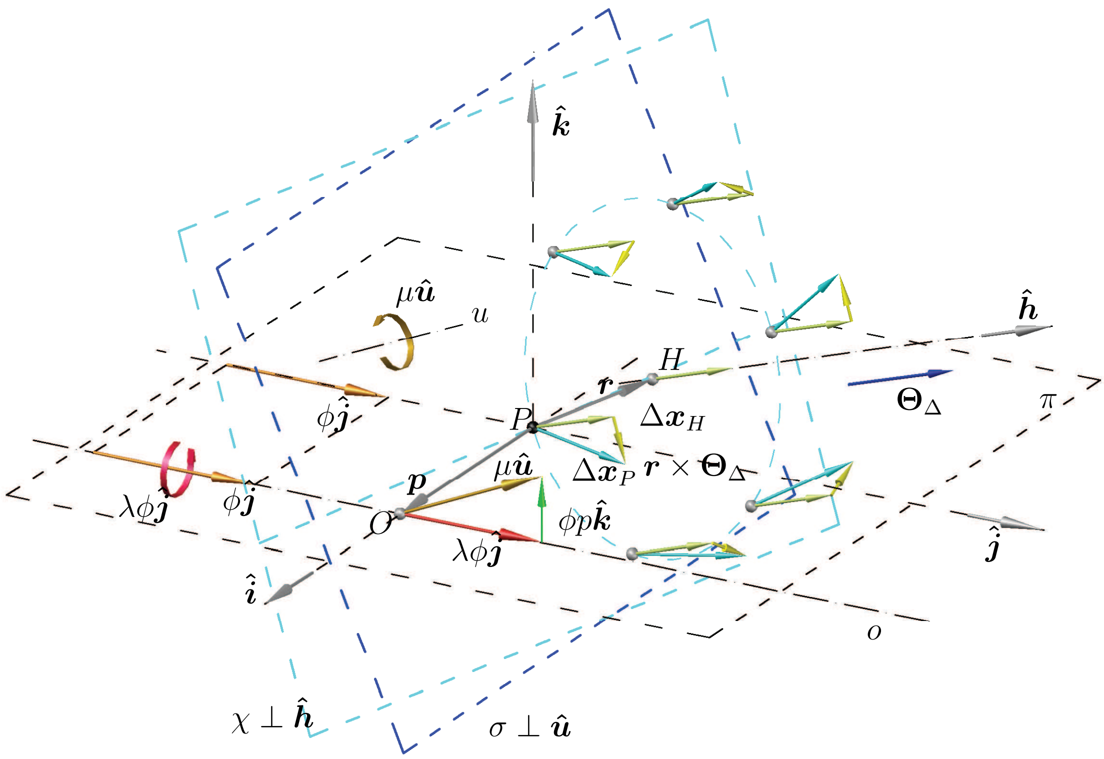


Figure 2: Wrench $\boldsymbol{w}$ and its consequent screw displacement $\boldsymbol{S}_\Delta$ in the general case

5. Condition for parallelism

Introducing a basis in $I\!R^6$, namely $(\boldsymbol{v_1},\ldots,\boldsymbol{v_6})$, $\boldsymbol{v_i}\in I\!R^6$, $i=1,\ldots,6$, the generalized vector

$$\boldsymbol{a}=\{\boldsymbol{a_d};\ \boldsymbol{a_r}\}$$

with $\boldsymbol{a_d}=\{a_{d1};a_{d2};a_{d3}\}$ and $\boldsymbol{a_r}=\{a_{r1};a_{r2};a_{r3}\}$ can be written, with respect to this basis, as

$$\boldsymbol{a}=\sum_1^6\alpha_i\boldsymbol{v_i}=\mathbf{V}\boldsymbol{\alpha} \tag{6}$$

with

$$\mathbf{V}=[\boldsymbol{v_1}\mid\ \ldots\ \mid\boldsymbol{v_6}]\,,$$

$$\boldsymbol{\alpha}=\{\alpha_1\ ;\ \ldots\ \ ;\alpha_6\}\,.$$

Considering the generalized vector $\boldsymbol{b} = \{\boldsymbol{b_d};\, \boldsymbol{b_r}\}$ the isotropic compliance condition between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a_d} \parallel \boldsymbol{b_d} \ ,$$

$$\boldsymbol{a_r} \parallel \boldsymbol{b_r} \ ,$$

can be written as

$$\begin{Bmatrix} \boldsymbol{b_d} \\ \boldsymbol{b_r} \end{Bmatrix} = \begin{bmatrix} \lambda_d \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda_r \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{a_d} \\ \boldsymbol{a_r} \end{Bmatrix} \tag{7}$$

where λ_d, λ_r are scalars, $\mathbf{I}$ is the 3×3 Identity matrix and $\mathbf{0}$ is the 3×3 zero matrix. Generally, if any relationship between $\boldsymbol{a}$ and $\boldsymbol{b}$ holds, i.e.

$$\boldsymbol{b} = \mathbf{M}\boldsymbol{a} \tag{8}$$

where $\mathbf{M}$ is a 6×6 matrix, by making use of Eqns. (6) and (7) such relationship can be written as

$$\left(\begin{bmatrix} \lambda_d \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda_r \mathbf{I} \end{bmatrix} - \mathbf{M} \right) \mathbf{V}\boldsymbol{\alpha} = 0 \ , \tag{9}$$

which must be true for any $\boldsymbol{\alpha} \in I\!R^6$, and implies

$$\mathbf{M} = \begin{bmatrix} \lambda_d \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \lambda_r \mathbf{I} \end{bmatrix} \ . \tag{10}$$

In the present investigation, the notation λ_d and λ_r will be replaced by $\tilde{c}_d$ and $\tilde{c}_r$, with the aim of representing the end-effector compliance in terms of displacement and rotation, respectively. Hence, $\tilde{c}_d$ will have units of mN^{-1}, while $\tilde{c}_r$ will have units of $\mathrm{N}^{-1}\mathrm{m}^{-1}$.

6. Isotropic compliance in $\boldsymbol{SE}(3)$: *local* formulation

6.1. Definition

The *local* isotropic compliance in *SE(3)* consists in imposing that the force $\phi\hat{\boldsymbol{j}}$ be parallel to the displacement of the tip point P, and that the moment $\mu\hat{\boldsymbol{u}}$ be parallel to the rotation axis $\hat{\boldsymbol{h}}$, i.e.

$$\hat{\boldsymbol{v}} = \hat{\boldsymbol{j}} \ , \tag{11}$$

$$\hat{\boldsymbol{h}} = \hat{\boldsymbol{u}} \ . \tag{12}$$

or

$$\Delta P \hat{\boldsymbol{v}} \parallel \phi \hat{\boldsymbol{j}} \iff \Theta_\Delta \left(\nu \hat{\boldsymbol{h}} + \vec{r} \times \hat{\boldsymbol{h}} \right) \parallel \phi \hat{\boldsymbol{j}} \ , \tag{13}$$

$$\Theta_\Delta \hat{\boldsymbol{h}} \parallel \mu \hat{\boldsymbol{u}} \iff \Theta_\Delta \hat{\boldsymbol{h}} \parallel \phi \left(\lambda \hat{\boldsymbol{j}} + \vec{p} \times \hat{\boldsymbol{j}} \right) \ . \tag{14}$$

The local isotropic compliance condition is illustrated in Fig. 3. Assuming

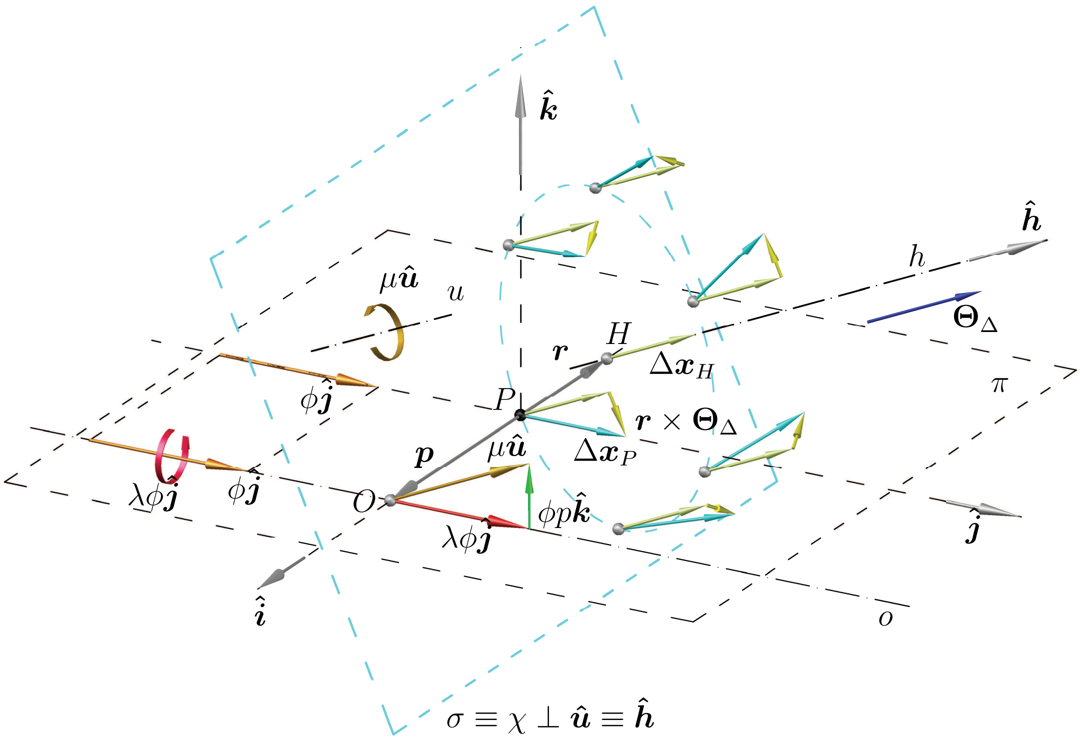


Figure 3: Local isotropic compliance. Wrench $\boldsymbol{w}$ and its consequent screw displacement $\boldsymbol{S}_\Delta$ when $\hat{\boldsymbol{j}} \equiv \hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{u}} \equiv \hat{\boldsymbol{h}}$

