An SMT-based Approach to Satisfiability Checking of MITL[☆,☆☆]

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Abstract

We present a satisfiability-preserving reduction from MITL interpreted over finitely-variable continuous behaviors to Constraint LTL over clocks, a variant of CLTL that is decidable, and for which an SMT-based bounded satisfiability checker is available. The result is a new complete and effective decision procedure for MITL. Although decision procedures for MITL already exist, the automata-based techniques they employ appear to be very difficult to realize in practice, and, to the best of our knowledge, no implementation currently exists for them. A prototype tool for MITL based on the encoding presented here has, instead, been implemented and is publicly available.

1. Introduction

Computer systems are inherently discrete-time objects, but their application to real-time control and monitoring often requires to deal with external asynchronous events that may not always happen at integer-valued times. Hence, a discrete-time assumption requires to approximate continuous time by choosing some fixed minimal interval, thus limiting the accuracy of modeling, verification and validation of such systems. To overcome this restriction, many continuous-time models have been developed, most notably Timed Automata [4], a dense-time operational model based on finite-state machines, but also descriptive models such as the continuous-time temporal logics MTL (Metric Temporal Logic) [5, 6] and MITL (Metric Interval Temporal Logic) [6]. In general, the role of temporal logics in verification and validation is two-fold. First, temporal logic allows abstract, concise and convenient expression of required properties of

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a system. Linear Temporal Logic (LTL) is often used with this goal in the verification of finite-state models, e.g., in model checking [7]. Second, temporal logic allows a descriptive approach to specification and modeling (see, e.g., [8, 9]). A descriptive model is based on axioms, written in some (temporal) logic, defining a system by means of its general properties, rather than by an operational model based on some kind of machine (e.g., a Timed Automaton) behaving in the desired way. In this case, verification typically consists of satisfiability checking of the conjunction of the model and of the (negation of) its desired properties. An example of the latter approach is Bounded Satisfiability Checking (BSC) [10], where MTL specifications and properties on *discrete* time are translated into Boolean logic, in an approach similar to Bounded Model Checking of LTL properties of finite-state machines.

In general, verification of continuous-time temporal logics is not as well supported as for discrete-time. Uppaal [11] is the de-facto standard tool for verification of Timed Automata, but its query language, a simplification of TCTL, falls short of being a full continuous-time temporal logics: not only satisfiability checking is not available in Uppaal, but even the formalization of general system properties in temporal logic may not be possible, aside from invariants, reachability and simple liveness and safety properties. Rather, non-trivial properties to be verified on an operational model must be expressed as other Timed Automata, i.e., at a lower level of abstraction. The main technique for the translation of MITL formulae into Timed Automata was first proposed in [6], with a more recent solution in [12]. Alternatively, in [13] it is shown that any formula of MITL can be translated into a formula of Event-Clock Logic (ECL) whose satisfiability is decidable. All these works deal with with the continuous semantics of MITL, based on finitely variable signals. A signal (also called a timed state sequence) is a mapping associating nonnegative real numbers with states. Finite variability is a very common requirement for continuous-time systems, ruling out only pathological behaviors (e.g., Zeno [9]) which do not have much practical interest.

However, to the best of our knowledge, neither proposal has been implemented, raising some doubts over the possibility of their actual application. Both [12] and [6] leverage on a fundamental property of MITL, namely that "Future" formulae $\mathbf{F}_{\langle a,b\rangle}$ may only finitely vary over any interval of length $b-a$. Hence, the delay between change points of Future can be measured by a finite set of clocks, whose cardinality depends on the constants a, b. Recent works by [14, 15] propose an automata-based approach to MITL by a translation into alternating timed automata, which could allow for efficient implementations in the case of the less general pointwise semantics (i.e., when interpreting formulae over timed words rather than over signals).

Rather than going through a translation into Timed Automata, in this paper we propose a new logic-based approach, which still exploits the previous property of the finite number of clocks for Future (and Until) formulae. Our work also assumes the continuous semantics of MITL over finitely-variable models. Our technique is based on generalizing BSC to MITL, by reducing satisfiability of MITL to satisfiability of *Constraint LTL over clocks* (CLTLoc), a decidable

$$M, t \models p \Leftrightarrow p \in M(t) \qquad p \in AP$$
$$M, t \models \neg\phi \Leftrightarrow M, t \not\models \phi$$
$$M, t \models \phi \wedge \psi \Leftrightarrow M, t \models \phi \text{ and } M, t \models \psi$$
$$M, t \models \phi \mathbf{U}_I \psi \Leftrightarrow \exists t' > t,\ t' - t \in I,\ M, t' \models \psi \text{ and } \ \forall t'' \in (t, t')\ M, t'' \models \phi$$

Table 1: Semantics of MTL.

variant of CLTL [16]. CLTLoc allows explicit clocks that, similarly to clocks of Timed Automata, can be compared with integer constants and reset to 0. In particular, an MITL formula is encoded into an equisatisfiable CLTLoc formula, which can be solved through the same techniques presented in [17, 18, 19]. The approach in generalizes BSC to CLTL, generating an encoding suitable for verification with standard Satisfiability Modulo Theories (SMT) solvers such as Z3 [20]. In [21], we show the decidability of CLTLoc and the modifications to the the procedure for CLTL of [19] to deal with clocks and time progress. An open-source prototype tool [22] implements our technique of BSC for MITL.

The paper is organized as follows. Section 2 defines MITL and its relevant variants, and Section 3 defines CLTLoc. Sections 4, 5, 6 and 7 define reductions from MITL and its variants to CLTLoc. Section 8 presents some experimental results with the prototype tool implementing the encodings, which shows the feasibility in practice of our approach. Section 9 concludes.

2. MTL, MITL, $\mathbf{MITL_{0,\infty}}$, past operators and counting modalities

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_{>0}$ the set of positive reals and $\mathbb{R}_+$ the set of nonnegative reals. An *interval* I is a convex subset of $\mathbb{R}_+$ of the form $\langle a, b\rangle$ or $\langle a, \infty)$, where $a \leqslant b$ are nonnegative integers, symbol $\langle$ is either (or [and symbol $\rangle$ is either) or].

Let AP be a finite set of atomic propositions. The syntax of (well-formed) formulae of MTL is defined by the grammar, with $p \in AP$:

$$\phi := \ p \mid \phi \wedge \phi \mid \neg\phi \mid \phi \mathbf{U}_I \phi$$

The *globally* $\mathbf{G}_I$ and *eventually* $\mathbf{F}_I$ modalities can be defined as usual: $\mathbf{F}_I(\phi) = \top \mathbf{U}_I \phi$ and $\mathbf{G}_I(\phi) = \neg \mathbf{F}_I(\neg\phi)$.

The semantics of MTL is defined in Table 1 with respect to a *signal* and a real number. A signal is a function $M : \mathbb{R}_+ \to \wp(AP)$, that throughout the paper is assumed to be *finitely variable* (f.v. for short), i.e., such that in every bounded interval there is a finite number of changes in the value of atomic propositions in AP. An MTL formula ϕ is *(f.v.) satisfiable* if there exists a (f.v.) signal M such that $M, 0 \models \phi$ (in this case, M is called a *model* of ϕ). Note that the semantics of Table 1 uses the "strict" version of the until operator, hence the values of ϕ and ψ in the current instant do not influence the truth of $\phi \mathbf{U}_I \psi$.

Hence, the modalities $\mathbf{U}_{[0,b\rangle}$ and $\mathbf{U}_{[0,\infty)}$ are equivalent, respectively, to $\mathbf{U}_{(0,b\rangle}$ and to $\mathbf{U}_{(0,\infty)}$, for all $b \geqslant 0$. To include also the current instant, we can define $\phi\mathbf{U}^i_I\psi$ as an abbreviation for $\phi \wedge \phi\mathbf{U}_I\psi$.

We denote with MITL [6] the syntactic fragment of MTL such that the intervals of the form $\langle a, b\rangle$, with $a, b \in \mathbb{N}$, are such that $b > a$. We denote with $\text{MITL}_{0,\infty}$ [6] the syntactic fragment of MITL such that the only allowed intervals have the form $\langle a, \infty)$, for $a \geqslant 0$, or the form $(0, b\rangle$, for $b > 0$. Therefore, in $\text{MITL}_{0,\infty}$ bounded intervals with nonzero left end points are prohibited.

MITL can be extended with the "since" $\mathbf{S}_I$ past modality [23], obtaining the language MITL+Past. The definition of $\mathbf{S}_I$ is symmetric to $\mathbf{U}_I$:

$$M, t \models \phi\mathbf{S}_I\psi \Leftrightarrow \exists t' < t,\ t - t' \in I,\ M, t' \models \psi \text{ and } M, t'' \models \phi\ \forall t'' \in (t', t)$$

The *historically* $\mathbf{H}_I$ and *eventually in the past* $\mathbf{P}_I$ operators can be defined symmetrically to their corresponding future modalities:

$\mathbf{P}_I(\phi) = \top\mathbf{S}_I\phi$ and $\mathbf{H}_I(\phi) = \neg\mathbf{P}_I(\neg\phi)$.

The relations among various logics are recalled in the following proposition, assuming, as everywhere in this paper, the continuous semantics (i.e., signals):

Proposition 1.

1. *MITL is as expressive as* $\mathit{MITL}_{0,\infty}$ *[13], but it is exponentially more succinct.*
2. *MITL+Past is strictly more expressive than MITL [24, 25].*
3. *Satisfiability is EXPSPACE-complete for MITL [6] when constants are encoded in binary, and it is PSPACE when the constants are encoded in unary; it is PSPACE-complete for* $\mathit{MITL}_{0,\infty}$ *[6], also in the case of the binary encoding of constants.*

2.1. Normal forms

Define two MTL formulae ϕ, ψ to be equivalent, written $\phi \equiv \psi$, if for every signal M, for every instant $t \geqslant 0$, we have $M, t \models \phi$ if, and only if, $M, t \models \psi$. As in [12, 26], it is convenient to introduce a normal form, where $\mathbf{U}_{(0,\infty)}$ and $\mathbf{F}_I$ (and their past counterparts) are the only temporal modalities, and consider the "metric" until $\mathbf{U}_I$ as derived.

By Lemma 4.1.1.2 of [6], for every MITL formula ϕ there exists an equivalent MITL formula ϕ' that uses only the temporal modalities $\mathbf{U}_{(0,\infty)}$, $\mathbf{F}_{(0,b\rangle}$, with $b > 0$, and $\mathbf{U}_{\langle a,b\rangle}$, with $0 < a < b$. Also, the number of distinct syntactic subformulae of ϕ' is linearly related to the size of ϕ, defined as the number of propositions, Boolean connectives and temporal modalities occurring in ϕ. A symmetrical result clearly holds also for past modalities in an MITL+Past formula. This property, together with the following well-known lemma, makes it possible to confine metric issues only to the operators $\mathbf{F}_I$ and $\mathbf{P}_I$, with I bounded, whose translation into CLTLoc is much simpler than the general case of $\mathbf{U}_I$ and $\mathbf{S}_I$.

Lemma 1. *For all $0 < a < b$, the following equivalences hold for all MTL formulae ϕ, ψ:*

$$(1)\ \phi \mathbf{U}_{[a,b\rangle}\psi \equiv \mathbf{G}_{(0,a)}\left(\phi \mathbf{U}^i_{(0,\infty)}\psi\right) \wedge \mathbf{G}_{(0,a]}\left(\psi \vee \phi \mathbf{U}^i_{(0,\infty)}\psi\right) \wedge \mathbf{F}_{(a,b\rangle}(\psi)$$

$$(2)\ \phi \mathbf{U}_{(a,b\rangle}\psi \equiv \mathbf{G}_{(0,a]}\left(\phi \mathbf{U}^i_{(0,\infty)}\psi\right) \wedge \mathbf{F}_{(a,b\rangle}(\psi)$$

Symmetrical results hold for $\mathbf{S}_{\langle a,b\rangle}$.

The following corollary of Lemma 1 allows for the elimination of $\mathbf{U}_{\langle a,b\rangle}$ in favor of $\mathbf{U}_{(0,\infty)}$ and $\mathbf{F}_{(0,b\rangle}$ (and similarly for $\mathbf{S}_{\langle a,b\rangle}$).

Corollary 1.
(1) For every $MITL_{0,\infty}$ formula ϕ there exists an equivalent $MITL_{0,\infty}$ formula ϕ' that uses only the temporal modalities $\mathbf{U}_{(0,\infty)}$ and $\mathbf{F}_{(0,b\rangle}$, for $b > 0$.
(2) For every MITL+Past formula ψ there exists an equivalent MITL+Past formula ψ' that uses only the temporal modalities $\mathbf{U}_{(0,\infty)}$, $\mathbf{F}_{\langle a,b\rangle}$, $\mathbf{S}_{(0,\infty)}$, $\mathbf{P}_{\langle a,b\rangle}$, for $0 \leqslant a < b$.
The number of distinct syntactic subformulae of ϕ' and ψ' are linearly related to the size of ϕ and ψ, respectively.

By the proof of Lemma 38 of [27], an MITL+Past formula $\mathbf{F}_{(a,b)}(\phi)$ may be replaced by a sequence of a alternations of $\mathbf{F}_{(0,1)}\mathbf{G}_{(0,1)}$ in front of $\mathbf{F}_{(0,b-a)}(\phi)$; more precisely, it is equivalent to the formula $\mathbf{F}_{(0,1)}\mathbf{G}_{(0,1)} \dots \mathbf{F}_{(0,1)}\mathbf{G}_{(0,1)}\mathbf{F}_{(0,b-a)}(\phi)$. An analogous result holds for $\mathbf{P}_{(a,b)}$. Moreover, by the same proof in [27], all temporal modalities in MITL+Past may be assumed to consider only open intervals of the form (a, b) or $(0, b)$. For instance, $\mathbf{F}_{(0,b]}(\phi)$ is equivalent to $\mathbf{F}_{(0,b)}(\phi) \vee \big((\neg\phi \mathbf{U}_{(0,\infty)}\phi) \wedge \mathbf{G}_{(0,1)}\mathbf{F}_{(0,b)}(\phi)\big)$. In fact, either ϕ occurs in the interval $(0, b)$, hence $\mathbf{F}_{(0,b)}(\phi)$ holds, or ϕ does not occur in $(0, b)$, but it must occur exactly in b. In the latter case, since formula $\neg\phi\mathbf{U}_{(0,\infty)}\phi$ captures the fact that ϕ becomes true in a left-closed manner, then $\mathbf{F}_{(0,b)}(\phi)$ and $\neg\phi\mathbf{U}_{(0,\infty)}\phi$ hold for 1 time unit, therefore $\mathbf{G}_{(0,1)}(\phi)$ holds at the current position.

Therefore, if we call $\mathrm{MITL}_{0,\infty}$+Past the fragment of MITL+Past where the only allowed temporal modalities are $\mathbf{U}_{(0,\infty)}$, $\mathbf{S}_{(0,\infty)}$, $\mathbf{F}_{(0,b)}$ and $\mathbf{P}_{(0,b)}$, for $b > 0$, the following result holds:

Proposition 2. *For every MITL+Past formula ψ there exists an equivalent $MITL_{0,\infty}$+Past formula ψ'. The size of ψ' is linear in the size of ϕ and in the unary encoding of the the maximum constant K occurring in ϕ, but it is exponential in the binary encoding of K.*

2.2. Counting modalities

Pnueli conjectured that logics such as MITL are unable to express naturally-occurring constraints, such as "Event θ_1 will occur, followed by event θ_2, both within the next time unit". This has led [28] to define a new "Pnueli modality" $\boldsymbol{\mathcal{P}}_n$: for every natural number $n > 0$, the modality $\boldsymbol{\mathcal{P}}_n(\theta_1, \dots, \theta_n)$ holds at time

t if there exists an increasing sequence of time instants $t < t_1 < t_2 < \cdots < t_n < t+1$ such that θ_i holds at t_i, for all $1 \leqslant i \leqslant n$. The Pnueli modalities were introduced for a syntactic fragment of MITL+Past, namely Quantified Temporal Logic (QTL for short) [29], where only $\mathbf{U}_{(0,\infty)}, \mathbf{S}_{(0,\infty)}, \mathbf{F}_{(0,1)}$ and $\mathbf{P}_{(0,1)}$ are allowed, but which is as expressive as MITL+Past. Embedding Pnueli modalities into QTL induces a hierarchy with respect to n. In fact, given $n > 0$, all modalities $\boldsymbol{\mathcal{P}}_h$, with $0 < h < n$, can be expressed in terms of $\boldsymbol{\mathcal{P}}_n$ as $\boldsymbol{\mathcal{P}}_h(\theta_1, \ldots, \theta_h) = \boldsymbol{\mathcal{P}}_n(\theta_1, \ldots, \theta_h, true, \ldots, true)$ by simply considering *true* as a formula. The hierarchy is strict, since QTL (hence also MITL+Past and $\text{MITL}_{0,\infty}$) augmented with the $\boldsymbol{\mathcal{P}}_n$ modality is strictly more expressive than QTL with the modality $\boldsymbol{\mathcal{P}}_{n-1}$ only (Theorem 7 of [28]). A simpler "counting" modality $\boldsymbol{\mathcal{C}}_n$, also introduced in [28], is defined as $\boldsymbol{\mathcal{C}}_n(\theta) = \boldsymbol{\mathcal{P}}_n(\theta, \ldots, \theta)$, i.e., θ must hold in at least n time instants in the open unit interval ahead. For every $n \geqslant 1$, the semantics of counting modality $\boldsymbol{\mathcal{C}}_n$ is:

$$M, t \models \boldsymbol{\mathcal{C}}_n(\phi) \Leftrightarrow \exists t_1 \ldots \exists t_n : t < t_1 < \cdots < t_n < t+1 \text{ and } M, t_k \models \phi \ \forall k \in \{1, \ldots, n\}.$$

Each $\boldsymbol{\mathcal{C}}_n$ modality is strictly more powerful than the $\boldsymbol{\mathcal{C}}_{n-1}$ modality; moreover, QTL extended with every counting modality is as expressive as QTL extended with every Pnueli modality [30]. Satisfiability of QTL with counting modalities is PSPACE-complete when each index n in a modality $\boldsymbol{\mathcal{C}}_n$ is encoded in unary, although it is EXPSPACE-complete if n is encoded in binary [31].

In the transformation into CLTLoc defined in this paper, we will also consider a language called *MITL+Past with counting*, which is the logic MITL+Past extended with counting modalities.

3. Constraint LTL over clocks

Constraint LTL (CLTL [16, 18]) is an extension of LTL allowing atomic formulae over a *constraint system* $\mathcal{D} = (D, \mathcal{R})$, where D is a specific domain of interpretation for a finite set of variables V and for constants, and $\mathcal{R}$ is a finite family of relations on D (of various arities). CLTLoc is a special case of CLTL, where the domain D is $\mathbb{R}_+$, the set $\mathcal{R}$ of relations is $\{<, =\}$ and the variables in V are interpreted as *clocks*.

Let AP be a finite set of atomic propositions. Well-formed CLTLoc formulae are defined as follows:

$$\phi := \ p \mid \alpha \sim \alpha \mid \phi \wedge \phi \mid \neg\phi \mid \mathbf{X}(\phi) \mid \mathbf{Y}(\phi) \mid \phi\mathbf{U}\phi \mid \phi\mathbf{S}\phi$$

where $p \in AP$, symbol $\sim$ stands for $<$ or $=$, α is a constant $c \in \mathbb{N}$ or a clock $x \in V$, and $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{U}$ and $\mathbf{S}$ are the usual "next", "previous", "until" and "since" operators of LTL. Boolean operators $\vee, \top, \bot, \Rightarrow$ can be introduced as usual; the "globally" $\mathbf{G}$, "eventually" $\mathbf{F}$, "release" $\mathbf{R}$, and "trigger" $\mathbf{T}$ operators may be defined as in LTL, i.e., $\phi\mathbf{R}\psi$ is $\neg(\neg\phi\mathbf{U}\neg\psi)$, $\phi\mathbf{T}\psi$ is $\neg(\neg\phi\mathbf{S}\neg\psi)$, $\mathbf{G}\phi$ is $\bot\mathbf{R}\phi$ and $\mathbf{F}\phi$ is $\top\mathbf{U}\phi$.

