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Tricomi Continuants

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Abstract: In this paper, we introduce and study the Tricomi continuants, a family of tridiagonal determinants forming a Sheffer sequence closely related to the Tricomi polynomials and the Laguerre polynomials. Specifically, we obtain the main umbral operators associated with these continuants and establish some of their basic relations. Then, we obtain a Turan-like inequality, some congruences, some binomial identities (including a Carlitz-like identity), and some relations with the Cayley continuants. Furthermore, we show that the infinite Hankel matrix generated by the Tricomi continuants has an LDU-Sheffer factorization, while the infinite Hankel matrix generated by the shifted Tricomi continuants has an LTU-Sheffer factorization. Finally, by the first factorization, we obtain the linearization formula for the Tricomi continuants and its inverse.

Keywords: exponential Riordan matrices; Sheffer polynomials; umbral operators; congruences; binomial identities; Cayley continuants; Hankel matrices; Sheffer factorizations

MSC: 05A19; 05A15; 05A40; 15A23; 15A15; 11B83

1. Introduction

In 1951, Tricomi [1] considered a sequence $\{\ell_n^{(\alpha)}(x)\}_{n\in\mathbb{N}}$ of non-orthogonal polynomials related to the generalized Laguerre polynomials. These polynomials satisfy the three-term recurrence

$$(n+1)\ell_{n+1}^{(\alpha)}(x) - (\alpha+n)\ell_n^{(\alpha)}(x) + x\ell_{n-1}^{(\alpha)}(x) = 0$$

with the initial values $\ell_0^{(\alpha)}(x) = 1$ and $\ell_1^{(\alpha)}(x) = \alpha$ and have ordinary generating series

$$\ell^{(\alpha)}(x;t) = \sum_{n\geq 0} \ell_n^{(\alpha)}(x)\, t^n = (1-t)^{x-\alpha} e^{xt}\,. \tag{1}$$

They have degree $\lfloor n/2 \rfloor$ (for a generic parameter α) and can be expressed as

$$\ell_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{x-\alpha}{k} (-1)^k \frac{x^{n-k}}{(n-k)!}\,.$$

Tricomi polynomials have been studied from several points of view [2–5]. Here, we will show that they are related to the following tridiagonal determinants (continuants [6,7]). Consider the $n \times n$ determinants

$$T_n^{(\nu)}(x) = \begin{vmatrix} x & 1 & & & & \\ \nu & x+1 & 1 & & & \\ & 2\nu & x+2 & 1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & (n-2)\nu & x+n-2 & 1 \\ & & & & (n-1)\nu & x+n-1 \end{vmatrix}_{n\times n}.$$

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For the first values of n, we have

$$\begin{aligned}
T_0^{(\nu)}(x) &= 1 \\
T_1^{(\nu)}(x) &= x \\
T_2^{(\nu)}(x) &= x^2 + x - \nu \\
T_3^{(\nu)}(x) &= x^3 + 3x^2 + (2 - 3\nu)x - 2\nu \\
T_4^{(\nu)}(x) &= x^4 + 6x^3 + (11 - 6\nu)x^2 + (6 - 14\nu)x + 3\nu(\nu - 2) \\
T_5^{(\nu)}(x) &= x^5 + 10x^4 + 5(7 - 2\nu)x^3 + 50(1 - \nu)x^2 + (15\nu^2 - 70\nu + 24)x + 4\nu(5\nu - 6)\,.
\end{aligned}$$

These determinants are monic polynomials of degree n (for any parameter ν) and satisfy the recurrence

$$T_{n+2}^{(\nu)}(x) = (x + n + 1)\, T_{n+1}^{(\nu)}(x) - (n+1)\nu\, T_n^{(\nu)}(x) \tag{2}$$

with the initial values $T_0^{(\nu)}(x) = 1$ and $T_1^{(\nu)}(x) = x$. From this recurrence, it is straightforward to obtain the exponential generating series

$$T^{(\nu)}(x;t) = \sum_{n\geq 0} T_n^{(\nu)}(x)\frac{t^n}{n!} = \frac{\mathrm{e}^{\nu t}}{(1-t)^{x-\nu}}\,. \tag{3}$$

