AIP | Physics of Fluids

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Citation: Physics of Fluids (1994-present) **26**, 041702 (2014); doi: 10.1063/1.4871292
View online: http://dx.doi.org/10.1063/1.4871292
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/26/4?ver=pdfcov
Published by the AIP Publishing

PHYSICS OF FLUIDS **26**, 041702 (2014)

On the extension of the eddy viscosity model to compressible flows

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(Received 15 January 2014; accepted 27 March 2014; published online 14 April 2014)

In the paper the authors examine the extension of the eddy viscosity modeling approach to compressible large eddy simulation. On the basis of formal algebraic relations among the generalized central moments and the filtered Favre terms, a new compressible eddy viscosity formulation is derived. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4871292]

One main problem in compressible LES, the Large Eddy Simulation of compressible or variable density flows, is to model the filtered quantities $\overline{\varrho u_i}$ and $\overline{\varrho u_i u_j}$, where the overline stands for a generic filtered average, ϱ is the density and u_i is the velocity field. In incompressible LES, the basic problem is to model $\overline{u_i u_j}$ and the simplest modeling assumption is based on the so-called eddy viscosity concept or gradient diffusion hypothesis

$$\begin{aligned} \overline{u_i u_j} &= \overline{u}_i \, \overline{u}_j + \tau(u_i, u_j), \\ \tau(u_i, u_j) &= -\nu_u \left(\partial_j \overline{u}_i + \partial_i \overline{u}_j \right), \end{aligned} \tag{1}$$

where $\tau(u_i, u_j)$ and ν_u are, respectively, the subgrid generalized central moment and the subgrid eddy viscosity associated with the turbulent transport of the velocity field. The usual transposition of this modeling approximation to compressible flows is composed of two steps. The first is to introduce the filtered Favre averages $\tilde{u}_i$ and the Favre subgrid stress $\vartheta(u_i, u_j)$, defined as

$$\begin{aligned} \overline{\varrho u_i} &= \overline{\varrho} \tilde{u}_i \\ \overline{\varrho u_i u_j} &= \overline{\varrho} \tilde{u}_i \tilde{u}_j + \overline{\varrho} \vartheta(u_i, u_j) \end{aligned} \tag{2}$$

and the second is to write

$$\vartheta(u_i, u_j) = -\nu_u \left(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i \right). \tag{3}$$

We remark that usually this modeling relation is only applied to the anisotropic part of the tensors $\overline{u_i u_j}$ and $\overline{\varrho u_i u_j}$, but as regards the following discussion this is unessential.

The extension of relation (1) to relation (3) for compressible flows is obviously very reasonable, and some theoretical arguments can be found in the seminal papers of Yoshizawa[1] and of Speziale *et al.*[2] From a formal point of view, and in order to have more insight, we write explicitly the standard mass unweighted filtered expansions of $\overline{\varrho u_i}$ and $\overline{\varrho u_i u_j}$

$$\overline{\varrho u_i} = \overline{\varrho} \tilde{u}_i = \overline{\varrho} \, \overline{u}_i + \tau(\varrho, u_i),$$

$$\begin{aligned} \overline{\varrho u_i u_j} = \overline{\varrho} \tilde{u}_i \tilde{u}_j + \overline{\varrho} \vartheta(u_i, u_j) &= \overline{\varrho} \, \overline{u}_i \, \overline{u}_j + \overline{\varrho} \tau(u_i, u_j), \\ &+ \overline{u}_i \tau(\varrho, u_j) + \overline{u}_j \tau(\varrho, u_i) + \tau(\varrho, u_i, u_j), \end{aligned} \tag{4}$$

where $\tau(\varrho, u_i)$ and $\tau(\varrho, u_i, u_j)$ are the generalized subgrid moments associated to the turbulent transport of the density. Very little work has been done to model directly all these terms in order to develop a LES approach to compressible flows without density weighting. To our knowledge only the papers[3,4] deal with this approach, and the modeling hypotheses made in these papers are mainly in the framework of the eddy viscosity approach, and can be summarized as

$$\begin{aligned} \tau(u_i, u_j) &= -\nu_u \left(\partial_j \overline{u}_i + \partial_i \overline{u}_j\right), \\ \tau(\varrho, u_j) &= -\nu_\varrho \partial_j \overline{\varrho}, \\ \tau(\varrho, u_i, u_j) &= 0, \end{aligned} \tag{5}$$

where ν_u and ν_ϱ are the eddy viscosity associated with the transport of the velocity field and the density, respectively.

