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On the regular decomposition of the inverse gravimetric problem in non-L^2 spaces

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Abstract The paper deals with the inverse gravimetric problem generalizing a classical decomposition of mass distributions into a harmonic component and another component that produces a zero external field. After a review and an extension of the well-known L^2 theory, mass distributions in L^p and in $H^{-s,2}$ are considered and proved to undergo an analogous decomposition. Examples will make the theory easier to grasp. Conclusions follow.

Keywords Gravity field · Inverse problems · Regular decomposition

Mathematics Subject Classification 31B20

1 An introduction to the inverse gravimetric problem

Assume we have a body $\overline{B}$ in R^3, with $\overline{B} = B \cup S$, B the interior of $\overline{B}$ and S its boundary. We shall agree $\overline{B}$ to be compact and simply connected, and we will denote by Ω the exterior of $\overline{B}$; so Ω is an (unbounded) open set, with boundary S too.

To be specific, we shall assume from now on that S is at least a Lipschitz surface (see Miranda 1970; McLean 2000), introducing later on more restrictive conditions.

Assume now that f is some mass distribution on $\overline{B}$. For the moment we can think of f as a measure of finite variation with support in $\overline{B}$ or even as a measure with a measurable density $f(x)(x \in \overline{B})$. If we call

$$N(x) = \frac{1}{4\pi}|x|^{-1} \tag{1}$$

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the Newton kernel, then the external Newtonian potential of f is

$$x \in \Omega,\; u(x) =< N(x-y), f >\equiv N(f), \tag{2}$$

where the coupling (2) can be thought of as an integral over the measure f, or, when it exists, as the Lebesgue integral on $\overline{B}$ of $N(x-y)f(y)$,

Remark 1 The theory developed in this paper has an obvious generalization to R^n, $n > 3$; nevertheless since we like to frame our problem within the geophysical or geodetic point of view, thinking of $u(x)$ as the exterior gravity potential of the Earth, we will carry out our proofs in R^3.

The inverse gravimetric problem is to determine f from u, i.e. to find the mass source f generating the external potential u. As a matter of fact this is one of the oldest improperly posed problems treated in the literature on inverse problems (Lavrentiev 1967; Tichonov and Arsenin 1977); probably first met in 1867 in a paper by Stokes (1867).

From the mathematical point of view the interesting feature comes from the large indetermination of the solution of (2), due to the existence of a wide class of functions (mass densities) f that produce a zero outer potential, i.e. the Newton operator $N(\cdot)$ has a large null space.

This effect has already been highlighted in a number of papers in the 19th century and at the beginning of the 20th century.

To better appreciate it let us start with an obvious proposition, (see for instance Gilbarg and Trudinger 1983, §2.4), that will be useful in the sequel too. We use the standard notation $\mathcal{D}(B)$ for the Schwartz space $C_0^\infty(B)$.

Proposition 1 *The class of mass distributions*

$$\Delta[\mathcal{D}(B)] \equiv \{f = \Delta\varphi;\, \varphi \in \mathcal{D}(B)\} \tag{3}$$

is contained in the null space of N, with respect with any space considered later on in the paper, for instance in $H^{-s,2}$, $\forall s > 0$.

Proof The proof is trivial and it doesn't require any regularity hypothesis on S.

If $\varphi \in \mathcal{D}(B)$, there is a compact set $k_\varphi \subset B$ with very regular $\widetilde{S}$ as boundary such that $\varphi \equiv 0$ outside (and on) $\widetilde{S}$ and $\widetilde{S}$ itself is a very smooth surface.

Call $\widetilde{B}$ the interior of $\widetilde{S}$, then

$$\begin{aligned}\forall x \in \Omega\,;\; \int_B N(x-y)\Delta\varphi(y)dy &= \int_{\widetilde{B}} N(x-y)\Delta\varphi(y)dy\\ &= \int_{\widetilde{B}} \Delta_y N(x-y)\varphi(y)dy + \int_{\widetilde{S}} [N(x-y)\frac{\partial}{\partial n_y}\varphi(y)\\ &\quad -\frac{\partial}{\partial n_y}N(x-y)\cdot\varphi(y)]dS \equiv 0.\end{aligned} \tag{4}$$

□

It follows that, as a minimum, the general solution of (2) has the form

$$f = \overline{f} + \Delta\varphi, \ (\varphi \in \mathcal{D}(B)) \tag{5}$$

with $\overline{f}$ one particular solution of the problem. A lot of literature exists, trying to identify conditions suitable to fix uniquely $\overline{f}$. Frequently we find a minimum norm condition, when a Hilbert space structure is given to the space F of the admissible densities. For instance, when $F \equiv L^2(B)$ the theory is well understood (see Sansò 1980; Sansò et al. 1986; Barzaghi and Sansò 1986, but also for instance Marussi 1980; Ballani and Stromeyer 1983; Michel and Fokas 2008; Moritz 1990 and the bibliographic discussion therein). Assume that u is generated by an $L^2(B)$ density f; then there is a unique minimum norm density $\overline{f}$ equivalent to f (i.e. generating the same external potential) and the result is that $\overline{f}$ is harmonic in B. In addition there is one and only one $\varphi \in H_0^{2,2}(B)$ such that the decomposition

$$f = \overline{f} + \Delta\varphi \quad (\Delta\overline{f} = 0 \text{ in } B, \varphi \in H_0^{2,2}(B)) \tag{6}$$

holds true. In a sense the result can be put in a more abstract form by saying that when F is a certain space of admissible densities, that in this paper will be assumed to be Banach, adopting the notation HF for the subspace

$$HF = \{f \in F, \ \Delta f = 0 \text{ in } B\} \tag{7}$$

and

$$F_0 \equiv [\{\Delta\varphi\ ; \ \varphi \in \mathcal{D}(B)\}]_F \tag{8}$$

the decomposition (6) holds with $\overline{f} \in HF$ and $\Delta\varphi \in F_0$. In addition when $F \equiv L^2(B)$ we get that HF and F_0 are closed, orthogonal complementary spaces. More generally when we can write

$$F = HF \oplus F_0 \quad (HF \cap F_0 \neq \{0\}) \tag{9}$$

with HF, F_0 closed subspaces of F and with a projection of F onto HF along F_0, that is a bounded operator, we say that we have a "regular decomposition of the solution" of the inverse gravimetric problem. So basically the question is whether the subspace HF of harmonic functions in F and the kernel F_0 of Newton's operator in F do perform a regular decomposition of F into complementary subspaces.

It has been stressed that when we consider N as an operator acting on a certain space F, $F_0 = \ker(N)$ will depend on F. In the rest of the paper we shall use different symbols for F_0, specifying each time the space to which the $\ker(N)$ refers.

The focus of this paper is to generalize the $L^2(B)$ result to other spaces. Considering that the true mass density of the earth is bounded we will prove the regular decomposition of $L^p(B)$, $\forall p > 2$. Indeed it could be nice to prove directly such a property for $L^\infty(B)$, yet since we will exploit a theorem related to the reflexivity of F, we will be satisfied with the case $F = \cap_{p>2} L^p(B)$ which, by the way is a much more restrictive condition than $F = L^2(B)$. On the other hand there is certainly a value in studying the regular decomposition of spaces larger than $L^2(B)$, for instance $H^{-s,2}(\overline{B})$, $(s > 0)$, because to such spaces belong some very useful (and utilized) distributions like point masses, surface layers etc. So, a similar result will be obtained for $f \in H^{-s,2}(\overline{B})$, for $s > 0$, integer. Again we will not go to the general case $F \equiv \mathcal{D}^*(\overline{B})$ in order to be able to use Banach space (in this case even Hilbert space) techniques, and we shall be content to consider $F = \cap_{s>2} H^{-s,2}(\overline{B})$.

It seems worth mentioning here that a completely different approach can be found in (Anger 1977, 1981), where the problem is reconducted to the identification of extremal points in a convex set in a space of measures.

Other approaches are discussed, e.g. in Michel and Fokas (2008), where numerical approximation methods are presented too.

