INSTABILITY RESULTS FOR A HILL EQUATION COUPLED WITH AN ASYMMETRICALLY NONLINEAR OSCILLATOR

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Abstract. We consider a Hill equation whose potential depends on the solution of a nonlinear oscillator. The nonlinearity of the oscillator is given by a function $f(x)$ which has polynomial growth as $x \to +\infty$ and is asymptotically constant as $x \to -\infty$. We provide explicit conditions on a set of 4 parameters for the stability of the Hill equation as the energy of the oscillator approaches infinity. In the case when the ratio of the angular frequencies of the linearized system (around the null solution) is an integer, we recover the same instability intervals as in the case in which f was extended by symmetry to an odd function. When this ratio is not an integer, the system is essentially unstable at high energies. Finally, we consider the case where f has different polynomial growth orders to $+\infty$ and to $-\infty$, and generalize previous results of Cazenave and Weissler concerning the stability of a nonlinear mode of the Kirchhoff string equation.

The problem and the choice of the assumptions on the function f are motivated by the (linear) stability analysis of a coupled nonlinear system of ODEs which is a simplified model for the interaction of flexural and torsional modes of vibration along the deck of a suspended bridge.

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1. Introduction

Let us consider the Hill equation ($\dot v = dv/dt$),

$$\ddot v(t) + Q(t; q)v(t) = 0, \tag{1.1}$$

whose periodic potential ($f'(x) = df/dx$),

$$Q(t, q) = \beta + \gamma f'(u(t; q)) \tag{1.2}$$

is given as a function of $u = u(t; q)$ which is the periodic solution of a nonlinear oscillator equation such as

$$\ddot u + \alpha u + f(u) = 0, \tag{1.3}$$

with initial data parameterized by the positive number q:

$$u(0) = q, \quad \dot u(0) = 0. \tag{1.4}$$

The numbers α, β, γ are real parameters, α, β are assumed to be positive. We assume the following conditions on the function $f : \mathbb{R} \to \mathbb{R}$:

(H1) $f \in C^1(\mathbb{R})$, f increasing, and $f(0) = 0$;

(H2) there exist $A > 0$, and $\nu > 1$, such that $\displaystyle\lim_{x\to+\infty} \frac{f'(x)}{\nu\, x^{\nu-1}} = A$;

(H3) $\lim_{x\to-\infty} f(x) = -h \ (h \geq 0), \quad f'(x) = O(1/x),\ x \to -\infty.$

The problem we want to address is the stability of the null solution of the equation (1.1) when the parameter $q \to +\infty$.

The equation (1.3) has a conserved energy,

$$E(q) = \frac{\dot{u}^2(t)}{2} + J(u(t)) \qquad (t \in \mathbb{R}),$$

where $J(x) = \alpha\frac{x^2}{2} + F(x)$ ($F(x) = \int_0^x f(r)dr$) is the potential energy. The assumption (H1), implies that the potential energy is convex, so that every solution u of (1.3) is periodic.

Moreover, from the initial conditions (1.4), we get $E(q) = J(q)$. Thanks to assumption (H2), we have $E(q) \to +\infty$ as $q \to +\infty$, therefore the problem of the present work can be seen as the study of the stability of the null solution of (1.1) at high energies of the solution u of the equation (1.3).

We refer to the end of this section for the applicative motivations of the problem, and for the reasons that led us to the choice of the assumptions on the nonlinearity f, in particular for the hypothesis (H3). Here we observe that very similar problems have been studied by Dickey [8], Cazenave-Weissler [7], Ghisi-Gobbino [13, 14], in relation to the stability of a nonlinear single mode of the Kirchhoff string equation at high energies; see also [4] for the stability of nonlinear modes for the Woinowsky–Krieger beam equation. The main difference with respect to the problem we face in the present work is that in all those papers the nonlinearity f was essentially symmetric (odd), having the same behavior as $x \to \pm\infty$. To better clarify this point, the typical nonlinearity in the stability theory of the Kirchhoff string is $f(x) = x^3$, whereas a simple example of nonlinearity falling within the assumptions (H1)-(H3) is $f(x) = (x^+)^3$ ($(\cdot)^+$ = positive part).

As we shall see, the results show analogies but highlight the decisive role of the ratio β/α in the asymmetric case, while it was irrelevant in the symmetric case.

Our main result is the following theorem.

Theorem 1. *Let us assume that f satisfies hypotheses (H1)-(H3), and define the following sequences,*

$$\gamma_n^- = \frac{1}{2\nu}\left[(\nu+1)n^2 - (\nu-1)n\right] \quad (n \geq 1), \qquad \gamma_n^+ = \frac{1}{2\nu}\left[(\nu+1)n^2 + (\nu-1)n\right] \quad (n \geq 0). \tag{1.5}$$

In the case when $\sqrt{\beta}$ is an integer multiple of $\sqrt{\alpha}$, then there exists $q_0 > 0$ such that, for every $q > q_0$, the solutions of the Hill equation (1.1) are unstable if

$$\gamma \in \cup_{n=1}^{\infty}(\gamma_n^-, \gamma_n^+) \cup (-\infty, 0),$$

and are stable if

$$\gamma \in \cup_{n=1}^{\infty}(\gamma_{n-1}^+, \gamma_n^-).$$

If $\sqrt{\beta}$ is not an integer multiple of $\sqrt{\alpha}$, then there exists $q_0 > 0$ such that, for every $q > q_0$, the solutions of the Hill equation (1.1) are unstable with at most the exception of a discrete set of values of the parameter γ.

A few remarks are in order.

In the case when $\sqrt{\beta}/\sqrt{\alpha}$ is not an integer, the exceptional values of γ for which there is stability, under a further condition on the nonlinearity f, are reported in Theorem 2 below.

The most relevant fact to observe is how much the ratio between the two angular frequencies $\sqrt{\beta}/\sqrt{\alpha}$ affects the stability of the system, and our interpretation is the following. When this ratio is an integer, the coupling of the two equations asymptotically behaves as if the nonlinearity was extended by symmetry to negative values of u. In other words, the transition in the region $u < 0$ is synchronous with the solution of the symmetric system, as shown by comparison with the results of [7]: assuming that f is an increasing odd function satisfying a growth condition such as (H2) (obviously, for both $x \to \pm\infty$), Theorem 4.3 of [7] introduces the same sequence of intervals (γ_n^-, γ_n^+), and shows the instability of the Hill equation 1.1, as $q \to \infty$, if $\gamma \in \cup_{n=1}^{\infty}(\gamma_n^-, \gamma_n^+)$[1]. On the other hand, when the ratio $\sqrt{\beta}/\sqrt{\alpha}$ is not an integer, the transition in the region $u < 0$ is asynchronous, so that an increase of the energy for (1.3) eventually yields the instability of the Hill equation.

Another comparison can be done with our previous result in [18], where the same system is studied under the assumptions (H1)-(H3) with $\nu = 1$. We remark that in general the instability results in [18] cannot be obtained as limit, as $\nu \to 1$, of the instability intervals of Theorem 1. More precisely, if the ratio $\sqrt{\beta}/\sqrt{\alpha}$ is an integer and we set $\nu = 1$ in Theorem 1, we obtain the same result of [18], in the sense that the instability intervals vanish in both cases. Again, the comparison fails in the case when $\sqrt{\beta}/\sqrt{\alpha}$ is not an integer.

The stability analysis of (1.1) is carried out by means of Floquet's theorem, see [6, 17]. We recall the definition of the stability discriminant $\Delta = \Delta(q)$ of the Hill equation: let $\mathcal{V}(t,s)$ be the transition matrix of the equation (1.1), that is

$$\mathcal{V}(t,s) = \begin{pmatrix} v_1(t,s) & v_2(t,s) \\ \dot{v}_1(t,s) & \dot{v}_2(t,s) \end{pmatrix}$$

in which $v_1(t,s)$, $v_2(t,s)$ are the solutions of (1.1) such that

$$\mathcal{V}(s,s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

the discriminant is defined as

$$\Delta(q) = v_1(2T_q, 0) + \dot{v}_2(2T_q, 0),$$

where $2T_q$ the period of u.

If $|\Delta| > 2$ the non-trivial solutions of the Hill equation are unbounded, if $|\Delta| < 2$ are all bounded. In the case when $\Delta = 2$ there exists at least a non trivial $2T_q$-periodic solution, and when $\Delta = -2$ there exists at least a non-trivial $4T_q$-periodic solution.