$$\boldsymbol{b_d} = \Delta P \hat{\boldsymbol{v}} \ , \tag{15}$$

$$\boldsymbol{b_r} = \Theta_\Delta \hat{\boldsymbol{h}} \ , \tag{16}$$

$$\boldsymbol{a_d} = \phi \hat{\boldsymbol{j}} \ , \tag{17}$$

$$\boldsymbol{a_r} = \mu \hat{\boldsymbol{u}} \ . \tag{18}$$

Eqn. (7) can be rewritten as

$$\begin{Bmatrix} \Delta P \hat{\boldsymbol{v}} \\ \Theta_\Delta \hat{\boldsymbol{h}} \end{Bmatrix} = \begin{bmatrix} \tilde{c}_d \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \tilde{c}_r \mathbf{I} \end{bmatrix} \begin{Bmatrix} \phi \hat{\boldsymbol{j}} \\ \mu \hat{\boldsymbol{u}} \end{Bmatrix} \ . \tag{19}$$

From Eqn. (4) and Eqn. (19), it follows

$$\mathbf{C} = \begin{bmatrix} \tilde{c}_d \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \tilde{c}_r \mathbf{I} \end{bmatrix} . \tag{20}$$

Figure 4 show a second example of a practical application of the local isotropic compliance property in $SE(3)$. In particular, in Fig. 4a the local isotropic compliance condition implies a displacement along the force line of action, $\hat{\boldsymbol{j}}$, and a rotation about an axis parallel to the torque axis, $\hat{\boldsymbol{k}}$. Figure 4b shows the end-effector displacement and rotation when the local isotropic compliance condition is not verified.

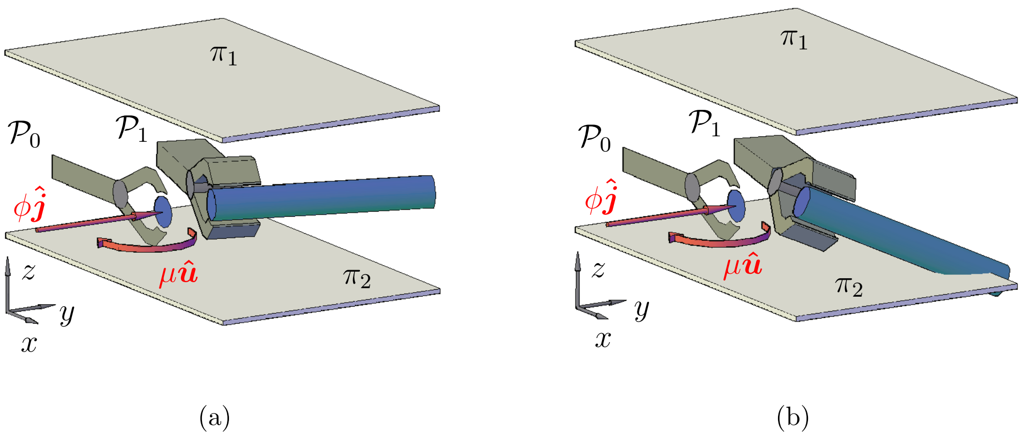


Figure 4: End-effector handling a tool between two parallel surfaces: (a) local isotropic compliance condition; (b) non isotropic compliance condition.

6.2. *Position of the twist screw axis*

The condition for the *local* isotropic compliance, Eqn. (19), can be rewritten as

$$\begin{Bmatrix} \Delta P \hat{\boldsymbol{v}} \\ \Theta_\Delta \hat{\boldsymbol{h}} \end{Bmatrix} = \begin{Bmatrix} \tilde{c}_d \phi \hat{\boldsymbol{j}} \\ \tilde{c}_r \mu \hat{\mathbf{u}} \end{Bmatrix} = \begin{Bmatrix} \tilde{c}_d \phi \hat{\boldsymbol{j}} \\ \tilde{c}_r \phi \left(\lambda \hat{\boldsymbol{j}} + p \hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} \right) \end{Bmatrix} , \tag{21}$$

and the unit vector representing the twist screw axis is equal to

$$\hat{\boldsymbol{h}} = \frac{\tilde{c}_r \left(\lambda \phi \hat{\boldsymbol{j}} + \phi p \hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} \right)}{\sqrt{(\tilde{c}_r \lambda \phi)^2 + (\tilde{c}_r \phi p)^2}} = \frac{\lambda \hat{\boldsymbol{j}} + p \hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}}{\sqrt{\lambda^2 + p^2}} . \tag{22}$$

The end-effector displacement $\Delta P\hat{\boldsymbol{v}}$ can be decomposed into a component parallel to the unit vector $\hat{\boldsymbol{h}}$, that is

$$\hat{\boldsymbol{h}} \otimes \hat{\boldsymbol{h}} \Delta P\hat{\boldsymbol{v}} = \frac{\tilde{c}_d \phi \lambda}{\lambda^2 + p^2} \left(\lambda \hat{\boldsymbol{j}} + p\hat{\boldsymbol{k}} \right) \ , \tag{23}$$

and a component perpendicular to the unit vector $\hat{\boldsymbol{h}}$, that is

$$\left(\mathbf{I} - \hat{\boldsymbol{h}} \otimes \hat{\boldsymbol{h}} \right) \Delta P\hat{\boldsymbol{v}} = \frac{\tilde{c}_d \phi p}{\lambda^2 + p^2} \left(p\hat{\boldsymbol{j}} - \lambda \hat{\boldsymbol{k}} \right) \ . \tag{24}$$

For the component perpendicular to the unit vector $\hat{\boldsymbol{h}}$, the relation

$$\left(\mathbf{I} - \hat{\boldsymbol{h}} \otimes \hat{\boldsymbol{h}} \right) \Delta P\hat{\boldsymbol{v}} = -\Theta_\Delta \hat{\boldsymbol{h}} \times \boldsymbol{r} \tag{25}$$

must hold. By cross multiplying each side of Eqn. (25) by $\hat{\boldsymbol{h}}$, it is possible to obtain

$$\boldsymbol{r} = -\frac{\tilde{c}_d \phi}{\Theta_\Delta^2} \hat{\boldsymbol{j}} \times \Theta_\Delta \hat{\boldsymbol{h}} \ . \tag{26}$$

By making use of $\Theta_\Delta \hat{\boldsymbol{h}}$ defined in Eqn. (21), Eqn. (26) becomes

$$\boldsymbol{r} = -\frac{\tilde{c}_d}{\tilde{c}_r} \frac{p}{\lambda^2 + p^2} \hat{\boldsymbol{i}} \ . \tag{27}$$

6.3. Particular case

Considering the wrench $\boldsymbol{w}$, a particular case arises when $\boldsymbol{p} = \mathbf{0}$ (and $\phi p\hat{\boldsymbol{k}} = 0$). This condition implies $\hat{\boldsymbol{u}} \parallel \hat{\boldsymbol{j}}$. In case of local isotropic compliance, $\mu\hat{\boldsymbol{u}} \parallel \hat{\boldsymbol{h}}$, and $\phi\hat{\boldsymbol{j}} \parallel \Delta P\hat{\boldsymbol{v}}$. Since $\hat{\boldsymbol{u}} \parallel \hat{\boldsymbol{j}}$, it follows that $\hat{\boldsymbol{v}} \parallel \hat{\boldsymbol{h}}$; thus $\boldsymbol{p} = \boldsymbol{r} = \mathbf{0}$. In other words, wrench screw axis, twist screw axis, moment, force, displacement and change of orientation vectors all have the same direction, passing through P. In this case, the local isotropic compliance condition reduces to the particular case screw-B isotropic compliance condition, as it will be clear in Section 7.3.2.