The paper is organized as follows: in §2 we shall briefly review the L^2 theory, adding the result that a solution exists whenever u is regular at infinity and $u \in H^{2,2}_{\text{loc}}(\Omega)$. In §3 we develop the theory for $u \in L^p(B)$, $p > 2$, and in §4 we analyze the case $u \in H^{-s,2}(\overline{B})$, with s a positive integer.

In §5 we try to make the theory more practical by presenting examples, some of which related to a spherical geometry. In doing so, the relation between regular decomposition and the use of reproducing kernels, in spaces of harmonic functions, is explored. Conclusions follow.

2 The L^2 theory: a summary and an existence result

We first assume that there is an $f \in L^2(B)$ such that, denoting with $(\,,\,)$ the $L^2(B)$ product,

$$x \in \Omega, \; u(x) = (N(x-y), f(y)) \tag{10}$$

for a given $u(x)$ harmonic in Ω. Indeed (10) would not be possible unless $u(x) \to 0$ for $|x| \to \infty$ and $u \in H^{2,2}_{\text{loc}}(\Omega)$, implying also that there are traces on S

$$u|_S = U \in H^{3/2}(S), \; \partial_n u|_S = V \in H^{1/2}(S). \tag{11}$$

We want to prove a result of uniqueness of the regular decomposition.

In the context of the present section by $F_0 = \ker(N)$ we mean the null space of the operator N in $L^2(B)$.

Theorem 1 *Suppose we are given a potential $u(x)$ in Ω and its $L^2(B)$ density f as in (10), then there is one and only one*

$$\overline{f} \in HL^2(B) \tag{12}$$

such that

$$f = \overline{f} + f_0, \tag{13}$$

with

$$f_0 \in F_0 \equiv \ker(N) \equiv HL^2(B)^{\perp}, \tag{14}$$

so that

$$\forall x \in \Omega \quad u(x) = N(f) \equiv N(\overline{f}). \tag{15}$$

In addition we have

$$f_0 = \Delta w, \ w \in H_0^{2.2}(B) \tag{16}$$

for a suitable w, i.e.

$$F_0 \equiv \Delta\{H_0^{2,2}\}. \tag{17}$$

Proof That the unique decomposition (13), with $\overline{f}$ as in (12), holds is just re-stating the orthogonal projection theorem for the subspace $HL^2(B)$, and this is true as soon as we know that $HL^2(B)$ is a closed subspace in $L^2(B)$.

In turn, this is a well-known result holding for all Bergman spaces $HL^p(B), \forall p \geq 2$ (see Proposition 8.3 in Axler et al. 2001). So we basically have to prove (14) and, in doing so, we shall prove (17) too. Note that if $v \in \ker(N)$, then

$$\forall x \in \Omega, \ 0 = (N(x-y), v(y)), \tag{18}$$

so that taking a sphere S_R of radius R enclosing B and some simple layer density $\lambda(x)$ over it, we can say that

$$0 = \left(\int_{S_R} N(x-y)\lambda(x)dS, v(y) \right) \tag{19}$$

By letting λ span some convenient space, e.g. $H^{-1/2}(S_R)$, the corresponding set of harmonic functions

$$u(y) = \int_{S_R} N(x-y)\lambda(x)dS \tag{20}$$

spans $H^{1,2}(B_R)$, of the ball B_R with boundary S_R. But then the restriction to the set B of the functions u of the type (20) is dense in $HL^2(B)$ because of Runge–Krarup's theorem (cf. Krarup (2006)).

It follows that

$$v \in \ker\ N \Rightarrow (v, h) = 0 \ \{\forall h \text{ in } HL^2(B)\} \Rightarrow v \in HL^2(B)^{\perp} \tag{21}$$

namely

$$\ker N \subseteq HL^2(B)^{\perp}. \tag{22}$$

On the other hand let $v \in HL^2(B)^{\perp}$, i.e.

$$0 = (v, u) \quad \forall u \in HL^2(B); \tag{23}$$

indeed (23) holds in particular for a set of $u \in HL^2(B)$, which is more regular than L^2, but still dense in $HL^2(B)$, e.g. $u \in HH^{1,2}(B)$.

Now, since $v \in L^2$, there is a function $w \in H^{2,2}(B)$ such that

$$\begin{cases} \Delta w = v \\ w|_S = 0, \end{cases} \tag{24}$$

so that (23) can be written as

$$\begin{aligned} & 0 \equiv (\Delta w, u) \equiv \int\limits_S w_n u dS \\ & \forall u \in HH^{1,2}(B). \end{aligned} \tag{25}$$

Since $w_n|_S$ is in $H^{1/2}(S)$ and $u|_S$ spans the same space too (McLean 2000, Chapter 3) we conclude that it has to be $w_n|_S = 0$, which together with $w|_S = 0$ implies $w \in H_0^{2,2}$. So we have proved that

$$v \in HL^2(B)^{\perp} \Rightarrow v = \Delta w, \;\; w \in H_0^{2,2}(B) \tag{26}$$

On the other hand $\Delta\{H_0^{2,2}(B)\} \subset \ker(N)$ as it follows from Proposition 1, recalling that $H_0^{2.2\cdot}(B)$ is in fact the closure of $\mathcal{D}(B)$ in $H^{2,2}$ and $\Delta\{\mathcal{D}(B)\} \subset \ker N$.

So

$$HL^2(B)^{\perp} \subset \Delta\{H_0^{2.2}(B)\} \subset \ker(N), \tag{27}$$

which, together with (22), proves (17). □

Remark 2 Now that we know that $HL^2(B)^{\perp} \equiv \ker(N)$, we can state our result in the form of a minimum norm principle too: indeed

$$\| \overline{f} \|_{L^2} \leq \| f \|_{L^2} \tag{28}$$

and we see that $\overline{f}$ is the solution of minimum $L^2(B)$ norm among all $L^2(B)$ densities generating the external $u(x)$. The fact that $\overline{f} \in HL^2(B)$, i.e. it is harmonic in B then becomes a property of this minimum norm solution.

Finally we close the paragraph with an existence theorem. To achieve its proof we need a result on be-harmonic problems which we can state in the form a proposition.

Proposition 2 *Consider the following bi-harmonic problem in B:*

$$\begin{aligned} &\Delta^2 \overline{u} = 0 \\ &\overline{u}|_S = U \\ &\partial_n \overline{u}|_S = V; \end{aligned} \tag{29}$$

assume that S is $C^{1,1}$ and that

$$U \in H^{3/2}(S), \;\; V \in H^{1/2}(S)\;; \tag{30}$$

then there is one and only one weak solution of (29)

$$\overline{u} \in H^{2,2}(B). \tag{31}$$

The result is known in the literature (see for instance Miranda 1970 §52); yet for the convenience of the reader a short proof is given in Appendix 1.

We are now in a position to prove the following existence theorem.

Theorem 2 *Let $u(x)$ be a given harmonic potential in $H^{2,2}_{loc}(\Omega)$, $u(x) \to 0$ for $|x| \to \infty$; let $S = \partial B$ be a $C^{1,1}$ surface; then there is a $\overline{u} \in H^{2,2}(B)$, such that*

$$\overline{f} = -\Delta \overline{u} \; (\in L^2(B)) \tag{32}$$

is a Newtonian L^2 density generating $u(x)$ in Ω.

Proof The proof is almost trivial. Given $u(x)$ one can compute the two traces

$$U = u|_S, V = \partial_n u|_S,$$

and by standard trace theorems (e.g. McLean 2000, Chapter 3), they both satisfy the hypotheses of Proposition (2).

Therefore there is one (and only one) solution $\overline{u}$ of (29) and we can compute $\overline{f}$ from (32); indeed $\overline{f} \in L^2(B)$.

Now take any $x \in \Omega$ and compute, by using the second Green identity,

$$x \in \Omega \quad \int_B N(x-y)\overline{f}(y)dB = -\int_B N(x-y)\Delta\overline{u}dB \tag{33}$$

$$= \int_S [N(x-y)V(y) - U(y)\partial_n N(x-y)]dS$$

$$\equiv u(x), \tag{34}$$

as it was to be proved. □

Remark 3 We note that because of (29) the $\overline{f}$ given by (32) is in fact the harmonic solution of the inverse problem in $L^2(B)$, i.e. the density belonging to $HL^2(B)$.