By comparing series (1) and (3), we have $T_n^{(\nu)}(x) = n!\ell_n^{(x)}(\nu)$. For this reason, we call these polynomials Tricomi continuants. Clearly, also the Tricomi continuants can be expressed in terms of the Laguerre polynomials. Specifically, considering the Laguerre polynomials as defined in [8], p. 108, we have the exponential generating series

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{\mathrm{e}^{-\frac{xt}{1-t}}}{(1-t)^{\alpha+1}} \tag{4}$$

and by formula (19) in [9], Volume II, p. 189,

$$\sum_{n\geq 0} L_n^{(\alpha-n)}(x)\,\frac{t^n}{n!} = (1+t)^{\alpha}\,\mathrm{e}^{-xt}\,. \tag{5}$$

Then, from series (3) and (5), we have

$$T_n^{(\nu)}(x) = (-1)^n L_n^{(-x+\nu-n)}(\nu)$$

or, more explicitly,

$$T_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k}(x-\nu)_k \tag{6}$$

where $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ is the Pochhammer symbol (rising factorial).

Furthermore, from series (3), we have that the Tricomi continuants form a Sheffer sequence. More precisely, they form the Steffensen sequence (cross-sequence) associated with the Sheffer matrix

$$\mathcal{T}^{(\nu)} = \left[\, T_{n,k}^{(\nu)} \,\right]_{n,k\geq 0} = \left((1-t)^{\nu}\mathrm{e}^{\nu t},\, \log\frac{1}{1-t} \right). \tag{7}$$

For $\nu \in \mathbb{Z}$, these matrices have integer entries (as can be easily deduced from formula (6)). For instance, for $\nu = -1, 1$, we have the (north-west partial) matrices

$$\mathcal{T}_5^{(-1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 \\ 9 & 20 & 17 & 6 & 1 & 0 \\ 44 & 109 & 100 & 45 & 10 & 1 \end{pmatrix}, \quad \mathcal{T}_5^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 3 & 1 & 0 & 0 \\ -3 & -8 & 5 & 6 & 1 & 0 \\ -4 & -31 & 0 & 25 & 10 & 1 \end{pmatrix}.$$

Moreover, if $\nu = 0$, we have $T_n^{(\nu)}(x) = (x)_n$ and then $\mathcal{T}^{(0)}$ is the Sheffer matrix of the Stirling numbers of the first kind.

In this paper, we will investigate various combinatorial and algebraic properties of the Tricomi continuants. More precisely, in Section 2 we recall the basic definitions and properties of the Sheffer sequences and matrices. In Section 3, we obtain the main umbral operators associated with the Tricomi continuants. Then, by using these operators, we establish some basic relations for these continuants and their derivatives. In Section 4, we obtain a binomial identity from which we will deduce a Turan-like inequality. In Section 5, we derive some congruences for the Tricomi continuants and their derivatives. In Section 6, we consider the polynomials $T_n^{(\nu+\mu n)}(x+\mu n)$ and obtain their exponential generating series, proving that they still form a Sheffer sequence. In Section 7, we establish a two-parameter binomial identity similar to the symmetric Carlitz identity for the Bernoulli numbers. In Section 8, we obtain some relations between the Tricomi continuants and the Cayley continuants (another classical family of continuants also forming a Sheffer sequence). In Section 9, we show that the infinite Hankel matrix generated by the Tricomi continuants admits an LDU-Sheffer factorization. Similarly, we show that the infinite Hankel matrix generated by the shifted Tricomi continuants admits an LTU-Sheffer factorization. Furthermore, by the first factorization, we obtain the linearization formula for the Tricomi continuants and its inverse. Finally, in Section 10, we obtain some representations in terms of the Stirling numbers.

2. Sheffer Sequences and Sheffer Matrices

Sheffer sequences form an important class of polynomial sequences appearing in several fields of mathematics, especially in analysis, combinatorics, and umbral calculus [8,10–16]. In what follows, we recall the main definitions and properties we will need in the present paper. See [17] for a historical account.

A Sheffer sequence [18] is a polynomial sequence $\{s_n(x)\}_{n\geq 0}$ having exponential generating series

$$s(x;t) = \sum_{n\geq 0} s_n(x)\,\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xf(t)}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ are two exponential series with $g_0 = 1$, $f_0 = 0$, and $f_1 \neq 0$. In this case, the Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ has spectrum $(g(t), f(t))$.