The work required to obtain the main theorem consists in calculating the limit of $\Delta(q)$ as $q \to +\infty$. From now on, we set

$$\Delta(\infty) = \lim_{q\to\infty} \Delta(q).$$

Theorem 1 is simply a consequence of this, more detailed

Theorem 2. *Assume that f satisfies hypotheses (H1)-(H3). Let the sequences $(\gamma_n^+)_{n\geq 0}$, and $(\gamma_n^-)_{n\geq 1}$ be defined as in (1.5).*

If $\sqrt{\beta}$ is an integer multiple of $\sqrt{\alpha}$, i.e $\sqrt{\beta} = m\sqrt{\alpha}$, for some $m \in \mathbb{N}$, then $\Delta(\infty)$ is always finite; more precisely

$$|\Delta(\infty)| > 2 \quad \text{if and only if} \quad \gamma \in \cup_{n=1}^{\infty}(\gamma_n^-, \gamma_n^+) \cup (-\infty, 0),$$
$$|\Delta(\infty)| < 2 \quad \text{if and only if} \quad \gamma \in \cup_{n=1}^{\infty}(\gamma_{n-1}^+, \gamma_n^-).$$

[1]The system is similar but not exactly the same, as regards the parameters. To obtain exactly the same intervals the parameter γ of the present work must be multiplied by ν.

If $\sqrt{\beta}$ is not an integer multiple of $\sqrt{\alpha}$, and $\gamma \notin \{\gamma_n^+ : n \geq 0\}$, then we have

$$|\Delta(\infty)| = \infty, . \tag{1.6}$$

If $\gamma = \gamma_n^+$, and the following condition is satisfied,

$$f'(x) = \nu A x^{\nu-1} + o(x^{\frac{\nu-1}{2}}), \quad \textit{for} \quad x \to \infty. \tag{1.7}$$

then

$$\Delta(\infty) = 2(-1)^n \cos(\sqrt{\beta/\alpha}\,\pi). \tag{1.8}$$

The proof of Theorem 2 is very technical and requires several steps. The main difficulties arise from the different time scales of the equation (1.3), depending on whether u is greater or less than zero. In the former case we have a hardening spring, so that we have a fast time scale; in the latter, the equation is asymptotically linear. In order to overcome the problem that arises at the transition between the two different behaviors, we essentially use a trick introduced by Cazenave-Weissler in [7] (see Section 4) which allows to reduce the limit equation in the hardening spring regime as $q \to +\infty$ to a hypergeometric equation. Actually, unlike [7], we need the exact values of the limit solutions at the transition time (Lemma 2.1), therefore we are forced to introduce, and use in a tricky way, a part of the machinery related to the hypergeometric functions. This is recalled in Appendix A.

Once the results in Theorem 1 are obtained, a natural question arises: what happens to the instability intervals, if the growth of $f(x)$ is super-linear both for $x \to +\infty$ and for $x \to -\infty$? In other words, we want to address the question of a hardening spring for both $u > 0$ and $u < 0$. In order to guarantee periodic solutions for the equation (1.3), we leave unchanged the Hypotheses (H1) and (H2) on $f(x)$, and substitute (H3) with the following:

(H3*) There exist $\bar{A} > 0$, and $\bar{\nu}$, with $1 < \bar{\nu} \leq \nu$, such that $\lim_{x \to -\infty} \frac{f'(x)}{\bar{\nu}\,|x|^{\bar{\nu}-1}} = \bar{A}$.

The following theorem highlights two very different behaviors of the discriminant $\Delta(q)$, depending on whether we have an *almost symmetric* case, in which the growth condition on the function $f(x)$ is the same at $\pm\infty$, and a *truly asymmetric* one, in which the order at infinity is different. In the first case, $\Delta(\infty)$ is always finite, independently of the values of α, β, and γ. Furthermore, if the two constants A and $\bar{A}$ are different, we observe the emergence of new instability intervals for the parameter γ as $q \to \infty$, which are not present when f has strictly symmetric asymptotic growth. In the second case, *i.e.* when $\nu \neq \bar{\nu}$, $\Delta(\infty)$ is always infinite, unless γ takes the values of a particular sequence.

We note that the case addressed in [18] can be viewed, taking into account the linear terms αu, and βv in (1.3)-(1.1), as an almost symmetric system with $\nu = 1$. Even in that paper, $\Delta(\infty)$ is always finite.

Finally we observe that the case previously analyzed in Theorem 1 is more complex, because both the additive parameter β and the multiplicative one γ in the periodic potential of the Hill equation (1.2) are taken in account; in Theorem 3 only γ is relevant.

Theorem 3. *Let us assume that f satisfies hypotheses (H1),(H2),(H3*).*

If $\bar{\nu} = \nu$, then $|\Delta(\infty)|$ is always finite, and is greater than 2, if and only if

$$\gamma \in \cup_{n=1}^{\infty}(\gamma_n^-, \gamma_n^+) \cup_{n=1}^{\infty} (\mu_n^-, \mu_n^+) \cup (-\infty, 0),$$

where $\gamma_{n-1}^+ < \mu_n^- < \mu_n^+ < \gamma_n^-$. The values of γ_n^-, γ_n^+ are defined in (1.5), while the values of μ_n^-, μ_n^+ depend on the ratio $A/\bar{A}$; in particular, $\mu_n^- = \mu_n^+$, if $\bar{A} = A$.

If $\bar{\nu} \neq \nu$, and if $\gamma \notin \{\gamma_n^+(\nu), \gamma_n^-(\bar{\nu})\}$, then $|\Delta(\infty)|$ is infinite.

Several remarks are in order.

The end-points of the new instability intervals (μ_n^-, μ_n^+) which emerge if $A \neq \bar{A}$ are implicitly defined by the formula (3.11) in Section 3.

We point out that γ_n^-, γ_n^+ are actually dependent not only on n but also on the exponents ν, $\bar{\nu}$. Accordingly, in the second part of the Theorem 3, we made evident such dependence, when it is needed, writing $\gamma_n^+(\nu)\ \gamma_n^-(\bar{\nu})$.

Theorem 3 can be seen as a generalization of Theorem 4.3 in [7] whose assumptions correspond, in our formulation, to the case $\nu = \bar{\nu}$, $A = \bar{A}$; actually, no symmetry (odd f) is needed.

A last simple remark is that Theorem 3 can be restated with $\bar{\nu} > \nu$ exchanging the roles of the two exponents.

1.1. **Motivations.** The problem and the choice of the assumptions on the function f are motivated by the stability analysis of the coupled nonlinear system,

$$\ddot{y} + \alpha y + \frac{1}{2}\left[f(y+z) + f(y-z)\right] = 0, \tag{1.9}$$

$$\ddot{z} + \beta z + \frac{\gamma}{2}\left[f(y+z) - f(y-z)\right] = 0, \tag{1.10}$$

which is a simplified model for the interaction of flexural and torsional modes of vibration along the deck of a suspended bridge. In this model, y is the amplitude of a single-mode flexural deflection, while z is the amplitude of a single-mode torsional deflection and the nonlinearity f represents the restoring action exerted by the hangers in addition to gravity, and is applied to both extremities of the deck. Several choices for f are possible, but typically it is assumed that the suspension cables do not resist compression, and behave as linear or nonlinear springs if stretched. As a consequence, any realistic assumption for the the action of the hangers is asymmetrical, as regards the growth of $f(x)$, as $x \to \pm\infty$.

The study of the dynamics of suspension bridges, with this kind of hypotheses on the nonlinearity, goes back to the most cited paper of Lazer, McKenna and related works [16, 22, 23] and to the so-called *fish-bone* model, proposed by K.S. Moore [25], where the nonlinearity is of the type u^+. This in not the only possible choice, see [24], where is proposed an exponential-type nonlinearity. These models are generalized and extensively studied by F. Gazzola and coworkers in a series of papers [2, 3, 5, 9, 10, 11] and in the book [12]; see also our previous works [18, 19, 20, 21] for more details on the derivation of the system, and for related results. The emerging problem in this context is the following: what are the conditions on the parameters α, β, and γ so that a transfer of energy from the flexural to the torsional mode may occur? The question can be addressed by linearization of the system (1.9)-(1.10) around a purely flexural periodic solution, i.e. a solution such that z is identically zero. In this way we arrive at the Hill equation (1.1) (linearized torsional mode), coupled with the equation (1.3) (flexural mode). The instability of the Hill equation can be interpreted as the instability of the flexural mode and, consequently, as a transfer of flexural energy into torsional energy.

The case when f is asymptotically linear for $x \to +\infty$, has been addressed in [18, 21]. The assumptions of the present paper can be summarized by saying that it has polynomial growth as $x \to +\infty$, and is asymptotically constant as $x \to -\infty$.

The plan of the paper is the following: in Section 2 we outline the proof of Theorem 2, in Section 3 the proof of Theorem 3, referring to Section 4 for the computation of limit solutions at the transition time, and to Section 5 for most of the required technicalities. Appendix A collects some well known useful formulas about hypergeometric functions.