3 Regular decomposition of L^p, $p > 2$

In this section we generalize the results of §2 to the case that $f \in L^p(B)$, or that $u(x) \in HH^{2,p}_{\mathrm{loc}}(\Omega)$, and it is regular at infinity.

In order to do this we need some preliminary results that we present in the form of a proposition.

Proposition 3 *Let us define the Bergman space*

$$HL^p(B) \equiv \{u\ ;\ u \in L^p(B), \Delta u = 0 \ in\ B\}; \tag{35}$$

then $HL^p(B)$ is Banach, i.e. it is closed in the $L^p(B)$ topology.

Proof The result can be proved in several ways, for instance see Axler et al. (2001), Proposition 8.3. Here we refer back to Weyl's Lemma (see Yosida 1978, Chapter 4, §7) claiming that any distributional solution $u \in L^2$ of $\Delta u = h$ is C^∞ in any subdomain where h is C^∞ too. In our case $f \equiv 0$ is $C^\infty(B)$. Then let $u_n \in HL^p(B)$, i.e. $(u_n, \Delta\varphi) = 0$, $\forall\varphi \in \mathcal{D}(B)$, and assume that $u_n \to u$ in $L^p(B)$.

Accordingly, one has to have

$$(u, \Delta\varphi) = 0,\ \ \forall\varphi \in \mathcal{D}(B), \tag{36}$$

i.e. $\Delta u = 0$ in distribution sense, so that $u \in C^\infty(B)$ and $\Delta u = 0$ in classical sense too, i.e. $u \in HL^p(B)$. □

Proposition 4 *The spaces $HL^p(B)$ are reflexive, i.e. one has isometrically*

$$HL^p(B)^{**} \equiv HL^p(B) \tag{37}$$

Proof In fact we can use Millman's theorem, claiming that a sufficient condition for a Banach space to be reflexive is that it is uniformly convex (cf. Yosida 1978, ch. V, §2).

On the other hand we know that $L^p(B)$ spaces are uniformly convex; this is Clarckson's theorem (see Yosida 1978 Chapt. V §2). Since the property of uniform convexity is that $\{\forall x, y(\| x \|\leq 1,\ \| y \|\leq 1, \| x - y \|> \varepsilon),\ \exists\delta(\varepsilon)$, independent of x and y such that

$$\| x + y \|\leq 2(1 - \delta), \tag{38}$$

one sees that a closed subspace of a uniformly convex Banach space, is uniformly convex too. The proposition is proved. □

The next result is to identify the dual spaces of $HL^p(B)$. Although intuitive and elementary, the author has not found the proof in the literature, so we provide our own here. Let us agree to call q the conjugate number of p, i.e.

$$\frac{1}{p}+\frac{1}{q}=1. \tag{39}$$

Proposition 5 *(Riesz representation) The dual $HL^p(B)^*$ can be represented by $HL^q(B)$ through the $L^p - L^q$ coupling; in particular*

$$\begin{aligned}&\forall F \in HL^p(B)^*,\ \exists v \in HL^q(B);\\ &\forall u \in HL^p(B)\ ;\ F(u) \equiv (v,u) \equiv \int_B v(x)u(x)dB.\end{aligned} \tag{40}$$

Moreover an equivalence norm relation holds

$$\| F \|_{HL^p(B)^*} \sim \| v \|_{HL^q(B)}. \tag{41}$$

The relation with inverted indexes holds too.

Proof Let $v \in HL^q(B)$ and consider the linear functional

$$F_v(u) = (v,u)\ ; \tag{42}$$

then indeed, by Hölder's inequality,

$$|F_v(u)| \leq \| v \|_{L^q} \| u \|_{L^q}, \tag{43}$$

i.e. $F_v$is a bounded linear functional on $HL^p(B)$, or

$$HL^q(B) \subseteq HL^p(B)^*. \tag{44}$$

Note that the same argument shows that

$$HL^p(B) \subseteq HL^q(B)^*, \tag{45}$$

namely if $v \in HL^p(B)$ and $F_v(u) = (v,u),\ \forall u \in HL^q(B)$ one has

$$\| F_v \|_{HL^q(B)^*} \leq \| v \|_{HL^p(B)}. \tag{46}$$

Moreover, from (46) it follows that the image of $HL^p(B)$ through F_v, is closed in $HL^q(B)^*$. On the other hand F_v is also one to one, because

$$F_v = 0 \Rightarrow (v,u) = 0,\ v \in HL^p(B),\ \forall u \in HL^q(B). \tag{47}$$

Since $HL^q(B) \supset HL^p(B)$ (with $q < 2 < p$), (47) implies that $v \equiv 0$. Therefore the set

$$\{F_v : v \in HL^p(B)\} \subseteq HL^q(B)^* \tag{48}$$

is closed in $HL^q(B)^*$.

To prove that the image of $HL^p(B)$ through F_v is onto, assume the inclusion (48) to be strict; this implies the existence of a $G \in HL^q(B)^{**}$ such that $\| G \|= 1$ in that space and on the same time

$$\forall v \in HL^p(B), \ < F_v, G >\equiv 0.$$

By Proposition (4) this implies that $\exists u \in HL^q(B)$, such that $\| u \|_{L^q(B)}= 1$ and

$$\forall v \in HL^p(B), \ (v, u) \equiv 0. \tag{49}$$

In particular this relation entails that if we define the potential

$$w(x) = (N(x - y), u(y))$$

we must have $w(x) \equiv 0$ in Ω so that

$$w|_S = \partial_n w|_S = 0.$$

On the other hand

$$\Delta^2 w = -\Delta u = 0 \text{ in } B$$

and therefore, by the uniqueness of the solution of the bi-harmonic equation (see Appendix 1), we must have also $w = 0$ in B, i.e. $u = \Delta w = 0$ in B.

This contradicts the fact that $\| u \|_{L^q(B)}= 1$.

Thus we have shown that the relation F_v, $v \in HL^p(B)$, is into and onto $HL^q(B)^*$, so that by a classical Banach theorem (cfr. Yosida 1978) one has to have

$$\| F_v \|_{HL^q(B)^*} \leq \| v \|_{HL^p(B)} \leq c \| F_v \|_{HL^q(B)^*}, \tag{50}$$

implying a norm equivalence between $HL^p(B)$ and $HL^q(B)^*$.

To come to (40) we note that what we have proved up to now can be summarized by the sequence of relations

$$HL^p \equiv HL^{q^*} \subset HL^2 \subset HL^q, \tag{51}$$

inclusions being dense so that (HL^p, HL^2, HL^q) can be viewed as a classical Gelfand triple with HL^2 as pivot space. This entails that the duality coupling can be represented by (an extension of) the L^2 product (40).

Now going to duals in (51) and recalling proposition (4) on reflexivity of such spaces, we find

$$HL^{p*} \equiv HL^q \supset HL^2 \equiv HL^{2*} \supset HL^{q*} \equiv HL^p, \tag{52}$$

all inclusions being dense one into the other.

Therefore, the final representation of HL^{p^*} by HL^q can be be realized by the L^2 coupling. The proof of the proposition is complete. □

We are now in a position to prove the regular decomposition theorem of $L^p(B)$, given a Newtonian potential $u(x)$ generated by an $L^p(B)$ density f,

$$x \in \Omega, \; u(x) = N(f) \equiv (N(x-y), f(y)). \tag{53}$$

In the context of this section by $F_0 = \ker(N)$ we mean the null space of the operator N in $L^p(B)$.