The semantics of CLTLoc is defined with respect to the constraint system $(\mathbb{R}, <, =)$ and the strict linear order $(\mathbb{N}, <)$ representing *positions* in time. The valuation of clocks is defined by a mapping $\sigma : \mathbb{N} \times V \to \mathbb{R}_+$, assigning, for every position $i \in \mathbb{N}$, a real value $\sigma(i, x)$ to each clock $x \in V$. Intuitively, a clock x measures the time elapsed since the last time when $x = 0$, i.e., the last "reset" of x. To ensure that time progresses at the same rate for every clock, σ must satisfy the following condition: for every position $i \in \mathbb{N}$, there exists a "time delay" $\delta_i > 0$ such that for every clock $x \in V$:

$$\sigma(i+1, x) = \begin{cases} \sigma(i, x) + \delta_i, & \text{progress} \\ 0 & \text{reset } x. \end{cases}$$

In this case, σ is called a *clock assignment*. In order to compare CLTLoc with MITL, in this paper we always assume that a clock assignment is such that $\sum_{i \in \mathbb{N}} \delta_i = \infty$ (i.e., time is always progressing).

An *interpretation* of CLTLoc is a pair (π, σ), where σ is a clock assignment and $\pi : \mathbb{N} \to \wp(AP)$ is a mapping associating a set of propositions $\pi(i)$ with each position i in $\mathbb{N}$. The semantics of CLTLoc at a position $i \in \mathbb{N}$ over an interpretation (π, σ) is defined in Table 2, where we assume that $\sigma(i, c) = c$ whenever c is a constant. A formula $\phi \in$ CLTLoc is *satisfiable* if there exists an

$$\begin{aligned}
(\pi, \sigma), i \models p &\Leftrightarrow p \in \pi(i) \text{ for } p \in AP \\
(\pi, \sigma), i \models \alpha_1 \sim \alpha_2 &\Leftrightarrow (\sigma(i, \alpha_1) \sim \sigma(i, \alpha_2)) \\
(\pi, \sigma), i \models \neg\phi &\Leftrightarrow (\pi, \sigma), i \not\models \phi \\
(\pi, \sigma), i \models \phi \wedge \psi &\Leftrightarrow (\pi, \sigma), i \models \phi \,\text{and}\, (\pi, \sigma), i \models \psi \\
(\pi, \sigma), i \models \mathbf{X}(\phi) &\Leftrightarrow (\pi, \sigma), i+1 \models \phi \\
(\pi, \sigma), i \models \mathbf{Y}(\phi) &\Leftrightarrow (\pi, \sigma), i-1 \models \phi \wedge i > 0 \\
(\pi, \sigma), i \models \phi \mathbf{U} \psi &\Leftrightarrow \exists\, j \geqslant i : (\pi, \sigma), j \models \psi \;\wedge\; \forall\, i \leqslant n < j \; (\pi, \sigma), n \models \phi \\
(\pi, \sigma), i \models \phi \mathbf{S} \psi &\Leftrightarrow \exists\, 0 \leqslant j \leqslant i : (\pi, \sigma), j \models \psi \;\wedge\; \forall\, j < n \leqslant i \; (\pi, \sigma), n \models \phi
\end{aligned}$$

Table 2: Semantics of CLTLoc.

interpretation (π, σ) such that $(\pi, \sigma), 0 \models \phi$. In this case, we say that (π, σ) is a *model* of ϕ and we write simply $(\pi, \sigma) \models \phi$.

By definition, the initial value $\sigma(0, x)$ of a clock x may be any non-negative value, but if needed any clock x may be initialized to 0 just by adding a constraint of the form $x = 0$. It is often convenient to assume that at every position there is at least one clock which is not reset. If this is the case, just add a new clock *Now*, which is never reset, except possibly at position 0. Hence, the time delay δ_i may uniquely be defined in each position i as $\sigma(i+1, Now) - \sigma(i, Now)$.

An example. Consider a simple channel, that receives an *in* event at one end and delivers it as an *out* event at the other end, with a variable delay of 3 to 5 time units. It is assumed that no other *in* event may occur until the corresponding *out* event has been issued. Let $AP = \{in, out\}$, $V = \{x\}$. The conjunction of the following formulae, within a **G** operator, specifies the system:

$$in \Rightarrow (x = 0 \wedge \mathbf{X}((x > 0 \wedge \neg out \wedge \neg in)\,\mathbf{U}(out \wedge 3 \leqslant x \leqslant 5)))$$
$$out \Rightarrow \mathbf{Y}(\neg out\ \mathbf{S}\ in)$$

When an *in* arrives, clock x is set to 0; we require that both no *out* and no *in* occur and also that the clock is not reset again until an *out* occurs; *out* can occur only at a position where the clock is between 3 and 5. To ensure that no spurious *out* without a corresponding *in* is generated, we also require that an occurrence of *out* is preceded by an occurrence of *in* that was not followed by a different occurrence of *out*.

4. Reducing finitely variable signals to CLTLoc interpretations

In this section, a formula ϕ is in general a formula of MITL+Past with counting. We denote with $sub(\phi)$ the set of all subformulae of ϕ.

Let M be a signal and let $\theta \in sub(\phi)$. We say that "θ becomes true", denoted e^u_θ, at instant $t \geqslant 0$ of signal M when θ holds right after t, but not before it, or t is the origin:

$$\exists\varepsilon > 0, \forall t' \in (t, t+\varepsilon) : M, t' \models \theta \text{ and}$$
$$t = 0 \text{ or } \exists\varepsilon' > 0, \forall t' \in (t-\varepsilon', t) : M, t' \models \neg\theta.$$

The opposite one "θ becomes false", denoted as e^d_θ, is simply the definition above with $\neg\theta$ instead of θ.

Formula θ has an "up-singularity" s^u_θ for signal M at instant t (in words, "θ becomes true in a singular manner") if the following holds:

$$t > 0,\ M, t \models \theta \text{ and } \exists\varepsilon > 0 \text{ s.t. } \forall t' \neq t \in (t-\varepsilon, t+\varepsilon) : M, t' \models \neg\theta.$$

Formula θ has a "down-singularity" s^d_θ (in words, "θ becomes false in a singular manner") at instant t for signal M if the formula above holds with $\neg\theta$ instead of θ. By definition, singularities do not occur in the origin.

If one of $e^u_\theta, e^d_\theta, s^u_\theta, s^d_\theta$ holds for M at t, we say that θ changes in M at t.

A *change point* in a signal M for a formula ϕ is a time instant t such that there exists a subformula $\theta \in sub(\phi)$ which changes in M at t.

When dealing with finitely variable signals, the following proposition is immediate:

Proposition 3. *For all f.v. signals M, the set of change points in M for ϕ is countable. Moreover, if the set is infinite, then it is also unbounded.*

Henceforth, also in the statements of lemmata and theorems, all signals are always assumed to be finitely variable.

Reducing MITL+Past with counting modalities to CLTLoc requires to represent continuous-time signals by CLTLoc models where positions in time are

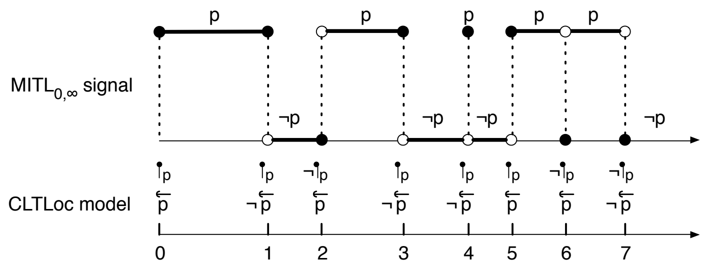


Figure 1: Example of a signal and a corresponding CLTLoc model (clocks not shown).

discrete. Positions in CLTLoc models represent change points and truth values of MITL formulae, while CLTLoc variables, behaving as clocks, measure the time progress between two consecutive change points. Every position in a CLTLoc model captures the "configuration" (the truth value of all the subformulae of ϕ) of one of the intervals in which the continuous-time signal is partitioned by the change points of ϕ. All the change points occurring in the signal are captured by the associated CLTLoc model and the value of a formula is stable between its change points (i.e., it does not vary). Figure 1 shows an example of a signal and a corresponding CLTLoc model. Our reduction thus defines the semantics of all subformulae occurring in ϕ by suitable CLTLoc formulae.

We now explain how to represent the value of MITL formulae and their change points on a signal through a CLTLoc model. For each subformula θ of ϕ we introduce two CLTLoc atomic propositions, $\upharpoonleft_\theta$ and $\overleftharpoonup{\theta}$, called *first* and *rest*, to represent the value of θ in, respectively, the first instant and the rest of an interval $[t, t')$ such that there are no change points in (t, t'). We also introduce two clocks, z^0_θ and z^1_θ, with the intended meaning to measure the time elapsed since the last two change points of θ.

In Table 3, we introduce some abbreviations, through combinations of the basic predicates $\upharpoonleft_\xi$ and $\overleftharpoonup{\xi}$, with the goal of representing various conditions on the values of an MITL+Past formula ξ. In particular, the CLTLoc formulae $\lrcorner\!\ulcorner_\xi$, $\urcorner\!\llcorner_\xi$, $\lrcorner\!\llcorner_\xi$ and $\urcorner\!\ulcorner_\xi$ are intended to represent, respectively, e^u_ξ, e^d_ξ, s^u_ξ and s^d_ξ. Notice that at position 0, both $\mathbf{Y}(\overleftharpoonup{\xi})$ and $\mathbf{Y}(\neg\overleftharpoonup{\xi})$ are false, no matter ξ, and elsewhere $\neg\mathbf{Y}(\neg\overleftharpoonup{\xi}) \equiv \mathbf{Y}(\overleftharpoonup{\xi})$, Therefore, at position 0 $\lrcorner\!\ulcorner_\xi$ holds if, and only if, $\overleftharpoonup{\xi}$ holds in 0, while $\lrcorner\!\llcorner_\xi$ can never hold at 0; similarly for $\urcorner\!\llcorner_\xi$ and $\urcorner\!\ulcorner_\xi$. The other shorthands may informally be explained as follows. Formula O is true only in the origin. Formula $\overset{\xi}{\lrcorner\!\uparrow}$ denotes that formula ξ has become true, possibly in a singular manner, or that the current position is the origin and ξ is true; symmetrically for $\overset{\xi}{\urcorner\!\downarrow}$. Formula $\overset{\xi}{\ulcorner\!\rightarrow}$ (resp. $\overset{\xi}{\llcorner\!\rightarrow}$) holds if ξ holds (resp. does not hold) in an interval starting from the current position. Also, formula $\overleftarrow{\xi}$ (resp.

$\lrcorner\!\ulcorner_\xi$	$= \neg\mathbf{Y}(\overleftarrow{\xi}) \wedge \overleftarrow{\xi}$	$\lrcorner\!\sqcap\!\llcorner_\xi$	$= \mathbf{Y}(\neg \overleftarrow{\xi}) \wedge \overset{\bullet}{\vert}_\xi \wedge \neg \overleftarrow{\xi}$
$\urcorner\!\llcorner_\xi$	$= \neg\mathbf{Y}(\neg \overleftarrow{\xi}) \wedge \neg \overleftarrow{\xi}$	$\urcorner\!\sqcup\!\ulcorner_\xi$	$= \mathbf{Y}(\overleftarrow{\xi}) \wedge \neg \overset{\bullet}{\vert}_\xi \wedge \overleftarrow{\xi}$
O	$= \neg\mathbf{Y}(\top)$		
$\overset{\xi}{\lrcorner\!\uparrow}$	$= \lrcorner\!\ulcorner_\xi \vee \lrcorner\!\sqcap\!\llcorner_\xi \vee (O \wedge \overset{\bullet}{\vert}_\xi)$	$\overset{\xi}{\ulcorner\!\rightarrow}$	$= \lrcorner\!\ulcorner_\xi \vee \urcorner\!\sqcup\!\ulcorner_\xi$
$\overset{\xi}{\urcorner\!\downarrow}$	$= \urcorner\!\llcorner_\xi \vee \urcorner\!\sqcup\!\ulcorner_\xi \vee (O \wedge \neg \overset{\bullet}{\vert}_\xi)$	$\overset{\xi}{\llcorner\!\rightarrow}$	$= \urcorner\!\llcorner_\xi \vee \lrcorner\!\sqcap\!\llcorner_\xi$
$\overset{\bullet\!-}{\xi}$	$= \overset{\bullet}{\vert}_\xi \wedge \overleftarrow{\xi}$	$\underset{\bullet\!-}{\xi}$	$= \neg \overset{\bullet}{\vert}_\xi \wedge \neg \overleftarrow{\xi}$

Table 3: CLTLoc shorthands used in the encoding.

$\underset{\bullet\!-}{\xi}$) holds if ξ is true (resp. false) throughout the current interval, including the current position.

We need also recall some preliminary results, whose trivial proofs are omitted. The following two lemmata characterize the change points of $\mathbf{U}_{(0,+\infty)}$ and $\mathbf{S}_{(0,+\infty)}$ formulae. When the value of formula $\mathbf{U}_{(0,+\infty)}$ (resp. $\mathbf{S}_{(0,+\infty)}$) changes on a signal M, the signal only varies in a left-closed (resp. left-open) manner because the formula holds in t if, and only if, it holds over a non-empty interval including t. This also guarantees that no singularity can occur for these formulae.

Lemma 2. *If $\theta = \gamma\mathbf{U}_{(0,+\infty)}\psi$ and M is a signal, then for all $t \in \mathbb{R}_+$ there is $\varepsilon \in \mathbb{R}_{>0}$ such that $M, t \models \theta$ if, and only if, $M, t' \models \theta$ for all $t' \in (t, t+\varepsilon]$.*

Lemma 3. *If $\theta = \gamma\mathbf{S}_{(0,+\infty)}\psi$ and M is a signal, then, for all $t \in \mathbb{R}_+$ there is $\varepsilon \in \mathbb{R}_{>0}$ such that $M, t \models \theta$ if, and only if, $M, t' \models \theta$ for all $t' \in [t-\varepsilon, t)$.*

Note that, in $t = 0$, $\gamma\mathbf{S}_{(0,+\infty)}\psi$ is false, and, for any $\varepsilon \in \mathbb{R}_{>0}$, $[-\varepsilon, 0)$ is not an interval of $\mathbb{R}_+$, so the lemma is trivially true. Singularities s^u cannot occur in signals associated with formulae $\mathbf{F}_{(a,b)}(\gamma)$ or $\boldsymbol{\mathcal{C}}_n(\gamma)$. In fact, the next Lemma 4 states that $\mathbf{F}_{(a,b)}(\gamma)$ or $\boldsymbol{\mathcal{C}}_n(\gamma)$ holds in t if, and only if, it holds over two nonempty left and right intervals including t.

Lemma 4. *Let θ be $\mathbf{F}_{(a,b)}(\gamma)$ (with $0 \leqslant a < b$) or $\boldsymbol{\mathcal{C}}_n(\gamma)$. If $M, t \models \theta$ then there is $\varepsilon \in \mathbb{R}_{>0}$ such that, for all $t' \in [t, t+\varepsilon]$ we have $M, t' \models \theta$ and, when $t > 0$, there is also $\varepsilon \in \mathbb{R}_{>0}$ such that $\varepsilon < t$ and for all $t' \in [t-\varepsilon, t]$ we have $M, t' \models \theta$.*

A similar result can be given also for formulae $\mathbf{P}_{(a,b)}(\gamma)$.

Lemma 5. *Let θ be $\mathbf{P}_{(a,b)}(\gamma)$ (with $0 \leqslant a < b$). If θ holds for a signal M in an instant t (i.e., $M, t \models \theta$), then there is $\varepsilon \in \mathbb{R}_{>0}$ such that $M, t' \models \theta$ for all $t' \in [t-\varepsilon, t+\varepsilon]$.*

Finally, the following result from [12] shows that formulae of the form $\mathbf{F}_{(a,b)}(\gamma)$ have inherent bounded variability.

Lemma 6 ([12])**.** *Let θ be $\mathbf{F}_{(a,b)}(\gamma)$, let M be a signal and let $0 < t_1 < t_2$ be two instants such that $M, t_1 \models e^u_\theta$ and $M, t_2 \models e^d_\theta \vee s^d_\theta$. Then, $t_2 - t_1 \geqslant b - a$.*

By Lemma 6, the distance between a change point where "θ becomes true" outside the origin and one where "θ becomes false" (possibly in a singular manner) for formulae $\theta = \mathbf{F}_{(a,b)}(\gamma)$ cannot be less than $b-a$. However, this property does not hold when e^u_θ occurs at $t = 0$ and γ becomes false before b. For instance, let $M, a \models p$ and $M, a + \varepsilon \models e^d_p$, where $\varepsilon > 0$ is such that $a + \varepsilon < b$; assume for simplicity that p remains false, i.e., for all $t \in [a + \varepsilon, +\infty)$, $M, t \not\models p$. Then, if $\theta = \mathbf{F}_{(a,b)}(p)$ we have that $M, 0 \models e^u_\theta$ and $M, \varepsilon \models e^d_\theta$. This property will be exploited below to define the translation of the $\mathbf{F}$ operator. It is fundamental in our construction, because it allows us to introduce a finite number of clocks that measure time elapsing between change points.

A property analogous to Lemma 6 holds for $\mathbf{P}_{(a,b)}$.

Lemma 7. *Let θ be $\mathbf{P}_{(a,b)}(\gamma)$ (with $0 < a < b$), M be a signal and $0 < t_1 < t_2$ two instants such that $M, t_1 \models e^u_\theta \vee s^d_\theta$ and $M, t_2 \models e^d_\theta$. Then, $t_2 - t_1 \geqslant b - a$.*

Corollary 2. *Let θ be $\mathbf{F}_{(a,b)}(\gamma)$ or $\mathbf{P}_{(a,b)}(\gamma)$, with $a \geqslant 0, b > 0$, and let t be an instant of time. If M is a signal such that $M, t \models \theta$, then, in $[t, t + b]$ there are at most $d = 2\left\lceil \frac{b}{b-a} \right\rceil$ change points in M.*

We have a similar result for operator $\boldsymbol{\mathcal{C}}_n$.

Lemma 8. *Let $\theta = \boldsymbol{\mathcal{C}}_n(\gamma)$, let t be an instant of time and M be a signal. Then, in interval $[t, t + 1]$ of M there are at most $2n$ change points.*

Proof. Let t be an instant where e^u_θ holds. Then, either there are exactly n singularities s^u_γ in $(t, t + 1]$, including one in $t + 1$, or γ becomes true in $t + 1$ in a non-singular manner. In the latter case, θ is true throughout $(t, t + 1]$. In the former case, θ can become false only at each singularity, and become true again in between two of them, hence there can only be $2n$ change points. Similar reasoning holds in the other cases, when e^d_θ, s^d_θ or none of them hold in t. □

5. Reducing $\mathbf{MITL_{0,\infty}}$ to CLTLoc

We define the translation from $\text{MITL}_{0,\infty}$ to CLTLoc, preserving the satisfiability over finitely variable signals. First, Section 5.1 introduces a set of general formulae, for every subformula θ of ϕ, defining constraints to guarantee that clock resets occur at suitable points. Then, in Section 5.2, we provide the operator-specific CLTLoc formulae that capture the semantics of $\text{MITL}_{0,\infty}$ connectives and temporal operators.

5.1. General Constraints on Clocks

This section describes the behavior of clocks in relation to change points. In general, clocks in CLTLoc are very similar to clocks of Timed Automata, but with one difference: a clock in CLTLoc is a variable, hence it cannot be reset and tested at the same time. Therefore, if we need to test a clock x against a positive constant and then to start a new time measure, we may introduce a pair of clocks, x^0, x^1 instead of one clock x, which are alternatively reset and tested. For instance, x^0 is reset at a position; at a later position x^0 is tested while at the same position x^1 may be reset; at a later position, x^1 is tested and x^0 is reset, and then later x^0 is tested and x^1 is reset, etc.

To represent the semantics of the temporal modality $\theta = \mathbf{F}_{(0,b)}(\gamma)$, we introduce two pairs of clocks, z_θ^0, z_θ^1 and z_γ^0, z_γ^1. Each pair of clocks is alternatively reset, with the technique described above. By Corollary 2, θ may vary at most once in any interval of length b: if θ becomes true at t then γ must become true at $t+b$ and, over the interval $(t, t+b)$, θ does not change its value anymore. Clocks z_θ^0 and z_θ^1 are used to measure the time elapsing between two consecutive change points of θ. Hence, if z_θ^i ($i = 0$ or $i = 1$) is reset in a position corresponding to time t, then there exists a position (corresponding to time $t+b$) where $z_\theta^i = b$ and also γ becomes true. Clocks z_γ^0 and z_γ^1 are used to measure the time elapsing between two consecutive change points of γ: a change point of γ may influence the truth value of θ only if the previous change point of γ occurred more than b time units earlier.