In order to have an idea of the effect that these modeling hypotheses have on the Favre subgrid stresses $\vartheta(u_i, u_j)$, we remark that we can easily derive from (4) the following basic relations among the standard filtered quantities and the Favre filtered terms[5]

$$\begin{aligned} \tilde{u}_i &= \overline{u}_i + \frac{\tau(\varrho, u_i)}{\overline{\varrho}}, \\ \vartheta(u_i, u_j) &= \tau(u_i, u_j) - \frac{\tau(\varrho, u_i)\tau(\varrho, u_j)}{\overline{\varrho}^2} + \frac{\tau(\varrho, u_i, u_j)}{\overline{\varrho}} . \end{aligned} \tag{6}$$

These relations are well known in the framework of the Reynolds and Favre averages, see, for example, Eq. (5.33) of Ref. 6, and they are usually expressed in terms of the associated Reynolds fluctuations ϱ', u_i', and Favre fluctuations u_i'' as

$$\begin{aligned} \tilde{u}_i &= \overline{u}_i + \frac{\overline{\varrho' u_i'}}{\overline{\varrho}}, \\ \widetilde{u_i'' u_j''} &= \overline{u_i' u_j'} - \frac{\overline{\varrho' u_i'}\;\overline{\varrho' u_j'}}{\overline{\varrho}^2} + \frac{\overline{\varrho' u_i' u_j'}}{\overline{\varrho}}. \end{aligned} \tag{7}$$

Less known is that if we introduce the generalized central moments,[7] they can easily be extended to a generic filtering operator not provided with the property $\overline{u}_i = \overline{\overline{u}}_i$, and this remark is the starting point of our investigation. Let us notice that from the first of the identities (6)

$$\tilde{u}_i = \overline{u}_i + \frac{\tau(\varrho, u_i)}{\overline{\varrho}}$$

we have

$$\begin{aligned} \partial_j \tilde{u}_i + \partial_i \tilde{u}_j = \partial_j \overline{u}_i + \partial_i \overline{u}_j &- \frac{\tau(\varrho, u_i)\partial_j \overline{\varrho} + \tau(\varrho, u_j)\partial_i \overline{\varrho}}{\overline{\varrho}^2} \\ &+ \frac{\partial_j \tau(\varrho, u_i) + \partial_i \tau(\varrho, u_j)}{\overline{\varrho}} , \end{aligned} \tag{8}$$

so that if we assume

$$\tau(u_i, u_j) = -\nu_u \left(\partial_j \overline{u}_i + \partial_i \overline{u}_j\right) \tag{9}$$

we can derive from the second of the identities (6)

$$\vartheta(u_i, u_j) = \tau(u_i, u_j) - \frac{\tau(\varrho, u_i)\tau(\varrho, u_j)}{\overline{\varrho}^2} + \frac{\tau(\varrho, u_i, u_j)}{\overline{\varrho}}$$

the following exact relation

$$\begin{aligned} \vartheta(u_i, u_j) = &-\nu_u \left(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i\right) \\ &- \nu_u \frac{\tau(\varrho, u_i)\partial_j \overline{\varrho} + \tau(\varrho, u_j)\partial_i \overline{\varrho}}{\overline{\varrho}^2} - \frac{\tau(\varrho, u_i)\tau(\varrho, u_j)}{\overline{\varrho}^2} \end{aligned}$$

$$+ \nu_u \frac{\partial_j \tau(\varrho, u_i) + \partial_i \tau(\varrho, u_j)}{\overline{\varrho}} + \frac{\tau(\varrho, u_i, u_j)}{\overline{\varrho}}. \tag{10}$$

By consequence the standard extension of the eddy viscosity model to compressible flows in terms of the Favre averages

$$\vartheta(u_i, u_j) = -\nu_u \left(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i\right) \tag{11}$$

can be derived on the basis of the following eddy viscosity modeling hypotheses

$$\begin{aligned} \tau(u_i, u_j) &= -\nu_u \left(\partial_j \overline{u}_i + \partial_i \overline{u}_j\right), \\ \tau(\varrho, u_j) &= -\nu_\varrho \partial_j \overline{\varrho}, \\ \tau(\varrho, u_i, u_j) &= -\nu_{\varrho u} \left(\partial_j \tau(\varrho, u_i) + \partial_i \tau(\varrho, u_j)\right) \end{aligned} \tag{12}$$

when we assume that the three eddy viscosities, related to the different transport mechanisms, are exactly given by

$$\nu_{\varrho u} = \nu_u; \qquad \nu_\varrho = 2\nu_u. \tag{13}$$

Obviously these assumptions are unjustified. In particular, as regards the eddy viscosity ν_ϱ associated to the transport of the density, it is easy to verify that if we remove this hypothesis and if we introduce an eddy viscosity ν_ϱ different from $2\nu_u$ we have

$$\vartheta(u_i, u_j) = -\nu_u \left(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i\right) - \frac{\nu_\varrho(\nu_\varrho - 2\nu_u)}{\overline{\varrho}^2} \partial_i \overline{\varrho}\, \partial_j \overline{\varrho}. \tag{14}$$

Relation (14) is the newly derived eddy viscosity formulation. We remark that it contains an additional new term depending upon the gradient of the filtered density. It becomes zero only if the eddy viscosity ν_ϱ, associated to the transport of the density, is null or equal to two times the eddy viscosity ν_u associated to the transport of the velocity field. It depends on the product of two subgrid eddy viscosities that typically are scaled as Δ^α, where Δ is the subgrid length and α is two for the Smagorinsky model and one for the energy equation model. The dynamic approach could be implemented to find numerically the appropriate constants for the different viscosities. It is not immediately obvious when the new contribution will be important. Probably for very strong gradients of the averaged densities its effects cannot be discarded, and it is the intention of the authors to explore in more detail the importance and the effect of this new term with numerical computations. It seems that in the framework of the eddy-viscosity modeling approach the theoretical result by itself could be of some general interest in the field.

We thank finally one Referee for the detailed and useful examination of the paper.

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