Theorem 3 *Given $u \in HH^{2,p}_{loc}(\Omega), \; p > 2$, and regular at infinity, and f as in (53) there is one and only one regular decomposition of $L^p(B)$, i.e.*

$$f = \overline{f} + f_0 \tag{54}$$

with

$$\overline{f} \in HL^p(B) \tag{55}$$

and

$$f_0 \in F_0 \equiv \ker(N). \tag{56}$$

Moreover we have

$$f_0 = \Delta w, \; w \in H_0^{2,p}(B), \tag{57}$$

namely

$$F_0 \equiv \Delta\{H_0^{2,p}\}.$$

Proof Since $p > 2$ and $L^p(B) \subset L^2(B)$, we already know that there is one and only one $\overline{f} \in HL^2(B)$ such that

$$\forall x \in \Omega \quad (N(x-y), \overline{f}(y)) \equiv (N(x-y), f(y)); \tag{58}$$

the question is whether $\overline{f} \in HL^p(B)$ too. By the way, using the same reasoning as in Theorem 1, we know that (58) is equivalent to

$$(v, f) = (v, \overline{f}) \tag{59}$$

$\forall v$ harmonic in B_R, a sphere encompassing S.

Invoking the Runge–Krarup theorem, we can extend the functional

$$F(v) = (v, f)$$

to all $v \in HL^q(B)$, because indeed

$$|F(v)| = |(v, f)| \leq \| f \|_{L^p(B)} \| v \|_{HL^q(B)} . \tag{60}$$

This shows that $\overline{f}$ is not only harmonic but it must have a bounded norm in $HL^p(B)$ too, since it corresponds to a bounded linear functional on $HL^q(B)$. In particular, combining (59) with (60), one has

$$\begin{aligned} \| \overline{f} \|_{HL^p(B)} &= \sup_{\| v \|_{HL^q(B)} = 1} |(v, \overline{f})| \\ &\leq \| f \|_{L^p(B)} . \end{aligned} \tag{61}$$

Now, that $f_0 = f - \overline{f}$ belongs to ker (N) in $L^p(B)$ is obvious; it remains to prove (57).

As in Theorem 1, we define w by

$$\begin{cases} \Delta w = f - \overline{f} \\ w|_S = 0 \end{cases} \tag{62}$$

and, by standard theory on Dirichlet's problem (see for instance Theorem 9.15 in Gilbarg and Trudinger 1983) we know that w exists and is unique and (recall also (61))

$$\| w \|_{H^{2,p}(B)} \leq c \| f - \overline{f} \|_{L^p(B)} \leq 2c \| f \|_{L^p(B)} . \tag{63}$$

Moreover, from (59) we have

$$\begin{aligned} 0 = (v, f - \overline{f}) &= \int_B v(x) \Delta w(x) dB \\ &= \int_S v(x) \frac{\partial w}{\partial n}(x) dS \end{aligned} \tag{64}$$

for every v harmonic in a spherical domain B_R containing B. Since $\frac{\partial w}{\partial u}|_S \in H^{1-(1/p),p}(S)$ and the traces $v|_S$ are dense in the dual of such a space, (64) implies

$$\frac{\partial w}{\partial n}|_S = 0 \tag{65}$$

which, together with $w|_S = 0$, implies

$$w \in H_0^{2,p}(B). \tag{66}$$

The theorem is proved. □

Remark 4 Although we have proved (61) one should not think that $\overline{f}$ is a minimum norm solution, as it happened in $L^2(B)$.

In fact, if $\ker N \equiv \Delta\{H_0^{2,p}(B)\}$, a minimum norm solution $\widetilde{f} = f - \Delta w$ is defined by

$$\int |f - \Delta w|^p dB = \min, \quad w \in H_0^{2,p}(B). \tag{67}$$

But, assuming p to be even, a simple variational reasoning then shows that one must have

$$\Delta \widetilde{f}^{p-1} = \Delta(f - \Delta w)^{p-1} \equiv 0, \tag{68}$$

namely $\widetilde{f}$ is not harmonic.

We turn now to the question of the existence of a density $f \in L^p(B)$ generating a potential u regular at infinity and such that $u \in HH^{2,p}_{\text{loc}}(\Omega)$. As in the case of $L^2(B)$, we find directly $\overline{f} \in HL^p(B)$.

Theorem 4 *Given u as above, there is one density $\overline{f} \in HL^p(B)$ generating u.*

Proof As for Theorem 2, we look for a function $\overline{u}$ such that

$$\Delta\overline{u} = -\overline{f}$$

implying also that

$$\Delta^2\overline{u} = 0. \tag{69}$$

The bi-harmonic Eq. (69) has to be complemented with the boundary conditions

$$\begin{aligned} \overline{u}|_S &= u|_S = U \in H^{2-(1/p),p}(S) \\ \partial_n\overline{u}|_S &= \partial_n u|_S = V \in H^{1-(1/p),p}(S). \end{aligned}$$

As explained in the Appendix, such a problem has a unique weak solution in $H^{2,p}(B)$, and so we are allowed to put

$$\overline{f} = -\Delta\overline{u} \in L^p(B) \tag{70}$$

Therefore we can compute

$$
\begin{aligned}
x \in \Omega, \quad & \int N(x-y)\overline{f}(y)dB \\
& = -\int N(x-y)\Delta\overline{u}dB \\
& = \int [N(x-y)V(y) - U(y)\partial_n N(x-y)]dS \\
& \equiv u(x),
\end{aligned}
$$

which is permitted because $N(x-y)$, when $x \in \Omega$, is a smooth function of y on S and in B.

The proof is complete. □

4 Regular decomposition of $H^{-s,2}$, $s > 0$

In this section we will generalize the regular decomposition theorem to Sobolev spaces of distributions, assuming that $f \in H^{-s,2}$, with s a positive integer value. But before we do so, let us observe that all the problems we will face are coming from a set "close to the boundary S". In fact the following proposition holds.

Proposition 6 *Assume $S \in C^{1,\lambda}$ and let f be any distribution in $\mathcal{D}^*(B)$, with compact support $K \subset B$; then the corresponding potential*

$$
u = N(f) \tag{71}
$$

admits of an equivalent representation

$$
u = N(\overline{f}) \tag{72}
$$

by an $L^2(B)$ density $\overline{f}$.

Proof Since $K \subset B$ there is a smooth $\widetilde{S}$, with interior $\widetilde{B}$, $K \subset \widetilde{B} \subset B$, and exterior $\widetilde{\Omega} \supset \Omega$.

Indeed u is very smooth in $\widetilde{\Omega}$ and therefore we can apply Theorem 2 and find a function $\widetilde{f}$ harmonic and L^2 in $\widetilde{B}$ such that $u = N(\widetilde{f})$ in $\widetilde{\Omega}$ and therefore in Ω too. The function $\overline{f} = \{\overline{f} = \widetilde{f}(x), x \in \widetilde{\Omega},\ \overline{f} = 0,\ x \in \widetilde{\Omega}\backslash\Omega\}$ does satisfy (72). Notice that $\overline{f}$ is not unique, however we can also apply Theorem 2 directly to u in Ω and then we can find the unique $\overline{f} \in HL^2(B)$ satisfying (72). □

We turn now to the main point of the section, namely the regular decomposition of $H^{-s,2}$. We give an explicit proof for $s = 1$, since the proof for higher s follows the same lines, on condition that S satisfies suitable regularity hypotheses to use trace and regularization theorems. To simplify further discussion on this point, we shall assume in this section that S is C^∞.

We shall derive the regular decomposition of $H^{-1,2}(B)$ from the analogous decomposition of $H^{1,2}(B)$, by applying the isometric isomorphism J that relates $H^{1,2}(B)$ to $H^{-1,2}(B)$ by means of the $L^2(B)$ representation, namely

$$\begin{aligned} &Ju \equiv w \; in H^{-1,2} \Leftrightarrow < u, v >_{H^{1,2}} \equiv (Ju, v) = \int (w)(v) dB \\ &\forall u, v \in H^{1,2}, \end{aligned} \tag{73}$$

So we start from the following theorem of regular decomposition of $H^{1,2}$.