2. The hardening-linear spring case: proof of Theorem 2

First, we show how it is possible to compute the discriminant $\Delta(q)$ from the values of v_1, v_2, and of their derivatives, at half the period T_q. Indeed, we can refer to the formulas in [17] p. 8 for Hill equations in the case when the potential $Q(t;q)$ is even in the first variable. This is actually the case, since by the initial condition (1.4), $u(-t) = u(t)$. Precisely, we have

$$v_1(2T_q, 0) = \dot{v}_2(2T_q, 0),$$

$$v_1(2T_q, 0) = 2v_1(T_q, 0)\dot{v}_2(T_q, 0) - 1 = v_1(T_q, 0)\dot{v}_2(T_q, 0) + \dot{v}_1(T_q, 0)v_2(T_q, 0),$$

so that

$$\frac{\Delta(q)}{2} = v_1(T_q, 0)\dot{v}_2(T_q, 0) + \dot{v}_1(T_q, 0)v_2(T_q, 0). \tag{2.1}$$

We denote by τ_q the first zero of u. Clearly, we have

$$u(t) > 0 \quad \text{for } 0 < t < \tau_q, \qquad u(t) < 0 \quad \text{for } \tau_q < t < 2T_q - \tau_q.$$

Then, we separate the analysis of the hardening spring regime, corresponding to the time interval $[0, \tau_q]$, from that of the linear spring regime corresponding to $[\tau_q, T_q]$, as the time scale is different in such intervals. In fact, we will show in the following section, that $\tau_q \to 0$, while $T_q - \tau_q \to \frac{\pi}{2\sqrt{\alpha}}$ as $q \to \infty$. As a consequence, the approximate computation of the transition matrix

$$\mathcal{V}(T_q, 0) = \mathcal{V}(T_q, \tau_q)\mathcal{V}(\tau_q, 0) \tag{2.2}$$

will be performed separately in the two intervals.

2.1. Asymptotic analysis of τ_q and T_q.

A crucial role in our argument is played by the asymptotic expansion of the two different time scales. The basic result is provided by the following Proposition. For the moment is all we need to prove the first part of Theorem 2. A more refined expansion is contained in the proof of Lemma 5.2, but it will be useful only to prove the last part of the Theorem, the one concerning the exceptional values of γ, in the case when the ratio $\sqrt{\beta}/\sqrt{\alpha}$ is not an integer.

Proposition 2.1. *Under the assumptions (H1)-(H3), we have*

$$\lim_{q\to\infty} q^{(\nu-1)/2}\, \tau_q = \sqrt{(\nu+1)/2A}\, B_\nu\,, \quad \text{where } B_\nu = \int_0^1 \frac{dx}{\sqrt{1 - x^{\nu+1}}}. \tag{2.3}$$

About the next interval $[\tau_q, T_q]$, there exists a constant $C = C(\alpha, h) > 0$, such that

$$\frac{\pi}{2\sqrt{\alpha}} - \frac{C}{|u(T_q)|} \le T_q - \tau_q \le \frac{\pi}{2\sqrt{\alpha}}. \tag{2.4}$$

Proof. From the energy identity, we obtain

$$\tau_q = \int_0^q \frac{du}{\sqrt{2(J(q) - J(u))}} = q\int_0^1 \frac{dx}{\{2(J(q) - J(xq))\}^{1/2}},$$

so that

$$q^{-1}\sqrt{2J(q)}\, \tau_q = \int_0^1 \left(1 - \frac{J(xq)}{J(q)}\right)^{-1/2} dx. \tag{2.5}$$

By Cauchy's mean value theorem, we have

$$\frac{J(xq)}{J(q)} = \frac{J(xq) - J(0)}{J(q) - J(0)} = \frac{xJ'(x\bar{q})}{J'(\bar{q})},$$

for some $\bar{q} \in (0, q)$. Being $J'(x)$ an increasing function due to assumption (H1), the last term is less than x, so that the integrand in (2.5) is bounded above by $(1-x)^{-1/2}$.

The asymptotic equivalence in (2.3) follows by observing that assumption (H2) yields

$$\lim_{x\to+\infty} \frac{F(x)}{x^{\nu+1}} = \lim_{q\to+\infty} \frac{J(q)}{q^{(\nu+1)}} = \frac{A}{\nu+1},$$

thus

$$q^{-1}\sqrt{2J(q)} \sim \sqrt{2A/(\nu+1)}q^{(\nu-1)/2} \quad q \to +\infty;$$

finally, for $x \geq 0$, by De'Hôpital rule we have

$$\lim_{q\to+\infty} \frac{J(xq)}{J(q)} = x^{\nu+1}$$

so that the last integral in (2.5) converges to B_ν.

To prove (2.4) we simplify the notation by putting $s = -u(T_q) > 0$. Recalling that $u(\tau_q) = 0$, again by the energy equation, we get

$$T_q - \tau_q = -\int_0^{-s} \frac{du}{\sqrt{2(J(-s) - J(u))}} = \frac{1}{\sqrt{\alpha}} \int_0^1 \frac{dx}{\sqrt{1 - x^2 + G(x,s)}},$$

in which we have set

$$G(x,s) = \frac{2(F(-s) - F(-sx))}{\alpha s^2}.$$

The function $G(x,s)$ is nonnegative, thus

$$T_q - \tau_q \leq \frac{1}{\sqrt{\alpha}} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2\sqrt{\alpha}}.$$

On the other hand, we have

$$\begin{aligned} T_q - \tau_q &= \frac{\pi}{2\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha}} \left(\int_0^1 \frac{dx}{\sqrt{1-x^2}} - \int_0^1 \frac{dx}{\sqrt{1-x^2+G(x,s)}} \right) \\ &\geq \frac{\pi}{2\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha}} \int_0^1 \frac{G(x,s)}{2(1-x^2)\sqrt{1-x^2}}\, dx, \end{aligned}$$

in which we have used the inequality $1/\sqrt{a} - 1/\sqrt{a+b} \leq b/2a\sqrt{a}$.

Since $-h \leq f(x)) \leq 0$, for $x < 0$, by the mean value theorem, it follows that

$$G(x,s) = \frac{-2f(\xi)s(1-x)}{\alpha s^2} \leq \frac{2h(1-x)}{\alpha s}. \tag{2.6}$$

In conclusion, we obtain the inequality (2.4) with

$$C = \frac{h}{\alpha^{3/2}} \int_0^1 \frac{dx}{(1+x)\sqrt{1-x^2}}\, dx.$$

□

Since $\dot{u}(T_q) = 0$, and $J(u(T_q)) = J(q)$, it follows that $\lim_{q\to\infty} u(T_q) = -\infty$. Thus, (2.4) yields

$$\lim_{q\to\infty} T_q - \tau_q = \frac{\pi}{2\sqrt{\alpha}}.$$

2.2. **Asymptotic analysis in the case $u > 0$.** In this case we need to stretch the time interval $[0, \tau_q]$ to $[0, 1]$ by using the rescaling,

$$w_q(t') = q^{-1}u(\tau_q t'), \qquad z_q(t') = v(\tau_q t'), \tag{2.7}$$

so that w_q solves the problem

$$\ddot{w}_q + \tau_q^2\,\alpha w_q + q^{-1}\tau_q^2 f(q w_q) = 0 \quad (0 < t' < 1), \quad w_q(0) = 1, \quad \dot{w}_q(0) = 0, \tag{2.8}$$

and z_q solves the Hill equation

$$\ddot{z}_q + \left(\tau_q^2\beta + \gamma\tau_q^2 f'(q w_q)\right) z_q = 0 \quad (0 < t' < 1). \tag{2.9}$$

Then we take the formal limits of (2.8), (2.9) as $q \to \infty$. From assumption (H2), and from formula (2.3), for every $x > 0$, we obtain

$$\lim_{q\to\infty} q^{-1}\tau_q^2 f(xq) = K_\nu x^\nu, \qquad \lim_{q\to\infty} \tau_q^2 f'(xq) = \nu K_\nu x^{\nu-1}, \qquad K_\nu = \frac{\nu+1}{2} B_\nu^2. \tag{2.10}$$

Recalling that $w_q(t') > 0$ for $0 \le t' < 1$, we are led to the limit problem,

$$\ddot{w} + K_\nu w^\nu = 0, \quad w(0) = 1, \quad \dot{w}(0) = 0, \qquad 0 < t' < 1, \tag{2.11}$$

and the limit Hill equation,

$$\ddot{z} + \gamma\,\nu\,K_\nu\,w^{\nu-1} z = 0, \qquad 0 < t' < 1. \tag{2.12}$$

The limits are not just formal, as shown by the following

Proposition 2.2. *Let w_q, w be the solution of the problems (2.8), (2.11), z_q, z be the solutions of (2.9), (2.12) respectively, corresponding to the same initial data. Then*

$$\lim_{q\to\infty} \left(\|w_q - w\|_{C^1([0,1])} + \|z_q - z\|_{C^1([0,1])}\right) = 0. \tag{2.13}$$

For ease of reading, we have postponed all the technical lemmas to the Section 5. In particular, the proof of Proposition 2.2 follows from a classic continuous dependence theorem, and from Lemma 5.1 which shows that both limits in (2.10) hold uniformly on $[0, 1]$.

2.3. **Asymptotic analysis in the case $u < 0$.** The main result in this case is given by the following Proposition that is crucial to prove the first part of Theorem 2. Roughly speaking, it says that, as $q \to \infty$, the equations (1.3) and (1.1), behave in this interval as two decoupled oscillators

$$\ddot{u} + \alpha u = 0, \quad \ddot{v} + \beta v = 0;$$

in addiction, it provides a suitable estimate of the rate of convergence.