For all $\theta \in sub(\phi)$, such that θ is $\mathbf{F}_{(0,b)}$ or it occurs as argument of $\mathbf{F}_{(0,b)}$ the following CLTLoc formula holds at position 0, simply stating that clock z_θ^0 is reset at 0 (while z_θ^1 can have any value):

$$z_\theta^0 = 0. \tag{1}$$

The other formulae of this section must hold at each position; for simplicity, the globally operator $\mathbf{G}$ is inserted explicitly only at the end of the section.

Whenever subformula θ changes its value (it becomes true or false, possibly in a singular way), one of its associated clocks z_θ^0 and z_θ^1 is reset:

$$\overset{\theta}{\upharpoonleft} \vee \overset{\theta}{\downharpoonright} \Leftrightarrow z_\theta^0 = 0 \vee z_\theta^1 = 0. \tag{2}$$

The clocks associated with θ are alternatively reset, i.e., between any two resets of clock z_θ^0 there must be a reset of clock z_θ^1, and vice-versa:

$$\bigwedge_{i \in \{0,1\}} (z_\theta^i = 0) \Rightarrow \mathbf{X}\Big((z_\theta^{(i+1) \bmod 2} = 0)\mathbf{R}(z_\theta^i \neq 0)\Big). \tag{3}$$

In the following, $\mathtt{ck}_\theta$ denotes the formula (1) $\wedge$ $\mathbf{G}$((2) $\wedge$ (3)).

5.2. Semantics of $MITL_{0,\infty}$ temporal modalities

This section presents the definition of $m(\theta)$, the translation of every subformula θ of an $\mathrm{MITL}_{0,\infty}$ formula into a suitable CLTLoc formula encoding its semantics. Essentially, $m(\theta)$ describes how θ becomes true and false depending on the value of its own subformulae.

The translation considers every possible case for θ, i.e., when θ has one of the forms $\neg\psi, \gamma \wedge \psi, \gamma\mathbf{U}_{(0,\infty)}\psi, \mathbf{F}_{(0,b)}(\gamma)$. The case of intervals of the form $(0, b]$ is omitted for brevity.

Case $\theta = \neg\psi$. The predicates for θ are the opposite ones of ψ:

$$m(\theta) = (\upharpoonleft_\theta \Leftrightarrow \neg \upharpoonleft_\psi) \wedge (\overleftarrow{\theta} \Leftrightarrow \neg \overleftarrow{\psi}). \tag{4}$$

Case $\theta = \gamma \wedge \psi$. The semantics of θ is the conjunction of the predicates for γ and ψ:

$$m(\theta) = (\upharpoonleft_\theta \Leftrightarrow \upharpoonleft_\gamma \wedge \upharpoonleft_\psi) \wedge (\overleftarrow{\theta} \Leftrightarrow \overleftarrow{\gamma} \wedge \overleftarrow{\psi}) \tag{5}$$

Case $\theta = \gamma \mathbf{U}_{(0,\infty)} \psi$. $\mathbf{U}$ formulae cannot have singularities, as this would violate Lemma 2. This means that when a $\mathbf{U}$ formula changes its value, it must do so in a left-closed manner (i.e., the value at the change point is the same as the one after the change point). Then, we have (6) below.

$$m(\theta) = \left(\upharpoonleft_\theta \Leftrightarrow \overleftarrow{\theta}\right) \wedge \left(\overleftarrow{\theta} \Leftrightarrow \overleftarrow{\gamma} \wedge \left(\overleftarrow{\psi} \vee \mathbf{X}\left(\overset{\bullet\!-}{\gamma}\mathbf{U}\left((\overset{\bullet\!-}{\gamma} \wedge \overleftarrow{\psi}) \vee \upharpoonleft_\psi\right)\right)\right)\right) \tag{6}$$

In particular, the second conjunct of Formula (6) states that θ holds in an interval if, and only if, either both ψ and γ hold in it, or there is a future interval in which ψ holds (either throughout the interval, or in its first instant), and γ holds throughout all intervals (including their first instants) in between.

Case $\theta = \mathbf{F}_{(0,b)}(\gamma)$. By Lemma 4, an up-singularity $\sqcap_\theta$ can never occur for a formula of the form $\mathbf{F}_{(0,b)}(\gamma)$. Also, if θ holds at the beginning of an interval (i.e., $\upharpoonleft_\theta$ holds), then it must hold also in the rest of the interval and, if $t > 0$, it must also hold in the interval before. Then, the following constraint holds in every position:

$$\upharpoonleft_\theta \Rightarrow \overleftarrow{\theta} \wedge (\mathbf{Y}(\overleftarrow{\theta}) \vee O) \tag{7}$$

Formula (8) states that, when θ becomes true with a rising edge $\ulcorner_\theta$, in an instant other than the origin, a clock z^j_θ is reset, and $\overset{\gamma}{\uparrow}$ will eventually be true exactly after time b from the reset of clock z^j_θ; if θ becomes true in the origin, then either it does so in a left-closed manner, and γ becomes true before clock z^0_θ becomes b, or it becomes true in a left-open manner, and γ becomes true exactly after time b.

$$\ulcorner_\theta \Leftrightarrow \left(\begin{array}{c} O \wedge \left(\begin{array}{l} \upharpoonleft_\theta \wedge (O \vee z^0_\theta > 0)\,\mathbf{U}\left(\overset{\gamma}{\uparrow} \wedge 0 < z^0_\theta < b \;\vee\; \overleftarrow{\gamma} \wedge O\right) \vee \\ \neg \upharpoonleft_\theta \wedge \mathbf{X}\left(z^0_\theta > 0\ \mathbf{U}\left(\overset{\gamma}{\uparrow} \wedge z^0_\theta = b \wedge \bigvee_{i \in \{0,1\}} z^i_\gamma \geqslant b\right)\right) \end{array}\right) \vee \\ \neg O \wedge \neg \upharpoonleft_\theta \wedge \bigvee_{j \in \{0,1\}} \left(z^j_\theta = 0 \wedge \mathbf{X}\left(z^j_\theta > 0\ \mathbf{U}\left(\overset{\gamma}{\uparrow} \wedge z^j_\theta = b \wedge \bigvee_{i \in \{0,1\}} z^i_\gamma > b\right)\right)\right) \end{array}\right) \tag{8}$$

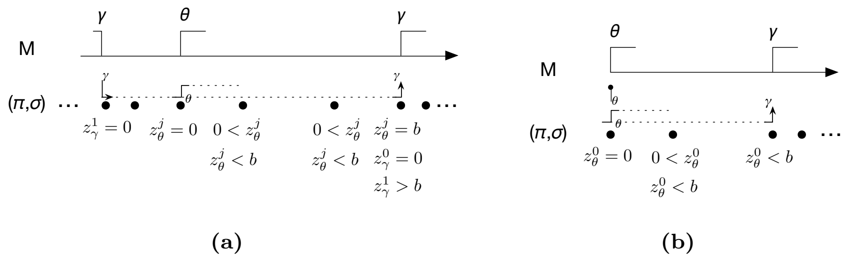


Figure 2: Examples of conditions for rising edges for $\theta = \mathbf{F}_{(0,b)}(\gamma)$ at $t > 0$ (a) and $t = 0$ (b).

Fig. 2 shows a graphical depiction of a pair of conditions for having a rising edge in an instant t, and in particular one for the case $t > 0$ and one for the case $t = 0$. More precisely, Fig. 2(a) shows a situation in which the second disjunct of the right-hand side of Formula (8) holds: one of the clocks associated with θ, say z^j_θ, is reset at position $k > 0$ of the CLTLoc interpretation corresponding to instant t, and when this clock takes value b (without having been reset in the meantime) γ becomes true, hence one of its z^i_γ clocks (z^0_γ in the example depicted) is also reset there, and the other one, which is not reset, has value $> b$; this, in turn, entails that the last time that γ became false was before t, hence at t formula θ has a rising edge. Similarly, Fig. 2(b) depicts a case in which the first condition of the right-hand side of Formula (8) holds: t is the origin (hence z^0_θ is reset there) and there is an instant before z^0_θ takes value b in which γ becomes true. In this case θ holds in the origin, too, so $\upharpoonleft_\theta$ is true there.

Formula (9) below states that, if γ becomes true at a time $t \geqslant b$ (i.e., when clock *Now* introduced in Section 3 has value $\geqslant b$), and γ was false in the interval of length b preceding t, at t one of the clocks associated with θ has value b, since $\mathbf{F}_{(0,b)}(\gamma)$ started holding b time units before time t. The formula is necessary to make sure that, if γ becomes true (and it was false for the last b time units, hence θ must have also become true b time units before), the right hand side of Formula (8) holds when θ becomes true, forcing $\ulcorner_\theta$ to hold there.

$$\left(\mathit{Now} \geqslant b \wedge \overset{\gamma}{\uparrow} \wedge \bigvee_{i \in \{0,1\}} z^i_\gamma \geqslant b \right) \Rightarrow \bigvee_{j \in \{0,1\}} z^j_\theta = b \tag{9}$$

When θ becomes false, hence γ becomes false and a clock z^i_γ is reset, it is either with a falling edge ($\urcorner_\theta$) or in a singular manner ($\urcorner\!\ulcorner_\theta$). In the former case, Formula (10), then γ cannot become true again as long as the clock that is reset with $\overset{\gamma}{\hookrightarrow}$ is not greater than b. In the latter case, Formula (11), γ must become true again exactly when the clock that is reset with $\overset{\gamma}{\hookrightarrow}$ is equal to b.

$$\urcorner_\theta \Leftrightarrow \overset{\gamma}{\hookrightarrow} \wedge \neg \mathbf{X} \left(\overset{\gamma}{\not\uparrow} \, \mathbf{U} \left(\overset{\gamma}{\uparrow} \wedge \bigvee_{i \in \{0,1\}} 0 < z^i_\gamma \leqslant b \right) \right) \tag{10}$$

$$
\top\!\!\top_{\theta} \Leftrightarrow \neg O \wedge \overset{\gamma}{\hookrightarrow} \wedge \mathbf{X}\left(\overset{\gamma}{\not\Lsh} \, \mathbf{U} \left(\overset{\gamma}{\Lsh} \wedge \bigvee_{i \in \{0,1\}} z_{\gamma}^{i} = b \right)\right) \tag{11}
$$

Then, for $\theta = \mathbf{F}_{(0,b)}(\gamma)$, $m(\theta)$ is (7) $\wedge$ (8) $\wedge$ (9) $\wedge$ (10) $\wedge$ (11).

Finally, $\mathrm{MITL}_{0,\infty}$ formula ϕ is satisfiable if, and only if, it holds in the first instant of the interval starting at 0, i.e., $\mathtt{init}_{\phi} = \upharpoonleft_{\phi}$. Then, for an $\mathrm{MITL}_{0,\infty}$ formula ϕ, the corresponding CLTLoc formula is:

$$
\mathtt{init}_{\phi} \wedge \bigwedge_{\theta \in sub(\phi)} \left(\mathtt{ck}_{\theta} \wedge \mathbf{G}\left(m(\theta) \right) \right). \tag{12}
$$

The next section shows the correctness of the translation.

5.3. Correctness and complexity of the reduction

To complete the results of this section, we need to show that an $\mathrm{MITL}_{0,\infty}$ formula ϕ is satisfiable if, and only if, there exists a pair (π, σ) that satisfies Formula (12).

First of all, we define a correspondence between $\mathrm{MITL}_{0,\infty}$ signals and CLTLoc interpretations. Given a finitely variable signal M and a finite set $\mathcal{F}$ of $\mathrm{MITL}_{0,\infty}$ formulae, we define function $r_{\mathcal{F}}(M)$ which associates with M the set of corresponding CLTLoc interpretations, where each formula of $\mathcal{F}$ is considered as an atomic proposition, thus disregarding its subformulae.

Definition 1. *Let M be a finitely variable signal, $I \subset \mathbb{N}$ be a nonempty finite set and $\mathcal{F} = \{\theta_i\}_{i \in I}$ be a finite set of $MITL_{0,\infty}$ formulae. Let $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}})$ be a CLTLoc interpretation such that $\pi_{\mathcal{F}} : \mathbb{N} \to \wp(\{\upharpoonleft_{\theta_i}, \overleftarrow{\theta_i}\}_{i \in I})$ and $\sigma_{\theta} : \mathbb{N} \times \{z_{\theta_i}^0, z_{\theta_i}^1\}_{i \in I} \cup \{Now\} \to \mathbb{R}_+$, where Now is the clock defined in Section 3. In the following we call t_k the timestamp corresponding to position $k \in \mathbb{N}$ in $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}})$, i.e., $t_k = \sigma_{\mathcal{F}}(k, Now)$, and we call T the set of timestamps, i.e., $T = \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+$.*

We have that $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}) \in r_{\mathcal{F}}(M)$ if the following conditions hold.

1. *If t is a change point in M for some $\theta_i \in \mathcal{F}$, then $t \in T$, i.e., there is $k \in \mathbb{N}$ such that $t = t_k = \sigma_{\mathcal{F}}(k, Now)$.*

   In addition, for all $\theta_i \in \mathcal{F}$:
2. *If $M, t_k \models \theta_i$, then $\upharpoonleft_{\theta_i} \in \pi_{\mathcal{F}}(k)$, otherwise $\upharpoonleft_{\theta_i} \notin \pi_{\mathcal{F}}(k)$.*
3. *If for all $t' \in (t_k, t_{k+1})$ it holds that $M, t' \models \theta$, then $\overleftarrow{\theta}_i \in \pi_{\mathcal{F}}(k)$, otherwise $\overleftarrow{\theta}_i \notin \pi_{\mathcal{F}}(k)$.*
4. *If $t_k \in T$ is a change point for θ_i, then either $\sigma_{\mathcal{F}}(k, z_{\theta_i}^0) = 0$ or $\sigma_{\mathcal{F}}(k, z_{\theta_i}^1) = 0$.*
5. *$\sigma_{\mathcal{F}}(0, z_{\theta_i}^0) = 0$.*
6. *After 0, the clocks associated with θ_i are reset modulo 2, i.e., if $\sigma_{\mathcal{F}}(k, z_{\theta_i}^j) = 0$, and $\sigma_{\mathcal{F}}(k', z_{\theta_i}^j) = 0$, where $j \in \{0,1\}$ and $k' > k$, then there is a $k < j' < k'$ s.t. $\sigma_{\mathcal{F}}(j', z_{\theta_i}^{(j+1) \bmod 2}) = 0$.*

Note that, in Definition 1, sequence T is well-defined, as by Proposition 3 the set of change points in M for each $\theta_i \in \mathcal{F}$ is countable. In addition, if $\overleftarrow{\theta_i} \notin \pi_{\mathcal{F}}(k)$, then for all $t' \in (t_k, t_{k+1})$ it holds that $M, t' \not\models \theta_i$ since, by condition 1 in Definition 1, there cannot be a change point in (t_k, t_{k+1}).

It is easy to see that the following holds.

Proposition 4. *Let M be a finitely variable signal and $\mathcal{F}$ a finite set of MITL$_{0,\infty}$ formulae. If $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}) \in r_{\mathcal{F}}(M)$, then for all $k \in \mathbb{N}$ and $\theta \in \mathcal{F}$ we have that $M, t_k \models e^u_\theta$ if, and only if, $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k \models \lrcorner\!\ulcorner_\theta$. Similarly for e^d_θ, s^u_θ, s^d_θ and the corresponding $\urcorner\!\llcorner_\theta$, $\lrcorner\!\ulcorner\!\urcorner\!\llcorner_\theta$, $\urcorner\!\llcorner\!\lrcorner\!\ulcorner_\theta$.*

It is clear from Proposition 4 that if in $t_k \in T$ there are no change points for $\theta \in \mathcal{F}$, then none of $\{\lrcorner\!\ulcorner_\theta, \urcorner\!\llcorner_\theta, \lrcorner\!\ulcorner\!\urcorner\!\llcorner_\theta, \urcorner\!\llcorner\!\lrcorner\!\ulcorner_\theta\}$ holds at position k in $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}})$ (so $\overleftarrow{\theta} \in \pi_{\mathcal{F}}(k-1)$ if, and only if, $\overset{\bullet}{|}_\theta, \overleftarrow{\theta} \in \pi_{\mathcal{F}}(k)$).

Note that, for any signal M, $r_{\mathcal{F}}(M)$ contains more than one CLTLoc interpretation; for example, given a signal in which $AP = \{p\}$ and p is always true, $r_{\{p\}}(M)$ contains both an interpretation in which $t_k = k$ and one in which $t_k = 2k$, and so on.

Not all CLTLoc interpretations $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}})$ represent MITL$_{0,\infty}$ signals. For example, for an interpretation $(\pi_{\{p, \mathbf{F}_{(0,1)}p\}}, \sigma_{\{p, \mathbf{F}_{(0,1)}p\}})$ in which, for all $k \in \mathbb{N}$, $\overset{\bullet}{|}_p \in \pi(k)$ and $\overleftarrow{p} \in \pi(k)$, but $\overset{\bullet}{|}_{\mathbf{F}_{(0,1)}p} \notin \pi(k)$ and $\overleftarrow{\mathbf{F}_{(0,1)}p} \notin \pi(k)$ there is no signal M such that $(\pi_{\{p, \mathbf{F}_{(0,1)}p\}}, \sigma_{\{p, \mathbf{F}_{(0,1)}p\}}) \in r_{\{p, \mathbf{F}_{(0,1)}p\}}(M)$. We indicate by $r^{-1}_{\mathcal{F}}((\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}))$ the – possibly empty – set of signals such that for each M in the set we have $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}) \in r_{\mathcal{F}}(M)$.

The following result shows that formulae $\mathtt{ck}_\theta$ impose that the clocks associated with each $\theta \in \mathcal{F}$ are reset in a way that respects mapping $r_{\mathcal{F}}$.

Lemma 9. *Let M be a signal and $\mathcal{F}$ be a finite set of MITL$_{0,\infty}$ formulae. For all interpretations $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}) \in r_{\mathcal{F}}(M)$ we have $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), 0 \models \bigwedge_{\theta \in \mathcal{F}} \mathtt{ck}_\theta$.*

Proof. The lemma is a consequence of the definition of the map $r_{\mathcal{F}}(M)$ and of the sequence of change points occurring in M for each $\theta \in \mathcal{F}$.

In fact, by definition of $r_{\mathcal{F}}(M)$, when a change point for some $\theta \in \mathcal{F}$ occurs in M, one of the two clocks z^0_θ, z^1_θ is reset. Note also that by Proposition 4 if $M, t_k \models e^u_\theta \vee s^u_\theta$ then $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k \models \overset{\theta}{\lrcorner\!\uparrow}$, and if $M, t_k \models e^d_\theta \vee s^d_\theta$ then $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k \models \overset{\theta}{\urcorner\!\downarrow}$, hence, we have Formula (2). Moreover, resets are defined circularly modulo 2, i.e., if $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k \models z^i_\theta = 0$ then

- either no reset of z^i_θ occurs after k (i.e., $\forall k' > k$: $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k' \models z^i_\theta \neq 0$),
- or there exists a position $k' > k$ such that $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k' \models z^i_\theta = 0$, and there is $k < j < k'$ $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), j \models z^{(i+1) \bmod 2}_\theta = 0$.

Then, we have $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), k+1 \models (z^{(i+1) \bmod 2}_\theta = 0)\mathbf{R}(z^i_\theta \neq 0)$, so Formula (3) holds.

Finally, by condition 5 of Definition 1, z^0_θ is reset in the origin, so we have Formula (1). □

Given a formula ϕ, each $(\pi, \sigma) \in r_{sub(\phi)}(M)$ partitions $\mathbb{R}_+$ in intervals $\{[t_k, t_{k+1})\}_{k \in \mathbb{N}}$ and, for each of them, it captures the value that each subformula of ϕ has in M there. Dually, if $M \in r^{-1}_{sub(\phi)}((\pi, \sigma))$, then M is a signal such that, for each $t \in \mathbb{R}_+$ and for each subformula θ of ϕ, the value of θ at t in M is captured by (π, σ). We have the following result.

Lemma 10. *Let M be a signal, and ϕ be an $MITL_{0,\infty}$ formula. For all $(\pi, \sigma) \in r_{sub(\phi)}(M)$ we have $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta$ and for all $k \in \mathbb{N}, \theta \in sub(\phi)$ we have $(\pi, \sigma), k \models m(\theta)$. Conversely, if $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta \wedge \mathbf{G}\,(m(\theta))$, then there is a signal M such that $(\pi, \sigma) \in r_{sub(\phi)}(M)$.*

Proof. The proof is split into two parts. More precisely, we show that:

1. for all $(\pi, \sigma) \in r_{sub(\phi)}(M)$ we have $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta$ and for all $k \in \mathbb{N}, \theta \in sub(\phi)$ we have $(\pi, \sigma), k \models m(\theta)$.
2. if $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta \wedge \mathbf{G}(m(\theta))$, then there is $M \in r^{-1}_{sub(\phi)}((\pi, \sigma))$.