Theorem 5 *Let $f \in H^{1,2}$, then there are unique functions $\overline{f}$ and f_0 in $H^{1,2}$ such that*

$$f = \overline{f} + f_0, \tag{74}$$

with $\overline{f} \in HH^{1.2}$ (i.e. $\Delta \overline{f} = 0$ in B, $\overline{f} \in H^{1,2}(B)$), and

$$f_0 \in F_0 \equiv \ker(N)(in\ H^{1,2}); \tag{75}$$

moreover, in $H^{1,2}(B)$,

$$\begin{aligned} F_0 &= \widetilde{H}^{1,2} \equiv \Delta\{\widetilde{H}_0^{3,2}\} \\ \widetilde{H}_0^{3,2} &\equiv H^{3,2} \cap H_0^{2,2} \equiv \{w \in H^{3,2}\ ;\ w|_S = w_n|_S = 0\}. \end{aligned} \tag{76}$$

Proof We first of all note that, since $H^{1,2} \subset L^2$, the unique decomposition (74) certainly holds with $\overline{f}$, $f_0 \in L^2$; so if we can show that one of the two is in $H^{1,2}$, the theorem is proved. On the other hand, by dint of Theorem 1

$$f_0 = \Delta w,\ w \in H_0^{2,2}. \tag{77}$$

Therefore we have

$$\begin{cases} \Delta^2 w = \Delta f_0 = \Delta f \equiv g \in H^{-1,2} \\ w|_S = 0,\ w_n|_S = 0, \end{cases} \tag{78}$$

because the Laplacian of a function in $H^{1,2}$ is a distribution in $H^{-1,2}$.

At this point it is enough to apply a regularization Lemma (see for instance Lemma 4.1 in Neĉas 1967), to conclude that

$$w \in H^{3,2} \Rightarrow f_0 = \Delta w \in H^{1,2}. \tag{79}$$

We note too that (79) implies

$$\begin{aligned} \| f_0 \|_{H^{1,2}} &\equiv \| \Delta w \|_{H^{1,2}} \leq c_1 \| w \|_{H^{3,2}} \leq c_2 \| \Delta f \|_{H^{-1,2}} \\ &\leq c_3 \| f \|_{H^{1,2}}. \end{aligned} \tag{80}$$

We further observe that $\overline{f}$ and f_0 are L^2 orthogonal, because of (14).

$$(f_0, \overline{f}) = 0. \tag{81}$$

□

Corollary 1 *The Theorem 3 has a geometric interpretation in $H^{1,2}$, namely*

$$H^{1,2} \equiv HH^{1,2} \oplus \widetilde{H}^{1,2}, \tag{82}$$

$$\begin{cases} HH^{1,2} = QH^{1,2} \\ \widetilde{H}^{1,2} = PH^{1,2} \end{cases} \tag{83}$$

$$PQ = QP = 0, \tag{84}$$

where P, Q are non-orthogonal, complementary, bounded projectors.

Proof That P and Q are projectors is a consequence of the uniqueness of (74) implying

$$P^2 = P, \, Q^2 = Q;$$

that $P + Q = I$ is again a consequence of Theorem 5. That Q is bounded comes from the regularization Lemma and (80); that P is bounded comes from $P = I - Q$. That $PQ = QP = 0$ is intrinsic to the definition of such operators.

That they are non-orthogonal, i.e. $HH^{1,2}$ and $\widetilde{H}^{1,2}$ are non-orthogonal in $H^{1,2}$, is proved in the following way: let $\overline{f}$ be non-constant in $HH^{1,2}$, so that $\overline{f}_n|_S \neq 0$. Then, $\forall f_0 = \Delta w, \; w \in \widetilde{H}_0^{3,2}$,

$$\begin{aligned} < \overline{f}, f_0 >_{H^{1,2}} &= \int_B (\overline{f} f_0 + \nabla \overline{f} \cdot \nabla f_0) dB \\ &= \int_B \overline{f} f_0 dB + \int_S \overline{f}_n f_0 dS \\ &= \int_S \overline{f}_n f_0 dS, \end{aligned} \tag{85}$$

the last equality being a consequence of (81). Now $f_0 = \Delta w$ and expressing Δ in local coordinates adapted to S one sees (see e.g. Miranda 1970) that, since $w_n|_S = 0$ one has

$$\Delta w|_S = w_{nn} \in H^{1/2}(S).$$

Since $\forall g \in H^{1/2}(S)$ one can find a $w \in H^{3,2}$ such that (see Miranda 1970)

$$w|_S = 0, \; w_n|_S = 0, \; w_{nn}|_S = g,$$

we can conclude that if $\overline{f}$ is orthogonal to $\widetilde{H}^{1,2}$ in $H^{1,2}$, then

$$\forall g \in H^{1/2}(S), \quad \int_S \overline{f}_n g dS = 0,$$

namely one should have

$$\overline{f}_n = 0,$$

which contradicts the fact that $\overline{f}$ is not constant and harmonic. □

It is interesting to note that, also due to the above Corollary, the decomposition (82) has nothing to do with the classical decomposition of $H^{1,2}$ into the sum of $H_0^{1,2}$ and the subspace of solutions of the homogenous equation

$$(I - \Delta)u = 0,$$

which, contrary to the above, are orthogonal. In particular the $\overline{f} = Qf$ given by (74) is not the minimum $H^{1,2}$ norm density that generates a given outer potential.

We shall map now this result in $H^{-1,2}(B)$, considering the representation (73) of the duality coupling. In order to do that, we need a more precise representation of the operator J defined in that formula.

Proposition 7 *Let J be the mapping $H^{1,2} \to H^{-1,2}$ defined by*

$$\forall u, v \in H^{1,2} \ < u, v >_{H^{1,2}} = (Ju, v), \tag{86}$$

then one can write

$$Ju = (I - \Delta)u + u_n \delta_S \tag{87}$$

where $(I - \Delta)$ is the distributional operator defined in $\mathcal{D}^*(B)$, i.e.

$$((I - \Delta)u, \varphi) \equiv (u, (I - \Delta)\varphi), \quad \forall \varphi \in \mathcal{D}(B) \tag{88}$$

and $u_n \delta_S$ is a distribution in $\mathcal{S}^*$, with support on S, defined by

$$(u_n \delta_S, \ \psi) \equiv \int_S u_u \psi dS, \quad \forall \psi \in \mathcal{S}. \tag{89}$$

In particular when $u \in H^{1,2}(B)$, both $(I - \Delta)u$ and $u_n \delta_S$ are in $H^{-1,2}(\overline{B})$, namely are distributions in $H^{-1,2}(R^3)$ with support in $\overline{B}$, and indeed

$$\| Ju \|_{H^{-1,2}(\overline{B})} \equiv \| u \|_{H^{1,2}(B)}. \tag{90}$$

The proof of this proposition is rather standard, however, for the convenience of the reader, it is added to this paper in Appendix 2.

One further proposition is useful to prepare the main result, namely the regular decomposition theorem in $H^{-1,2}(B)$.

Proposition 8 *Let us consider the transpose projectors P^T, Q^T. They are indeed complementary bounded projectors too, and in particular*

$$\{P^T u\ ;\ u \in H^{1,2}\} \equiv \{HH^{1,2}\}^{\perp}_{(H^{1,2})} \tag{91}$$

$$\{Q^T u\ ;\ u \in H^{1,2}\} \equiv \{\widetilde{H}^{1,2}\}^{\perp}_{(H^{1,2})} \tag{92}$$

where the symbol $\frac{1}{(H)}$ means orthogonal complement in H.

Furthermore we have

$$\widetilde{H}^{-1,2} \stackrel{\text{def}}{=} \{JP^T u\ ;\ u \in H^{1,2}\} \equiv \{HH^{1,2}\}^{\perp}_{(L^2)} \tag{93}$$

$$HH^{-1,2} \stackrel{\text{def}}{=} \{JQ^T u\ ;\ u \in H^{1,2}\} \equiv \{\widetilde{H}^{1,2}\}^{\perp}_{(L^2)} \tag{94}$$

where the L^2-orthogonal complement to a subspace in $H^{1,2}$, is orthogonal to the above in the sense of the L^2 duality coupling (73).

Proof That P^T, Q^T do not coincide with P, Q is a consequence of the fact that P and Q are non-orthogonal projectors.