Proposition 2.3. *Let us set*

$$\phi = \frac{\pi}{2}\sqrt{\frac{\beta}{\alpha}}.$$

Under assumptions (H1)-(H3), we have that

$$\mathcal{V}(T_q, \tau_q) = \begin{pmatrix} \cos\phi & \dfrac{\sin\phi}{\sqrt{\beta}} \\ -\sqrt{\beta}\sin\phi & \cos\phi \end{pmatrix} + o(\tau_q). \tag{2.14}$$

Proof. Let us set $T = \tau_q + \pi/2\sqrt{\alpha}$, and let $Q(t,q)$ be the potential as defined in (1.2). Since $f'(x) > 0$, Lemma 5.4 can be restated by saying that

$$\|Q(\cdot,q) - \beta\|_{L^1([\tau_q,T])} \le K\gamma\,\nu\,\tau_q/q.$$

Therefore, owing to a general lemma (see Lemma 5.1 in [18], or [26]), we obtain that

$$\mathcal{V}(t,\tau_q) = \begin{pmatrix} \cos\sqrt{\beta}(t-\tau_q) & \dfrac{\sin\sqrt{\beta}(t-\tau_q)}{\sqrt{\beta}} \\ -\sqrt{\beta}\sin\sqrt{\beta}(t-\tau_q) & \cos\sqrt{\beta}(t-\tau_q) \end{pmatrix} + O(\tau_q/q),$$

uniformly w.r.t. $t \in [\tau_q, T]$.

Noting that

$$\mathcal{V}(T,\tau_q) = \begin{pmatrix} \cos\phi & \dfrac{\sin\phi}{\sqrt{\beta}} \\ -\sqrt{\beta}\sin\phi & \cos\phi, \end{pmatrix} + O(\tau_q/q),$$

the assertion is proved once we know that $\mathcal{V}(T_q,\tau_q) = \mathcal{V}(T,\tau_q) + o(\tau_q)$. But this easily follows from the estimate (2.4) in Proposition 2.1,

$$0 \le \frac{\pi}{2\sqrt{\alpha}} - (T_q - \tau_q) \le \frac{C}{|u(T_q)|},$$

and Lemma 5.3, since

$$|u(T_q)| > \frac{\dot{w}(1)}{2\sqrt{\alpha}}\frac{q}{\tau_q},$$

for sufficiently large q. □

2.4. **Conclusion of the proof of Theorem 2.** Recalling the rescaling (2.7) in the interval $[0,\tau_q]$, we get

$$\mathcal{V}(\tau_q,0) = \begin{pmatrix} z_{1,q}(1) & \tau_q z_{2,q}(1) \\ \dot{z}_{1,q}(1)/\tau_q & \dot{z}_{2,q}(1), \end{pmatrix} \tag{2.15}$$

where $z_{i,q}(t)$, $\dot{z}_{i,q}(t)$, $i = 1,2$ are the entries of the transition matrix $\mathcal{Z}_q(t,0)$ of the equation (2.9). After some simple calculations, from (2.1), (2.2), (2.15) we get the following decomposition of the discriminant $\Delta(q)$:

$$\frac{\Delta(q)}{2} = A_1(q) + A_2(q) + A_3(q),$$

where

$$A_1(q) = \frac{2}{\tau_q}\,\dot{v}_2(T_q,\tau_q)v_2(T_q,\tau_q)\dot{z}_{1,q}(1)\dot{z}_{2,q}(1) \tag{2.16}$$

$$A_2(q) = \left(v_1(T_q,\tau_q)\dot{v}_2(T_q,\tau_q) + \dot{v}_1(T_q,\tau_q)v_2(T_q,\tau_q)\right)\left(z_{1,q}(1)\dot{z}_{2,q}(1) + \dot{z}_{1,q}(1)z_{2,q}(1)\right) \tag{2.17}$$

$$A_3(q) = 2\tau_q\,v_1(T_q,\tau_q)\dot{v}_1(T_q,\tau_q)z_{1,q}(1)z_{2,q}(1). \tag{2.18}$$

Let $z_i(t)$, $\dot{z}_i(t)$ be the entries of the transition matrix $\mathcal{Z}(t,0)$ of the limit Hill equation (2.12). Owing to Proposition 2.2, and Proposition 2.3, we conclude that

$$A_1(q) = \frac{\tau_q^{-1}}{\sqrt{\beta}}\left(\dot{z}_1(1)\dot{z}_2(1) + R_q\right)\left(\sin(2\phi) \,+\, o(\tau_q)\right) \tag{2.19}$$

$$A_2(q) = \left(z_1(1)\dot{z}_2(1) + \dot{z}_1(1)z_2(1)\right)\cos(2\phi) \,+\, o(1) \tag{2.20}$$

$$A_3(q) = o(1), \tag{2.21}$$

where in (2.19) we have set

$$R_q = (\dot z_{1,q}(1) - \dot z_1(1))\dot z_2(1) + (\dot z_{2,q}(1) - \dot z_2(1))\dot z_1(1) + (\dot z_{1,q}(1) - \dot z_1(1))(\dot z_{2,q}(1) - \dot z_2(1)). \tag{2.22}$$

Since $R_q = o(1)$, from the asymptotic formulas (2.19)-(2.21), it is clear that $|\Delta(\infty)| = \infty$, unless one of the following cases is satisfied:

$$\text{I)}\ \sqrt{\beta} = m\sqrt{\alpha}; \qquad \text{II)}\ \dot z_1(1)\dot z_2(1) = 0 \ \text{ and } \ R_q = O(\tau_q).$$

In case I), we obtain $\lim_{q\to\infty} A_1(q) = 0$, due to Proposition 2.3 and

$$\frac{\Delta(\infty)}{2} = \lim_{q\to\infty} A_2(q) = (-1)^m (z_1(1)\dot z_2(1) + z_2(1)\dot z_1(1)).$$

Thanks to the Liouville theorem, the determinant of the transition matrix $\mathcal{Z}(t, 0)$ is identically equal to 1, for $0 \le t \le 1$, so that we can rewrite,

$$\frac{\Delta(\infty)}{2} = (-1)^m (2z_1(1)\dot z_2(1) - 1) = (-1)^m (1 - 2z_2(1)\dot z_1(1)). \tag{2.23}$$

To complete the calculation of $\Delta(\infty)$, we need the explicit values of $z_i(1)$, $\dot z_i(1)$, $i = 1, 2$, which are provided by the following crucial Lemma whose proof is postponed to Section 4.

Lemma 2.1. *Let us define the following numbers,*

$$c_1 = \frac{1}{2}, \quad a_1, b_1 = \frac{1}{4(\nu+1)}\left[\nu - 1 \pm \sqrt{(\nu-1)^2 + 8\gamma\nu(\nu+1)}\right] \tag{2.24}$$

$$c_2 = \frac{3}{2}, \quad a_2, b_2 = \frac{1}{4(\nu+1)}\left[3\nu + 1 \pm \sqrt{(\nu-1)^2 + 8\gamma\nu(\nu+1)}\right] \tag{2.25}$$

Under assumptions (H1), (H2), we have that (B_ν is defined in (2.3))

$$z_1(1) = \frac{\Gamma(1/2)\Gamma(1/(\nu+1))}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)}, \qquad z_2(1) = \frac{2}{(\nu+1)B_\nu}\,\frac{\Gamma(3/2)\Gamma(1/(\nu+1))}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)}, \tag{2.26}$$

$$\dot z_1(1) = (\nu+1)B_\nu\,\frac{\Gamma(1/2)\Gamma(\nu/(\nu+1))}{\Gamma(a_1)\Gamma(b_1)}, \qquad \dot z_2(1) = 2\,\frac{\Gamma(3/2)\Gamma(\nu/(\nu+1))}{\Gamma(a_2)\Gamma(b_2)}. \tag{2.27}$$

Let us come back to the proof of Theorem 2.