Part 1.
The fact that $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta$ is a direct application of Lemma 9.

The rest of the proof is by case analysis on the kinds of subformulae that can appear in ϕ. Suppose $t_k \in T$. The case of $\theta \in AP$, for which no constraint $m(\theta)$ is defined, is trivial.

The rest of the cases is listed below.

Cases $\theta = \neg\gamma$ and $\theta = \gamma \wedge \psi$. These cases are a straightforward consequence of conditions 1 and 2 of Definition 1.

Case $\theta = \gamma \mathbf{U}_{(0,\infty)} \psi$. The first conjunct of Formula (6) holds in t_k by conditions 1 and 2 of Definition 1 and Lemma 2. The second conjunct is also a consequence of conditions 1-2 of Definition 1: for θ to hold in t_k, either both γ and ψ hold throughout (t_k, t_{k+1}) (i.e., $(\pi, \sigma), k \models \overleftarrow{\gamma} \wedge \overleftarrow{\psi}$), or there is a future interval $(t_{k'}, t_{k'+1})$, with $t_{k'} > t_k$ such that:

- either $M, t_{k'} \models \psi$ and γ holds throughout $(t_k, t_{k'})$
- or for all $t' \in (t_{k'}, t_{k'+1})$ we have $M, t' \models \psi$ and γ holds throughout $(t_k, t_{k'+1})$ (hence including the whole interval $(t_{k'}, t_{k'+1})$).

By definition of $r_{sub(\phi)}(M)$, in the first case $(\pi, \sigma), k \models \overleftarrow{\gamma} \wedge \mathbf{X}(\overset{\bullet\!-}{\gamma} \mathbf{U} \upharpoonleft_\psi)$ holds. In the second case $\overleftarrow{\gamma} \wedge \mathbf{X}(\overset{\bullet\!-}{\gamma} \mathbf{U}(\overset{\bullet\!-}{\gamma} \wedge \overleftarrow{\psi}))$ holds. All in all, Formula (6) holds in k.

Case $\theta = \mathbf{F}_{(0,b)}(\gamma)$. By Lemma 4, we have that in this case θ cannot become true in a singular manner, so s^u_θ never holds in M. Also, by the same lemma we have that if θ holds in t_k, it must also hold in (t_{k-1}, t_k) (if it exists, i.e., if $t_k \neq 0$) and in (t_k, t_{k+1}), so $\upharpoonleft_\theta \Rightarrow \overleftarrow{\theta} \wedge (\mathbf{Y}(\overleftarrow{\theta}) \vee O)$ holds in k. In addition, when θ switches value in a non-singular manner, the way it changes (left-open or left-closed) is always the same for falling edges (in which case it is left-closed), and for rising edges the only instant in which it is undetermined is the origin (in all other instants it is left-open). Hence, we have to consider only three cases: θ becomes true in t_k (i.e., $M, t_k \models e^u_\theta$); θ becomes false in t_k (i.e., $M, t_k \models e^d_\theta$); θ is false in t_k in a singular manner (i.e., $M, t_k \models s^d_\theta$). We consider them separately.

Subcase $\mathbf{e^u_\theta}$. Suppose e^u_θ holds in t_k. We separate the cases $t_k > 0$ and $t_k = 0$.

If $t_k > 0$, for θ to become true in t_k, it must be that γ holds at $t_k + b$, or there is $\varepsilon > 0$ such that γ holds throughout the interval $(t_k + b, t_k + b + \varepsilon)$ and it does not hold throughout the interval $[t_k - \varepsilon, t_k + b)$, for some $\varepsilon > 0$. Hence, in $t_k + b$ γ either e^u_γ holds, or s^u_γ does, so by definition of $r_{sub(\theta)}(M)$ there is $t_{k'} = t_k + b$ and we have $(\pi, \sigma), k' \models \lrcorner\!\ulcorner_\gamma \vee \lrcorner\!\ulcorner\!\llcorner_\gamma$, which is $\overset{\gamma}{\lrcorner\!\uparrow}$. In addition, one of the clocks associated with θ must be equal to b in k'. Since in $t_{k'}$ γ has a change point, by definition of $r_{sub(\theta)}(M)$ one of z^0_γ, z^1_γ is reset in k'; the other clock in k' must be $> b$, since in $[t_k, t_k + b)$ there are no change points for γ. Then, one of the clocks associated with θ is reset in k, and since θ stays true throughout the interval $t_k, t_k + b$, none of them is reset between k and k', so the clock that is reset in k has value > 0 throughout (k, k'), and it has value b in k'. Finally, by Lemma 4, we have $M, t_k \not\models \theta$, so we also have $(\pi, \sigma), k \models \neg \overset{\bullet}{|}_\theta$.

Hence, $\neg O \wedge \neg \overset{\bullet}{|}_\theta$ and also $z^j_\theta = 0 \wedge \mathbf{X}(z^j_\theta > 0 \; \mathbf{U}(\overset{\gamma}{\lrcorner\!\uparrow} \wedge z^j_\theta = b \wedge \bigvee_{i \in \{0,1\}} z^i_\gamma > b))$ holds in k for some $j \in \{0, 1\}$.

If $t_k = 0$, we have two cases. The case when γ becomes true at time b, but it is false throughout $(0, b)$, is very similar to the one for $t_k > 0$ (except that the clock that is reset in t_k is z^0_θ, and that γ must have a falling edge in 0). If, instead, γ is true for some $0 < t < b$, then θ in 0 becomes true in a left-closed manner (i.e., $\overset{\bullet}{|}_\theta$ holds in 0). For γ to be true sometime in $(0, b)$, it must be that either it has a rising edge in 0 (i.e., $M, t_k \models e^u_\gamma$, so it holds in an interval $(0, \varepsilon)$ for some $\varepsilon > 0$), or there is a $0 < t_{k'} < b$ in which γ becomes true, i.e., we have $M, t_{k'} \models e^u_\gamma \vee s^u_\gamma$. Then, by definition of $r_{sub(\theta)}(M)$ we have $(\pi, \sigma), k' \models \overset{\gamma}{\lrcorner\!\uparrow}$. As θ stays true throughout $[0, t_{k'})$, clock z^0_θ, which by definition of $r_{sub(\theta)}(M)$ is reset in 0, is positive throughout interval $(0, t_{k'})$, and it is still $< b$ in $t_{k'}$. Then, the formula $\overset{\bullet}{|}_\theta \wedge (O \vee z^0_\theta > 0) \, \mathbf{U}(\overset{\gamma}{\lrcorner\!\uparrow} \wedge 0 < z^0_\theta < b \;\; \vee \;\; \overleftarrow{\gamma} \wedge O)$ holds in the origin. All in all, Formula (8) holds at position k.

If instead $M, t_k \not\models e^u_\theta$, then if $t_k > 0$ no clock associated with θ is reset in k; if, instead, $t_k = 0$, then γ remains false throughout $(0, b]$. In both cases the right hand side of Formula (8) does not hold, so Formula (8) does, since, by definition of $r_{sub(\theta)}(M)$ we have $(\pi, \sigma), k \not\models \lrcorner\!\ulcorner_\theta$.

Finally, Formula (9) holds in all $k \in \mathbb{N}$, since if the antecedent holds in k, then in $t_k - b$, which is $\geqslant 0$ because $Now \geqslant 0$ in k, θ becomes true, so there is k' such that $t_{k'} = t_k - b$ where one of the clocks z^j_θ is reset, and $z^j_\theta = b$ in k.

Subcase $\mathbf{e^d_\theta}$. Suppose θ has a falling edge in t_k. In this case it must be that if $t_k \neq 0$, γ holds in t_k or throughout the interval $(t_k - \varepsilon, t_k)$ for some $\varepsilon > 0$, and it does not become true throughout interval $(t_k, t_k + b]$. Note that γ cannot become true in $t_k + b$ either, or θ is true right after t_k, whereas we are assuming that in t_k it has a falling edge. Then, we have $M, t_k \models e^d_\gamma \vee s^u_\gamma$, so by definition of $r_{sub(\theta)}(M)$ we have $(\pi, \sigma), k \models \overset{\gamma}{\hookrightarrow}$. Also, since γ has a change point in t_k, one of z^0_γ, z^1_γ is reset in k and, for γ not to become true in $(t_k, t_k + b]$ by definition of $r_{sub(\theta)}(M)$ the next time after t_k that either e^u_γ or s^u_γ occurs the clock that is reset at k cannot still be less than or equal to b. Then, we have that $(\pi, \sigma), k \models \neg\mathbf{X}(\overset{\gamma}{\not\nearrow} \mathbf{U}(\overset{\gamma}{\lrcorner\!\!\uparrow} \wedge \bigvee_{i\in\{0,1\}} 0 < z^i_\gamma \leqslant b))$, and Formula (10) holds at position k.

If θ does *not* have a falling edge in t_k (i.e., $M, t_k \not\models e^d_\theta$), then either γ does not become false in t_k, or, if it does, it becomes true anew in $(t_k, t_k + b]$. In all these cases, the right hand side of Formula (10) does not hold in k, so Formula (10) does, as $(\pi, \sigma), k \not\models \urcorner_\theta$ by definition of $r_{sub(\phi)}(M)$.

Subcase $\mathbf{s^d_\theta}$. Suppose θ has a down-singularity in t_k. In this case it must be that $t_k \neq 0$ (i.e., $\neg O$ holds in k) and (i) γ holds in t_k or throughout the interval $(t_k - \varepsilon, t_k)$ for some $\varepsilon > 0$; (ii) at $t_k + b$ either e^u_γ or s^u_γ occur; and (iii) throughout interval $(t_k, t_k + b)$ there are no change points where γ becomes true . This case is similar to Subcase $\mathbf{e^d_\theta}$, except that we require γ to become true in $t_k + b$, i.e., when the clock related to γ that is reset at k takes value b. Hence, we have that $(\pi, \sigma), k \models \overset{\gamma}{\hookrightarrow} \wedge \mathbf{X}(\overset{\gamma}{\not\nearrow} \mathbf{U}(\overset{\gamma}{\lrcorner\!\!\uparrow} \wedge \bigvee_{i\in\{0,1\}} z^i_\gamma = b))$, and Formula (11) holds at position k.

If θ does *not* have a down-singularity in t_k (i.e., $M, t_k \not\models s^d_\theta$), then one can show, as for case Subcase $\mathbf{e^d_\theta}$, that the right hand side of Formula (11) does not hold at k, so the whole formula does.

Part 2.
Let us consider (π, σ) such that $(\pi, \sigma), 0 \models \bigwedge_{\theta\in sub(\phi)} \mathtt{ck}_\theta \wedge \mathbf{G}(m(\theta))$. Since, as mentioned in Section 3, time is progressing in (π, σ), for all $t \in \mathbb{R}_+$ there is $k \in \mathbb{N}$ such that $t \in [t_k, t_{k+1})$ (we recall that $t_k = \sigma(k, \mathit{Now})$).

To show that there exists $M \in r^{-1}_{sub(\phi)}((\pi, \sigma))$ we first describe how M is obtained, then we show that, for each $t \in \mathbb{R}_+$, where $t \in [t_k, t_{k+1})$, and for each $\theta \in sub(\phi)$, $M, t \models \theta$, if, and only if, either $t = t_k$ and $(\pi, \sigma), k \models \upharpoonleft_\theta$, or $t \in (t_k, t_{k+1})$ and $(\pi, \sigma), k \models \overleftarrow{\theta}$.

To define M, we impose that, for each $p \in AP$ and $t \in \mathbb{R}_+$, where $t \in [t_k, t_{k+1})$, $p \in M(t)$ if, and only if, either $t = t_k$ and $\upharpoonleft_p \in \pi(k)$, or $t \in (t_k, t_{k+1})$ and $\overleftarrow{p} \in \pi(k)$.

The rest of the proof is carried out by induction on the structure of ϕ.

The base case is given by $\theta \in AP$, for which the result holds by construction.

The cases $\theta = \neg\gamma$, $\theta = \gamma \wedge \psi$ are straightforward.

The case for $\theta = \gamma\mathbf{U}_{(0,\infty)}\psi$ is also easily shown, when one considers that, because of Lemma 2, for $t \in [t_k, t_{k+1})$ we have that $M, t, \models \theta$ if, and only if, $M, t_k \models \theta$.

Finally, we consider the case $\theta = \mathbf{F}_{(0,b)}(\gamma)$.

To achieve the desired goal, we show that θ in M has a change point if, and only if, the corresponding propositions hold in (π, σ). More precisely, we show that, for all $t \in \mathbb{R}_+$, $M, t \models e^u_\theta$ if, and only if, there is $t_k = t$ such that $(\pi, \sigma) \models \lrcorner\!\!\urcorner_\theta$, and similarly for e^d_θ, s^d_θ (by Lemma 4 s^u_θ never occurs).

We sketch the first case, the others are similar.

We first show that, if t is such that there is $t_k = t$ and $\lrcorner\!\ulcorner_\theta \in \pi(k)$, then $M, t \models e^u_\theta$. In this case, the right hand side of Formula (8) holds in k. Let us consider the case $k > 0$ (i.e., $(\pi, \sigma), k \not\models O$). This entails that a clock z^j_θ is reset at k, and there is $k' > k$ such that $(\pi, \sigma), k' \models \overset{\gamma}{\lrcorner\!\!\uparrow} \wedge z^j_\theta = b$, and γ stays false throughout $[k, k')$, because in k' the clock of γ that is not reset is $> b$. Then, $t_{k'} = t_k + b$, and by inductive hypothesis $M, t_{k'} \models e^u_\gamma \vee s^u_\gamma$ and for all $t' \in [t_k, t_k + b)$ we have $M, t' \not\models \gamma$; hence, there is $\varepsilon > 0$ such that in $(t_k - \varepsilon, t_k)$ θ does not hold, but it holds in $(t_k, t_k + \varepsilon)$. Hence, $M, t_k \models e^u_\theta$. The case $k = 0$ is similar.

Suppose there is no $k \in \mathbb{N}$ such that $t = t_k$ or, if such k exists, then $\lrcorner\!\ulcorner_\theta \notin \pi(k)$. Suppose $t > 0$. We show that it cannot be that $M, t \models e^u_\theta$. In fact, suppose $M, t \models e^u_\theta$; then γ becomes true in $t + b$, and it is false in $[t, t + b)$; hence, by inductive hypothesis there is k' such that $t_{k'} = t + b$ and the antecedent of Formula (9) holds in k'. As a consequence, one of the clocks z^j_θ has value b in $t_{k'}$, so there is $k \in \mathbb{N}$ where $\sigma(k, z^j_\theta) = 0$ and $\sigma(k, Now) = t_{k'} - b = t$, so it is not true that there is no $k \in \mathbb{N}$ such that $t = t_k$. Then, $t = t_k$ for some k, and the second disjunct in the right hand side of Formula (8) holds in k, and so does the whole formula on the right hand side, thus contradicting the assumption $\lrcorner\!\ulcorner_\theta \notin \pi(k)$.
If, instead, $t = 0$, then $t = t_0$; if $M, t_0 \models e^u_\theta$, then the first disjunct of the right hand side of Formula (8) would hold, which again would entail a contradiction with the assumption $\lrcorner\!\ulcorner_\theta \notin \pi(0)$. □

Finally, from Lemma 10 the following theorem descends by observing that signal M is model for ϕ if, and only if, $M, 0 \models \phi$, which means that $\upharpoonleft_\phi$ holds in 0.

Theorem 1. *An MITL$_{0,\infty}$ formula ϕ is f.v. satisfiable if, and only if, Formula* (12) *is satisfiable.*

Consider an MITL$_{0,\infty}$ formula ϕ. The translation provided in this section introduces, for each $\theta \in sub(\phi)$, 2 atomic propositions $\upharpoonleft_\theta, \overleftarrow{\theta}$ and 2 variables z^0_θ, z^1_θ. The size of every CLTLoc formula $m(\theta)$ does not depend on $|\theta|$, but only (when θ has the form $\mathbf{F}_{(0,b)}(\gamma)$) on the binary encoding of constant b. Hence, the size of Formula (12) linearly depends on the size of ϕ. [21] shows that satisfiability for a CLTLoc formula is PSPACE in the number of subformulae and in the size of the binary encoding of the maximum constant occurring in it. Then our translation preserves the PSPACE complexity of the satisfiability of MITL$_{0,\infty}$.

6. Reduction of MITL+Past to CLTLoc

We first extend the encoding of Section 5.3 to deal with MITL$_{0,\infty}$+Past, by including also subformulae with past modalities of the forms: $\gamma \mathbf{S}_{(0,\infty)} \psi$ and $\mathbf{P}_{(0,b)}(\gamma)$. By Proposition 2, this also gives an encoding for the full MITL+Past.

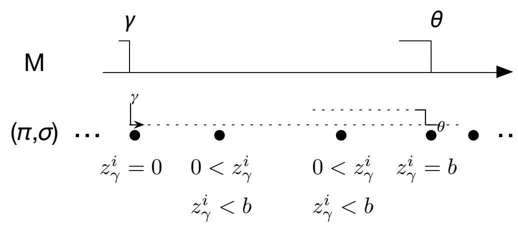


Figure 3: An example of falling edge for $\theta = \mathbf{P}_{(0,b)}(\gamma)$.

However, we then show also a direct encoding of MITL+Past, without resorting to an intermediate translation into $\mathrm{MITL}_{0,\infty}$+Past.

Case $\theta = \gamma \mathbf{S}_{(0,+\infty)} \psi$. **S** formulae cannot have singularity points, as they would violate Lemma 3. In addition, when an **S** formula changes its value after the origin, it must do so in a left-open manner (i.e., the value at the change point is the same as the one before the change point). In the origin, instead, θ is false. Then, we have

$$m(\theta) = \left(\overset{\bullet}{|}_\theta \Leftrightarrow \mathbf{Y}(\overleftarrow{\theta}) \right) \wedge \left(\overleftarrow{\theta} \Leftrightarrow \overset{\bullet\!-}{\gamma}\ \mathbf{S}((\overset{\bullet}{|}_\psi \vee \overleftarrow{\psi}) \wedge \overleftarrow{\gamma}) \right). \tag{13}$$

Case $\theta = \mathbf{P}_{(0,b)}(\gamma)$. Note that $\mathbf{P}_{(0,b)}(\gamma)$ is false in $t = 0$, no matter γ. As for **F** formulae, Lemma 5 implies that $\lrcorner\sqcap\llcorner_\theta$ can never occur for θ. In addition, by Lemma 5, if θ holds in the first instant of an interval t (i.e., $\overset{\bullet}{|}_\theta$), it must also hold in the intervals before and after t. Then, the following constraint holds:

$$\overset{\bullet}{|}_\theta \Rightarrow \overleftarrow{\theta} \wedge \mathbf{Y}\left(\overleftarrow{\theta}\right). \tag{14}$$

Formula (15) states that for θ to become true with a rising edge in t, γ must also become true (possibly in a singular manner). This is sufficient if $t = 0$. If $t > 0$, there are two cases: either γ was never true before t (so it was false in the origin and it stayed so), or the last change point of γ before t was before $t - b$, so the clock associated with γ that is not reset in t is greater than b.

$$\lrcorner\!\ulcorner_\theta \Leftrightarrow \overset{\gamma}{\lrcorner\!\uparrow} \wedge \left(O \vee \mathbf{Y}\left(\overset{\gamma}{\not\lrcorner\!\uparrow}\, \mathbf{S}\ (O \wedge \underset{\bullet\!-}{\gamma}) \right) \vee \bigvee_{i \in \{0,1\}} z^i_\gamma > b \right) \tag{15}$$

Formula (16) states that θ has a falling edge in t if, and only if, either $t = 0$ and there is ε such that γ is false in $[0, \varepsilon)$, or the last time γ became true was at $t - b$ (and it continues being false also after t). This corresponds to the condition (depicted in Fig. 3) that there is z^i_γ that is equal to b in t, and the last time γ had a change point it was $z^i_\gamma = 0$ and γ became false. γ cannot become true in t, or θ would not have a falling edge; if γ becomes true in t, then θ has a down-singularity, as specified by Formula (17).

$$\rceil\!\lfloor_\theta \Leftrightarrow \bigvee_{i \in \{0,1\}} \left(z_\gamma^i = b \wedge \left(\overset{\gamma}{\not\nearrow} \mathbf{S} \left(\overset{\gamma}{\hookrightarrow} \wedge z_\gamma^i = 0 \wedge \neg(O \wedge \underset{\leftarrow}{\gamma}) \right) \right) \right) \vee (O \wedge \underset{\leftarrow}{\gamma}) \tag{16}$$

$$\rceil\!\!\top_\theta \Leftrightarrow \overset{\gamma}{\uparrow} \wedge \bigvee_{i \in \{0,1\}} \left(z_\gamma^i = b \wedge \mathbf{Y} \left(\overset{\gamma}{\not\nearrow} \mathbf{S} \left(\overset{\gamma}{\hookrightarrow} \wedge z_\gamma^i = 0 \wedge \neg(O \wedge \underset{\leftarrow}{\gamma}) \right) \right) \right) \tag{17}$$

Finally, we introduce the analogous of Formula (9) for the eventuality in the past. More precisely, following Formula (18) specifies that if γ becomes false and there are no change points for γ for at least b time units, the CLTLoc model includes a position in which the clock that is reset with the falling edge of γ hits value b. Formula (18) is necessary to make sure that, if γ becomes false (and it does not become true again for b time units, hence θ must also become false after b), eventually the right hand side of Formulae (16) and (17) holds.

$$\bigwedge_{i \in \{0,1\}} \left(\overset{\gamma}{\hookrightarrow} \wedge z_\gamma^i = 0 \wedge \neg(O \wedge \underset{\leftarrow}{\gamma}) \Rightarrow \mathbf{X} \left(z_\gamma^i > 0 \; \mathbf{U} \left(z_\gamma^i = b \vee (\overset{\gamma}{\uparrow} \wedge 0 < z_\gamma^i < b) \right) \right) \right). \tag{18}$$

Then, for $\theta = \mathbf{P}_{(0,b)}(\gamma)$, define $m(\theta)$ as (14) $\wedge$ (15) $\wedge$ (16) $\wedge$ (17) $\wedge$ (18).