An Appell sequence [19] is a Sheffer sequence with spectrum $(g(t), t)$. A Steffensen sequence [16] is a Sheffer sequence $\{s_n^{(\nu)}(x)\}_{n\geq 0}$, depending on a parameter ν, with spectrum $(g(t)h(t)^\nu, f(t))$, where $h(t) = \sum_{n\geq 0} h_n \frac{t^n}{n!}$ is an exponential series with $h_0 = 1$. Each Steffensen sequence is a cross-sequence [13,14], since it satisfies the binomial identity

$$\sum_{k=0}^{n} \binom{n}{k} s_k^{(\mu)}(x) s_{n-k}^{(\nu)}(y) = s_n^{(\mu+\nu)}(x+y)\,.$$

A Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$ is an infinite lower triangular matrix such that

$$\sum_{n\geq k} s_{n,k}\frac{t^n}{n!} = g(t)\frac{f(t)^k}{k!}$$

for two exponential series $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$, with $g_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$. The entries of the matrix S satisfy the recurrence

$$s_{n+1,k+1} = \frac{n+1}{k+1} \sum_{i=0}^{n-k} \binom{k+i}{k} a_i s_{n,k+i} \tag{8}$$

where the a_n are the coefficients of the series $a(t) = t/\widehat{f}(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!}$, where $\widehat{f}(t)$ is the compositional inverse of $f(t)$.

Clearly, the row polynomials of the Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$ form the Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ with spectrum $(g(t), f(t))$, and vice versa.

The Sheffer matrices form a group with respect to the matrix multiplication (as happens for the analogous Riordan matrices [20,21]). More precisely, the product of two Sheffer matrices $S_1 = (g_1(t), f_1(t))$ and $S_2 = (g_2(t), f_2(t))$ is given by

$$S_1 S_2 = (g_1(t) g_2(f_1(t)), f_2(f_1(t)))\,,$$

the identity matrix is $I = (1, t)$ and the inverse of a Sheffer matrix $S = (g(t), f(t))$ is the Sheffer matrix

$$S^{-1} = (g(\widehat{f}(t))^{-1}, \widehat{f}(t))\,.$$

Given a Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0}$ and its inverse $S^{-1} = [\widehat{s}_{n,k}]_{n,k\geq 0}$, we have the inversion theorem: given any two sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, we have

$$u_n = \sum_{k=0}^{n} s_{n,k} v_k \qquad \Longleftrightarrow \qquad v_n = \sum_{k=0}^{n} \widehat{s}_{n,k} u_k\,.$$

Consider, for instance, the generalized Stirling numbers of the first kind $s_{n,k}^{(\mu)}$ and the generalized Stirling numbers of the second kind $S_{n,k}^{(\mu)}$, defined as the entries of the following Sheffer matrices [22,23]

$$\left(\frac{1}{(1-t)^{\mu}}, \log\frac{1}{1-t}\right) = \left[\, s_{n,k}^{(\mu)} \,\right]_{n,k\geq 0} \qquad \text{and} \qquad (e^{\mu t}, e^t - 1) = \left[\, S_{n,k}^{(\mu)} \,\right]_{n,k\geq 0}. \tag{9}$$

These matrices are related to each other by the following identities

$$\left(\frac{1}{(1-t)^{\mu}}, \log\frac{1}{1-t}\right)^{-1} = (e^{-\mu t}, 1 - e^{-t}) = \left[\, (-1)^{n-k} S_{n,k}^{(\mu)} \,\right]_{n,k\geq 0}$$

$$(e^{\mu t}, e^t - 1)^{-1} = \left(\frac{1}{(1+t)^{\mu}}, -\log\frac{1}{1+t}\right) = \left[\, (-1)^{n-k} s_{n,k}^{(\mu)} \,\right]_{n,k\geq 0}.$$