7. Isotropic compliance in $SE(3)$: *screw* formulation

7.1. Definition

The *screw isotropic compliance* condition in *SE(3)* consists in imposing the parallelism between the wrench screw axis o and the twist axis h, i.e.

$$\hat{\boldsymbol{j}} \parallel \hat{\boldsymbol{h}}, \tag{28}$$

or

$$\Theta_\Delta \nu \hat{\boldsymbol{h}} \parallel \phi \hat{\boldsymbol{j}} , \tag{29}$$

$$\Theta_\Delta \hat{\boldsymbol{h}} \parallel \lambda \phi \hat{\boldsymbol{j}} , \tag{30}$$

as depicted in Fig. 5. By invoking Eqn. (7), it follows

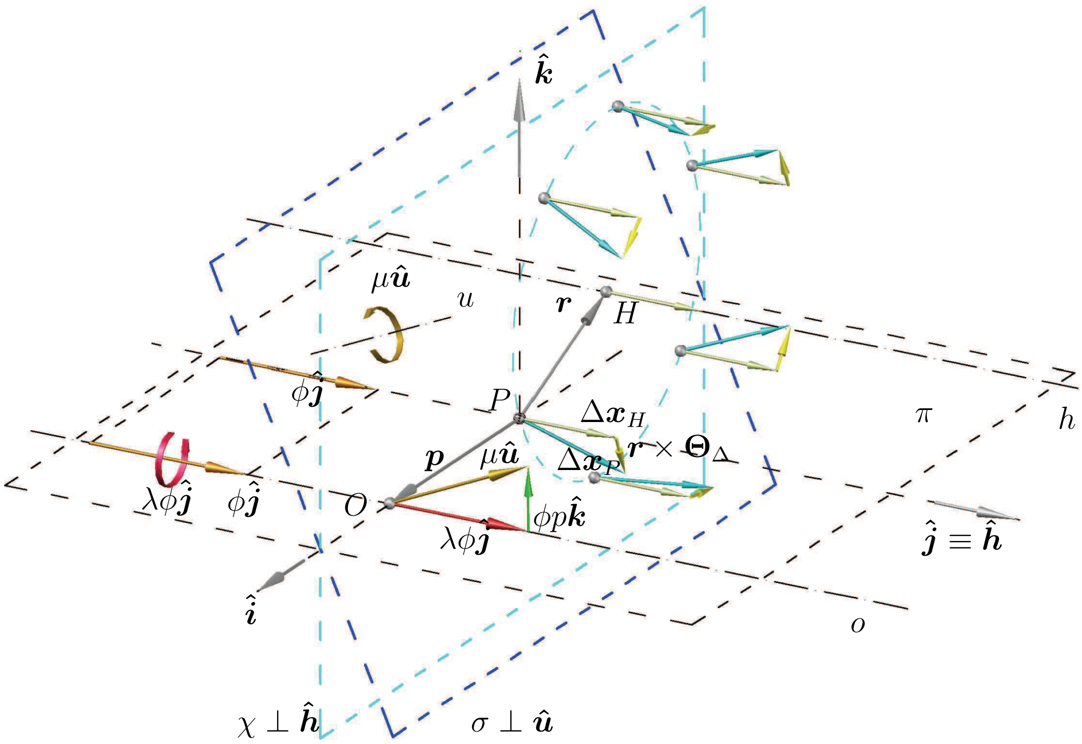


Figure 5: Screw isotropic compliance. Wrench $\boldsymbol{w}$ and its consequent screw displacement $\boldsymbol{S}_\Delta$ when the wrench and twist screw axes o and h are parallel, and so $\hat{\boldsymbol{j}} \equiv \hat{\boldsymbol{h}}$

$$\begin{Bmatrix} \nu\Theta_\Delta \hat{\boldsymbol{h}} \\ \Theta_\Delta \hat{\boldsymbol{h}} \end{Bmatrix} = \begin{bmatrix} \tilde{c}_d \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \tilde{c}_r \mathbf{I} \end{bmatrix} \begin{Bmatrix} \phi \hat{\boldsymbol{j}} \\ \lambda\phi\hat{\boldsymbol{j}} \end{Bmatrix} , \tag{31}$$

or, equivalently,

$$\nu\Theta_\Delta \hat{\boldsymbol{h}} = \tilde{c}_d \phi \hat{\boldsymbol{j}} , \tag{32}$$

$$\Theta_\Delta \hat{\boldsymbol{h}} = \tilde{c}_r \lambda \phi \hat{\boldsymbol{j}} . \tag{33}$$

From the definitions of wrench and twist, it follows

$$\phi\hat{\boldsymbol{j}} = \left[\mathbf{S}_{\boldsymbol{p}} + \lambda\mathbf{I}\right]^{-1}\mu\hat{\boldsymbol{u}} \;, \tag{34}$$

$$\Theta_\Delta\hat{\boldsymbol{h}} = \left[\mathbf{S}_{\boldsymbol{r}} + \nu\mathbf{I}\right]^{-1}\Delta P\hat{\boldsymbol{v}} \;. \tag{35}$$

Substituting (35) and (34) in (32) and (33), respectively, yields

$$\nu\left[\mathbf{S}_{\boldsymbol{r}} + \nu\mathbf{I}\right]^{-1}\Delta P\hat{\boldsymbol{v}} = \tilde{c}_d\phi\hat{\boldsymbol{j}} \;, \tag{36}$$

$$\Theta_\Delta\hat{\boldsymbol{h}} = \tilde{c}_r\lambda\left[\mathbf{S}_{\boldsymbol{p}} + \lambda\mathbf{I}\right]^{-1}\mu\hat{\boldsymbol{u}} \;. \tag{37}$$

In matrix form

$$\begin{Bmatrix}\Delta P\hat{\boldsymbol{v}}\\ \Theta_\Delta\hat{\boldsymbol{h}}\end{Bmatrix} = \begin{bmatrix}{}^{\tilde{c}_d}\!/\!_{\nu}\left[\mathbf{S}_{\boldsymbol{r}} + \nu\mathbf{I}\right] & \mathbf{0}\\ \mathbf{0} & \tilde{c}_r\lambda\left[\mathbf{S}_{\boldsymbol{p}} + \lambda\mathbf{I}\right]^{-1}\end{bmatrix}\begin{Bmatrix}\phi\hat{\boldsymbol{j}}\\ \mu\hat{\boldsymbol{u}}\end{Bmatrix} \;, \tag{38}$$

or, equivalently,

$$\boldsymbol{S}_\Delta = \mathbf{C}\boldsymbol{w} \;,$$

where

$$\mathbf{C} = \begin{bmatrix}{}^{\tilde{c}_d}\!/\!_{\nu}\left[\mathbf{S}_{\boldsymbol{r}} + \nu\mathbf{I}\right] & \mathbf{0}\\ \mathbf{0} & \tilde{c}_r\lambda\left[\mathbf{S}_{\boldsymbol{p}} + \lambda\mathbf{I}\right]^{-1}\end{bmatrix} \tag{39}$$

is the compliance matrix.

Analogously,

$$\boldsymbol{w} = \mathbf{K}\boldsymbol{S}_\Delta \;,$$

where

$$\mathbf{K} = \mathbf{C}^{-1} = \begin{bmatrix}{}^{\nu}\!/\!_{\tilde{c}_d}\left[\mathbf{S}_{\boldsymbol{r}} + \nu\mathbf{I}\right]^{-1} & \mathbf{0}\\ \mathbf{0} & {}^{1}\!/\!_{(\tilde{c}_r\lambda)}\left[\mathbf{S}_{\boldsymbol{p}} + \lambda\mathbf{I}\right]\end{bmatrix} \tag{40}$$

is the stiffness matrix.

7.2. Position of the twist screw axis

The condition for the screw isotropic compliance condition, Eqn. (31), can be rewritten as

$$\nu\Theta_\Delta\hat{\boldsymbol{h}} = \tilde{c}_d\phi\hat{\boldsymbol{j}} \;, \tag{41}$$

$$\Theta_\Delta\hat{\boldsymbol{h}} = \tilde{c}_r\lambda\phi\hat{\boldsymbol{j}} \;. \tag{42}$$

From the previous equations it can be seen that

$$\lambda\nu = \frac{\tilde{c}_d}{\tilde{c}_r} \,, \tag{43}$$

so the pitch of the twist depends on the compliance coefficients $\tilde{c}_d$ and $\tilde{c}_d$, other than on the wrench pitch λ. If Eqns. (29) and (30) hold, the generalized displacement $\boldsymbol{S}_\Delta$, or

$$\begin{Bmatrix} \Delta P \hat{\boldsymbol{v}} \\ \Theta_\Delta \hat{\boldsymbol{h}} \end{Bmatrix} = \begin{Bmatrix} \Theta_\Delta \nu \hat{\boldsymbol{h}} + \boldsymbol{r} \times \Theta_\Delta \hat{\boldsymbol{h}} \\ \Theta_\Delta \hat{\boldsymbol{h}} \end{Bmatrix} \,, \tag{44}$$

can be rewritten, by making use of Eqns. (41), (42), and (43), as

$$\begin{Bmatrix} \Delta P \hat{\boldsymbol{v}} \\ \Theta_\Delta \hat{\boldsymbol{h}} \end{Bmatrix} = \begin{Bmatrix} \tilde{c}_d \phi \hat{\boldsymbol{j}} + \tilde{c}_r \lambda \boldsymbol{r} \times \phi \hat{\boldsymbol{j}} \\ \tilde{c}_r \lambda \phi \hat{\boldsymbol{j}} \end{Bmatrix} \,. \tag{45}$$

The position of the twist screw axis can be set by the control system, by defining the position vector $\boldsymbol{r}$. Two particular screw isotropic compliance cases, that depend on the position of the twist screw axis and thus on the selection of vector $\boldsymbol{r}$, are discussed in Section 7.3.