We recall some well known properties of the Γ function:

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}; \tag{2.28}$$

from which we get

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \qquad \Gamma(1/(\nu+1))\Gamma(\nu/(\nu+1)) = \frac{\pi}{\sin(\pi/(\nu+1))}.$$

To simplify notations, let us momentarily set

$$\mathcal{A} = \frac{\sqrt{(\nu-1)^2 + 8\gamma\nu(\nu+1)}}{4(\nu+1)}, \qquad \mathcal{B} = \frac{\nu+3}{4(\nu+1)} \tag{2.29}$$

so that $\quad c_1 - a_1 = \mathcal{B} - \mathcal{A}, \quad c_1 - b_1 = \mathcal{B} + \mathcal{A}$

We observe that, from (2.24), (2.25), $(c_1 - a_1) + a_2 = 1$, $(c_1 - b_1) + b_2 = 1$, therefore by using the third property in (2.28),

$$\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)\Gamma(a_2)\Gamma(b_2) = \frac{\pi^2}{\sin(\pi(\mathcal{B} - \mathcal{A}))\sin(\pi(\mathcal{B} + \mathcal{A}))}.$$

After the obvious simplifications in the expression of $z_1(1)\dot{z}_2(1)$ given by the formulas (4.2)-(2.27), we obtain

$$z_1(1)\dot{z}_2(1) = \frac{\sin(\pi(\mathcal{B} - \mathcal{A}))\sin(\pi(\mathcal{B} + \mathcal{A}))}{\sin(\pi/(\nu+1))}.$$

Since,

$$\sin(\pi(\mathcal{B} - \mathcal{A}))\sin(\pi(\mathcal{B} + \mathcal{A})) = \frac{1}{2}\cos(2\pi\mathcal{A}) - \frac{1}{2}\cos(2\pi\mathcal{B}) = \frac{1}{2}\cos(2\pi\mathcal{A}) + \frac{1}{2}\sin(\pi/(\nu+1)),$$

we conclude that, in case I),

$$\frac{\Delta(\infty)}{2} = (-1)^m(2z_1(1)\dot{z}_2(1) - 1) = (-1)^m \frac{\cos(2\pi\mathcal{A})}{\sin(\pi/(\nu+1))}. \tag{2.30}$$

Observing that,

$$\sin\left(\frac{\pi}{\nu+1}\right) = \cos\left(2\pi\frac{\nu-1}{4(\nu+1))}\right),$$

in the case when $\gamma \geq -\frac{(\nu-1)^2}{8\nu(\nu+1)}$, that is when $\mathcal{A}$ is a nonnegative real number, the instability condition $|\Delta(\infty)| > 2$ is satisfied if and only if

$$-\frac{\nu-1}{4(\nu+1)} + n < \mathcal{A} < \frac{\nu-1}{4(\nu+1)} + n, \qquad -\frac{\nu-1}{4(\nu+1)} + \frac{1}{2} + n < \mathcal{A} < \frac{\nu-1}{4(\nu+1)} + \frac{1}{2} + n \tag{2.31}$$

for some $n = 0, 1, 2, ...$ The first condition in (2.31) for $n = 0$ is satisfied if and only if $-\frac{(\nu-1)^2}{8\nu(\nu+1)} \leq \gamma < 0$. When $\gamma \geq 0$, solving with respect to γ the set of inequalities in (2.31), we obtain

$$2\,\frac{\nu+1}{\nu}\,n^2 - \frac{\nu-1}{\nu}\,n < \gamma < 2\,\frac{\nu+1}{\nu}\,n^2 + \frac{\nu-1}{\nu}\,n, \quad n \geq 1, \tag{2.32}$$

and

$$2\,\frac{\nu+1}{\nu}\,n^2 + \frac{\nu+3}{\nu}\,n + \frac{1}{\nu} < \gamma < 2\,\frac{\nu+1}{\nu}\,n^2 + \frac{3\nu+1}{\nu}\,n + 1, \quad n \geq 0. \tag{2.33}$$

Let us also consider the case $\gamma < -\frac{(\nu-1)^2}{8\nu(\nu+1)}$, *i.e.* the case when $\mathcal{A}$ is purely imaginary. In this case we have $\cos(2\pi\mathcal{A}) = \cosh(2\pi\mathcal{A}i) > 1$, thus, directly from (2.30), we get $|\Delta(\infty)| > 2$. Putting all the information together, the first unbounded interval of instability is $(-\infty, 0)$ while the sequence of the other instability intervals starts from (2.33) with $n = 0$, then alternates (2.32) and (2.33).

To conclude the proof of the first part of the theorem, we observe that γ_{2n}^-, γ_{2n}^+ are the endpoints of (2.32); similarly γ_{2n+1}^-, γ_{2n+1}^+ are the endpoints of (2.33). In addition, if $m = \sqrt{\beta/\alpha}$ is even, the intervals (2.32) correspond to $\Delta(\infty) > 2$, whereas the intervals (2.33) correspond to $\Delta(\infty) < -2$. The reverse happens if m is odd.

More information is provided by the study of the endpoints of the intervals (γ_n^-, γ_n^+). Precisely, we have $|\Delta(\infty)| = 2$ if and only if one of the following quantities $z_1(1)$ $\dot{z}_1(1)$, $z_2(1)$, $\dot{z}_2(1)$ vanishes, that is respectively:

$$z_1(1) = 0 \iff (c_1 - a_1) = -n \iff \gamma = \gamma^-_{2n+1} \tag{2.34}$$

$$\dot z_1(1) = 0 \iff b_1 = -n \iff \gamma = \gamma^+_{2n} \tag{2.35}$$

$$z_2(1) = 0 \iff (c_2 - a_2) = -n \iff \gamma = \gamma^-_{2(n+1)} \tag{2.36}$$

$$\dot z_2(1) = 0 \iff b_2 = -n \iff \gamma = \gamma^+_{2n+1} \tag{2.37}$$

We prove the first two equivalences, leaving the other to the reader.

We have $z_1(1) = 0$ if and only if the denominator in formula (4.2) is singular, that is if $(c_1 - a_1)$ takes a integer value $-n \leq 0$. Solving such equation,

$$(c_1 - a_1) = \frac{1}{4(\nu+1)}\left[\nu + 3 - \sqrt{(\nu-1)^2 + 8\gamma\nu(\nu+1)}\right] = -n,$$

we obtain

$$\gamma = 2\,\frac{\nu+1}{\nu}\,n^2 + \frac{\nu+3}{\nu}\,n + \frac{1}{\nu}$$

which is the left endpoint in (2.33), and corresponds to γ^-_{2n+1}.

We have $\dot z_1(1) = 0$ if and only if b_1 takes a integer value $-n \leq 0$. The equation

$$b_1 = \frac{1}{4(\nu+1)}\left[\nu - 1 - \sqrt{(\nu-1)^2 + 8\gamma\nu(\nu+1)}\right] = -n,$$

leads to

$$\gamma = 2\,\frac{\nu+1}{\nu}\,n^2 + \frac{\nu-1}{\nu}\,n,$$

which corresponds to γ^+_{2n} and so on.

Now, let us consider the remaining case II): $\dot z_1(1)\dot z_2(1) = 0$. From (2.35), (2.37), it is clear that $\dot z_1(1)\dot z_2(1) = 0$ if and only if $\gamma = \gamma^+_n$, for some integer $n \geq 1$.

Finally, we conclude the proof of the theorem: under assumption (1.7), Lemma 5.2 yields $R_q = o(\tau_q)$. Therefore, from (2.20) we obtain

$$\Delta(\infty)/2 = (z_1(1)\dot z_2(1) + \dot z_1(1) z_2(1))\cos(2\phi).$$

If $\dot z_1(1) = 0$, by using the Liouville Theorem, we get

$$\Delta(\infty)/2 = z_1(1)\dot z_2(1)\cos 2\phi = \cos 2\phi,$$

while, if $\dot z_2(1) = 0$,

$$\Delta(\infty)/2 = \dot z_1(1) z_2(1)\cos 2\phi = -\cos 2\phi.$$

3. The hardening spring case: proof of Theorem 3

Proof. The argument follows the lines of the proof of Theorem 2, the main difference being that the approximate computation of $\mathcal{V}(T_q, \tau_q)$ provided by Proposition 2.14 for the linear spring regime is no longer valid. Therefore, we need a new expression $\mathcal{V}(T_q, \tau_q)$ which takes into account that the hardening spring regime holds also for $u < 0$. The argument is based on an adaptation of the previous Proposition 2.1.

As in the proof of Proposition 2.1, let us set $u(T_q) = -s$, and consider the transition time for negative values of u in the half cycle $[0, T_q]$:

$$\tau_s = T_q - \tau_q.$$

Since $J(-s) = J(q)$, we get $s \to +\infty$ as $q \to +\infty$. More precisely, since $J(x) \sim \frac{\bar{A}}{\bar{\nu}+1}|x|^{\bar{\nu}+1}$, as $x \to -\infty$, we have

$$\frac{q^{\nu+1}}{s^{\bar{\nu}+1}} = \frac{J(-s)}{s^{\bar{\nu}+1}}\frac{q^{\nu+1}}{J(q)} \sim \frac{\bar{A}(\nu+1)}{A(\bar{\nu}+1)}, \qquad q \to +\infty,$$

which yields

$$s \sim \left(\frac{A(\bar{\nu}+1)}{\bar{A}(\nu+1)}\right)^{1/(\bar{\nu}+1)} q^{(\nu+1)/(\bar{\nu}+1)}, \qquad q \to +\infty.$$

Again, by the same technique used in Proposition 2.1, we obtain

$$\lim_{s\to\infty} s^{(\bar{\nu}-1)/2}\,\tau_s = \sqrt{(\bar{\nu}+1)/2\bar{A}}\,B_{\bar{\nu}}. \tag{3.1}$$