Hence, we have the Stirling inversion theorem

$$u_n = \sum_{k=0}^{n} S_{n,k}^{(\mu)} v_k \qquad \Longleftrightarrow \qquad v_n = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k}^{(\mu)} u_k\,.$$

or

$$u_n = \sum_{k=0}^{n} s_{n,k}^{(\mu)} v_k \qquad \Longleftrightarrow \qquad v_n = \sum_{k=0}^{n} (-1)^{n-k} S_{n,k}^{(\mu)} u_k\,. \tag{10}$$

Given a Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$, we have

$$\overline{S} = (g(-t), -f(-t)) = \left[\, (-1)^{n-k} s_{n,k} \,\right]_{n,k\geq 0}$$

or $\overline{S} = MSM$, where $M = (1, -t)$. A Sheffer matrix S is a pseudo-involution when $S^{-1} = \overline{S}$, or, equivalently, when SM is an involution. For these matrices, we have the following inversion theorem: given any two sequences $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, we have

$$u_n = \sum_{k=0}^{n} s_{n,k} v_k \qquad \Longleftrightarrow \qquad v_n = \sum_{k=0}^{n} (-1)^{n-k} s_{n,k} u_k \,.$$

For instance, the binomial matrix

$$B(q) = \left[\binom{n}{k} q^{n-k} \right]_{n,k\geq 0} = \left(\frac{1}{1-qt}, \frac{t}{1-qt} \right)$$

is a pseudo-involution. In this case, we have the binomial inversion theorem

$$u_n = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} v_k \qquad \Longleftrightarrow \qquad v_n = \sum_{k=0}^{n} \binom{n}{k} (-q)^{n-k} u_k \,. \tag{11}$$

Also the Lah matrix

$$L = \left[\left| \begin{matrix} n \\ k \end{matrix} \right| \right]_{n,k\geq 0} = \left(1, \frac{t}{1-t} \right), \tag{12}$$

whose entries are the (signless) Lah numbers, is a pseudo-involution. In this case, we have the Lah inversion theorem

$$u_n = \sum_{k=0}^{n} \left| \begin{matrix} n \\ k \end{matrix} \right| v_k \qquad \Longleftrightarrow \qquad v_n = \sum_{k=0}^{n} \left| \begin{matrix} n \\ k \end{matrix} \right| (-1)^{n-k} u_k \,. \tag{13}$$

Given a Sheffer matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$, the Sheffer transform $\mathcal{T}_S$ of an exponential series $h(t) = \sum_{n\geq 0} h_n \frac{t^n}{n!}$ is the exponential series given by

$$\mathcal{T}_S h(t) = (g(t), f(t))\, h(t) = g(t)\, h(f(t)) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} s_{n,k} h_k \right) \frac{t^n}{n!} \,.$$

By Pincherle's theorem [24], every linear operator $L : \mathbb{R}[x] \to \mathbb{R}[x]$ can be represented by means of an exponential series in the derivative D with respect to x. More precisely, there exists a unique polynomial sequence $\{L_n(x)\}_{n\geq 0}$, where $L_n(x) \in \mathbb{R}[x]$ for every $n \in \mathbb{N}$, such that

$$Lp(x) = \sum_{k\geq 0} \frac{L_k(x)}{k!} D^k p(x) = \sum_{k=0}^{n} \frac{L_k(x)}{k!} p^{(k)}(x)$$

for every polynomial $p(x) \in \mathbb{R}[x]$ of degree n.

Given a polynomial sequence $\{p_n(x)\}_{n\geq 0}$, where each $p_n(x) \in \mathbb{R}[x]$ has degree n, we can consider the linear operators $J, M, N, A : \mathbb{R}[x] \to \mathbb{R}[x]$ defined, for every $n \in \mathbb{N}$, by

$$\begin{aligned} Jp_n(x) &= np_{n-1}(x)\,, & Mp_n(x) &= p_{n+1}(x)\,, \\ Np_n(x) &= np_n(x)\,, & Ap_n(x) &= \frac{p'_{n+1}(x)}{n+1}\,. \end{aligned} \tag{14}$$

The operator J is the umbral derivative (or lowering operator, or annihilation operator), the operator M is the umbral shift (or raising operator, or creation operator), the operator N is the umbral theta operator, and the operator A is the Appell operator associated with the sequence $\{p_n(x)\}_{n\geq 0}$.