That they are projectors comes from transposing the two relations $P^2 = P$, $Q^2 = Q$ and that they are complementary, from transposing the relation $P + Q = I$, so that

$$(P^T)^2 = P^T,\ (Q^T)^2 = Q^T \tag{95}$$

$$P^T + Q^T = I. \tag{96}$$

That they are bounded is due to the fact that the transpose of a bounded operator has the same norm (e.g. see McLean 2000, Lemma 2.9). The relation (91) comes from

$$\forall u, v \in H^{1,2},\ < P^T u, Qv >_{H^{1,2}} = < u, PQv >_{H^{1,2}} = 0 \tag{97}$$

noting also that $\{Qv, v \in H^{1,2}\} \equiv HH^{1,2}$. The relation (92) comes from the symmetric reasoning.

The relation (93) is just a reformulation of (97) because

$$(JP^T u, Qv) = < P^T u, Qv > = 0. \tag{98}$$

The same holds for (94). □

We are now ready to prove the main theorem of this section.

Theorem 6 *The space $H^{-1,2}(\overline{B})$ admits of a unique, regular decomposition*

$$\begin{aligned} & f = \overline{f} + f_0, \\ & f \in H^{-1,2}(\overline{B}), \overline{f} \in HH^{-1,2}(\overline{B}), \;\; f_0 \in \widetilde{H}^{-1,2}, \end{aligned} \tag{99}$$

such that $\overline{f}$ is harmonic in B and $f_0 \in F_0 \equiv \ker(N)$ (in $H^{-1,2}(B)$), i.e.

$$(f_0, h) = 0, \; \forall h \in HH^{1,2}\,, \tag{100}$$

so that

$$\forall x \in \Omega \quad (f, N(x-y)) \equiv (\overline{f}, N(x-y)). \tag{101}$$

More precisely, one has

$$\overline{f} = JQ^T u \tag{102}$$
$$f_0 = JP^T u \tag{103}$$

for $u \in HH^{1,2}$ such that

$$Ju = f. \tag{104}$$

Proof Given any $f \in H^{-1,2}$ there is a solution $u \in H^{1,2}$ of (104) because J is an isometric isomorphism between the two spaces (cfr. Proposition 7). On the other hand

$$\left(Q^T + P^T\right) u = u$$

so that the decomposition

$$f = Ju = JQ^T u + JP^T u = \overline{f} + f_0$$

is unique and covers the whole $H^{-1,2}$.

What we need to prove is that $\overline{f}$ is harmonic in B and that f_0 can be seen as the $H^{-1,2}$ limit of $\Delta\varphi_n, \varphi_n \in \mathcal{D}(B)$.

The first statement is just a consequence of Proposition 8, relation (94). In fact (94) implies that a fortiori

$$\forall \varphi \in \mathcal{D}(B), \; (JQ^T u, \Delta\varphi)_{L^2} = (\overline{f}, \Delta\varphi)_{L^2} = 0, \tag{105}$$

because $\varphi \in \widetilde{H}_0^{3,2}$ and then $\Delta\varphi \in \widetilde{H}^{1,2}$ (see 76); hence $\overline{f}$ is harmonic in B.

As for the second claim, we fist note that the set $\Phi \equiv \Delta\{\mathcal{D}(B)\}$ is contained in $\widetilde{H}^{-1,2} \equiv (HH^{1,2})^{\perp}_{(L^2)}$. In fact, for every fixed $\varphi \in \mathcal{D}(B)$ and $\forall h \in HH^{1,2}$

$$(\Delta\varphi, h) \equiv 0. \tag{106}$$

On the other hand we want to prove that Φ is dense in $\widetilde{H}^{-1,2}$, in the topology of $H^{-1,2}$, or that

$$\overline{\Phi} = [\Phi]_{H^{-1,2}} \equiv \widetilde{H}^{-1,2} \tag{107}$$

But if (107) is not true, then there is an $\widetilde{f} \in \widetilde{H}^{-1,2}$ such that

$$\forall \varphi \in \mathcal{D}(B), \ < \Delta\varphi, \widetilde{f} >_{H^{-1,2}} = 0. \tag{108}$$

On the other hand, since $\widetilde{f} \in \widetilde{H}^{-1,2}$, there must be a $\widetilde{u} \in H^{1,2}$ such that

$$\widetilde{f} = J P^T \widetilde{u}, \tag{109}$$

so that

$$\forall \varphi \in \mathcal{D}(B), \ < \Delta\varphi, J P^T \widetilde{u} >_{H^{-1,2}} \equiv 0. \tag{110}$$

But, due to the definition of J and the fact that this operator is an isometry of $H^{1,2}$, we have the identity

$$\forall u, v \in H^{1,2} \quad < u, v >_{H^{1,2}} = (Ju, v)_{L^2} = < Ju, Jv >_{H^{-1,2}}; \tag{111}$$

by applying the second of (111) to (110) with $Ju = \Delta\varphi$ and $v = P^T \widetilde{u}$, we see that $(\Delta\varphi, P^T \widetilde{u})_{L^2} = 0$, namely

$$P^T \widetilde{u} \in HH^{1,2}. \tag{112}$$

However $P^T \widetilde{u}$ is orthogonal in $H^{1,2}$ to $HH^{1,2}$ by the definition (91), so that (112) implies

$$P^T \widetilde{u} = 0 \Rightarrow \widetilde{f} = J P^T \widetilde{u} = 0.$$

Therefore (107) is proved and the theorem is complete. □

The same reasoning applies for larger integers s, as far as regularization theorems can be derived from the general theory of elliptic equations. However, here we mention explicitly only the case $s = 2$, for which the regular decomposition of $H^{2,2,}(B)$ can be first derived by using the Lemma 4.2 in Neĉas (1967), and then mapped as above to $H^{-2,2}(\overline{B})$.

5 Examples

We present here a few examples where an explicit computation of $\overline{f}$, and hence of f_0, can be performed. The first two are relative to a spherical domain of radius R, the third is more general and has the purpose of highlighting the relation between f, $\overline{f}$ and the reproducing kernel of the relevant space.

Let us note before starting that the spherical harmonics $\{Y_{nm}(\sigma),\ n = 0, 1 \ldots; 0 \leq |m| \leq n\}$ will be used with the standard normalization $\frac{1}{4\pi}\int Y_{nm}^2(\sigma)d\sigma = 1$.

Example 1 A point mass at x_0 in a ball $\overline{B}$ is basically a distribution in $H^{-2,2}(\overline{B})$

$$f = m\delta_{x_0}(x) = m\delta(x - x_0). \tag{113}$$

This can be seen because the support of f is $x_0 \in \overline{B}$ and

$$\widehat{f}(p) = m\int e^{i2\pi p\cdot x}\delta_{x_0}d_3x = me^{i2\pi p\cdot x_0} \tag{114}$$

so that

$$\int \frac{|\widehat{f}(p)|^2}{(1+p^2)^s}d_3p = 4\pi m^2\int_0^{+\infty}\frac{p^2dp}{(1+p^2)^s} < +\infty$$

for an integer s, only if $s \geq 2$.

To find $\overline{f}$ we simply note that the potential generated by m outside S can be written as

$$u(x) = \frac{m}{r_{x_0x}} = \frac{m}{R}\sum\left(\frac{r_0}{R}\right)^n\frac{Y_{nm}(\sigma_0)}{2n+1}\left(\frac{R}{r_x}\right)^{n+1}Y_{nm}(\sigma_x), \tag{115}$$

where (r_0, σ_0) are the spherical coordinates of x_0, and in particular $\sigma_0 = (\lambda_0, \vartheta_0)$. On the other hand for any harmonic $\overline{f}(y)$

$$\overline{f}(y) = \sum\overline{f}_{nm}\left(\frac{r_y}{R}\right)^nY_{nm}(\sigma_y),$$

one can easily compute, for $r_x \geq R$,

$$\left(\overline{f}, \frac{1}{r_{xy}}\right) = (4\pi R^2)\sum\overline{f}_{nm}\frac{1}{(2n+1)(2n+3)}\cdot\left(\frac{R}{r_x}\right)^{n+1}Y_{nm}(\sigma_x). \tag{116}$$

A direct comparison gives

$$\overline{f}_{nm} = \frac{m}{4\pi R^3}(2n+3)\left(\frac{r_0}{R}\right)^nY_{nm}(\sigma_0). \tag{117}$$

It is remarkable that, since the norm in $H^s(B)$ of f can be equivalently written as

$$\| f \|_{H^s(B)}^2 \sim R^3\sum f_{nm}^2\frac{(1+n)^{2s}}{2n+3}, \tag{118}$$

one has, for negative s too,

$$\| \overline{f} \|^2_{H^{-s}(\overline{B})} \sim \frac{m^2}{(4\pi)^2} \sum \frac{(2n+3)^2(2n+1)}{(2n+3)(1+n)^{2s}} \left(\frac{r_0}{R}\right)^{2n}.$$

As one can see, such a sum is convergent for any positive or negative s when $r_0 < R$, i.e. x_0 is inside B, thus confirming Proposition 6.