7.3. Screw-A and screw-B isotropic compliance

In this paragraph we consider two particular cases:

- screw-A isotropic compliance: the screw axes of the wrench and of the twist are *coincident*;
- screw-B isotropic compliance: the screw axes of the wrench and of the twist are *coincident* and *pass through P*.

7.3.1. Screw-A isotropic compliance

If the wrench and twist screw axes are coincident, as depicted in Fig. 6,

$$\mathbf{S}_{\boldsymbol{r}} = \mathbf{S}_{\boldsymbol{p}},$$

and, with the notation $\mathbf{S}_{\boldsymbol{r}} = \mathbf{S}_{\boldsymbol{p}} = \mathbf{S}_{\mathbf{rp}}$, from Eqns. (39) and (40) it follows that

$$\mathbf{C} = \begin{bmatrix} \tilde{c}_d/\nu\, [\mathbf{S}_{\mathbf{rp}} + \nu \mathbf{I}] & \mathbf{0} \\ \mathbf{0} & \tilde{c}_r \lambda\, [\mathbf{S}_{\mathbf{rp}} + \lambda \mathbf{I}]^{-1} \end{bmatrix} \,, \tag{46}$$

$$\mathbf{K} = \begin{bmatrix} \nu/\tilde{c}_d\, [\mathbf{S}_{\mathbf{rp}} + \nu \mathbf{I}]^{-1} & \mathbf{0} \\ \mathbf{0} & 1/(\tilde{c}_r \lambda)\, [\mathbf{S}_{\mathbf{rp}} + \lambda \mathbf{I}] \end{bmatrix} \,, \tag{47}$$

respectively.

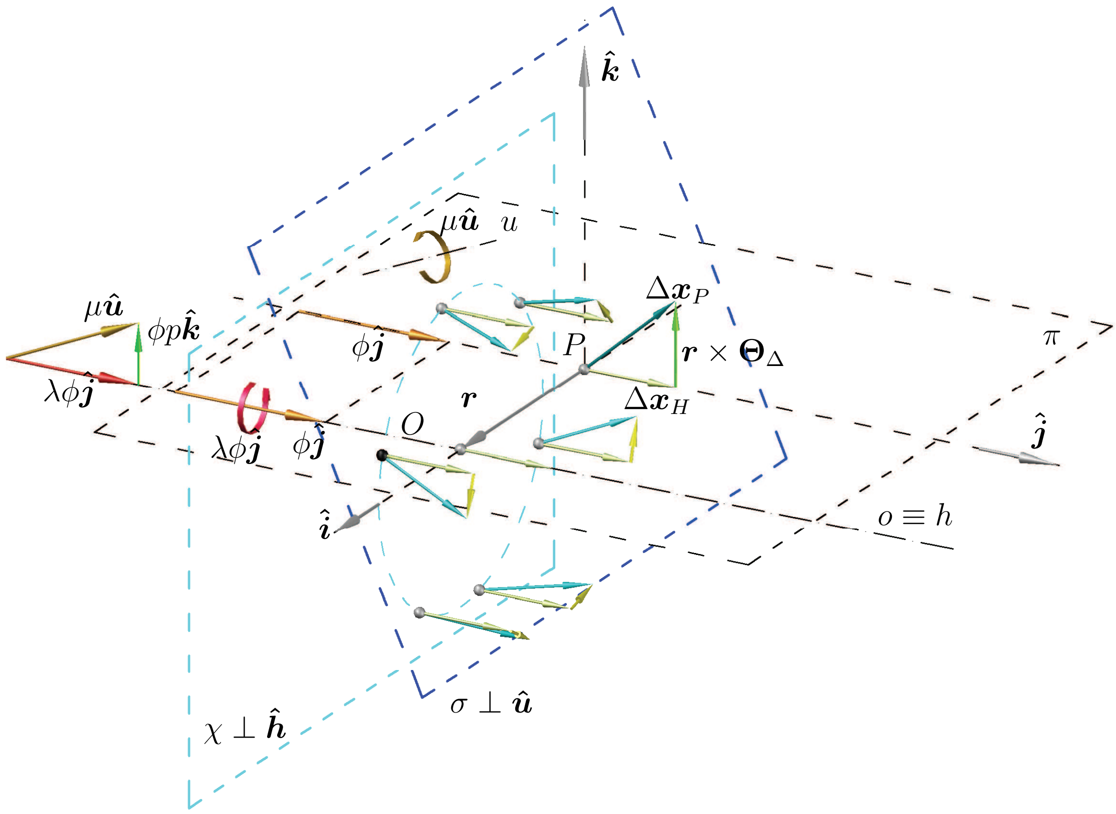


Figure 6: Screw-A isotropic compliance. Wrench $\boldsymbol{w}$ and its consequent screw displacement $\boldsymbol{S}_\Delta$ when the wrench and twist screw axes o and h are coincident, and so $\hat{\boldsymbol{j}}\equiv\hat{\boldsymbol{h}}$. Furthermore, points O and H are coincident, as are vectors $\boldsymbol{r}$ and $\boldsymbol{p}$

7.3.2. Screw-B isotropic compliance

If wrench and twist screw axes are coincident and pass through the end-effector tip point P, as depicted in Fig. 7, then

$$\mathbf{S}_{\boldsymbol{r}}=\mathbf{S}_{\boldsymbol{p}}=\mathbf{0},$$

and, from Eqns. (46) and (47), it follows that

$$\mathbf{C}=\begin{bmatrix}\tilde{c}_d\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \tilde{c}_r\mathbf{I}\end{bmatrix}, \tag{48}$$

$$\mathbf{K}=\begin{bmatrix}{}^{1}/_{\tilde{c}_d}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & {}^{1}/_{\tilde{c}_r}\mathbf{I}\end{bmatrix}, \tag{49}$$

respectively.

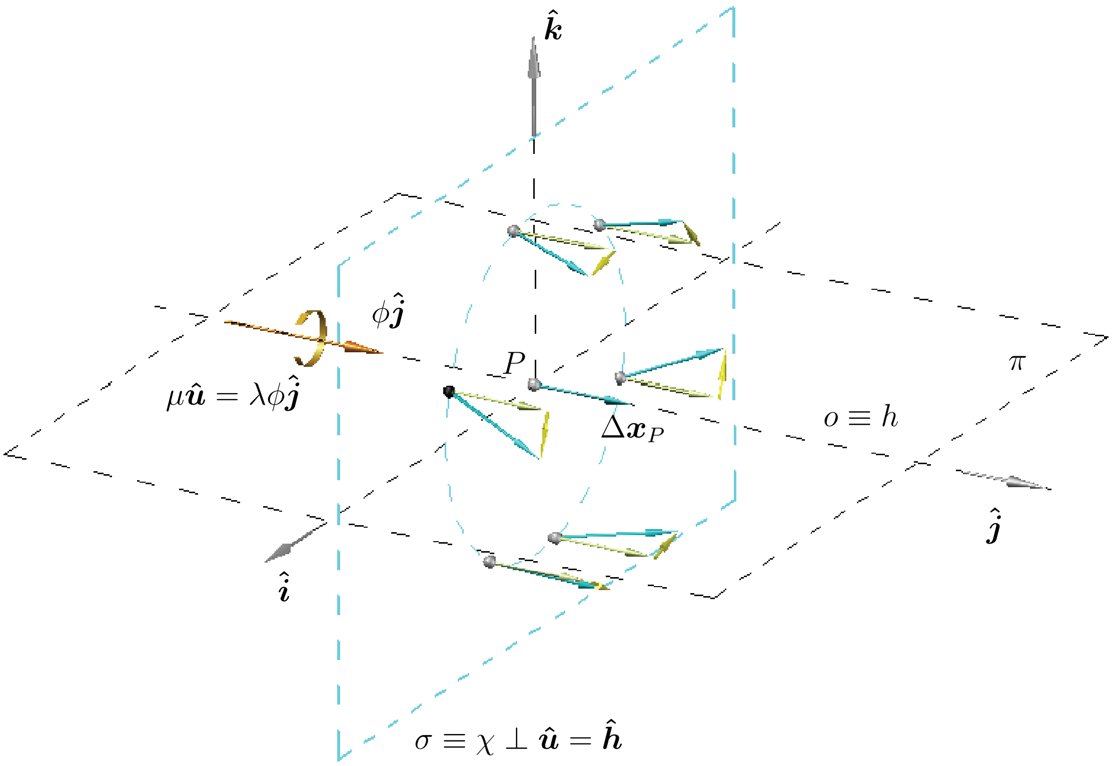


Figure 7: Screw-B isotropic compliance. Wrench $\boldsymbol{w}$ and its consequent screw displacement $\boldsymbol{S}_\Delta$ when the wrench and twist screw axes o and h are coincident and passing through P. Since $\boldsymbol{r} = \boldsymbol{p} = \boldsymbol{0}$, points O, H, and P are coincident

8. Isotropic compliance in brief

In this Section, the main characteristics of the different kinds of isotropic compliance are briefly described, focusing on the conditions of parallelism between the screw axes, the tip displacement vector, the change of orientation vector, the applied force and moment vectors.