Given an MITL$_{0,\infty}$+Past formula ϕ and the translation $m(\theta)$ as extended in this section, the corresponding CLTLoc formula is still (12).

Correctness and complexity of the translation of the new modalities

Lemma 11. *Lemma 10 holds also when $m(\theta)$ is extended to subformulae θ of the form $\gamma \mathbf{S}_{(0,\infty)} \psi$ and $\mathbf{P}_{(0,b)}(\gamma)$.*

Proof. The proof follows the same structure as the one for Lemma 10, since the statement is just an extension. It is hence enough to prove the result for the kinds of subformulae considered in this section.

Part 1.
Suppose $t_k \in T$. We analyze the different types of subformulae introduced, and we show that for all $(\pi, \sigma) \in r_{sub(\phi)}(M)$ and for all $k \in \mathbb{N}, \phi \in sub(\phi)$ we have $(\pi, \sigma), k \models m(\phi)$.

Case $\theta = \gamma \mathbf{S}_{(0,\infty)} \psi$. The case for formulae for the form $\gamma \mathbf{S}_{(0,\infty)} \psi$ is similar to the one for $\gamma \mathbf{U}_{(0,\infty)} \psi$, so we do not detail it for brevity.

Case $\theta = \mathbf{P}_{(0,b)}(\gamma)$. By Lemma 5, in this case θ cannot become true in a singular manner, so s^u_θ never occurs. Also by Lemma 5, we have that if θ holds in t_k, it must also hold in (t_{k-1}, t_k) (which must exist) and in (t_k, t_{k+1}), so $\overset{\bullet}{|}_\theta \Rightarrow \overleftarrow{\theta} \wedge \mathbf{Y}(\overleftarrow{\theta})$ holds in k. In addition, when θ changes value in a non-singular manner, the way it changes (left-open or left-closed) is always the same (left-open for a rising edge, and left-closed for a falling one). Hence, there are only three cases: θ becomes true in t_k (i.e., $M, t_k \models e^u_\theta$); θ becomes false in t_k (i.e., $M, t_k \models e^d_\theta$); θ is false in t_k in a singular manner (i.e., $M, t_k \models s^d_\theta$). We consider them separately.

Subcase $\mathbf{e^u_\theta}$. Suppose θ has a rising edge in t_k. In this case, it must be that γ holds either in t_k or in an interval $(t_k, t_k + \varepsilon)$, for some $\varepsilon > 0$, and it does not hold throughout $(t_k - b, t_k)$, or $t_k = 0$. Then, it must be that $M, t_k \models e^u_\gamma$, or $M, t_k \models s^u_\gamma$, i.e., $(\pi, \sigma), k \models \overset{\gamma}{\lrcorner\!\uparrow}$. Since t_k is a change point for γ, either z^0_γ or z^1_γ is reset at position k, by definition of $r_{sub(\theta)}(M)$. Then, the condition that γ is not true in $(t_k - b, t_k)$, since it becomes true in t_k, corresponds to there not being change points for γ in $[t_k - b, t_k)$, i.e., the clock between z^0_γ and z^1_γ that is not reset in k must be greater than b in k. If, however, $t_k < b$, but γ has remained false since the origin, the clock that is not reset in t_k was reset in 0, so it is less than b even if γ never became true before. Hence, we have to consider also the special case in which γ is false throughout $[0, t_k)$; this occurs if in the origin we have $M, 0 \models e^d_\gamma \wedge \neg\gamma$, and e^u_γ and s^u_γ no dot hold since. This corresponds to having $(\pi, \sigma), k \models \mathbf{Y}(\overset{\gamma}{\not\uparrow} \mathbf{S}(O \wedge \underleftarrow{\gamma}))$. All in all, at position k in (π, σ) we have that $\overset{\gamma}{\lrcorner\!\uparrow} \wedge (O \vee \mathbf{Y}(\overset{\gamma}{\not\uparrow} \mathbf{S}(O \wedge \underleftarrow{\gamma})) \vee \bigvee_{i \in \{0,1\}} z^i_\gamma > b)$ so Formula (15) holds at k.

If, instead $M, t_k \not\models e^u_\theta$, then none of the conditions above occurs. In particular, at t_k γ either does not become true or there is $0 < \varepsilon < b$ such that γ becomes true in $t_k - \varepsilon$. In both cases, the right hand side of Formula (15) does not hold, so Formula (15) does, since, by definition of $r_{sub(\theta)}(M)$ we have $(\pi, \sigma), k \not\models \lrcorner\!\ulcorner_\gamma$.

Subcase $\mathbf{e^d_\theta}$. Suppose θ has a falling edge in t_k. We separate the case $t_k = 0$ from $t_k > 0$. In the former case, the falling edge simply corresponds to the fact that θ starts false, hence also γ starts false; that is, $(\pi, \sigma), 0 \models O \wedge \underleftarrow{\gamma}$.

In the latter case, it must be that γ holds in $t_k - b$, or in an interval $(t_k - 1 - \varepsilon, t_k - b)$, but it does not hold in $(t_k - b, t_k)$. Then, γ must have a change point in $t_k - b$, so, by definition of $r_{sub(\theta)}(M)$, there must be a position k' with $t_{k'} = t_k - b$ where one of z^0_γ, z^1_γ is reset, and its value at position k is b (it cannot be reset between k' and k, as γ does not have change points there, and if γ has a change point in t_k, then the clock that is reset at k is not the one reset at k'). Either $M, t_{k'} \models e^d_\gamma$, or $M, t_{k'} \models s^u_\gamma$ holds, i.e., we have $(\pi, \sigma), k' \models \overset{\gamma}{\llcorner\!\!\to}$.

Overall, for one of z^0_γ, z^1_γ we have, at position k in (π, σ), $z^i_\gamma = b \wedge (\overset{\gamma}{\not\uparrow} \mathbf{S}(\overset{\gamma}{\llcorner\!\!\to} \wedge z^i_\gamma = 0))$ so Formula (16) holds at position k.

If θ does *not* have a falling edge in t_k (i.e., $M, t_k \not\models e^d_\theta$), then either γ does not become false in $t_k - b$, or, if it does, it becomes true anew in $[t_k - b, t_k)$. In all these cases, the right hand side of Formula (16) does not hold in k, so Formula (16) does, as $(\pi, \sigma), k \not\models \urcorner_\theta$.

Subcase $\mathbf{s^d_\theta}$. Suppose θ has a down-singularity in t_k, hence $t_k > 0$. then, γ holds in $t_k - b$, or in an interval $(t_k - b - \varepsilon_1, t_k - b)$, and also in t_k, or in an interval $(t_k, t_k + \varepsilon_2)$, but not in $(t_k - b, t_k)$. Then, γ must have a change point (where either e^d_γ or s^u_γ occurs) in $t_k - b$ (so there must be a position k' with $t_{k'} = t_k - b$), and one (where either e^u_γ or s^u_γ occurs) in t_k, but none in between. Also, one of z^0_γ, z^1_γ is reset in $t_{k'}$ and the other is reset in t_k. This is just a combination of the conditions for Subcases $\mathbf{e^u_\theta}$ and $\mathbf{e^d_\theta}$, so $\overset{\gamma}{\uparrow} \wedge \bigvee_{i \in \{0,1\}} \left(z^i_\gamma = b \wedge \mathbf{Y} \left(\overset{\gamma}{\not\uparrow} \mathbf{S} \left(\overset{\gamma}{\hookrightarrow} \wedge z^i_\gamma = 0 \right) \right) \right) \wedge \neg O$ holds at position k, hence also Formula (17) holds at k.

If θ does *not* have a down-singularity in t_k (i.e., $M, t_k \not\models s^d_\theta$), then one can show, as for Subcase $\mathbf{e^d_\theta}$, that the right hand side of Formula (17) does not hold at k, so the whole formula does.

Finally, Formula (18) holds in each $k \in \mathbb{N}$ since if γ becomes false in t_k (hence $\overset{\gamma}{\hookrightarrow}$ holds in k), then either it becomes true again before $t_k + b$, or it does not, in which case θ becomes false at $t_k + b$; in the latter case, there is k' such that $t_{k'} = t_k + b$, and the clock z^j_γ that is reset in k has value b in k'. In both cases, the right hand side of Formula (18) holds.

Part 2.
The proof is similar to the one of Part 2 of Lemma 10. We briefly sketch the case $\theta = \mathbf{P}_{(0,b)}(\gamma)$, focusing on change points of the form e^d_θ. More precisely, we show that, for all $t \in \mathbb{R}_+$, $M, t \models e^d_\theta$ if, and only if, there is $t_k = t$ such that $(\pi, \sigma) \models \urcorner_\theta$.

If $t_k = t$ and $\urcorner_\theta \in \pi(k)$, the right hand side of Formula (16) holds in k. Let us focus on the case $k > 0$. Then, for some $j \in \{0, 1\}$, $z^j_\gamma = b$ holds in k, and there is $k' < k$ such that $(\pi, \sigma), k' \models \overset{\gamma}{\hookrightarrow} \wedge z^j_\gamma = 0$, and γ stays false throughout $(k', k + \varepsilon)$, for some $\varepsilon > 0$. Then, $t_{k'} = t_k - b$, and by inductive hypothesis $M, t_{k'} \models e^d_\gamma \vee s^u_\gamma$ and for all $t' \in (t_k - b, t_k + \varepsilon)$ we have $M, t' \not\models \gamma$. Hence, $M, t_k \models e^d_\theta$.

Suppose there is no $k \in \mathbb{N}$ such that $t = t_k$ or, if such k exists, then $\urcorner_\theta \notin \pi(k)$. Let us focus on the case $t > 0$. Suppose $M, t \models e^d_\theta$; then γ becomes false in $t - b$, and it is false throughout $(t - b, t + \varepsilon)$, for some $\varepsilon > 0$; hence, by inductive hypothesis there is k' such that $t_{k'} = t - b$ and the antecedent of Formula (18) holds in k'. As a consequence, one of the clocks z^j_γ has value b in $t_{k'} + b$, so there is $k \in \mathbb{N}$ where $\sigma(k, z^j_\gamma) = b$ and $\sigma(k, \mathit{Now}) = t_{k'} + b = t$, so it is not true that there is no $k \in \mathbb{N}$ such that $t = t_k$. Then, $t = t_k$ for some k, and the first disjunct in the right hand side of Formula (16) holds in k, and so does the whole formula on the right hand side, thus contradicting the assumption $\urcorner_\theta \notin \pi(k)$.

□

Theorem 2. *An $MITL_{0,\infty}$ + Past formula ϕ is f.v. satisfiable if, and only if, Formula* (12) *is satisfiable.*

We now show that our reduction of $\text{MITL}_{0,\infty}$+Past to CLTLoc induces a PSPACE decision procedure also when the constants are encoded in binary.

In fact, consider an $\text{MITL}_{0,\infty}$+Past formula ϕ, and the corresponding equisatisfiable CLTLoc Formula (12). As in the case of the translation for $\text{MITL}_{0,\infty}$, even with the new modalities the size of Formula (12) linearly depends on the size of ϕ and on the binary encoding of constants. Since the satisfiability of a CLTLoc formula is PSPACE in the number of subformulae and in the binary encoding of the constants, the decision procedure induced by our encoding is in PSPACE.

Since, by Proposition 2, an MITL+Past formula may be translated into an equivalent $\text{MITL}_{0,\infty}$+Past formula, whose size is however exponential in the binary encoding of the maximum constant occurring in the formula, it follows:

Corollary 3. *The reduction of MITL+Past to CLTLoc induces an EXPSPACE decision procedure for the f.v. satisfiability problem, when the constants are encoded in binary, and a PSPACE procedure when the constants are encoded in unary.*

This is in line with the well-known fact that the satisfiability of MITL is EXPSPACE-complete [6] when the constants are encoded in binary.

6.1. A direct encoding for $\mathbf{F}_{(a,b)}$

Here we extend the encoding to include also subformulae of the form $\mathbf{F}_{(a,b)}(\gamma)$. The case of the form $\mathbf{P}_{(a,b)}(\gamma)$ is similarly defined and can be found in Appendix A. Although this direct encoding is not necessary for proving equisatisfiability and theoretical complexity results, nonetheless it may be smaller than the encoding obtained by eliminating $\mathbf{F}_{(a,b)}(\gamma)$ using Proposition 2. For instance, the elimination applied to formula $\mathbf{F}_{(10,11)}(\gamma)$ actually produces 20 more subformulae, each one with a pair of clocks and three predicates (with the associated formulae), so in total 42 clocks and 63 predicates; our direct encoding only deals with one subformula, adding in total 22 clocks and only three predicates.

The encoding can easily be extended to provide direct support also for intervals of the form $[a,b]$, $(a,b]$ or $[a,b)$, but these cases are omitted for brevity.

First of all, we remark that Lemma 4 holds also for formulae of the form $\mathbf{F}_{(a,b)}(\gamma)$. As a consequence, change points where s^u_θ holds cannot occur for these kinds of formulae. In case of subformulae of the form $\theta = \mathbf{F}_{(a,b)}(\gamma)$ we introduce, in addition to clocks z^0_θ, z^1_θ of Section 5, $d = 2\left\lceil \frac{b}{b-a} \right\rceil$ *auxiliary clocks* $\{x^j_\theta\}_{j\in\{0,\ldots,d-1\}}$ which, by Corollary 2, are enough to store the time elapsed since the occurrence of change points for θ between the current time instant t and $t+b$. Observe that clocks z^0_θ and z^1_θ could be removed, since in any discrete position their value can equivalently be obtained from the value of the last auxiliary clock that has been reset. However, we still use them to obtain a simpler translation. The behavior of the auxiliary clocks is defined by the following formulae.

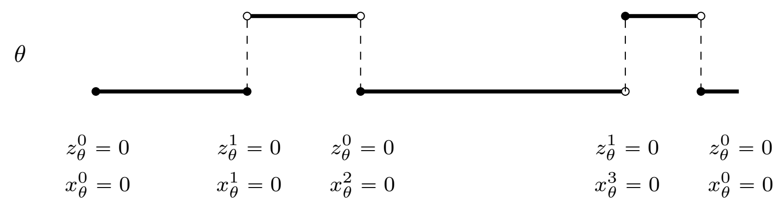


Figure 4: Sequence of circular resets for a formula $\mathbf{F}_{(a,b)}(\gamma)$ with four auxiliary clocks.

Each reset $x_\theta^i = 0$ entails one of $e_\theta^u, e_\theta^d, s_\theta^d$ and each change point is marked by a single reset $x_\theta^i = 0$ (Formula (19)).

$$\left(\lrcorner\!\ulcorner_\theta \vee \overset{\theta}{\urcorner\!\llcorner} \Leftrightarrow \bigvee_{i=0}^{d-1} x_\theta^i = 0 \right) \wedge \left(\bigwedge_{i=0}^{d-1} \bigwedge_{j=0, i \neq j}^{d-1} \neg (x_\theta^i = 0 \wedge x_\theta^j = 0) \right) \tag{19}$$

The occurrence of resets for clocks x_θ^i is circularly ordered and the sequence of resets starts from the origin by x_θ^0 (see an example in Figure 4). If $x_\theta^i = 0$, then, from the next position, all the other clocks are strictly greater than 0 until the next $x_\theta^{(i+1) \bmod d} = 0$ occurs.

$$\bigwedge_{i=0}^{d-1} \left(x_\theta^i = 0 \Rightarrow \mathbf{X} \left((x_\theta^{(i+1) \bmod d} = 0) \mathbf{R} \bigwedge_{j \in [0, d-1],\, j \neq i} (x_\theta^{(j+1) \bmod d} > 0) \right) \right) \tag{20}$$

Formula $x_\theta^0 = 0$, evaluated at position 0, sets the first reset of the sequence, constrained by formulae (19)-(20).

Formula $(x_\theta^0 = 0) \wedge \mathbf{G}\left((19) \wedge (20)\right)$ is denoted as $\texttt{auxck}_\theta$.

The next formulae capture the semantics of the $\mathbf{F}_{(a,b)}$ modality. For the sake of simplicity, the translation only considers the case $a > 0$ although a general translation including the case $a = 0$ of Section 5 can be devised. However, dealing with two different translations is simpler and allows one to obtain much more efficient decision procedures based on a direct translation of the metric modalities that actually occur in a formula. Because of Lemma 4, an up-singularity $\lrcorner\!\urcorner\!\llcorner_\theta$ can never occur for $\theta = \mathbf{F}_{(a,b)}(\gamma)$. Then, as for $\mathbf{F}_{(0,b)}$, Formula (7) holds in every instant.

Formula (21) is similar to (8); it differs from the latter in that it specifies that, for θ to have a rising edge, γ must become true after time b, but it only needs to be false in the $b - a$ instants before b, rather than throughout the interval up to time b.

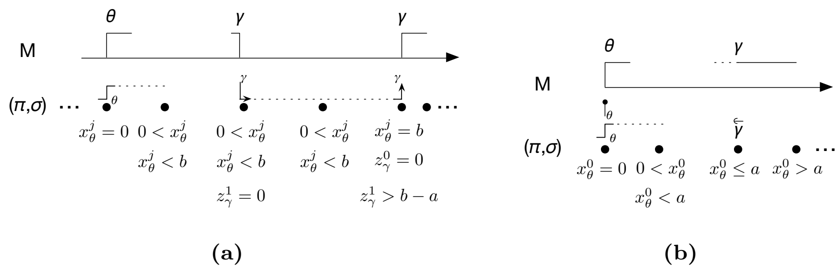


Figure 5: Examples of conditions for rising edges for $\theta = \mathbf{F}_{(a,b)}(\gamma)$ at $t > 0$ (a) and $t = 0$ (b).

$$\Lsh_\theta \Leftrightarrow \left(\begin{array}{c} O \wedge \left(\begin{array}{c} \upharpoonleft_\theta \wedge (O \vee x_\theta^0 > 0)\, \mathbf{U} \left(\begin{array}{c} \overset{\gamma}{\Lsh} \wedge a < x_\theta^0 < b \,\vee \\ \overleftarrow{\gamma} \wedge x_\theta^0 \leqslant a \wedge \mathbf{X}(x_\theta^0 > a) \end{array} \right) \vee \\ \neg \upharpoonleft_\theta \wedge \mathbf{X} \left(x_\theta^0 > 0\, \mathbf{U} \left(\overset{\gamma}{\Lsh} \wedge x_\theta^0 = b \wedge \bigvee_{i\in\{0,1\}} z_\gamma^i \geqslant b - a \right)\right) \end{array} \right) \vee \\ \neg O \wedge \neg \upharpoonleft_\theta \wedge \bigvee_{j=0}^{d-1} \left((x_\theta^j = 0) \wedge \mathbf{X} \left(x_\theta^j > 0\, \mathbf{U} \left(\overset{\gamma}{\Lsh} \wedge x_\theta^j = b \wedge \bigvee_{i\in\{0,1\}} z_\gamma^i > b - a \right)\right)\right) \end{array} \right) \quad (21)$$

Fig. 5 shows a pair of examples of conditions in which $\theta = \mathbf{F}_{(a,b)}(\gamma)$ has a rising edge. Fig. 5(a) depicts a case in which the second disjunct of the right-hand side of Formula (21) is true. In this case at position $k > 0$ of the CLTLoc interpretation, which corresponds to an instant $t > 0$ in the signal, one of the d auxiliary clocks of θ, say x_θ^j, is reset and exactly when it takes value b subformula γ becomes true (hence one of its associated clocks, say z_γ^0 is reset); in addition, the clock that is not reset when γ becomes true (z_γ^1 in our example) has value that is greater that $b - a$, which entails that the last time γ became false was more than $b-a$ instants before, hence θ is false before instant t, and it becomes true at t. Similarly, Fig. 5(b) depicts a case in which θ becomes true in the origin (when x_θ^0 is reset by definition), and corresponds to the first condition of the right-hand side of Formula (21). In the example represented, γ is true in an interval that starts before (or when) x_θ^0 takes value a, and ends when x_θ^0 is greater than a; hence, this interval includes an interval $(a, a + \varepsilon)$, so $\mathbf{F}_{(a,b)}(\gamma)$ has a rising edge in 0, and also it holds in 0 itself (i.e., $\upharpoonleft_\theta$ is true in 0).