Using again (2.3), and (3.1), we obtain the following asymptotics for the ratio τ_s/τ_q, which will be useful in the next steps of the proof,

$$\frac{\tau_s}{\tau_q} \sim \frac{B_{\bar{\nu}}}{B_\nu}\left(\frac{A(\bar{\nu}+1)}{\bar{A}(\nu+1)}\right)^{\frac{1}{\bar{\nu}+1}} q^{\frac{\nu-\bar{\nu}}{\bar{\nu}+1}} = cq^{\frac{\nu-\bar{\nu}}{\bar{\nu}+1}}. \tag{3.2}$$

In order to compute the new expression for $\mathcal{V}(T_q,\tau_q)$, we reverse the time in the equations (1.3), (1.1) during the interval $[\tau_q, T_q]$, by setting

$$\bar{u}(t') = -u(T_q - t'), \quad \bar{v}(t') = v(T_q - t') \qquad 0 \le t' \le \tau_s. \tag{3.3}$$

We get a new couple of equations on the interval $(0,\tau_s)$,

$$\ddot{\bar{u}} + \alpha\bar{u} + \bar{f}(\bar{u}) = 0, \quad \bar{u}(0) = s, \quad \dot{\bar{u}}(0) = 0. \tag{3.4}$$

$$\ddot{\bar{v}} + (\beta + \gamma\bar{\nu}\bar{f}'(\bar{u}))\bar{v} = 0, \tag{3.5}$$

in which the function $\bar{f}(x)$ is defined by $\bar{f}(x) = -f(-x)$, for $x > 0$.

Owing to assumption (H3*), $\bar{f}(x)$ satisfies the condition (H2) with $\bar{A}$, $\bar{\nu}$ in place of A, ν. With an obvious change in notations, formula (2.15) and Proposition 2.2 apply to the equations (3.4), (3.5). It follows that the transition matrix $\bar{\mathcal{V}}(\tau_s,0)$ for the equation (3.5) is, as in (2.15),

$$\bar{\mathcal{V}}(\tau_s,0) = \begin{pmatrix} \bar{z}_{1,s}(1) & \tau_s\bar{z}_{2,s}(1) \\ \dot{\bar{z}}_{1,s}(1)/\tau_s & \dot{\bar{z}}_{2,s}(1) \end{pmatrix}. \tag{3.6}$$

Now, we can recover the entries of $\mathcal{V}(T_q,\tau_q)$. First of all, from (3.3), we get

$$\mathcal{V}(T_q,\tau_q) = \begin{pmatrix} \bar{v}_1(0,\tau_s) & -\bar{v}_2(0,\tau_s) \\ -\dot{\bar{v}}_1(0,\tau_s) & \dot{\bar{v}}_2(0,\tau_s) \end{pmatrix};$$

then, since $\bar{\mathcal{V}}(0,\tau_s) = \bar{\mathcal{V}}(\tau_s,0)^{-1}$, we obtain (recall that $det(\bar{\mathcal{V}}(\tau_s,0)) = 1$),

$$\bar{\mathcal{V}}(0,\tau_s) = \begin{pmatrix} \dot{\bar{z}}_{2,s}(1) & -\tau_s\bar{z}_{2,s}(1) \\ -\dot{\bar{z}}_{1,s}(1)/\tau_s & \bar{z}_{1,s}(1) \end{pmatrix}.$$

We conclude that

$$\mathcal{V}(T_q,\tau_q) = \begin{pmatrix} \dot{\bar{z}}_{2,s}(1) & \tau_s\bar{z}_{2,s}(1) \\ \dot{\bar{z}}_{1,s}(1)/\tau_s & \bar{z}_{1,s}(1) \end{pmatrix}.$$

The next step consists in computing $\Delta(q)/2 = A_1(q) + A_2(q) + A_3(q)$, with $A_i(q)$ are defined as in (2.16), (2.17), (2.18). Substituting the new expression of the entries of $\mathcal{V}(T_q,\tau_q)$ into

(2.16),(2.17),(2.18), and recalling that from Proposition 2.2, we have $\|\bar{z}_s - \bar{z}\|_{C^1([0,1])} \to 0$ if $q \to +\infty$, we obtain

$$A_1(q) = 2\frac{\tau_s}{\tau_q}\ (\bar{z}_1(1)\bar{z}_2(1) + o(1))\,(\dot{z}_1(1)\dot{z}_2(1) + o(1)) \tag{3.7}$$

$$A_2(q) = (\bar{z}_1(1)\dot{\bar{z}}_2(1) + \dot{\bar{z}}_1(1)\bar{z}_2(1) + o(1))\,(z_1(1)\dot{z}_2(1) + \dot{z}_1(1)z_2(1) + o(1)) \tag{3.8}$$

$$A_3(q) = 2\frac{\tau_q}{\tau_s}\ (\dot{\bar{z}}_1(1)\dot{\bar{z}}_2(1) + o(1))\,(z_1(1)z_2(1) + o(1))\,. \tag{3.9}$$

in which $\bar{z}_1$, $\bar{z}_2$ are the fundamental solutions of the limit equation for (3.5).

Let us consider the first part of the Theorem, whose assumption is $\bar{\nu} = \nu$. From (3.2), we obtain

$$\lim_{q\to\infty}\frac{\tau_s}{\tau_q} = \left(\frac{A}{\bar{A}}\right)^{\frac{1}{\nu+1}} = c > 0.$$

Moreover, since the limit equations do not depend on the values A, $\bar{A}$, we have that $\bar{z}_1(1) = z_1(1)$ and so on. It follows that $\Delta(\infty)$ simplifies to

$$\frac{\Delta(\infty)}{2} = 2\left(c + \frac{1}{c}\right)(z_1(1)z_2(1)\dot{z}_1(1)\dot{z}_2(1)) + (z_1(1)\dot{z}_2(1) + \dot{z}_1(1)z_2(1))^2\,.$$

Keeping in mind that $\dot{z}_1(1)z_2(1) = z_1(1)\dot{z}_2(1) - 1$, and defining $x = z_1(1)\dot{z}_2(1)$, we get the simpler expression

$$\frac{\Delta(\infty)}{2} = \left[1 + \frac{1}{2}\left(c + \frac{1}{c}\right)\right](2x-1)^2 - \frac{1}{2}\left(c + \frac{1}{c}\right).$$

As is easily seen, $\Delta(\infty) > 2$ corresponds to $|2x-1| > 1$, and this leads exactly to the same calculation following (2.30) in Theorem 2, where the instability intervals correspond to the implicit relation

$$|\cos(2\pi\mathcal{A})| > \left|\cos\left(2\pi\frac{\nu-1}{4(\nu+1))}\right)\right|, \tag{3.10}$$

in which $\mathcal{A}$ defined in (2.29). Such inequality is solved explicitly, and leads to the intervals (γ_n^-, γ_n^+).

The case $\Delta(\infty) < -2$ corresponds to $|2x-1| < H$, where $H = H(A/\bar{A})$ is defined by,

$$H = \sqrt{\frac{(c+\frac{1}{c}) - 2}{(c+\frac{1}{c}) + 2}} \geq 0.$$

If $A = \bar{A}$, then $c = 1$ and $H = 0$, it follows that there are no other solutions of $|\Delta(\infty)| > 2$; if $A \neq \bar{A}$, then $0 < H < 1$, and we obtain another sequence of intervals (μ_n^-, μ_n^+) implicitly defined by

$$|\cos(2\pi\mathcal{A})| < H\left|\cos\left(2\pi\frac{\nu-1}{4(\nu+1))}\right)\right|. \tag{3.11}$$

We are not able to explicitly solve (3.11), but comparing (3.10) and (3.11) it is clear that the intervals (μ_n^-, μ_n^+) are intercalated with the intervals (γ_n^-, γ_n^+).

Let us come to the second part of the theorem. Under the hypothesis $\bar{\nu} < \nu$, by (3.2), we have that the ratio τ_s/τ_q goes to infinity. From formula (3.7), it is clear that also $\Delta(q)$ goes to infinity, unless $\bar{z}_1(1)\bar{z}_2(1)\dot{z}_1(1)\dot{z}_2(1) = 0$. We already know (see (2.35) and (2.37)) that

$$\dot{z}_1(1) = 0 \iff \gamma = \gamma_{2n}^+(\nu), \qquad \dot{z}_2(1) = 0 \iff \gamma = \gamma_{2n+1}^+(\nu).$$

Using (2.34), we have that $\bar{z}_1(1) = 0 \iff (c_1 - a_1) = -n$, where a_1, c_1 are defined in (2.24) with $\bar{\nu}$ instead of ν, that leads to $\gamma = \gamma^-_{2n+1}(\bar{\nu})$. Similarly (2.36) shows that $\bar{z}_2(1) = 0$ if and only if $\gamma = \gamma^-_{2n}(\bar{\nu})$. □

From the proof of the theorem, we obtain that if $\gamma = \gamma^+_k(\nu)$, or $\gamma = \gamma^-_k(\bar{\nu})$, for some positive integer k, the limit $\Delta(\infty)$ could assume a finite value. It is possible to get more precise results, in the spirit of Lemma 5.2, by giving sufficient growth conditions on f in order to guarantee that the limit of the discriminant of instability is finite, and get the value of such limit. We omit this part, but observe that is possible to find examples of functions f such that $|\Delta(\infty)|$ can be finite and either greater than or less than 2, for those particular values of γ.