- *isotropic compliance* in $E(3)$: the force $\phi\hat{\boldsymbol{\jmath}}$ is parallel to the tip point displacement $\Delta P\hat{\boldsymbol{v}}$, but, in general, the moment $\mu\hat{\boldsymbol{u}}$ is not parallel to the rotation axis $\hat{\boldsymbol{h}}$. Furthermore, neither the force $\phi\hat{\boldsymbol{\jmath}}$ is parallel to the moment $\mu\hat{\boldsymbol{u}}$, nor the tip point displacement $\Delta P\hat{\boldsymbol{v}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$.

- *local* in $SE(3)$: the wrench and twist screw axes, o and h, respectively, are skew lines ($\hat{\boldsymbol{h}} \nparallel \hat{\boldsymbol{\jmath}}$) not passing through the contact point P. The force $\phi\hat{\boldsymbol{\jmath}}$ is parallel to the tip point displacement $\Delta P\hat{\boldsymbol{v}}$, and the moment $\mu\hat{\boldsymbol{u}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$. In general, neither the force $\phi\hat{\boldsymbol{j}}$ is parallel to the moment $\mu\hat{\boldsymbol{u}}$, nor the tip point displacement $\Delta P\hat{\boldsymbol{v}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$.

- *screw* in $SE(3)$: wrench and twist screw axes, o and h, respectively, are parallel lines ($\hat{\boldsymbol{h}} \equiv \hat{\boldsymbol{j}}$), both of them not passing through the contact point P. The force $\phi\hat{\boldsymbol{j}}$ is not parallel to the tip point displacement $\Delta P\hat{\boldsymbol{v}}$, and the moment $\mu\hat{\boldsymbol{u}}$ is not parallel to the rotation axis $\hat{\boldsymbol{h}}$. Neither the force $\phi\hat{\boldsymbol{j}}$ is parallel to the moment $\mu\hat{\boldsymbol{u}}$, nor the tip point displacement $\Delta P\hat{\boldsymbol{v}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$.

- *screw-A* (task screw) in $SE(3)$: wrench and twist screw axes, o and h, respectively, are coincident ($o \equiv h$), and not passing through the contact point P. The force $\phi\hat{\boldsymbol{j}}$ is not parallel to the tip point displacement $\Delta P\hat{\boldsymbol{v}}$, and the moment $\mu\hat{\boldsymbol{u}}$ is not parallel to the rotation axis $\hat{\boldsymbol{h}}$. Neither the force $\phi\hat{\boldsymbol{j}}$ is parallel to the moment $\mu\hat{\boldsymbol{u}}$, nor the tip point displacement $\Delta P\hat{\boldsymbol{v}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$.

- *screw-B* (tool screw) in $SE(3)$: wrench and twist screw axes, o and h, respectively, are coincident ($o \equiv h$), and passing through the contact point P. The force $\phi\hat{\boldsymbol{j}}$ is parallel to the tip point displacement $\Delta P\hat{\boldsymbol{v}}$, and the moment $\mu\hat{\boldsymbol{u}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$. Furthermore, since $\boldsymbol{p} = \boldsymbol{r} = \boldsymbol{0}$, the force $\phi\hat{\boldsymbol{j}}$ is parallel to the moment $\mu\hat{\boldsymbol{u}}$, and the tip point displacement $\Delta P\hat{\boldsymbol{v}}$ is parallel to the rotation axis $\hat{\boldsymbol{h}}$.

The previous characteristics are summarized in Table 1.

9. The coupled control strategy

To achieve local or screw isotropic compliance, a coupled control strategy can be considered, based on active stiffness regulation. A coupled, active stiffness matrix may overcome the limitations of the diagonal, passive joint stiffness matrix. For example, in Ref. [33], variable stiffness actuation (VSA) technology was used in combination with an active impedance controller to extend the achievable Cartesian stiffness range.

Table 1: Comparison of generalized isotropic compliance

	Isotropic Compliance	o, h	P, o	P, h	$\hat{j}$, $\hat{v}$	$\hat{u}$, $\hat{h}$	$\hat{j}$, $\hat{u}$	$\hat{v}$, $\hat{h}$
$E(3)$		$\nparallel$	$\notin$	$\notin$	$\parallel$	$\nparallel$	$\nparallel$	$\nparallel$
$SE(3)$	Local	$\nparallel$	$\notin$	$\notin$	$\parallel$	$\parallel$	$\nparallel$	$\nparallel$
	Screw	$\parallel, \not\equiv$	$\notin$	$\notin$	$\nparallel$	$\nparallel$	$\nparallel$	$\nparallel$
	Screw-A	$\parallel, \equiv$	$\notin$	$\notin$	$\nparallel$	$\nparallel$	$\nparallel$	$\nparallel$
	Screw-B	$\parallel, \equiv$	$\in$	$\in$	$\parallel$	$\parallel$	$\parallel$	$\parallel$

Screw-A: wrench and twist screw axis coincident

Screw-B: wrench and twist screw axis coincident and passing through P

In this Section, the control system is considered in two different arrangements. In fact, active stiffness can work in parallel or in series with the passive stiffness, depending on the mechanism under consideration [19]. The two arrangements are here considered separately, focusing on the stiffness matrix in the parallel case, and on the compliance matrix in the serial arrangement. The control stiffness or compliance matrix are determined for all the cases.

Partitioning the passive compliance matrix in the form

$$\mathbf{C_p} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{D}^{\mathrm{T}} & \mathbf{B} \end{bmatrix} , \tag{50}$$

the passive stiffness matrix can be written as

$$\mathbf{K_p} = \begin{bmatrix} \mathbf{E} & \mathbf{G} \\ \mathbf{G}^{\mathrm{T}} & \mathbf{H} \end{bmatrix} , \tag{51}$$

where

$$\mathbf{E} = \left(\mathbf{A} - \mathbf{D}\mathbf{B}^{-1}\mathbf{D}^{\mathrm{T}}\right)^{-1} , \tag{52}$$

$$\mathbf{G} = -\left(\mathbf{A} - \mathbf{D}\mathbf{B}^{-1}\mathbf{D}^{\mathrm{T}}\right)^{-1}\mathbf{D}\mathbf{B}^{-1} , \tag{53}$$

$$\mathbf{H} = \left(\mathbf{B} - \mathbf{D}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{D}\right)^{-1} . \tag{54}$$

The control system is designed to transform the Cartesian stiffness or compliance matrix in a diagonal matrix, exerting two simultaneous actions:

- it decouples the effects of forces and moments on the end-effector displacement and rotation, by setting equal to zero the elements of the off-diagonal blocks;
- it achieves the isotropic compliance condition by separately transforming the diagonal blocks in particular submatrices, whose form depends on the type of isotropic compliance.

9.1. Passive and Active Stiffness in Parallel Configuration

Considering the spectral decomposition of the diagonal blocks $\mathbf{E}$ and $\mathbf{H}$, the passive stiffness matrix can be written as

$$\mathbf{K_p} = \begin{bmatrix} \mathbf{U_E}\boldsymbol{\Sigma}_\mathbf{E}\mathbf{U_E}^{\mathrm{T}} & \mathbf{G} \\ \mathbf{G}^{\mathrm{T}} & \mathbf{U_H}\boldsymbol{\Sigma}_\mathbf{H}\mathbf{U_H}^{\mathrm{T}} \end{bmatrix}, \tag{55}$$

where the columns of $\mathbf{U_E}$, $\mathbf{U_H}$ form a set of orthonormal eigenvectors of $\mathbf{E}$ and $\mathbf{H}$, respectively, and $\boldsymbol{\Sigma}_\mathbf{E}$, $\boldsymbol{\Sigma}_\mathbf{H}$ are diagonal matrices having the eigenvalues of $\mathbf{E}$ and $\mathbf{H}$ on the diagonal, respectively.