However, even for $r_0 = R$, one finds that (118) is finite only if the integer s is at least equal to 2, i.e. we find that $\delta_{x_0}(x)$ is in $H^{-2,2}$ even for $x_0 \in S$, as it should be.

Example 2 In this example we want to consider the case of a single layer on a sphere S;

$$f = f_S(\sigma)\delta(r-R) = \left(\sum f_{nm} Y_{nm}(\sigma)\right)\delta(r-R). \tag{119}$$

We first of all claim that

$$f \in H^{-1,2}(\overline{B}) \Leftrightarrow f_S(\sigma) \in H^{-1/2}(S).$$

For this purpose we note that the support of f is $S \subset \overline{B}$. Then we observe that $f \in H^{-1,2}(\overline{B})$, if and only if the coupling

$$(f, v), \quad v \in H^{1,2}(B)$$

is bounded by the norm of v in $H^{1,2}(B)$.

As for any $v \in H^{1,2}(B)$

$$v = \sum v_{nm}(r) Y_{nm}(\sigma)$$

we have a trace on S which also belongs to $H^{1/2}(S)$, i.e.

$$\begin{aligned} \| v|_S \|_{H^{1/2}(S)} &\sim \left\{\sum v_{nm}(R)^2(1+n)\right\}^{1/2} \\ &\leq c \| v \|_{H^{1,2}(B)} < +\infty, \end{aligned}$$

we find then

$$\begin{aligned} |(f, v)| &= 4\pi R^2 \left|\sum f_{nm} v_{nm}(R)\right| \\ &\leq c \left\{\sum \frac{f_{nm}^2}{1+n}\right\}^{1/2} \| v \|_{H^{1,2}(B)} \end{aligned}$$

which is bounded if $f_S \in H^{-1/2}(S)$, i.e.

$$\| f_S \|_{H^{-1/2}(S)} \sim \left\{\sum \frac{f_{nm}^2}{1+n}\right\} < +\infty. \tag{120}$$

Thus we note that the outer potential generated by f is just

$$r \geq R,\ u(x) = 4\pi R \sum f_{nm} \frac{1}{(2n+1)} \cdot \left(\frac{R}{r_x}\right)^{n+1} Y_{nm}(\sigma_x), \tag{121}$$

which is in $H^{1,2}_{\text{loc}}(\Omega)$, because its trace on S is in $H^{1/2}(S)$. In fact, recalling (120),

$$\begin{aligned} \| u(x)|_S \|^2_{H^{1/2}(S)} &= (4\pi R)^2 \sum \frac{f^2_{nm}}{(2n+1)^2}(1+n) \\ &\sim (4\pi R^2)^2 \| f_S \|^2_{H^{-1/2}(S)} . \end{aligned} \tag{122}$$

On the other hand comparing (121) with (116) we find immediately that the equivalent harmonic distribution $\overline{f}$ is

$$\overline{f}(y) = \frac{1}{R} \sum (2n+3) f_{nm} \left(\frac{r_y}{R}\right)^n Y_{nm}(\sigma_y). \tag{123}$$

That $\overline{f}$ is in $H^{-1/2}(\overline{B})$, as it should be, derives from the formula (118) with $s = -1$, namely

$$\| \overline{f} \|^2_{H^{-1,2}(\overline{B})} = R \sum \frac{(2n+3)^2}{(2n+3)(1+n)^2} f^2_{nm}$$

which is bounded by $\| f_S \|^2_{H^{-1/2}(S)}$, thanks to (120).

Example 3 We return now to the example of a square integrable mass distribution f but this time in a domain B with smooth boundary S and with an arbitrary shape. The purpose is to discuss explicitly the shape of $\overline{f}$ and of $f_0 = \Delta w$ too. The first statement is that HL^2 has a reproducing kernel $k(x, y)$, such that

$$k(x, y) = \sum \psi_n(x) \psi_n(y)$$

with $\{\psi_n(x)\}$ any orthonormal sequence complete in $HL^2(B)$ and therefore

$$k(x, y) = k(y, x)$$

$$\Delta_y k(x, y) = \Delta_x k(x, y) = 0 \text{ in } B \tag{124}$$

$$(k(x, y), h(y)) \equiv h(x),\ \forall h \in HL^2(B). \tag{125}$$

That such a kernel exists for Bergman spaces, including $HL^2(B)$, is known in the literature (see Axler et al. 2001, Chapter 8) and derives from the fact that the evaluation functional, at any $x \in B$, $(x \neq S)$ is bounded. That $\forall f(y) \in L^2(B)$,

$$\overline{f}(x) = (k(x, y), f(y)) \tag{126}$$

is known too and it can also be derived at once from the decomposition

$$f(x) = \overline{f}(x) + f_0(x)$$

by observing that f_0 is L^2 orthogonal to harmonic functions, so that, thanks to (124) and (125), one gets

$$\begin{aligned}(k(x, y),\ f(y)) &= (k(x, y), \overline{f}(y)) + (k(x, y), f_0(y)) \\ &= \big(k(x, y), \overline{f}(y)\big) \equiv \overline{f}(x),\end{aligned}$$

i.e. (126).

All the above is known in the literature. However we would like now to get an explicit representation of

$$f_0 = \Delta w = f - \overline{f},\ w \in \widetilde{H}_0^{2,2}.$$

Let us consider the kernel

$$Q(x, y) = G(x, y) - \int_B G(x, z)k(z, y)dB_z; \tag{127}$$

where $G(x, y)$ is the Green function of the domain B.

We claim that

$$w(x) = \int_B Q(x, y)f(y)dB_y. \tag{128}$$

In fact, since

$$\Delta_x G(x, y) = \delta(x - y)$$

we get

$$\Delta_x Q(x, y) = \delta(x - y) - k(x, y)$$

so that, recalling also (126),

$$\begin{aligned}\Delta w(x) &= \int_B \{\delta(x - y) - k(x - y)\}f(y)dB_y \\ &\equiv f(x) - \overline{f}(x).\end{aligned}$$

At the same time, that $w|_S = 0$ is immediately seen from (127) because $G(x, y) = 0, x \in S$, and so $Q(x, y) = 0, x \in S$.

The fact that $\frac{\partial w}{\partial n}|_S = 0$ is derived from a different form of $Q(x, y)$. All we need is to recall that

$$G(x, y) = -N(x - y) + H(x, y)$$

where $H(x, y)$ is symmetric, harmonic and such that

$$H(x, y) = N(x - y), \; x \in S.$$

Accordingly, since

$$H(x, y) - \int_B H(x, z)k(z, y)dB_y = 0,$$

we can rewrite (127) as

$$Q(x, y) = -N(x - y) + \int_B N(x - z)k(z - y)dB_z. \tag{129}$$

Now we have

$$x \in S, \; \frac{\partial}{\partial n_x} Q(x, y) = -\frac{\partial}{\partial n_x} N(x - y) + \int_B \frac{\partial}{\partial n_x} N(x - z)k(z, y)dB_z. \tag{130}$$

But the kernel

$$x \in S, \; -\frac{\partial}{\partial n_x} N(x - y) = \frac{n_x \cdot (x - y)}{|x - y|^3}$$

is well-known to be $O\left(\frac{1}{|x-y|}\right)$ (see Miranda 1970) if S is smooth as we have assumed; so this kernel is, $\forall x \in S$, in $L^2(B)$.