4. Computation of $z_i(1)$, and $\dot{z}_i(1)$, $i = 1, 2$

In this section we prove Lemma 2.1. To proceed with the calculations of $z_i(t)$ for $t \in (0, 1)$, we exploit the idea of [7]: quite surprisingly, the solution of the Hill equation (2.12) can be written as a local function of w, and $\dot{w}$. To do this, we introduce

$$x(t) = 1 - w(t)^{\nu+1} \qquad (0 \le t \le 1), \tag{4.1}$$

then we assume the following *ansatz*:

$$z_1(t) = P(x(t)), \qquad z_2(t) = \dot{w}(t)R(x(t)),$$

where R and P are functions to be determined. Note that, by the initial condition in(2.11), we have $x(0) = 0$, and $x(1) = 1$, therefore the initial conditions for z_1, z_2, are satisfied provided that

$$P(0) = 1, \qquad R(0) = -\frac{1}{K_\nu},$$

where K_ν is defined in (2.10).

It turns out that both $P(x)$, and $R(x)$ satisfy a hypergeometric differential equation, see (A.1):

$$x(1-x)P''(x) + \left[\frac{1}{2} - \left(1 + \frac{\nu - 1}{2(\nu+1)}\right)x\right]P'(x) - \frac{-\gamma\nu}{2(\nu+1)}P(x) = 0,$$
$$x(1-x)R''(x) + \left[\frac{3}{2} - \left(1 + \frac{3\nu + 1}{2(\nu+1)}\right)x\right]R'(x) - \frac{(1-\gamma)\nu}{2(\nu+1)}R(x) = 0.$$

The verification is straightforward but long and tedious, so we have decided to skip it. By comparison with the hypergeometric equation (A.1), we obtain

$$z_1(t) = F\left(a_1, b_1; c_1; x(t)\right), \qquad z_2(t) = -\frac{1}{K_\nu}\,\dot{w}(t)\,F\left(a_2, b_2; c_2; x(t)\right) \quad (0 \le t \le 1), \tag{4.2}$$

where the numbers a_i, b_i, c_i are defined in (2.24), (2.25).

Note that, in both cases $i = 1, 2$, the coefficient of the hypergeometric function satisfy the relation,

$$c_i - a_i - b_i = \frac{1}{\nu+1}. \tag{4.3}$$

The computation for $z_1(1)$ follows directly from (4.2), (A.4), and $x(1) = 1$. As for $z_2(1)$, we proceed in the same way, just observe that, by the energy identity,

$$\frac{\dot{w}(t)^2}{2} + K_\nu \frac{w(t)^{\nu+1}}{\nu+1} = \frac{K_\nu}{\nu+1} \qquad (0 < t < 1),$$

and the definition of K_ν, we get $\dot{w}(1) = -B_\nu$.

As for the derivatives, by the definition (4.1) of $x(t)$, we have for $0 \leq t < 1$,

$$\dot{z}_1(t) = -(\nu+1)\dot{w}(t)w(t)^\nu \frac{d}{dx}F(a_1,b_1;c_1;x(t)) = -(\nu+1)\dot{w}(t)(1-x(t))^{\nu/(\nu+1)}\frac{d}{dx}F(a_1,b_1;c_1;x(t)).$$

Now we can use formula (A.6), (note that $a_i + b_i - c_i + 1 = \nu/(\nu+1)$), to get the value at $t = 1$:

$$\dot{z}_1(1) = -(\nu+1)\dot{w}(1)\lim_{x\to 1}(1-x)^{\nu/(\nu+1)}\frac{d}{dx}F(a_1,b_1;c_1;x) = (\nu+1)B_\nu\frac{\Gamma(1/2)\Gamma(\nu/(\nu+1))}{\Gamma(a_1)\Gamma(b_1)}.$$

We conclude with the computation of $\dot{z}_2(1)$. We have

$$\begin{aligned}
\dot{z}_2(t) &= -\frac{1}{K_\nu}\ddot{w}(t)F(a_2,b_2;c_2;x(t)) + \frac{\nu+1}{K_\nu}\dot{w}(t)^2 w(t)^\nu\frac{d}{dx}F(a_2,b_2;c_2;x(t)).\\
&= w(t)^\nu\left[F(a_2,b_2;c_2;x(t)) + \frac{\nu+1}{K_\nu}\dot{w}(t)^2\frac{d}{dx}F(a_2,b_2;c_2;x(t))\right]\\
&= (1-x(t))^{\nu/(\nu+1)}\left[F(a_2,b_2;c_2;x(t)) + \frac{\nu+1}{K_\nu}\dot{w}(t)^2\frac{d}{dx}F(a_2,b_2;c_2;x(t))\right].
\end{aligned}$$

Thus, since $\dot{w}(1)^2 = 2K_\nu/(\nu+1)$,

$$\dot{z}_2(1) = \lim_{x\to 1}(1-x)^{\nu/(\nu+1)}\left[F(a_2,b_2;c_2;x) + 2\frac{d}{dx}F(a_2,b_2;c_2;x)\right].$$

By using formula (A.3), we have

$$\lim_{x\to 1}(1-x)^{\nu/(\nu+1)}F(a_2,b_2;c_2;x) = 0$$

while, by (A.6),

$$2\lim_{x\to 1}(1-x)^{\nu/(\nu+1)}\frac{d}{dx}F(a_2,b_2;c_2;x) = 2\frac{\Gamma(3/2)\Gamma(\nu/(\nu+1))}{\Gamma(a_2)\Gamma(b_2)},$$

which provides the desired value of $\dot{z}_2(1)$.

5. Technical Lemmas preliminary to Theorem 2

In this section we have included all the technical Lemmas necessary for the proof of Propositions 2.2 and 2.3. The following Lemma is mainly used in the proof of Proposition 2.2, but also serves for a refined asymptotic expansion of τ_q under assumption (1.7), see Lemma 5.2.

Lemma 5.1. *Let be $g \in C^0([0,+\infty))$, $g(x) \geq 0$, $a > 0$ and $\lim_{x\to+\infty}\frac{g(x)}{x^a} = B$ finite. Then*

$$\lim_{q\to+\infty}\frac{g(xq)}{q^a} = Bx^a \quad \textit{uniformly w.r.t. } x \in [0,1], \quad .$$

Proof. First we define the function

$$G(r) = \max_{0\leq s\leq r} g(s).$$

Clearly, $G(r)$ is a increasing function, and $0 \leq g(r) \leq G(r)$, for every $r \geq 0$. We claim that there exists a constant M such that

$$\frac{G(r)}{r^a} \leq M, \quad \forall r \geq 1.$$

In order to prove it, we observe that there exist a constant M' such that $\frac{g(r)}{r^a} \leq M'$ for every $r \geq 1$. Thus we have $G(r) \leq \max_{0\leq s\leq 1} g(s) + M' r^a$ for every $r \geq 0$, so that

$$\frac{G(r)}{r^a} \leq \max_{0\leq s\leq 1} g(s) + M' = M, \quad \forall r \geq 1.$$

Let us fix $\delta \in (0,1)$. Since $\lim_{r\to+\infty} \frac{g(r)}{r^a} = B$, then there exist h such that

$$\left| \frac{g(r)}{r^a} - B \right| \leq (M+B)\,\delta^a, \quad \forall r \geq h.$$

If $0 \leq x \leq \delta$, we have

$$\left| \frac{g(xq)}{q^a} - Bx^a \right| \leq \frac{G(xq)}{q^a} + Bx^a \leq \frac{G(\delta q)}{(\delta q)^a}\delta^a + B\delta^a \leq (M+B)\delta^a, \quad \text{if} \quad q \geq 1/\delta,$$

while, for $\delta \leq x \leq 1$,

$$\left| \frac{g(xq)}{q^a} - Bx^a \right| = \left| \frac{x^a g(xq)}{(xq)^a} - Bx^a \right| \leq \left| \frac{g(xq)}{(xq)^a} - B \right| \leq (M+B)\,\delta^a \quad \text{if } q \geq h/\delta.$$

□

In order to prove the part II) of Theorem 2 we need a more accurate estimate of the rate of convergence of z_q to z.

Lemma 5.2. *Let R_q be defined as in (2.22). If condition (1.7) holds, then $R_q = o(\tau_q)$, $q \to \infty$.*

Proof. As a matter of fact, we shall prove that

$$\|w_q - w\|_{C^1([0,1])} + \|z_q - z\|_{C^1([0,1])} = o(\tau_q), \tag{5.1}$$

which, of course, yields the assertion.