In case of parallel configuration, the control stiffness matrix is added to the passive stiffness matrix. To achieve the isotropic compliance condition, the control system must exert a generalized force in the Cartesian space that decouples the effects of forces and moments on the end-effector displacement and rotation, and attains both the parallelism between the force and the displacement and the rotation about the axis of the external moment, namely,

$$\begin{Bmatrix} \boldsymbol{\Delta f_c} \\ \boldsymbol{\Delta M_c} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Delta}\mathbf{E} & -\mathbf{G} \\ -\mathbf{G}^{\mathrm{T}} & \boldsymbol{\Delta}\mathbf{H} \end{bmatrix} \begin{Bmatrix} \Delta P \hat{\boldsymbol{v}} \\ \Theta_\Delta \hat{\boldsymbol{h}}\,, \end{Bmatrix} \tag{56}$$

with

$$\boldsymbol{\Delta}\mathbf{E} = \mathbf{U_E}\left(\boldsymbol{\Gamma}_\mathbf{E} - \boldsymbol{\Sigma}_\mathbf{E}\right)\mathbf{U_E}^{\mathrm{T}}, \tag{57}$$

$$\boldsymbol{\Delta}\mathbf{H} = \mathbf{U_H}\left(\boldsymbol{\Gamma}_\mathbf{H} - \boldsymbol{\Sigma}_\mathbf{H}\right)\mathbf{U_H}^{\mathrm{T}}. \tag{58}$$

The structure of matrices $\boldsymbol{\Gamma}_\mathbf{E}$ and $\boldsymbol{\Gamma}_\mathbf{H}$ depends on the isotropic compliance property to be achieved.

9.1.1. Local isotropic compliance

$$\mathbf{\Gamma_E} = \frac{1}{\tilde{c}_d}\mathbf{I}, \tag{59}$$

$$\mathbf{\Gamma_H} = \frac{1}{\tilde{c}_r}\mathbf{I}, \tag{60}$$

9.1.2. Screw isotropic compliance

$$\mathbf{\Gamma_E} = \mathbf{U_E}^{\mathrm{T}} \left(\frac{\nu}{\tilde{c}_d} \left[\mathbf{S}\boldsymbol{r} + \nu \mathbf{I} \right]^{-1} \right) \mathbf{U_E}, \tag{61}$$

$$\mathbf{\Gamma_H} = \mathbf{U_H}^{\mathrm{T}} \left(\frac{1}{\tilde{c}_r \lambda} \left[\mathbf{S}\boldsymbol{p} + \lambda \mathbf{I} \right] \right) \mathbf{U_H}. \tag{62}$$

Furthermore, in this case, two particular conditions can be achieved.

- Screw-A isotropic compliance (twist and wrench screw axes coincident):

$$\mathbf{\Gamma_E} = \mathbf{U_E}^{\mathrm{T}} \left(\frac{\nu}{\tilde{c}_d} \left[\mathbf{S_{rp}} + \nu \mathbf{I} \right]^{-1} \right) \mathbf{U_E}, \tag{63}$$

$$\mathbf{\Gamma_H} = \mathbf{U_H}^{\mathrm{T}} \left(\frac{1}{\tilde{c}_r \lambda} \left[\mathbf{S_{rp}} + \lambda \mathbf{I} \right] \right) \mathbf{U_H}. \tag{64}$$

- Screw-B isotropic compliance (twist and wrench screw axes coincident and passing through P):

$$\mathbf{\Gamma_E} = \frac{1}{\tilde{c}_d}\mathbf{I}, \tag{65}$$

$$\mathbf{\Gamma_H} = \frac{1}{\tilde{c}_r}\mathbf{I}. \tag{66}$$

The action of the control force is then translated into the addition of a control matrix that sets equal to zero the off-diagonal block elements of the passive stiffness matrix, and transforms the diagonal blocks in scalar submatrices, that is

$$\mathbf{K_c} = \begin{bmatrix} \mathbf{\Delta E} & -\mathbf{G} \\ -\mathbf{G}^{\mathrm{T}} & \mathbf{\Delta H} \end{bmatrix} . \tag{67}$$

Then, the overall stiffness matrix is

$$\mathbf{K} = \mathbf{K_p} + \mathbf{K_c} = \begin{bmatrix} \tilde{k}_d\mathbf{I} & 0 \\ 0 & \tilde{k}_r\mathbf{I} \end{bmatrix}, \tag{68}$$

and the overall compliance matrix is

$$\mathbf{C} = \begin{bmatrix} \tilde{c}_d\mathbf{I} & 0 \\ 0 & \tilde{c}_r\mathbf{I} \end{bmatrix}. \tag{69}$$

From Eqn. (67), the control stiffness matrix in the joint space can be evaluated by means of Eqn. (3), in terms of the control matrices $\mathbf{k_c}$ and $\mathbf{K_c}$:

$$\mathbf{k_c} = \mathbf{J}^{\mathrm{T}}\mathbf{K_c}\mathbf{J}. \tag{70}$$

9.2. Passive and Active Stiffness in Serial Configuration

In case of serial configuration, the inverse of the overall stiffness matrix is equal to the sum of the inverse of the passive stiffness matrix and the inverse of the control stiffness matrix. Hence, the passive compliance matrix can be considered, in terms of the spectral decomposition of its diagonal blocks,

$$\mathbf{C_p} = \begin{bmatrix} \mathbf{U_A}\mathbf{\Sigma_A}\mathbf{U_A}^{\mathrm{T}} & \mathbf{D} \\ \mathbf{D}^{\mathrm{T}} & \mathbf{U_B}\mathbf{\Sigma_B}\mathbf{U_B}^{\mathrm{T}} \end{bmatrix}, \tag{71}$$

where the columns of $\mathbf{U_A}$, $\mathbf{U_B}$ form a set of orthonormal eigenvectors of $\mathbf{A}$ and $\mathbf{B}$, respectively, and $\mathbf{\Sigma_A}$, $\mathbf{\Sigma_B}$ are diagonal matrices having the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ on the diagonal, respectively.

To achieve the isotropic compliance condition, the control system must exert a generalized force in the Cartesian space that decouples the force and moment effects on the end-effector displacement and rotation, and attain both the parallelism between the force and the displacement and the rotation about the axis of the external moment, namely,

$$\begin{Bmatrix} \Delta P\hat{\boldsymbol{v}} \\ \Theta_\Delta\hat{\boldsymbol{h}} \end{Bmatrix} = \begin{bmatrix} \mathbf{\Delta A} & -\mathbf{D} \\ -\mathbf{D}^{\mathrm{T}} & \mathbf{\Delta B} \end{bmatrix} \begin{Bmatrix} \mathbf{\Delta}\boldsymbol{f_c} \\ \mathbf{\Delta}\boldsymbol{M_c} \end{Bmatrix}, \tag{72}$$

with

$$\mathbf{\Delta A} = \mathbf{U_A}\left(\mathbf{\Lambda_A} - \mathbf{\Sigma_A}\right)\mathbf{U_A}^{\mathrm{T}}, \tag{73}$$

$$\mathbf{\Delta B} = \mathbf{U_B}\left(\mathbf{\Lambda_B} - \mathbf{\Sigma_B}\right)\mathbf{U_B}^{\mathrm{T}}. \tag{74}$$

The structure of matrices $\mathbf{\Lambda_A}$ and $\mathbf{\Lambda_B}$ depends on the isotropic compliance property to be achieved.

9.2.1. Local isotropic compliance

$$\mathbf{\Lambda_A} = \tilde{c}_d\mathbf{I}, \tag{75}$$

$$\mathbf{\Lambda_B} = \tilde{c}_r\mathbf{I}, \tag{76}$$

9.2.2. Screw isotropic compliance

$$\mathbf{\Lambda_A} = \mathbf{U_E}^{\mathrm{T}}\left(\frac{\tilde{c}_d}{\nu}\left[\mathbf{S}\boldsymbol{r} + \nu\mathbf{I}\right]\right)\mathbf{U_E}, \tag{77}$$

$$\mathbf{\Lambda_B} = \mathbf{U_H}^{\mathrm{T}}\left(\tilde{c}_r\lambda\left[\mathbf{S}\boldsymbol{p} + \lambda\mathbf{I}\right]^{-1}\right)\mathbf{U_H}\,. \tag{78}$$

The particular screw cases are the following.

- Screw-A isotropic compliance (twist and wrench screw axes coincident):

$$\mathbf{\Lambda_A} = \mathbf{U_E}^{\mathrm{T}}\left(\frac{\tilde{c}_d}{\nu}\left[\mathbf{S_{rp}} + \nu\mathbf{I}\right]\right)\mathbf{U_E}, \tag{79}$$

$$\mathbf{\Lambda_B} = \mathbf{U_H}^{\mathrm{T}}\left(\tilde{c}_r\lambda\left[\mathbf{S_{rp}} + \lambda\mathbf{I}\right]^{-1}\right)\mathbf{U_H}\,. \tag{80}$$

- Screw-B isotropic compliance (twist and wrench screw axes coincident and passing through P):

$$\mathbf{\Lambda_A} = \tilde{c}_d\mathbf{I}\,, \tag{81}$$

$$\mathbf{\Lambda_B} = \tilde{c}_r\mathbf{I}\,. \tag{82}$$

The action of the control force is then translated into the addition of a control matrix that sets equal to zero the off-diagonal block elements of the passive compliance matrix and transforms the diagonal blocks in scalar submatrices, that is

$$\mathbf{C_c} = \begin{bmatrix} \mathbf{\Delta A} & -\mathbf{D} \\ -\mathbf{D}^{\mathrm{T}} & \mathbf{\Delta B} \end{bmatrix}. \tag{83}$$

Consequently, the overall compliance matrix,

$$\mathbf{C} = \mathbf{C_p} + \mathbf{C_c}, \tag{84}$$

assumes the form given by Eqn. (69). Once the control compliance matrix in the Cartesian space is determined with Eqn. (83), the control stiffness matrix in the joint space can be evaluated by means of Eqn. (70), assuming $\mathbf{K_c} = \mathbf{C_c}^{-1}$.