Formulae (22)-(25) are similar to (9)-(11).

$$Now \geqslant b \wedge \overset{\gamma}{\Lsh} \wedge \bigvee_{i\in\{0,1\}} z_\gamma^i \geqslant (b-a) \Rightarrow \bigvee_{j=0}^{d-1} x_\theta^j = b \quad (22)$$

$$\urcorner\!\llcorner_\theta \Leftrightarrow \left(\begin{array}{l} O \wedge \neg \lrcorner\!\ulcorner_\theta \qquad \vee \\ \bigvee_{j=0}^{d-1} \left(x_\theta^j = 0 \wedge \mathbf{X}\left((x_\theta^j > 0) \mathbf{U} \left(\overset{\gamma}{\text{↳}} \wedge x_\theta^j = a \wedge \neg \mathbf{X} \left(\not\overset{\gamma}{\text{⬏}} \, \mathbf{U} \left(\overset{\gamma}{\text{⬏}} \wedge a < x_\theta^j \leqslant b \right)\right)\right)\right)\right) \end{array} \right) \tag{23}$$

$$Now \geqslant a \wedge \overset{\gamma}{\text{↳}} \wedge \mathbf{X} \left(\overset{\gamma}{\text{⬏}} \, \mathbf{R} \neg \left(\overset{\gamma}{\text{⬏}} \wedge \bigwedge_{i=\{0,1\}} z_\gamma^i \leqslant (b-a) \right)\right) \Rightarrow \bigvee_{j=0}^{d-1} x_\theta^j = a \tag{24}$$

$$\urcorner\!\sqcup\!\ulcorner_\theta \Leftrightarrow \neg O \wedge \bigvee_{j=0}^{d-1} \left(x_\theta^j = 0 \wedge \mathbf{X}\left((x_\theta^j > 0) \mathbf{U} \left(\overset{\gamma}{\text{↳}} \wedge x_\theta^j = a \ \wedge \mathbf{X} \left(\not\overset{\gamma}{\text{⬏}} \, \mathbf{U} \left(\overset{\gamma}{\text{⬏}} \wedge x_\theta^j = b \right)\right)\right)\right)\right) \tag{25}$$

Then, $m(\theta)$ is (7) $\wedge$ (21) $\wedge$ (22) $\wedge$ (23) $\wedge$ (24) $\wedge$ (25).

Given an MITL+Past formula ϕ also with temporal modalities of the form $\mathbf{F}_{(a,b)}$, define the corresponding CLTLoc formula as:

$$\mathtt{init}_\phi \wedge \bigwedge_{\theta \in sub(\phi)} (\mathtt{ck}_\theta \wedge \mathbf{G}\,(m(\theta))) \wedge \bigwedge_{\substack{\theta \in sub(\phi) \\ \theta = \mathbf{F}_{(a,b)}(\gamma)}} \mathtt{auxck}_\theta. \tag{26}$$

Proof of correctness of the encoding for $\theta = \mathbf{F}_{(a,b)}(\gamma)$. To show the correctness of the translation we first extend mapping $r_{\mathcal{F}}(M)$ to include also the auxiliary clocks, which are introduced in a similar manner as z_θ^0, z_θ^1. First, for all positions $k \geqslant 0$, $\sigma_{\mathcal{F}}(k, z_\theta^0) = 0$ or $\sigma_{\mathcal{F}}(k, z_\theta^1) = 0$ if, and only if, $\bigvee_{j=0}^{d-1} \sigma_{\mathcal{F}}(k, x_\theta^j) = 0$, i.e, whenever a change point for θ occurs, an auxiliary clock is reset. To avoid simultaneous resets of different clocks, if x_θ^j is reset then no $x_\theta^{j'}$ is reset, for $j' \neq j$. Auxiliary clocks are circularly reset modulo d; i.e., if x_θ^j is reset at position k, then the next reset of x_θ^j, if it exists, occurs in a position $k' > k$ such that all other clocks $x_\theta^{j'}$ $(j' \neq j)$ are reset, in order, exactly once in (k, k'). Note that, if a clock x_θ^j is reset at position k, the next position k' when the clock is reset must be such that $t_{k'} > t_k + b$, i.e., given a formula $\theta = \mathbf{F}_{(a,b)}(\gamma)$, every clock x_θ^j is reset only once over intervals of length b. The sequence of resets starts with $x_\theta^0 = 0$.

The following lemma is the analogous of Lemma 9 for $\mathtt{auxck}_\theta$ (for brevity we omit its proof, which is similar to the one of Lemma 9).

Lemma 12. *Let M be a signal and $\mathcal{F}$ be a finite set of MITL+Past formulae. For all interpretations $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}) \in r_{\mathcal{F}}(M)$ we have $(\pi_{\mathcal{F}}, \sigma_{\mathcal{F}}), 0 \models \bigwedge_{\substack{\theta \in \mathcal{F} \\ \theta = \mathbf{F}_{(a,b)}(\gamma)}} \mathtt{auxck}_\theta$.*

Then, we have the following result, which extends Lemma 10.

Lemma 13. *Let M be a signal, and ϕ be an MITL+Past formula. For any $(\pi, \sigma) \in r_{sub(\phi)}(M)$ we have $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta \wedge \bigwedge_{\substack{\theta \in sub(\phi) \\ \theta = \mathbf{F}_{(a,b)}(\gamma)}} \mathtt{auxck}_\theta$ and*

for all $k \in \mathbb{N}, \theta \in sub(\phi)$ *we have* $(\pi, \sigma), k \models m(\theta)$.
Conversely, if $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} (\mathtt{ck}_\theta \wedge \mathbf{G}(m(\theta))) \wedge \bigwedge_{\substack{\theta \in sub(\phi) \\ \theta = \mathbf{F}_{(a,b)}(\gamma)}} \mathtt{auxck}_\theta$*, then there is a signal* M *such that* $(\pi, \sigma) \in r_{sub(\phi)}(M)$.

Proof. The proof has the same structure as those of Lemmata 10 and 11. We focus on the case $\theta = \mathbf{F}_{(a,b)}(\gamma)$.

Part 1.
Lemmata 9 and 12 guarantee that $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta \wedge \bigwedge_{\substack{\theta \in sub(\phi) \\ \theta = \mathbf{F}_{(a,b)}(\gamma)}} \mathtt{auxck}_\theta$.

Then, suppose $t_k \in T$. To deal with $\mathbf{F}_{(a,b)}(\gamma)$, we need to consider three cases: $M, t_k \models e^u_\theta$, $M, t_k \models e^d_\theta$ and $M, t_k \models s^d_\theta$. Since they are very similar to those of case $\mathbf{F}_{(0,b)}(\gamma)$ in Lemma 10, we only briefly sketch them.

Subcase $\mathbf{e^u_\theta}$. We focus on the case $M, t_k \models e^u_\theta$ (the case $M, t_k \not\models e^u_\theta$ is analogous to the one of Lemma 10).
Suppose $t_k > 0$. For θ to become true in t_k, it must be that γ holds at $t_k + b$, or there is $\varepsilon > 0$ such that γ holds in interval $(t_k + b, t_k + b + \varepsilon)$ and it does not hold throughout interval $(t_k + a, t_k + b)$. Hence, the case is similar to the one for $\mathbf{F}_{(0,b)}(\gamma)$, with the only difference that we use auxiliary clocks x^j_θ instead of z^j_θ, and the value of z^j_γ must be $> b - a$ instead of $> b$.
If $t_k = 0$, we have two cases. The case where γ becomes true in b, but it is false throughout (a, b), is very similar to the one for $t_k > 0$ (except that the clock that is reset in t_k is z^0_θ). The other case is when there is $a < t < b$ where γ is true, hence θ in 0 becomes true in a left-closed manner (i.e., $\upharpoonleft_\theta$ holds in 0). For γ to be true sometimes in (a, b), there must be a position k' such that either (i) $a < t_{k'} < b$ and γ becomes true in $t_{k'}$ (i.e., $(\pi, \sigma), k' \models \overset{\gamma}{\lrcorner\!\!\uparrow}$), or (ii) $t_{k'} \leqslant a < t_{k'+1}$ and γ holds throughout $(t_{k'}, t_{k'+1})$. By the usual arguments, this is captured by the second part of the first disjunct of Formula (21).
All in all, Formula (21) holds at position k if $M, t_k \models e^u_\theta$.

Also, by similar arguments as those used for Formula (9) in the proof of Lemma 10, Formula (22) holds at all positions $k \in \mathbb{N}$.

Subcase $\mathbf{e^d_\theta}$. Suppose θ has a falling edge in t_k. If $t_k = 0$, this is equivalent e^u_θ holding, since singularities cannot happen there, i.e., in $k = 0$ we have $O \wedge \neg \ulcorner_\theta$. Otherwise, it must be that γ holds in $t_k + a$ or in an interval $(t_k - \varepsilon + a, t_k + a)$ for some $\varepsilon > 0$, and it does not become true throughout interval $(t_k + a, t_k + b]$. Formula γ cannot become true in $t_k + b$, or θ is true right after t_k, whereas we are assuming that in t_k it has a falling edge. Then, we have $M, t_k + a \models e^d_\gamma \vee s^u_\gamma$, so by definition of $r_{sub(\theta)}(M)$ there is k' such that $t_{k'} = t_k + a$ and we have $(\pi, \sigma), k' \models \overset{\gamma}{\llcorner\!\!\rightarrow}$. In addition, one of the x^j_θ clocks is reset in k, which is not reset again until after k', hence $x^j_\theta = a$ in k' and the next time γ becomes true again (i.e., $\overset{\gamma}{\lrcorner\!\!\uparrow}$ holds), x^j_θ cannot be $\leqslant b$. Then, Formula (23) holds at k.

If instead $M, t_k \not\models e^d_\theta$, then if $t_k > 0$ no auxiliary clock associated with θ is reset in k; if, instead, $t_k = 0$, then γ is true is $(0, \varepsilon)$, for some $\varepsilon > 0$. In both cases the right hand side of Formula (23) does not hold, so Formula (23) does, since, by definition of $r_{sub(\theta)}(M)$ we have $(\pi,\sigma),k \not\models \urcorner_\theta$.

Similar arguments as those used for Formula (9) in the proof of Lemma 10, show that Formula (24) holds at all positions $k \in \mathbb{N}$.

Subcase $\mathbf{s}^{\mathbf{d}}_{\theta}$. Suppose θ has a down-singularity in t_k. Then, by definition we have $t_k > 0$. In this case the conditions are similar to those for e^d_θ, except that exactly at $t_{k'} = t_k + b$ formula γ becomes true again (i.e., $\overset{\gamma}{\lrcorner} \wedge x_j = b$ holds in k'). The truth of Formula (25) descends from there.

Part 2.

The proof that, if $(\pi,\sigma),0 \models \bigwedge_{\theta\in sub(\phi)} \mathtt{ck}_\theta \wedge \mathbf{G}\,(m(\theta)) \wedge \bigwedge_{\substack{\theta\in sub(\phi)\\ \theta=\mathbf{F}_{(a,b)}(\gamma)}} \mathtt{auxck}_\theta$, then there is a signal M such that $M = r^{-1}_{sub(\phi)}((\pi,\sigma))$ is analogous to the corresponding ones in Lemmata 10 and 11, so we omit it for brevity. □

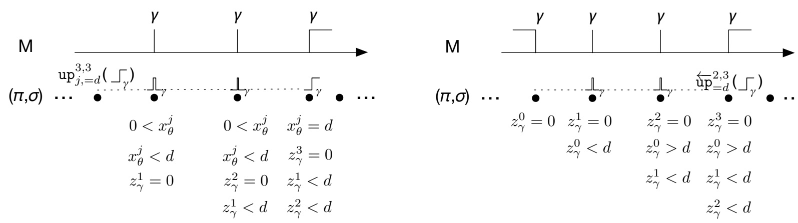
Complexity of the translation. Consider an MITL+Past formula ϕ with occurrences of $\mathbf{F}_{(a,b)}$ and the corresponding equisatisfiable CLTLoc Formula (26). Let K be the maximum constant appearing in ϕ. Then, the size of the CLTLoc translation is $O(|\phi|K)$, i.e., it is exponential in the size of the binary encoding of K. Since the satisfiability of a CLTLoc formula is PSPACE in the size of the formula and in the binary encoding of the constants, the decision procedure induced by our encoding is in EXPSPACE, as expected.

7. Reduction of counting modalities to CLTLoc

The $\boldsymbol{\mathcal{C}}_n$ operator is a generalization of $\mathbf{F}_{(0,1)}$, since $\boldsymbol{\mathcal{C}}_1(\gamma) = \mathbf{F}_{(0,1)}(\gamma)$. To capture its semantics, we need to introduce more clocks than used in describing the semantics of $\mathbf{F}_{(0,1)}$, both for $\theta = \boldsymbol{\mathcal{C}}_1(\gamma)$ and γ. Precisely, we introduce n_γ clocks $z^0_\gamma, \dots, z^{n_\gamma-1}_\gamma$ for subformula γ, with $n_\gamma \geqslant n+1$, and $c_\theta = 2n+1$ clocks $x^0_\theta, \dots, x^{c_\theta-1}_\theta$ for subformula θ. Note that the exact number of necessary n_γ clocks depends on the operators in which γ appears. For example, if n' is the largest number such that there is a subformula of the form $\boldsymbol{\mathcal{C}}_{n'}(\gamma)$, $n_\gamma = n'+1$. If γ does not appear in a formula of the form $\boldsymbol{\mathcal{C}}_n(\gamma)$ or of the form $\mathbf{F}_I(\gamma)$, then $n_\gamma = 2$. Similarly, if formula $\theta = \boldsymbol{\mathcal{C}}_n(\gamma)$ itself appears in a formula of the form $\boldsymbol{\mathcal{C}}_{n'}(\theta)$, then it will be associated with both $n_\theta = n'+1$ clocks z^j_θ and c_θ clocks x^j_θ. Since clocks z^j_θ and x^j_θ play very similar roles, one could introduce a single set of clocks with cardinality the maximum of n_θ and c_θ. However, in the following encoding for reasons of clarity we keep the sets separate.

Consider now a formula ϕ that includes counting modalities. Its translation to CLTLoc has, in addition to the parts introduced in Section 5 and 6 (with the necessary adjustments to take into account the fact that the number of z^j_γ clocks can be more than 2), the translation of the counting modalities. In the rest of this section we show the translation $m(\theta)$ for $\boldsymbol{\mathcal{C}}_n$, with $n \geqslant 1$.

Clocks x^j_θ play a role similar to those with the same name introduced in Section 6, so their behavior is similarly governed by formula $\mathtt{auxck}_\theta$. In addition,



(a) $\mathtt{up}^{3,3}_{j,=d}(\lrcorner\!\ulcorner_\gamma)$ **(b)** $\overleftarrow{\mathtt{up}}^{2,3}_{=d}(\lrcorner\!\ulcorner_\gamma)$

Figure 6: A first batch of abbreviations:

$\mathtt{up}^1_{j,\sim d}(B_\gamma) = \mathbf{X}\Big(x^j_{\theta,\gamma} > 0\ \mathbf{U}\ \Big(B_\gamma \wedge 0 < x^j_\theta \wedge x^j_\theta \sim d\Big)\Big)$

$\mathtt{up}^n_{j,\sim d}(B_\gamma) = \mathbf{X}\ \Big(x^j_{\theta,\gamma} > 0\ \mathbf{U}\Big(\lrcorner\!\llcorner_\gamma \wedge 0 < x^j_\theta < d \wedge \mathtt{up}^{n-1}_{j,\sim d}(B_\gamma)\Big)\Big)$ for $n \geqslant 2$

$\mathtt{up}^{n,p}_{j,\sim d}(B_\gamma) = \mathtt{up}^n_{j,\sim d}(B_\gamma \wedge z^{\hat{p}}_\gamma \leqslant d)$ for $n \geqslant 1$

$\overleftarrow{\mathtt{nspikes}}_1(\gamma) = \mathbf{Y}(\overset{\gamma}{\not\nearrow}\mathbf{S}(\lrcorner\!\llcorner_\gamma \vee (\uparrow_\gamma \wedge \neg \overleftarrow{\gamma} \wedge O)))$

$\overleftarrow{\mathtt{nspikes}}_n(\gamma) = \mathbf{Y}(\overset{\gamma}{\not\nearrow}\mathbf{S}(\lrcorner\!\llcorner_\gamma \wedge \overleftarrow{\mathtt{nspikes}}_{n-1}(\gamma)))$ for $n \geqslant 2$

$\overleftarrow{\mathtt{up}}^{0,p}_{=d}(B_\gamma) = B_\gamma \wedge z^{\hat{p}}_\gamma < d$

$\overleftarrow{\mathtt{up}}^{n,p}_{=d}(B_\gamma) = B_\gamma \wedge z^{\hat{p}}_\gamma < d \wedge \overleftarrow{\mathtt{nspikes}}_n(\gamma)$ for $n \geqslant 1$

since Lemma 4 holds also for the $\boldsymbol{\mathcal{C}}_n$ modalities, an up-singularity $\lrcorner\!\llcorner_\theta$ can never occur for a formula of the form $\boldsymbol{\mathcal{C}}_n(\gamma)$, and Formula 7 is introduced as for the **F** modality.

For the sake of readability, some shorthands are useful. Let $x^j_{\theta,\gamma} > 0$ stand for $(x^j_\theta > 0 \wedge \bigwedge_{i\in\{0,\dots,n_\gamma-1\}} z^i_\gamma > 0)$, where n_γ is the number of clocks introduced for γ. We also write $z^{\hat{p}}_\gamma \sim d$ (where $\sim \in \{<, \leqslant, =, \geqslant, >\}$) to state that there are exactly p clocks of γ satisfying $\sim d$. The following Formula (27) defines $z^{\hat{p}}_\gamma \sim d$ (where $\oplus$ is the sum modulo n_γ).

$$z^{\hat{p}}_\gamma \sim d = \bigvee_{i=0}^{n_\gamma-1}\left(\bigwedge_{j\in\{i\oplus 1,\dots,i\oplus p\}} (z^i_\gamma \sim d) \wedge \bigwedge_{j\in\{i,\dots,i\oplus(p+1)\}} (z^j_\gamma \not\sim d)\right) \qquad (27)$$

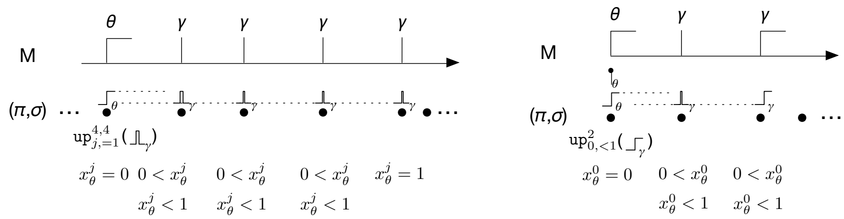
Let B_γ denote a CLTLoc formula associated with γ (e.g., a Boolean combination of $\lrcorner\!\ulcorner_\gamma$, $\overset{\gamma}{\lrcorner\!\uparrow}$, $z^{\hat{p}}_\gamma \sim d$, etc.). Fig. 6 recursively defines $\mathtt{up}^n_{j,\sim d}(B_\gamma)$, whose intuitive meaning is that it holds in every instant such that: 1) there is a future time instant t such that clock x^j_θ has value $\sim d$, B_γ holds; 2) γ has at least $n-1$ up-singularities (i.e., instants where $\lrcorner\!\llcorner_\gamma$ holds) before t. To ensure that, moreover, in instant t above there are only p clocks associated with γ whose value is $\leqslant d$ (i.e., γ has changed value p times between the instants in which x^j_θ was 0 and d), we also define $\mathtt{up}^{n,p}_{j,\sim d}(B_\gamma)$. Fig. 6(a) depicts a situation in which $\mathtt{up}^{3,3}_{j,=d}(\lrcorner\!\ulcorner_\gamma)$ holds. Notice that the formula $\mathtt{up}^n_{0,\sim d}(B_\gamma)$ will be used only when evaluated in the origin O: the number of change points of γ before B_γ holds is certainly n, so parameter p is unnecessary.