Furthermore $-\frac{\partial}{\partial n_x} N(x - y)$ is obviously harmonic in y; we therefore conclude that

$$\int_B \frac{\partial}{\partial n_x} N(x - z)k(z - y)dB_z = \frac{\partial}{\partial n_x} N(x - y).$$

From (130) we then conclude that $\frac{\partial}{\partial n_x} Q(x, y) = 0$ for $x \in S$ and so, from (128), we find that w satisfies $\frac{\partial}{\partial n} w(x) = 0$, $x \in S$ too.

6 Conclusions

In conclusion we could say that the classical decomposition of mass distributions into a harmonic component and a component generating a zero external field, which can be approximated by $\Delta\varphi$, $\varphi \in \mathcal{D}(B)$, has been proved to be applicable to a considerable variety of Banach and Hilbert spaces, other than $L^2(B)$. At the same time the L^2 theory has been amplified by existence theorems and by a representation of the decomposition that refers back to the application of a reproducing kernel. The same concept could be usefully generalized to other spaces too, in future research.

Furthermore, an interesting generalization of the inverse gravimetric problem, could include some constraints on the solution for instance to have mass distribution with a layered structure, and a fixed geometry. In this case the explicit description of the subspaces F and F_0 should become more complex but interesting.

Appendix 1

Consider the problem of Proposition (2)

$$\begin{aligned} \Delta^2\overline{u} &= 0 \qquad \text{in } B \\ \overline{u}|_S &= U \in H^{3/2}(S) \\ \partial_n\overline{u}|_S &= V \in H^{1/2}(S) \\ S &\in C^{1,1}; \end{aligned} \tag{131}$$

as it is known, under the above regularity conditions, one can always find (see McLean 2000, Appendix A) a $w \in H^{2,2}(B)$ such that

$$w|_S = U,\ \partial_n w|_S = V.$$

So we can transform (131) into

$$\overline{u} = w - u, \tag{132}$$

$$\Delta^2 u = \Delta^2 w \in H^{-2,2}(B) \tag{133}$$

$$\begin{aligned} u|_S &= 0 \\ \partial_n u|_S &= 0 \\ u &\in H_0^{2,2}. \end{aligned}$$

The problem (133) has the following equivalent weak formulation

$$\begin{aligned} &\forall\varphi \in \mathcal{D}(B), \quad \int \Delta u \Delta\varphi dB \equiv \int \Delta w \Delta\varphi dB \\ &(u \in H_0^{2,2}). \end{aligned} \tag{134}$$

Since $\mathcal{D}(B)$ is dense in $H_0^{2,2}$, we can extend (134) to all functions in this space. Notice that in $H_0^{2,2}$ the product

$$< u, v >_{H_0^{2,2}} = \int \Delta u \Delta v dB \tag{135}$$

provides an equivalent topology to that inherited from $H^{2,2}$. This follows e.g. from Gilbarg and Trudinger (1983), Lemma 9.17, where it is stated that when $u \in H_0^{2,2}$ one has

$$\| u \|_{H^{2,2}} \leq c \| \Delta u \|_{L^2}, \tag{136}$$

the reverse inequality (with a different constant) being obvious. Therefore

$$\left| \int \Delta w \Delta \varphi dB \right| \leq \left(\int (\Delta w)^2 dB \right)^{1/2} \| \varphi \|_{H_0^{2,2}}, \tag{137}$$

so that, recalling that $w \in H^{2,2}(B)$, we find that

$$F(\varphi) = \int \Delta w \Delta \varphi dB$$

is a bounded functional on $H_0^{2,2}$.

Accordingly, (134), written as

$$< u, \varphi >_{H_0^{2,2}} \equiv F(\varphi) \tag{138}$$

has one and only one solution $u \in H_0^{2,2}$ in accordance to Riesz's theorem. The function $\overline{u} = w - u$ is then the solution of (131).

Remark 5 The same result holds if we go over from (131), which comes from boundary values of an $H^{2,2}_{\mathrm{loc}}(\Omega)$ function, onto the more general problem

$$\begin{aligned} \Delta^2 \overline{u} &= 0 \text{ in } B \\ \overline{u}|_S &= U \in H^{2-(1/p),p}(S) \\ \partial_n \overline{u}|_S &= V \in H^{1-(1/p),p}(S), \end{aligned} \tag{139}$$

which derives form the assumption $u(x) \in H^{2,p}_{\mathrm{loc}}(\Omega)$.

In this case we are looking for a weak solution $\overline{u} \in H^{2,p}(B)$.

The existence of w in $H^{2,p}(B)$ agreeing with the given boundary values, or equivalently extending u from Ω into B, is guaranteed by the same theorems (see McLean 2000, Appendix A), the weak form of (139) is then the same as (134), with $\varphi \in \mathcal{D}(B)$. Moreover (136) generalizes to

$$\| u \|_{H^{2,p}(B)} \leq C \| \Delta u \|_{L^p(B)}$$

for $u \in H_0^{2,p}(B)$, always by dint of Lemma 9.17 in Gilbarg and Trudinger (1983), so that the existence of the solution comes by extending (134) to $\varphi \in H_0^{2,p}$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, and then by applying the Riesz theorem to the duality coupling between $H_0^{2,p}$ and $H_0^{2,q}$.

Appendix 2

In this Appendix we give proof of Proposition 7. We first establish the representation (87) when $u = \chi, v = \psi \in C^\infty$. In this case in fact

$$
\begin{aligned}
< \chi, \psi >_{H^{1,2}} &= \int\limits_R (\chi\psi + \nabla\chi \cdot \nabla\psi) dB \\
&\equiv \int\limits_B [(I - \Delta)\chi]\psi dB + \int\limits_S \chi_n \psi dS \\
&\equiv (J\chi, \psi).
\end{aligned}
\tag{140}
$$

We prove that the two bilinear forms

$$
B_1(\chi, \psi) = \int_B [(I - \Delta)\chi]\psi dB \tag{141}
$$

$$
B_2(\chi, \psi) = \int_S \chi_n \psi dS \tag{142}
$$

are both jointly continuous in the $H^{1,2}$ norm of χ and ψ and finally we extend (140) letting χ tend to any $u \in H^{1,2}$ and ψ tend to any $v \in H^{1,2}$, considering that this is possibile because C^∞ is dense in $H^{1,2}$.

Since (140) can be written as

$$
< \chi, \psi >_{H^{1,2}} = B_1(\chi, \psi) + B_2(\chi, \psi)
$$

it is enough to prove that $B_1(\chi, \psi)$ is continuous on $H^{1,2}(B) \otimes H^{1,2}(B)$ to conclude that the same holds for $B_2(\chi, \psi)$ because $< \chi, \psi >_{H^{1,2}}$ is obviously continuous on the above product space.

Now we note that

$$
B_1(\chi, \psi) = (\chi, \psi) - (\Delta\chi, \psi). \tag{143}
$$

On the other hand

$$
|(\Delta\chi, \psi)| = \left| \int_B \Delta\chi\psi dB \right| \leq \| \Delta\chi \|_{H^{-1,2}} \| \psi \|_{H^{1,2}}
$$

and since Δ is a continuous operator $H^{1,2} \to H^{-1,2}$ one has

$$
|(\Delta\chi, \psi)_{L^2}| \leq c \| \chi \|_{H^{1,2}} \| \psi \|_{H^{1,2}}
$$

so that, returning to (94),

$$
|B_1(\chi, \psi)| \leq \| \chi \|_{L^2} \| \psi \|_{L^2} + c \| \chi \|_{H^{1,2}} \| \psi \|_{H^{1,2}} \tag{144}
$$

and since $H^{1,2}$ is continuously embedded in L^2 the continuity of B_1 on $H^{1,2} \otimes H^{1,2}$ is proved.

Finally, the relation (90) is just the consequence of the definition of the norm in $H^{-1,2}(\overline{B})$ as dual to $H^{1,2}(B)$, namely

$$\begin{aligned} \| \; Ju \; \|_{H^{-1,2}} &= \sup_{\|v\|_{H^{1,2}}=1} \int (Ju) v dB \\ &= \sup_{\|v\|_{H^{1,2}}=1} < u, v >_{H^{1,2}} = \| \; u \; \|_{H^{1,2}} \; . \end{aligned}$$

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