10. Algorithm

In order to illustrate how the isotropic compliance property can be achieved, it is possible to consider a serial manipulator in a posture, possibly not near to a kinematic singularity, defined by means of the joint coordinate vector $\boldsymbol{q}$. The following operations can be performed.

1. Assign the tip compliance by means of the coefficients $\tilde{c}_d$, $\tilde{c}_r$;
2. evaluate the Jacobian matrix and the compliance matrix in the Cartesian space with Eqn. (5);

in case of parallel arrangement:

3p) calculate the matrices $\mathbf{\Gamma_E}$ and $\mathbf{\Gamma_H}$;

4p) evaluate the stiffness matrix $\mathbf{K_p}$ in the Cartesian space, Eqn. (51) and its submatrices $\mathbf{E}$, $\mathbf{G}$, $\mathbf{H}$, Eqns. (52), (53), (54), respectively;

5p) evaluate the control stiffness matrix $\mathbf{K_c}$ in the Cartesian space, Eqn. (67);

in case of serial arrangement:

3s) evaluate the compliance matrix submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}$;

4s) evaluate the control compliance matrix $\mathbf{C_c}$ in the Cartesian space, Eqn. (83);

5s) evaluate the control stiffness matrix in the Cartesian space, $\mathbf{K_c} = \mathbf{C_c}^{-1}$;

6) evaluate the control stiffness matrix $\mathbf{k_c}$ in the joint space, Eqn. (70).

10.1. Singular configuration

The isotropic compliance property cannot be achieved whenever the posture of the manipulator corresponds to a kinematic singularity, i.e. when the Jacobian matrix $\boldsymbol{J}$ is singular. In that case, the passive stiffness along the eigenvectors in the nullspace of $\boldsymbol{J}$ would be infinite, whereas the control authority in that same direction would be zero.

11. Example of application

In this section an application of the proposed algorithm to a particular PUMA manipulator with parallel active and passive stiffness arrangement is presented. The purpose is showing how local isotropic compliance property in $SE\,(3)$ can be achieved for the analyzed robot by using coupled active control. The manipulator under study is depicted in Fig. 8a in its reference posture ($\boldsymbol{q} = \mathbf{0}$). Table 2 lists the Denavit-Hartenberg parameters and the joint passive stiffness coefficients $k_{p1}, \ldots, k_{p6}$. The perturbation load is applied to a point of the end-effector which is located in correspondence of the origin of joint 6, as shown in Table 2 and in Fig. 8.

11.1. Control matrix computation

The posture selected in this example is defined by the joint coordinates vector $\boldsymbol{q_e} = [0, 15, -40, 30, 44, 66]\,\mathrm{deg}$, as illustrated in Fig. 8b. Following the algorithm presented in Section 10, the compliance of the end-effector is chosen by imposing the coefficients (see

Table 2: PUMA manipulator: Denavit-Hartenberg parameters and joint passive stiffness coefficients

Joint	d (m)	a (m)	α (deg)	k_p (Nm/rad)
1	0	0	90	550
2	0	0.4318	0	470
3	0.15	0.0203	-90	630
4	0.4318	0	90	770
5	0	0	-90	660
6	0	0.2	0	530

Eqn. (10))

$$\begin{aligned} \tilde{c}_d &= 2.0 \cdot 10^{-3}\,\mathrm{mN}^{-1}\,, \\ \tilde{c}_r &= 1.7 \cdot 10^{-3}\,\mathrm{N}^{-1}\mathrm{m}^{-1}\mathrm{rad}\,, \end{aligned} \tag{85}$$

and by defining the matrices $\mathbf{\Gamma_E}$ and $\mathbf{\Gamma_H}$ accordingly (see Eqs. (59) and (60), respectively). Once the Jacobian matrix is calculated in the assigned posture,

$$\mathbf{J}(\boldsymbol{q_e}) = \begin{bmatrix} -0.0375 & -0.5629 & -0.4512 & -0.1699 & -0.0196 & -0.1937 \\ 0.6050 & 0.0000 & 0.0000 & -0.0407 & -0.0283 & 0.0047 \\ 0.0000 & 0.6050 & 0.1879 & 0.0792 & 0.0737 & -0.0497 \\ 0.0000 & 0.0000 & 0.0000 & 0.4226 & 0.4532 & -0.2412 \\ 0.0000 & -1.0000 & -1.0000 & -0.0000 & -0.8660 & -0.3473 \\ 1.0000 & 0.0000 & 0.0000 & 0.9063 & -0.2113 & 0.9062 \end{bmatrix}, \tag{86}$$

the passive compliance matrix in the Cartesian space is given by $\mathbf{C_p} = \mathbf{J}\mathbf{k_p}^{-1}\mathbf{J}^{\mathrm{T}}$, and the passive stiffness matrix can be obtained by means of Eq. (51):

$$\mathbf{K_p} = \begin{bmatrix} 22809.98 & -4897.07 & 14565.55 & -5086.14 & -4716.60 & 3817.78 \\ -4897.07 & 3496.69 & -3364.11 & 1161.45 & 888.50 & -1409.03 \\ 14565.55 & -3364.11 & 12491.22 & -3113.42 & -1953.61 & 2581.29 \\ -5086.14 & 1161.45 & -3113.42 & 2771.94 & 1236.83 & -832.45 \\ -4716.60 & 888.50 & -1953.61 & 1236.83 & 1537.31 & -714.35 \\ 3817.78 & -1409.03 & 2581.29 & -832.45 & -714.35 & 1007.12 \end{bmatrix}. \tag{87}$$

Hence, by making use of matrices $\mathbf{K_p}$, $\mathbf{\Gamma_E}$, and $\mathbf{\Gamma_H}$, the control stiffness matrix can be obtained by means of Eqs. (57), (58), and (67):

$$\mathbf{K_c} = \begin{bmatrix} -22309.98 & 4897.07 & -14565.55 & 5086.14 & 4716.60 & -3817.78 \\ 4897.07 & -2996.69 & 3364.11 & -1161.45 & -888.50 & 1409.03 \\ -14565.55 & 3364.11 & -11991.22 & 3113.42 & 1953.61 & -2581.29 \\ 5086.14 & -1161.45 & 3113.42 & -2183.70 & -1236.83 & 832.45 \\ 4716.60 & -888.50 & 1953.61 & -1236.83 & -949.07 & 714.35 \\ -3817.78 & 1409.03 & -2581.29 & 832.45 & 714.35 & -418.89 \end{bmatrix}. \quad (88)$$

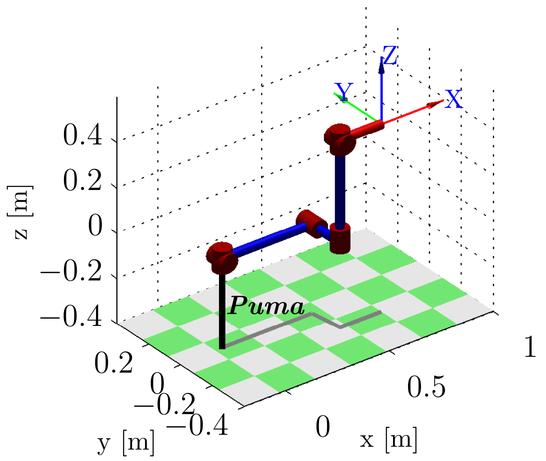
Finally, the control stiffness matrix in the joint space for the parallel case can be obtained by means of Eq. (3):

$$\mathbf{k_c} = \begin{bmatrix} 221.94 & 10.55 & 8.46 & 524.00 & -132.48 & 538.11 \\ 10.55 & 459.68 & 772.06 & 71.80 & 537.25 & 243.77 \\ 8.46 & 772.06 & 77.67 & 45.78 & 520.78 & 243.33 \\ 524.00 & 71.80 & 45.78 & -163.36 & 5.16 & 437.53 \\ -132.48 & 537.25 & 520.78 & 5.16 & -68.46 & 0.00 \\ 538.11 & 243.77 & 243.33 & 437.53 & 0.00 & 78.24 \end{bmatrix}. \quad (89)$$

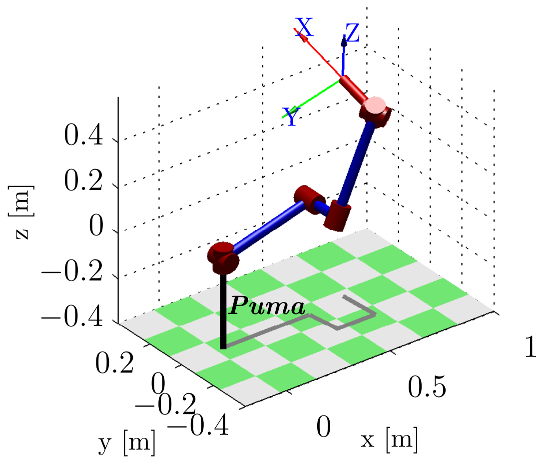
11.2. Verification by Multibody Simulation

The system response has been verified by means of a dynamic simulation performed using MBDyn, a free general-purpose multibody solver developed at Politecnico di Milano [34, 35]. The multibody model includes six rigid bodies and six revolute joints. The constitutive law of each joint is characterized by the nominal torsional stiffness k_p, listed in Table 2, and by an additional damping term, arbitrarily set to $10^{-3} \cdot k_p$ Nms/rad, that stabilizes the analysis. Arbitrary (although realistic) inertia properties have been given to each link. The values are not relevant in this context, since only quasi-static problems have been addressed by gently applying the loads.