Fig. 6 also introduces shortands similar to $\mathtt{up}^{n,p}_{j,=d}(B_\gamma)$, but which refer to the interval *before* B_γ holds. Formula $\overleftarrow{\texttt{nspikes}}_n(\gamma)$ holds if the last n times when γ changed value before the current instant are of the form $\lrcorner\!\sqcap\!\llcorner_\gamma$. Then, formula $\overleftarrow{\texttt{up}}^{n,p}_{=d}(B_\gamma)$ holds if B_γ holds, the last n times when γ changed value were up-singularities, and the number of clocks associated with γ that are less than d is p, hence, if $p = n+1$, all n "spikes" occurred within the last d time units. Fig. 6(b) shows an example of $\overleftarrow{\texttt{up}}^{2,3}_{=d}(\lrcorner\!\ulcorner_\gamma)$ holding.

Using the abbreviations of Fig. 6, we capture through CLTLoc formulae the conditions that make $\theta = \boldsymbol{\mathcal{C}}_n(\gamma)$ have a rising edge (i.e., that corresponds to $\lrcorner\!\ulcorner_\theta$). Formula (28) describes that, when θ becomes true with a rising edge $\lrcorner\!\ulcorner_\theta$ in an instant $t > 0$, then it does so in a left-open manner (i.e., θ does not hold in t), a clock x^j_θ is reset, and (i) either γ has $n-1$ up-singularities before x^j_θ hits 1 and γ becomes true again also with an up-singularity when $x^j_\theta = 1$, or (ii) γ has a rising edge when $x^j_\theta = 1$ (hence it is true infinitely many times in a right neighborhood of that instant) and it also has *up to* $n-1$ (possibly 0) up-singularities before $x^j_\theta = 1$. If instead θ becomes true in $t = 0$ in a left-closed manner (i.e., θ holds in t; the left-open case is similar to the one above), before clock $x^0_\theta = 1$ either γ has a rising edge (so it is true infinitely many times before $x^0_\theta = 1$) preceded by up to $n-1$ (possibly 0) up-singularities, or there are n up-singularities before $x^0_\theta = 1$.

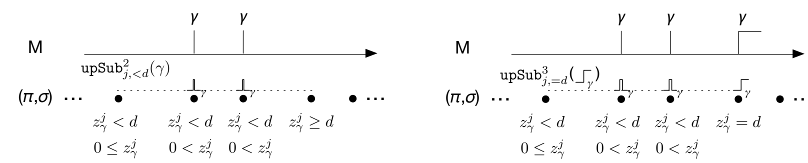
$$\lrcorner\!\ulcorner_\theta \Leftrightarrow \left(\begin{array}{l} O \wedge \left(\begin{array}{l} \text{⫯}_\theta \wedge \left(\lrcorner\!\ulcorner_\gamma \vee \texttt{up}^n_{0,<1}(\lrcorner\!\sqcap\!\llcorner_\gamma) \vee \bigvee\limits_{k \in \{1,\dots,n\}} \texttt{up}^k_{0,<1}(\lrcorner\!\ulcorner_\gamma) \right) \vee \\ \neg\, \text{⫯}_\theta \wedge \left(\texttt{up}^n_{0,=1}(\lrcorner\!\sqcap\!\llcorner_\gamma) \vee \bigvee\limits_{k \in \{1,\dots,n\}} \texttt{up}^k_{0,=1}(\lrcorner\!\ulcorner_\gamma) \right) \end{array} \right) \vee \\ \neg O \wedge \bigvee\limits_{j=0}^{c_\theta - 1} \left(\neg\, \text{⫯}_\theta \wedge x^j_\theta = 0 \wedge \left((\texttt{up}^{n,n}_{j,=1}(\lrcorner\!\sqcap\!\llcorner_\gamma) \wedge \not\lrcorner\!\sqcap\!\llcorner_\gamma) \vee \bigvee\limits_{k \in \{1,\dots,n\}} \texttt{up}^{k,k}_{j,=1}(\lrcorner\!\ulcorner_\gamma) \right) \right) \end{array} \right) \tag{28}$$

Fig. 7 shows a pair of examples of conditions corresponding to $\theta = \boldsymbol{\mathcal{C}}_4(\gamma)$ having a rising edge. In particular, Fig. 7(a) depicts a case in which the second disjunct of the right-hand side of Formula (28) holds. In this case, at $t > 0$, corresponding to position $k > 0$ of the CLTLoc interpretation, one of the c_θ clocks associated with θ, say x^j_θ, is reset (note that in this example $c_\theta = 9$, as we are considering $n = 4$). Also, between t and the instant in which x^j_θ takes value 1 there are exactly 3 other instants in which γ has an up-singularity, and when $x^j_\theta = 1$ subformula γ has another up-singularity. All in all $\texttt{up}^{4,4}_{j,=1}(\lrcorner\!\sqcap\!\llcorner_\gamma)$ holds at t, and so does the second disjunct of Formula (28). Fig. 7(b), instead, shows a situation in which $\theta = \boldsymbol{\mathcal{C}}_4(\gamma)$ has a rising edge in $t = 0$ (where x^0_θ is reset). More precisely, the case depicted corresponds to the first condition of Formula (28) being true. In fact, at an instant in which $x^0_\theta < 1$ subformula γ has a rising edge and between 0 an that instant there is one point in which γ has an up-singularity. Hence, in 0 formula $\texttt{up}^2_{0,<1}(\lrcorner\!\ulcorner_\gamma)$ holds, and so does the first condition of Formula (28).



(a) **(b)**

Figure 7: Examples of conditions for rising edges for $\theta = \boldsymbol{\mathcal{C}}_4(\gamma)$ at $t > 0$ (a) and $t = 0$ (b).



(a) $\mathtt{upSub}^2_{j,<d}(\gamma)$ **(b)** $\mathtt{upSub}^3_{j,=d}(\ulcorner_\gamma)$

Figure 8: A second batch of abbreviations:
$\mathtt{upSub}^0_{j,\lesssim d}(\gamma) = \neg\mathbf{X}(\overset{\gamma}{\not\uparrow}\,\mathbf{U}(\overset{\gamma}{\uparrow} \wedge 0 < z^j_\gamma \leqslant d))$
$\mathtt{upSub}^n_{j,\lesssim d}(\gamma) = \mathbf{X}(\overset{\gamma}{\not\uparrow}\,\mathbf{U}(\sqcup\!\sqcap\!\sqcup_\gamma \wedge 0 < z^j_\gamma \lesssim d \wedge \mathtt{upSub}^{n-1}_{j,\lesssim d}(\gamma)))$
$\mathtt{upSub}^1_{j,=d}(B_\gamma) = \mathbf{X}(\overset{\gamma}{\not\uparrow}\,\mathbf{U}(B_\gamma \wedge z^j_\gamma = d))$
$\mathtt{upSub}^n_{j,=d}(B_\gamma) = \mathbf{X}(\overset{\gamma}{\not\uparrow}\,\mathbf{U}(\sqcup\!\sqcap\!\sqcup_\gamma \wedge 0 < z^j_\gamma < d \wedge \mathtt{upSub}^{n-1}_{j,d}(B_\gamma)))$

Formula (29) states that if t is an instant (such that $t \geqslant 1$) in which either (i) in the preceding interval of length 1 γ has $n-1$ up-singularities and γ also becomes true in t with an up-singularity (i.e., $\overleftarrow{\mathtt{up}}^{n-1,n}_{<1}(\sqcup\!\sqcap\!\sqcup_\gamma)$ holds), or (ii) γ has a rising edge and in the preceding interval of length 1 γ has *at most* $n-1$ up-singularities (i.e., $\overleftarrow{\mathtt{up}}^{k-1,k}_{=1}(\ulcorner_\gamma)$ holds for some $k \leqslant n$), then in t one of the clocks associated with θ must be 1 (indeed, $\boldsymbol{\mathcal{C}}_n(\gamma)$ started to hold exactly 1 time unit before t, see also Fig. 6(b)), and all others are greater than 1. Formula (29) plays a similar role as Formula (9): it makes sure that, if in the interval of length 1 preceding t the conditions hold for $\boldsymbol{\mathcal{C}}_n(\gamma)$ to become true in $t-1$, then the right hand side of Formula (28) holds at the position corresponding to $t-1$, thus forcing $\ulcorner_\theta$ to hold there.

$$\mathit{Now} \geqslant 1 \;\wedge\; \left(\overleftarrow{\mathtt{up}}^{n-1,n}_{=1}(\sqcup\!\sqcap\!\sqcup_\gamma) \vee \bigvee_{k \in \{1,\ldots,n\}} \overleftarrow{\mathtt{up}}^{k-1,k}_{=1}(\ulcorner_\gamma)\right) \Rightarrow \bigvee_{i \in \{0,\ldots,c_\theta - 1\}} x^i_\theta = 1. \tag{29}$$

To describe the conditions under which θ becomes false, either with a falling edge (i.e., $\rceil\!\lfloor_\theta$ holds), or with a singularity (i.e., $\rceil\!\lceil_\theta$ holds) we introduce a pair of further shorthands, shown in Fig. 8. Formula $\mathtt{upSub}^0_{j,\lesssim d}(\gamma)$ (where $\lesssim \in \{<, \leqslant\}$) holds if, from the current instant (excluded) until the instant when clock z^j_γ hits value d (included), γ never becomes true. Then, $\mathtt{upSub}^n_{j,\lesssim d}(\gamma)$ holds if, in the interval that starts in the current instant and ends when clock $z^j_\gamma = d$ (both endpoints excluded if $\lesssim$ is $<$), γ has exactly n up-singularities. Fig. 8(a) exemplifies when $\mathtt{upSub}^2_{j,<d}(\gamma)$ holds. Note that if there are at least $n+1$ clocks associated with γ, it may be the case (if z^j_γ has been reset "recently") that $\mathtt{upSub}^n_{j,<d}(\gamma)$ holds and z^j_γ is not reset before it becomes d. Similarly, $\mathtt{upSub}^n_{j,=d}(B_\gamma)$ holds if, in the interval that starts in the current instant and ends when $z^j_\gamma = d$ (endpoints excluded), γ has $n-1$ up-singularities, and B_γ holds when $z^j_\gamma = d$. Fig. 8(b) depicts a case where $\mathtt{upSub}^3_{j,=d}(\rfloor\!\lceil_\gamma)$ holds.

When $\theta = \mathcal{C}_n(\gamma)$ becomes false with either a falling edge ($\rceil\!\lfloor_\theta$) or in a singular manner ($\rceil\!\lceil_\theta$), γ becomes false, and a clock z^i_γ is reset. Consider first the former condition (Formula (30)). There are two cases: γ becomes false with a falling edge $\rceil\!\lfloor_\gamma$, or it has an up-singularity $\rfloor\!\lfloor_\gamma$. In the former case, γ can have *up to* $n-1$ up-singularities before $z^i_\gamma = 1$: it can have less than $n-1$, since γ holds infinitely many times before it has a falling edge. In the latter case, γ must have *exactly* $n-1$ up-singularities before $z^i_\gamma = 1$, or θ does not have a falling edge.

$$\rceil\!\lfloor_\theta \Leftrightarrow \overset{\gamma}{\hookrightarrow} \wedge \bigwedge_{i=0}^{n_\gamma-1} \left(z^i_\gamma = 0 \Rightarrow \begin{pmatrix} \rceil\!\lfloor_\gamma \wedge \bigvee_{k\in\{0,\dots,n-1\}} \mathtt{upSub}^k_{i,\leqslant 1}(\gamma) & \vee \\ \rfloor\!\lfloor_\gamma \wedge \mathtt{upSub}^{n-1}_{i,<1}(\gamma) & \end{pmatrix} \right). \tag{30}$$

Finally, as captured by Formula (31), for θ to have a down-singularity $\rceil\!\lceil_\theta$, not only γ must become false with $\rceil\!\lceil_\theta$, but it must also become true again exactly when the clock z^i_γ, which is reset with $\rceil\!\lceil_\theta$, takes value 1.

$$\rceil\!\lceil_\theta \Leftrightarrow \neg O \wedge \overset{\gamma}{\hookrightarrow} \wedge \bigwedge_{i=0}^{n_\gamma-1} \left(z^i_\gamma = 0 \Rightarrow \begin{pmatrix} \mathtt{upSub}^n_{i,=1}(\rfloor\!\uparrow_\gamma) & \vee \\ \rceil\!\lfloor_\gamma \wedge \bigvee_{k=1}^{n-1} \mathtt{upSub}^k_{i,=1}(\rfloor\!\lceil_\gamma) & \end{pmatrix} \right). \tag{31}$$

Finally, for $\theta = \mathcal{C}_n(\gamma)$, $m(\theta)$ is (7) $\wedge$ (28) $\wedge$ (29) $\wedge$ (30) $\wedge$ (31), and we have the following result similar to Lemmata 10, 11, and 13.

Lemma 14. *Let M be a signal, and ϕ be an MITL+Past formula including counting modalities. For all $(\pi, \sigma) \in r_{sub(\phi)}(M)$ we have $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta \wedge \bigwedge_{\substack{\theta\in sub(\phi) \\ \theta=\mathbf{F}_{(a,b)}(\gamma) \text{ or } \theta=\mathcal{C}_n(\gamma)}} \mathtt{auxck}_\theta$ and for all $k \in \mathbb{N}, \theta \in sub(\phi)$ we have $(\pi, \sigma), k \models m(\theta)$.*
Conversely, if $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} (\mathtt{ck}_\theta \wedge \mathbf{G}(m(\theta))) \wedge \bigwedge_{\substack{\theta\in sub(\phi) \\ \theta=\mathbf{F}_{(a,b)}(\gamma) \text{ or } \theta=\mathcal{C}_n(\gamma)}} \mathtt{auxck}_\theta$, then there is a signal M such that $(\pi, \sigma) \in r_{sub(\phi)}(M)$.

Proof. The proof follows the same structure as those for Lemmata 10, 11, and 13. Here we focus on the case for subformulae of the form $\theta = \boldsymbol{\mathcal{C}}_n(\gamma)$.

Part 1.
Suppose $t_k \in T$. As for Lemma 10, Formula (7) holds by Lemma 4. As usual, we separately consider the cases $M, t_k \models e^u_\theta$, $M, t_k \models e^d_\theta$, and $M, t_k \models s^d_\theta$.

Subcase $\mathbf{e^u_\theta}$. Suppose that e^u_θ holds in t_k. Assume at first that $t_k > 0$. For θ to have a rising edge in t_k, it cannot be that γ has a rising edge e^u_γ in interval $[t_k, t_k + 1)$, or given a small enough $\varepsilon > 0$ in interval $(t_k - \varepsilon, t_k - \varepsilon + 1)$ there would be an infinite number of instants in which γ is true, hence θ would hold also before t_k, instead of having a rising edge. For the same reason, γ cannot have a falling edge in t_k. For θ to become true in t_k, then, γ must become true in $t_k + 1$. We have two cases: $M, t_k + 1 \models e^u_\gamma$ and $M, t_k + 1 \models s^u_\gamma$.

If $M, t_k + 1 \models e^u_\gamma$, this is enough to make θ become true in t_k. However, in $[t_k, t_k + 1)$ there can be up to $n - 1$ up-singularities s^u_γ.

If $M, t_k + 1 \models s^u_\gamma$, for θ to become true in t_k there must be exactly $n - 1$ up-singularities s^u_γ in $(t_k, t_k + 1)$, and none in t_k.

By definition of $r_{sub(\phi)}(M)$ we have that $t_k + 1 = t_{k'} \in T$, and also all instants between t_k and $t_k + 1$ where there are singularities are in T. In addition, in k' one of the clocks associated with θ must be 1; in fact, by Lemma 8 in interval $(t_k, t_k + 1]$ θ can change value at most $2n$ times, but there are $2n + 1$ clocks associated with the formula, so the clock that, by definition of $r_{sub(\phi)}(M)$, is reset at k can be reset again only after k'. Then, one of the clocks associated with θ is 0 in k. Also, for Lemma 4, θ cannot hold in t_k, so by definition of $r_{sub(\phi)}(M)$ we have $(\pi, \sigma), k \models \neg \,⫯_\theta$. Finally, by the reasoning above, if $M, t_k + 1 \models e^u_\gamma$, we have $(\pi, \sigma), k \models \bigvee_{k \in \{1,\ldots,n\}} \mathtt{up}^{k,k}_{j,=1}(\lrcorner\!\ulcorner_\gamma)$, while if $M, t_k + 1 \models s^u_\gamma$ we have $(\pi, \sigma), k \models \mathtt{up}^{n,n}_{j,=1}(\lrcorner\!\sqcap\!\llcorner_\gamma) \wedge \not{\lrcorner\!\sqcap\!\llcorner}_\gamma$, so the right hand side of Formula (28) holds.

If $t_k = 0$ we have two further cases: the interval in which there are n occurrences of γ is $(0, b]$, or it is a proper subset thereof (i.e., it is $(0, \varepsilon)$, with $\varepsilon < 1$). The former case is analogous to the case $t_k > 0$.
The latter case is also very similar, but the n occurrences of γ are all such that $x^0_\theta < 1$ (in particular, it can happen that γ has a rising edge in 0); also, in this case θ holds in 0, so we have $(\pi, \sigma), k \models ⫯_\theta$. Then, we have that formula $⫯_\theta \wedge (\lrcorner\!\ulcorner_\gamma \vee \mathtt{up}^n_{0,<1}(\lrcorner\!\sqcap\!\llcorner_\gamma) \vee \bigvee_{k \in \{1,\ldots,n\}} \mathtt{up}^k_{0,<1}(\lrcorner\!\ulcorner_\gamma))$ holds at k.

All in all, Formula (28) in this case holds in k if $M, t_k \models e^u_\theta$.

The case in which $M, t_k \not\models e^u_\theta$ is similar to the one in the proof of Lemma 10, so we omit if for brevity. Also, similar arguments as those used in the proof of Lemma 10 to show that Formula (9) holds in all $k \in \mathbb{N}$ are used to show the same thing for Formula (29).

Subcase $\mathbf{e^d_\theta}$. For θ to have a falling edge in t_k, γ must become false in t_k, but also there must be a $\varepsilon > 0$ such that in $(t_k - \varepsilon, t_k)$ θ holds (unless $t_k = 0$), and in $(t_k, t_k + \varepsilon)$ it does not. We separate two cases: γ becomes false with a falling edge (i.e., $M, t_k \models e^d_\gamma$), or it has an up-singularity (i.e., $M, t_k \models s^u_\gamma$). In both cases, it cannot be that γ has a rising edge in $(t_k, t_k + 1]$, or θ would be true also in $(t_k, t_k + \varepsilon)$. Let us consider the two cases separately.

If $M, t_k \models e^d_\gamma$ (which also includes the case in which $t_k = 0$), then γ can have up to $n-1$ singularities in $(t_k, t_k+1]$.

If $M, t_k \models s^u_\gamma$, then γ must have exactly $n-1$ singularities in (t_k, t_k+1), and it cannot change value again in t_k+1.

By definition of $r_{sub(\phi)}(M)$, all instants t_{k_i} in $(t_k, t_k+1]$ when γ has up-singularities are in T. Also, in both cases above there is a clock z^i_γ that is reset at k, and that clock is not reset again in $(t_k, t_k+1]$ because γ is associated with at least $n+1$ clocks. Hence, if $M, t_k \models e^d_\gamma$, there is $h \in \{0, \ldots, n-1\}$ such that $(\pi, \sigma), k \models \mathtt{upSub}^h_{i,\leqslant 1}(\gamma)$. Similarly, if $M, t_k \models s^u_\gamma$, then $(\pi, \sigma), k \models \mathtt{upSub}^{n-1}_{i,<1}(\gamma)$. In both cases, the right hand side of Formula (30) holds at k.

If instead $M, t_k \not\models e^d_\theta$, then either γ does not become false in t_k, or, if it does, it becomes true n times anew in $(t_k, t_k+1]$. In all these cases, the right hand side of Formula (30) does not hold in k, so Formula (30) does, as $(\pi, \sigma), k \not\models \urcorner_\theta$.