First we need a better asymptotic expansion of τ_q. For doing so, we mimic the proof in Proposition 2.1. From our assumption (1.7), we may set

$$J(x) = \frac{A}{\nu+1}(x^{\nu+1} + H(x)), \qquad \text{with} \quad H'(x) = o(x^{(\nu+1)/2}).$$

Then, from (2.5), we get

$$\sqrt{\frac{2A}{\nu+1}}\, q^{\frac{\nu-1}{2}}\, \tau_q = \int_0^1 \frac{dx}{\{1 - x^{\nu+1} + (H(q) - H(xq))/q^{\nu+1})\}^{1/2}}.$$

From the above formula and recalling the meaning of B_ν, we obtain

$$\sqrt{\frac{2A}{\nu+1}}\, q^{\frac{\nu-1}{2}}\, \tau_q - B_\nu = \int_0^1 \frac{1}{\sqrt{1-x^{\nu+1}}} \left(\frac{1}{\sqrt{1+G(x;q)}} - 1 \right) dx \tag{5.2}$$

where

$$G(x;q) = \frac{H(q) - H(xq)}{1 - x^{\nu+1}}\, q^{-(\nu+1)}.$$

Due to the mean value Theorem, we have

$$G(x;q) = \frac{q(1-x)H'(\xi q)}{1-x^{\nu+1}} q^{-(\nu+1)},$$

for some $0 < \xi < 1$, thus $|G(x;q)| \le |H'(\xi q)| q^{-\nu}$. From assumption (1.7), and thanks to Lemma 5.1, with $B = 0$, it follows that $\lim_{q\to\infty} \sup_{0\le\xi\le1} |H'(\xi q)| q^{-(\nu+1)/2} = 0$. Therefore $\sup_{0\le x\le 1} |G(x;q)| = o(q^{-(\nu+1)/2})$, so that we can conclude that

$$q^{\frac{\nu-1}{2}} \tau_q = \sqrt{\frac{\nu+1}{2A}}\,(B_\nu + o(q^{-(\nu+1)/2})). \tag{5.3}$$

On the other hand, using (1.7) ad Lemma 5.1, with $B = 0$, we have that

$$\frac{f'(xq) - \nu A (xq)^{\nu-1}}{q^{(\nu-1)/2}} \to 0$$

uniformly w.r.t. $x \in [0,1]$, so that

$$\sup_{x\in[0,1]} \left| f'(xq) - \nu A (xq)^{\nu-1} \right| = o(q^{(\nu-1)/2}). \tag{5.4}$$

Putting together (5.3) and (5.4), we obtain

$$\sup_{x\in[0,1]} \left| \tau_q^2 f'(xq) - \nu K_\nu x^{\nu-1} \right| = o(q^{-(\nu-1)/2}) = o(\tau_q).$$

An analogous calculation yields

$$\sup_{x\in[0,1]} \left| q^{-1}\tau_q^2 f(xq) - K_\nu x^{\nu} \right| = o(\tau_q).$$

Owing to a classical continuous dependence theorem (see [15, Thm 3, ch. XV]) we get the desired estimate (5.1). □

The following 2 lemmas are used in the analysis of the linear spring regime, and are necessary for the proof of Proposition 2.3.

Lemma 5.3. *Let w be the solution of problem (2.11), u the solution of (1.3) with the given initial data. For $\tau_q \le t \le 2T_q - \tau_q$, let us set*

$$\tilde{u}_q(t) = \frac{q}{\tau_q} \frac{\sin\sqrt{\alpha}(t-\tau_q)}{\sqrt{\alpha}}\, \dot{w}(1).$$

Then, we have

$$\|u - \tilde{u}_q\|_{C^1([\tau_q, 2T_q-\tau_q])} = o(q/\tau_q).$$

Proof. Let us set $y(t) = q^{-1}\tau_q u(t)$ for $\tau_q \le t \le 2T_q - \tau_q$. Then y solves the problem

$$\ddot{y} + \alpha y + q^{-1}\tau_q f(u) = 0, \qquad y(\tau_q) = 0, \quad \dot{y}(\tau_q) = \dot{w}_q(1).$$

Owing to Proposition 2.2, we have that

$$\dot{y}(\tau_q) = \dot{w}(1) + o(1).$$

On the other hand, since $u(t) \le 0$ in the interval $[\tau_q, 2T_q - \tau_q]$, by the assumptions (H1), (H3), we get the estimate

$$|q^{-1}\tau_q f(u(t))| \le q^{-1}\tau_q\, h \to 0, \qquad \text{as } q \to \infty.$$

From the continuous dependence theorem, we obtain that

$$y(t) = \frac{\sin\sqrt{\alpha}(t-\tau_q)}{\sqrt{\alpha}}\, \dot{w}(1) + o(1)$$

in the $C^1([\tau_q, 2T_q - \tau_q])$-norm. Thus, by definition of $y(t)$ and $\tilde{u}_q(t)$, we get

$$u(t) - \tilde{u}_q(t) = q/\tau_q \left(y(t) - \frac{\sin \sqrt{\alpha}(t - \tau_q)}{\sqrt{\alpha}} \right) = o(q/\tau_q)$$

in the $C^1([\tau_q, 2T_q - \tau_q])$-norm. □

Lemma 5.4. *Let us set $T = \tau_q + \frac{\pi}{2\sqrt{\alpha}}$. Then, there exists a constant K such that*

$$\int_{\tau_q}^{T} f'(u(s))\, ds \le K\, \tau_q/q. \tag{5.5}$$

Proof. Let us fix a time t_1, such that $\tau_q < t_1 < T_q$, for instance $t_1 = \tau_q + \pi/6\sqrt{\alpha}$. We split the integral into the sum $\int_{\tau_q}^{t_1} f'(u(s))\, ds + \int_{t_1}^{T} f'(u(s))\, ds$.

Let us start with the estimate of the first integral, which is not trivial given that $u(\tau_q) = 0$. Since $u'(s) < 0$ for $\tau_q \le s < T_q$, and $f(u(\tau_q)) = 0$, by integration by parts we obtain

$$\int_{\tau_q}^{t_1} f'(u(s)) ds = \int_{\tau_q}^{t_1} f'(u(s)) \frac{\dot{u}(s)}{\dot{u}(s)} ds = \frac{f(u(t_1))}{\dot{u}(t_1)} + \int_{\tau_q}^{t_1} f(u(s)) \frac{\ddot{u}(s)}{\dot{u}(s)^2}\, ds.$$

Since $\ddot{u}(s) = -\alpha u(s) - f(u(s)) > 0$, for $\tau_q \le s < T_q$, we conclude that

$$\int_{\tau_q}^{t_1} f'(u(s)) ds < \frac{f(u(t_1))}{\dot{u}(t_1)}. \tag{5.6}$$

Owing to Lemma (5.3), we have that

$$\dot{u}(t_1) = \frac{q}{\tau_q} \cos(\sqrt{\alpha}(t_1 - \tau_q)\dot{w}(1) + o(q/\tau_q),$$

therefore we get

$$|\dot{u}(t_1)| \ge \frac{q}{4\tau_q}, \quad \text{for sufficiently big } q.$$

From the inequality (5.6), and by assumption (H3), it follows that

$$\int_{\tau_q}^{t_1} f'(u(s)) ds \le 4hq^{-1}\tau_q.$$

The estimate for the second integral follows by the assumption (H3), and again by Lemma 5.3. □

Appendix A. Hypergeometric function toolbox

The formulas in this section can be retrieved *e.g.* in [1].

By $F(a, b; c; x)$ we denote the hypergeometric function which is defined for $|x| < 1$ by the power series

$$F(a, b; c; x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + ...$$

The hypergeometric differential equation satisfied by $F(a, b; c; x)$ is

$$x(1 - x)\frac{d^2y}{dx^2} + [c - (a + b + 1)x]\frac{dy}{dx} - aby = 0. \tag{A.1}$$

Throughout the article we use a couple of classical formulas on the hypergeometric function: the differential formula,

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x); \tag{A.2}$$

and the connection formula,

$$\begin{aligned} F(a,b;c;x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b+1-c;1-x) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}F(c-a,c-b;1+c-a-b;1-x). \end{aligned} \tag{A.3}$$

In particular, in the case when

$$c-a-b>0,$$

we obtain two formulas which are very useful for our purposes. Directly from (A.3), we have

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.4}$$

Combining (A.2), and (A.3), we get (we use $\Gamma(z+1) = z\Gamma(z)$)

$$\begin{aligned} \frac{d}{dx}F(a,b;c;x) &= ab\,\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}F(a+1,b+1;a+b+2-c;1-x) \\ &+ \frac{\Gamma(c)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b-1}F(c-a,c-b;c-a-b;1-x). \end{aligned} \tag{A.5}$$

From this last formula, we see that $\frac{d}{dx}F(a,b;c;x)$ is singular at $x=1$ in the case

$$c-a-b<1,$$

nevertheless (A.5) yields

$$(1-x)^{a+b-c+1}\frac{d}{dx}F(a,b;c;x) = \frac{\Gamma(c)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} + o(1), \qquad \text{as } x \to 1. \tag{A.6}$$

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