A sequence of three independent forces and three independent moments is applied to the end-effector. Each force and moment is directed along one of the coordinate axes; the magnitude of each force and moment is equal to 0.2 N and 0.5 Nm, respectively. These



(a)



(b)

Figure 8: PUMA manipulator in: (a) posture defined by $\boldsymbol{q} = \mathbf{0}$; (b) posture defined in $\boldsymbol{q_e}$.

values have been chosen to produce a configuration change small enough to neglect kinematic nonlinearities in the perturbed solution. Each force and moment is slowly increased following a regular pattern, then kept constant for some time, and subsequently slowly decreased to zero. The small applied loads induce correspondingly small changes of configuration; thus, the feedback gain matrix, Eq. (89), can remain constant.

The results presented in Fig. 9 show the end-effector loads, the corresponding displacement and the finite rotation vectors, $\boldsymbol{u}$ and $\boldsymbol{\Theta}$, computed with and without the control action. It is clear that when the feedback action is present, the solution provides almost perfectly the sought local isotropic compliance. Figure 10 shows, with more accuracy, the first 3 seconds of simulation, which refers to the loading cycle along the x axis. The response obtained when the end-effector is loaded with a force in the other directions, or with a moment, are qualitatively identical. The small deviations from the isotropic compliance behavior during the loading and unloading phases of the end-effector are due to the viscous dissipation in the joints, which was fictitiously added to stabilize the computation.

The passive stiffness matrix $\mathbf{K}_p$ in the Cartesian coordinates space has been indirectly verified by independently perturbing each component of the pose of the end-effector in the Cartesian space while constraining the remaining ones, and measuring the resulting reaction forces and moments. The matrix has been found in good agreement with the corresponding analytical values, within the desired accuracy. This may be very practical and helpful a procedure for the experimental determination of the control stiffness matrix in the joint space.

12. Conclusions

In this work, the concept of *isotropic compliance* has been extended from the Euclidean Space $E(3)$ to the Special Euclidean Group $SE(3)$. Such extension, which appears for the first time in literature, has required the adoption of screw vector notation, which has been useful to ascertain that in $SE(3)$ the concept of *isotropic compliance* is not as simple as in $E(3)$. In fact, four different types of such property had to be introduced to cope with the variety of all the possible practical applications, depending on the required force-displacement-moment-rotation parallelism in the Euclidean Space and on the position of the instantaneous rotation axis, with respect to the force action line and application point. They have been termed *local*, *screw*, *task screw* and *tool screw* isotropic compliance. It has also been disclosed how coupled active stiffness regulation can be used to achieve the four isotropic compliance conditions. The actuators can be embedded in the system in such a way to work either in parallel or as a series with the passive joints stiffness. Finally, the proposed computational procedure has been checked by dynamic simulation on a detailed example.

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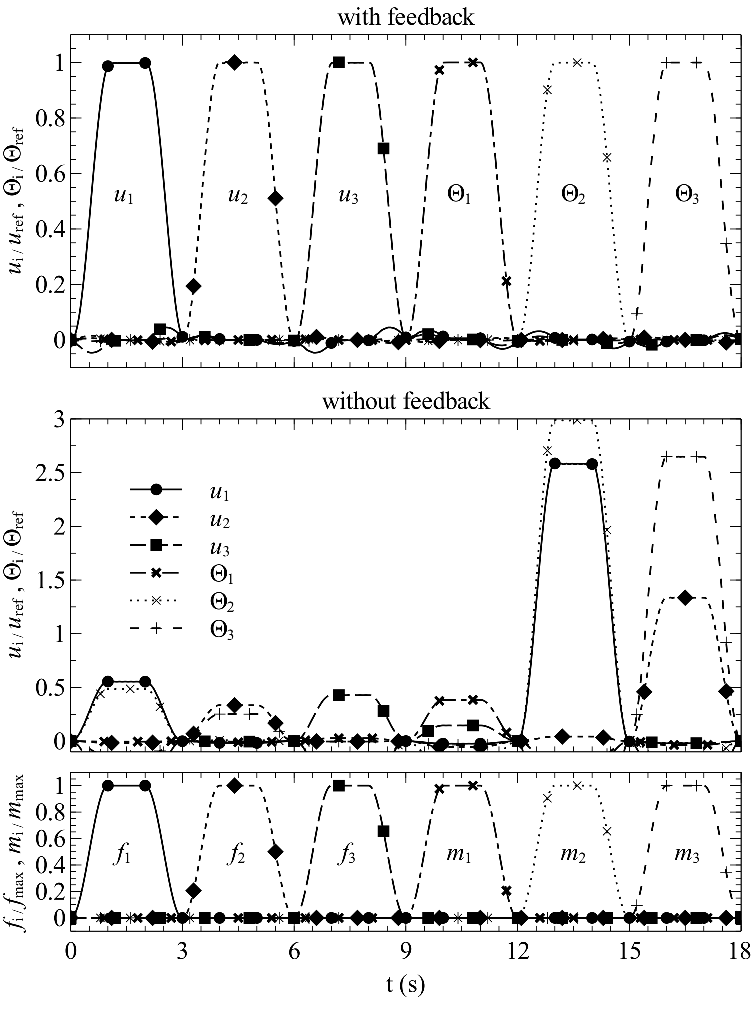

Figure 9: PUMA static response: time histories of the end-effector loads (bottom plot) and of the corresponding displacement and rotation components obtained with and without feedback (upper and middle plot, respectively).

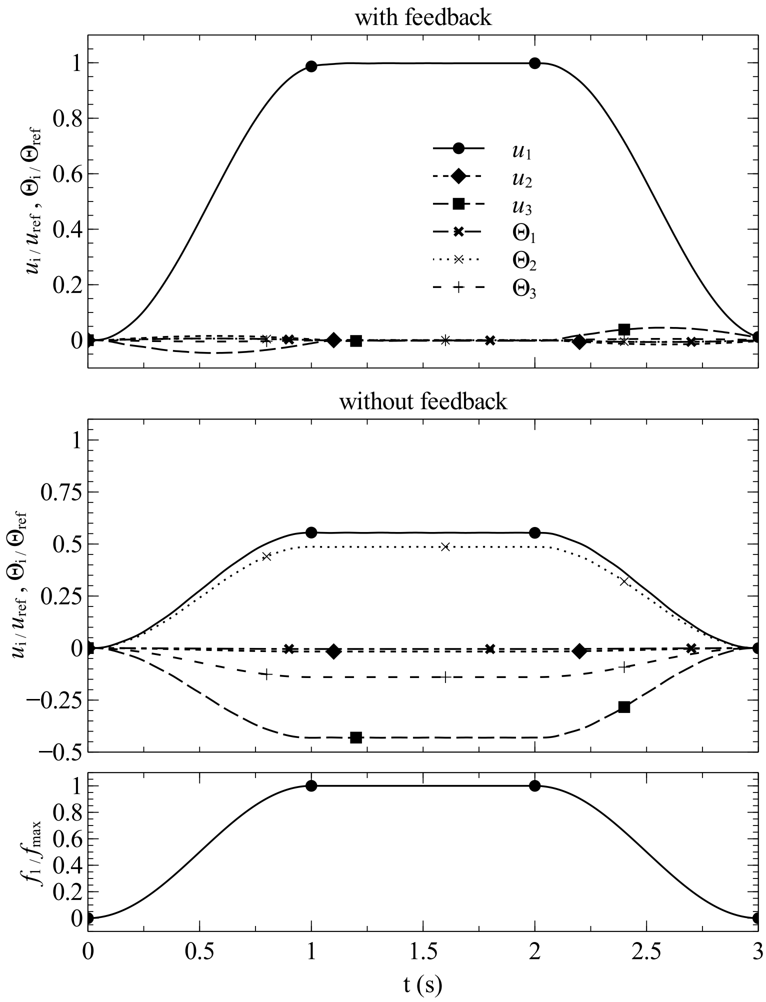

Figure 10: PUMA static response: time histories of end-effector load along the x direction (bottom plot), and of the corresponding displacement and rotation vector components obtained with and without feedback (upper and middle plot, respectively).

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