Subcase $\mathbf{s}^{\mathbf{d}}_\theta$. For θ to have a down-singularity in t_k, γ must become false in t_k and there must be $\varepsilon > 0$ such that θ holds in $(t_k - \varepsilon, t_k)$ (unless $t_k = 0$), but then for all $t' \in (t_k, t_k + \varepsilon)$ it must be that there are n instants in $(t', t'+1)$ when γ holds. This corresponds to having conditions in t_k similar to those of Subcase $\mathbf{e}^{\mathbf{d}}_\theta$, with the condition that γ becomes true again in t_k+1. We skip the rest of this case for brevity.

Part 2.
The proof that, if $(\pi, \sigma), 0 \models \bigwedge_{\theta \in sub(\phi)} \mathtt{ck}_\theta \wedge \mathbf{G}\,(m(\theta)) \wedge \bigwedge_{\substack{\theta \in sub(\phi) \\ \theta = \mathbf{F}_{(a,b)}(\gamma) \text{ or } \theta = \boldsymbol{\mathcal{C}}_n(\gamma)}} \mathtt{auxck}_\theta$, then there is a signal M such that $M = r^{-1}_{sub(\phi)}((\pi, \sigma))$ is similar to those in Lemmata 10 and 11, so we omit it for brevity. □

Given an MITL+Past with counting formula ϕ, define the corresponding CLTLoc formula as:

$$\mathtt{init}_\phi \wedge \bigwedge_{\theta \in sub(\phi)} (\mathtt{ck}_\theta \wedge \mathbf{G}\,(m(\theta))) \wedge \bigwedge_{\substack{\theta \in sub(\phi) \\ \theta = \mathbf{F}_{(a,b)}(\gamma) \text{ or } \theta = \boldsymbol{\mathcal{C}}_n(\gamma)}} \mathtt{auxck}_\theta. \quad (32)$$

Theorem 3. *An MITL+Past with counting formula ϕ is satisfiable if, and only if, Formula (32) is satisfiable.*

Consider an MITL+Past formula ϕ with counting, with K and n being, respectively, the largest constant in temporal modalities and the largest index of the counting modalities occurring in ϕ. The corresponding equisatisfiable CLTLoc Formula (32) differs from Formula (26), whose size is $O(|\phi|K)$, only because it also includes constraints for subformulae θ of the form $\boldsymbol{\mathcal{C}}_n(\gamma)$. For each θ of this form, the size and the number of clocks of $m(\boldsymbol{\mathcal{C}}_n(\gamma))$ polynomially depend on parameters c_θ and n_γ, which are $O(n)$. In particular, the size of $m(\boldsymbol{\mathcal{C}}_n(\gamma))$ is $O(n^4)$ because of Formula (28). In fact, formulae $\mathtt{up}^{k,k}_{j,=1}(\lrcorner\!\ulcorner_\gamma)$ are $O(n^2)$ as they include formulae $z^{\hat{p}}_\gamma \sim d$ (27), which are $O(n^2)$. Therefore, the size of Formula (32) depends linearly on $|\phi|K + n^4$ which is exponential in the size of the binary encoding of K and of n. Its satisfiability is then in EXPSPACE when considering a binary encoding. It is nonetheless in PSPACE by considering the unary encoding of both constants of temporal modalities and indexes of the counting modalities, which is consistent with the PSPACE complexity (with a unary encoding of the indexes) of QTL augmented with counting modalities [31].

A simple generalization of the counting operators is a counting modality $\mathcal{C}^b_n(\gamma)$ in which γ occurs at least n times in the interval $(0, b)$, instead of only $(0, 1)$. It is easy to see that our translation $m(\mathcal{C}_n(\gamma))$ can be adapted to this case simply by changing bounds 1 in formulae (28)-(31) to b. Hence, our translation shows that also the satisfiability of MITL+Past augmented with counting modalities $\mathcal{C}^b_n(\gamma)$ is PSPACE-complete when considering a unary encoding of constants and indexes.

8. Implementation & Experimental Results

A decision procedure for CLTLoc [21] is implemented in a plugin, called `ae²zot`, of our Zot toolkit [32], whereas all the reductions outlined in the paper are implemented in the `qtlsolver` tool, available from [22]. The tool translates MITL+Past into CLTLoc, which can be checked for satisfiability by `ae²zot`.

We carried out some experiments (available from the `qtlsolver` website [22]), on a desktop computer with a 2.8GHz AMD PhenomTMII processor and 8GB RAM; the solver was Microsoft Z3 3.2.

Table 4 shows a few examples of formulae, together with a short explanation. The abbreviation $\mathbf{G}^i(\phi) = \phi \wedge \mathbf{G}_{(0,\infty)}(\phi)$ enforces that ϕ holds also in the current instant. Table 5 shows the result (SAT or UNSAT), the bound and the approximate time taken by translation and solving.

Ref.	**Formula**	**Comment**
ϕ_1	$p \wedge \mathbf{G}_{(0,100)}\neg p \wedge \mathbf{G}^i(p \Rightarrow \mathbf{F}_{(0,200)}p)$ $\wedge \mathbf{G}^i(\mathbf{G}_{(0,100)}\neg p \Rightarrow \mathbf{G}_{(100,200)}\neg p)$	p occurs in isolated points with period 100, starting at 0.
ϕ_2	$\mathbf{G}_{(0,\infty)}\big(p \Rightarrow \mathbf{F}_{(0,1)}q \vee \mathbf{P}_{(0,1)}q\big)$	q must hold within 1 time unit before or after each p.
ϕ_3	$\mathbf{G}_{(0,\infty)}\big(q \Rightarrow \mathbf{G}_{(0,100)}\neg q\big)$	q occurs at isolated points, at least 100 time units apart.
ϕ_4	$\mathbf{G}_{(0,\infty)}\big(q \Rightarrow \mathbf{G}_{(0,100]}\neg q\big)$	Like ϕ_3, but strictly aperiodic.
ψ_1	$\mathbf{G}_{(0,\infty)}\big(\mathbf{F}_{(0,\infty)}q \wedge (q \rightarrow \mathcal{C}^2_2 q)\big)$	q occurs infinitely often; when it holds, $\exists$ 2 or more occurrences in the adjacent interval of length 2.
ψ_2	$\mathbf{G}_{(0,\infty)}\big(q \rightarrow \mathbf{F}_{(0,1)}q\big)$	property not necessarily holding for ψ_1
ψ_3	$\mathbf{G}_{(0,\infty)}\big(\mathbf{F}_{(0,\infty)}\big(q \wedge \mathbf{F}_{(0,1)}q\big)\big)$	property holding for ψ_1

Table 4: Examples of formulae (where $\mathcal{C}^2_2$ is defined at the end of Section 7).

Formula	Result	Bound	Time
ϕ_1	SAT	10	10 seconds
$\phi_1 \wedge \phi_2$	SAT	10	40 seconds
$\phi_1 \wedge \phi_2 \wedge \phi_3$	SAT	20	10 minutes
$\phi_1 \wedge \phi_2 \wedge \phi_3$ with periodic constraint on p, q	SAT	20	15 minutes
$\phi_1 \wedge \phi_2 \wedge \phi_4$	SAT	30	80 minutes
$\phi_1 \wedge \phi_2 \wedge \phi_4$ with periodic constraint on p, q	UNSAT	30	12 hours
ψ_1	SAT	25	24 seconds
$\psi_1 \wedge \neg\psi_2$	SAT	25	50 seconds
$\psi_1 \wedge \neg\psi_3$	UNSAT	25	57 minutes

Table 5: Experimental Results

Note that, even if the constants appearing in Formula ϕ_1 are in the order of the hundreds, a bound of 10 positions is enough for `qtlsolver` to satisfy ϕ_1, since events in the corresponding models occur only sparsely.

Our tool allows one to add constraints also at the CLTLoc or at the SMT levels. For example, in the experiments, we added SMT constraints imposing that the *values* of the clocks (instead of the clock regions) associated with propositions p and q be periodic; this allowed us to check that formula $\phi_1 \wedge \phi_2 \wedge \phi_3$ admits periodic models, while $\phi_1 \wedge \phi_2 \wedge \phi_4$ does not (i.e., it is unsatisfiable with the periodic constraint, at least with bound 30).

Counting modalities, in the general version $\mathcal{C}_n^b(\gamma)$, were tested over specification ψ_1, checking also a property ψ_2 that does not necessarily hold and a property ψ_3 that does instead hold.

9. Conclusions

We presented a satisfiability-preserving translation from continuous-time metric temporal logics over signals to CLTLoc. In particular, we considered MITL, $\text{MITL}_{0,\infty}$ and their extensions with past and counting modalities. As CLTLoc is naturally defined over a pointwise semantics (i.e., on timed words), the translation assumes that signals are finitely variable and leverage on the fundamental property for which all temporal modalities of MITL, and $\text{MITL}_{0,\infty}$, may only finitely vary in time. This allows representing the real line as an infinite sequence of intervals and, then, capturing the semantics of MITL and $\text{MITL}_{0,\infty}$ formulae through CLTLoc. The encoding has been implemented in a prototype tool [22]. Preliminary experiments are promising, as the tool was able to solve formulae representing significant temporal behaviors. To the best of our knowledge, our approach is the first allowing an effective implementation of a fully automated verification tool for continuous-time metric temporal logics. Verification of formulae requiring many clocks may be infeasible, since satisfiability of MITL is EXPSPACE-complete. However, in practice a large number of clocks is not very frequent, and several examples of MITL formulae could be verified.

The techniques presented in this paper for MITL can be tailored also to other logics. We consider an example here. In [33], $\mathrm{MITL}_{0,\infty}$ was shown to be equivalent to another temporal logic, called Event-Clock Logic (ECL), which is in PSPACE. Although our work only concerns MITL and $\mathrm{MITL}_{0,\infty}$ our results can directly be applied for solving the satisfiability of ECL as well, by means of the above equivalence of the languages. However, an explicit, possibly more efficient, encoding of ECL into CLTLoc may be devised, since only a finite number of explicit clocks are known to be enough to capture ECL semantics.

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Appendix A. Encoding of $\theta = \mathbf{P}_{(a,b)}(\gamma)$

By Lemma 5, singularities s^u_θ cannot occur in signals for formulae of the form $\theta = \mathbf{P}_{(a,b)}(\gamma)$.

By Lemma 7, the distance between a change point where "θ becomes true" (possibly in the origin, or with a down singularity) and a change point where "θ becomes false" for formulae $\theta = \mathbf{P}_{(a,b)}(\gamma)$ cannot be less than $b-a$, so Corollary 2 holds also for $\mathbf{P}_{(a,b)}$. As for the case $\mathbf{F}_{(a,b)}$, this property will be exploited below to define the translation of the $\mathbf{P}_{(a,b)}$ operator.

In case of subformulae of the form $\theta = \mathbf{P}_{(a,b)}(\gamma)$ we introduce, similarly for formulae of the form $\theta = \mathbf{F}_{(a,b)}(\gamma)$, in addition to clocks z^0_θ, z^1_θ of Section 5, $d = 2\left\lceil \frac{b}{b-a} \right\rceil$ *auxiliary clocks* $\{x^j_\theta\}_{0 \leqslant j \leqslant d-1}$, which are used to store the time elapsed since the occurrence of change points for γ that cause θ to change value (hence, not all change points regarding γ are taken into account) and the current time instant t. Note that in the case of $\mathbf{P}_{(a,b)}$ auxiliary clocks $\{x^j_\theta\}_j$ are reset not when θ has a change point, but when γ has a change point that leads, later on, to a change point of θ.

The behavior of the auxiliary clocks is defined by the following formulae.

Each reset $x^i_\theta = 0$ entails that the current instant is the origin, or one of $e^u_\gamma, e^d_\gamma, s^d_\gamma,$ s^u_γ occurs, but only in a situation where, a or b instants later, θ changes value (Formula (A.1)). More precisely, there are three cases in which one of the auxiliary clocks is reset:

1. In the origin.
2. When γ becomes true at t_k, and it was false throughout $(t_k - b, t_k)$. This corresponds to the clock that is not reset in k being $\geqslant b-a$, or to γ always being false from the origin O until k (second disjunt of the left hand side of Formula (A.1)).
3. When γ becomes false at t_k, and it stays false until at least $t_k + (b-a)$. This corresponds to there not being a $t_{k'}$ where γ become true such that $t_{k'} < t_k + (b-a)$, which in turn corresponds to there not being $k' > k$ where the clock z^i_γ that is reset at k has value $< b-a$ and $\overset{\gamma}{\lrcorner\!\uparrow}$ holds (third disjunct of the left hand side of Formula (A.1)).

$$\left(\begin{array}{ll} O & \vee \\ \overset{\gamma}{\lrcorner\!\uparrow} \wedge \left(\bigvee_{i=0}^{1} z^i_\gamma \geqslant (b-a) \ \vee \ (z^0_\gamma > 0)\mathbf{S}(O \wedge \underleftarrow{\gamma}) \right) & \vee \\ \overset{\gamma}{\llcorner\!\rightarrow} \wedge \bigvee_{i=0}^{1} \left(z^i_\gamma = 0 \wedge \neg \mathbf{X}\left(z^i_\gamma > 0 \mathbf{U}(\overset{\gamma}{\lrcorner\!\uparrow} \wedge z^i_\gamma < b-a) \right)\right) & \end{array}\right) \Leftrightarrow \bigvee_{j=0}^{d-1} x^j_\theta = 0 \tag{A.1}$$

Each change point of γ that leads to a change in the value of θ is marked by a single reset $x_\theta^i = 0$ (Formula (A.2)).

$$\left(\bigwedge_{i=0}^{d-1} \bigwedge_{j=0, i \neq j}^{d-1} \neg (x_\theta^i = 0 \wedge x_\theta^j = 0) \right) \tag{A.2}$$

The occurrence of resets for clocks x_θ^i is circularly ordered and the sequence of resets starts from the origin by x_θ^0. If $x_\theta^i = 0$, then, from the next position, all the other clocks are strictly greater than 0 until the next $x_\theta^{(i+1) \bmod d} = 0$ occurs.

$$\bigwedge_{i=0}^{d-1} \left(x_\theta^i = 0 \Rightarrow \mathbf{X} \left((x_\theta^{(i+1) \bmod d} = 0) \mathbf{R} \bigwedge_{j \in [0, d-1],\, j \neq i} (x_\theta^{(j+1) \bmod d} > 0) \right) \right) \tag{A.3}$$

Formula $x_\theta^0 = 0$, evaluated at position 0, sets the first reset of the sequence, constrained by formulae (A.1)-(A.3).

Define formula $\texttt{pauxck}_\theta$ as $(x_\theta^0 = 0) \wedge \mathbf{G}\,((\text{A.1}) \wedge (\text{A.2}) \wedge (\text{A.3}))$.

The next formulae capture the semantics of the $\mathbf{P}_{(a,b)}$ modality. For the sake of simplicity, the translation only considers the case $a > 0$. Because of Lemma 5, an up-singularity $\sqcup\!\sqcap\!\sqcup_\theta$ can never occur for $\theta = \mathbf{P}_{(a,b)}(\gamma)$. In addition, $\mathbf{P}_{(a,b)}(\gamma)$ is false in the origin, no matter γ. Then, as for $\mathbf{P}_{(0,b)}(\gamma)$, Formula 14 holds in every instant.

Formula (A.4) defines that θ has a rising edge in t_k if, and only if, there is an auxiliary clock x_θ^j that has value a in k, the last time x_θ^j was reset γ became true – which entails, by Formula (A.1), that γ is false throughout $(t_k - b, t_k - a)$ – and there is no $t_{k'} = t_k - b$ where γ becomes false. Note that, if there were $t_{k'} = t_k - b$ where γ becomes false, then θ in t_k would not have a rising edge, but a down-singularity.

$$\lrcorner\!\ulcorner_\theta \Leftrightarrow \bigvee_{j=0}^{d-1} x_\theta^j = a \wedge \left(x_\theta^j > 0 \ \mathbf{S}\ (\overset{\gamma}{\lrcorner\!\uparrow} \wedge x_\theta^j = 0) \right) \wedge \neg \bigvee_{i=0}^{d-1} x_\theta^i = b \wedge \left(x_\theta^i > 0 \ \mathbf{S}\ (\overset{\gamma}{\llcorner\!\rightarrow} \wedge x_\theta^i = 0) \right) \tag{A.4}$$

Formula (A.5) is similar, but for the falling edge. More precisely, θ has a falling edge in t_k if, and only if, either t_k is the origin, or there is a clock x_θ^j whose value is b in k, the last time x_θ^j was reset γ became false – which entails, by Formula (A.1), that γ is false throughout $(t_k - b, t_k - a)$ – and there is no $t_{k'} = t_k - a$ where γ becomes true. Note that, if there were $t_{k'} = t_k - a$ where γ becomes true, then θ in t_k would not have a falling edge, but a down-singularity.

$$
\rceil\!\lfloor_\theta \Leftrightarrow \left(O \quad \vee \left(\begin{array}{c} \bigvee_{j=0}^{d-1} x_\theta^j = b \wedge \left(x_\theta^j > 0 \ \mathbf{S} \ (\overset{\gamma}{\hookrightarrow} \wedge x_\theta^j = 0) \right) \\ \wedge \\ \neg \quad \bigvee_{i=0}^{d-1} x_\theta^i = a \wedge \left(x_\theta^i > 0 \ \mathbf{S} \ (\overset{\gamma}{\uparrow} \wedge x_\theta^i = 0) \right) \end{array} \right) \right) \tag{A.5}
$$

Formula (A.6) essentially combines the conditions of Formulae (A.4) and (A.5), and states that θ in t_k has a down-singularity if, and only if, γ becomes false in $t_k - b$, it becomes true in $t_k - a$, and it stays false in $(t_k - b, t_k - a)$.

$$
\rceil\!\lceil_\theta \Leftrightarrow \bigvee_{j=0}^{d-1} x_\theta^j = a \wedge \left(x_\theta^j > 0 \ \mathbf{S} \ (\overset{\gamma}{\uparrow} \wedge x_\theta^j = 0) \right) \wedge \bigvee_{i=0}^{d-1} x_\theta^i = b \wedge \left(x_\theta^i > 0 \ \mathbf{S} \ (\overset{\gamma}{\hookrightarrow} \wedge x_\theta^i = 0) \right) \tag{A.6}
$$

Constraint (A.7) is similar to Formulae (18) and (22), as it guarantees that, if in t_k the conditions are met for θ to become true (possibly with a down-singularity) at $t_k + a$, then there is a corresponding k' such that $t_{k'} = t_k + a$ where one of the auxiliary clocks associated with θ has value a, and where the right hand side of Formula (A.4) or the right hand side of Formula (A.6) hold, thus forcing $\rfloor\!\lceil_\theta$ or $\rceil\!\lceil_\theta$ to hold in k'.

$$
\overset{\gamma}{\uparrow} \wedge \left(\bigvee_{i=0}^{1} z_\gamma^i \geqslant (b-a) \vee z_\gamma^0 > 0 \mathbf{S} (O \wedge \underset{\leftarrow}{\gamma}) \right) \Rightarrow \bigvee_{j=0}^{d-1} \left(x_\theta^j = 0 \wedge \mathbf{X} \Big(x_\theta^j > 0 \mathbf{U} x_\theta^j = a \Big) \right) \tag{A.7}
$$

Formula (A.8) plays a similar role as Formula (A.7), but for the case where in $t_k + b$ θ becomes false, possibly with a down-singularity. Hence, it forces a k' such that $t_{k'} = t_k + b$ to exist in the CLTLoc interpretation, where the right hand side of Formula (A.5) or the right hand side of Formula (A.6) hold, thus forcing $\rceil\!\lfloor_\theta$ or $\rceil\!\lceil_\theta$ to hold in k'.

$$
\overset{\gamma}{\hookrightarrow} \wedge \bigvee_{i=0}^{1} \left(\begin{array}{l} z_\gamma^i = 0 \wedge \\ \neg \mathbf{X} \Big(z_\gamma^i > 0 \mathbf{U} \left(\overset{\gamma}{\uparrow} \wedge z_\gamma^i < b - a \right) \Big) \end{array} \right) \Rightarrow \bigvee_{j=0}^{d-1} \left(x_\theta^j = 0 \wedge \mathbf{X} \Big(x_\theta^j > 0 \mathbf{U} x_\theta^j = b \Big) \right) \tag{A.8}
$$

Then, $m(\theta)$ is (14) $\wedge$ (A.4) $\wedge$ (A.5) $\wedge$ (A.6) $\wedge$ (A.7) $\wedge$ (A.8).