The umbral operators J, M, N, and A of a Sheffer sequence $\{s_n(x)\}_{n\geq 0}$ with spectrum $(g(t), f(t))$ are given by [8]

$$J = \widehat{f}(D) \tag{15}$$

$$M = \frac{g'(\widehat{f}(D))}{g(\widehat{f}(D))} + xf'(\widehat{f}(D)) \tag{16}$$

$$N = MJ = \left(\frac{g'(\widehat{f}(D))}{g(\widehat{f}(D))} + xf'(\widehat{f}(D))\right)\widehat{f}(D) \tag{17}$$

$$A = a(D) = \frac{D}{\widehat{f}(D)} \,. \tag{18}$$

Some other important operators are the shift operator E^λ defined by $E^\lambda p(x) = p(x+\lambda)$ and represented by $E^\lambda = \mathrm{e}^{\lambda D}$, the forward difference operator $\Delta = E - 1 = \mathrm{e}^D - 1$, and the backward difference operator $\nabla = 1 - E^{-1} = 1 - \mathrm{e}^{-D}$. Moreover, we have

$$D = \log\frac{1}{1-\nabla} \tag{19}$$

$$\frac{D^k}{k!} = \frac{1}{k!}\left(\log\frac{1}{1-\nabla}\right)^k = \sum_{i\geq k}\begin{bmatrix} i \\ k \end{bmatrix}\frac{\nabla^i}{i!} \tag{20}$$

where the coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are the Stirling numbers of the first kind.

3. Umbral Operators

In this section, we obtain the main umbral operators for the Tricomi continuants and then we deduce some basic relations. First of all, we have the following representation theorem.

Theorem 1. *The umbral operators for the Tricomi continuants are given by*

$$J = 1 - \mathrm{e}^{-D} = 1 - E^{-1} \tag{21}$$

$$M = -\nu(\mathrm{e}^D - 1) + x\mathrm{e}^D = (x-\nu)E + \nu \tag{22}$$

$$N = (x-\nu)E - x + 2\nu - \nu E^{-1} \tag{23}$$

$$A = \frac{D}{1-\mathrm{e}^{-D}} = \frac{D}{1-E^{-1}} \tag{24}$$

or, equivalently, by

$$J = \nabla \tag{25}$$

$$M = (x-\nu)\Delta + x \tag{26}$$

$$N = (x-\nu)\Delta + \nu\nabla \tag{27}$$

$$A = \frac{1}{\nabla}\log\frac{1}{1-\nabla} \,. \tag{28}$$

Moreover, the operators J, M, and N satisfy the relation

$$\nu J + M - N = x \,. \tag{29}$$

Proof. From spectrum (7), we have

$$g(t) = (1-t)^\nu \mathrm{e}^{\nu t} \,, \qquad f(t) = \log\frac{1}{1-t} \,,$$

$$\widehat{f}(t) = 1 - \mathrm{e}^{-t} \,, \qquad g'(t) = -\nu t(1-t)^{\nu-1}\mathrm{e}^{\nu t} \,, \qquad f'(t) = \frac{1}{1-t} \,,$$

$$\frac{g'(t)}{g(t)} = -\frac{\nu t}{1-t} \,, \qquad \frac{g'(\widehat{f}(t))}{g(\widehat{f}(t))} = -\nu(\mathrm{e}^t - 1) \,, \qquad f'(\widehat{f}(t)) = \mathrm{e}^t \,, \qquad \frac{t}{\widehat{f}(t)} = \frac{t}{1-\mathrm{e}^{-t}} \,.$$

It follows that, by formulas (15)–(18), we have identities (21)–(24), respectively. Then, using the definition of the difference operators, we have at once identities (25)–(27). To obtain identity (28), just use (19). □

Using these representations of the umbral operators, we can obtain several identities for the Tricomi continuants. For instance, we have the following ones.

Theorem 2. *The Tricomi continuants satisfy the relations*

$$T_n^{(\nu)}(x) - T_n^{(\nu)}(x-1) = nT_{n-1}^{(\nu)}(x) \tag{30}$$

$$T_{n+1}^{(\nu)}(x) = (x-\nu)T_n^{(\nu)}(x+1) + \nu T_n^{(\nu)}(x) \tag{31}$$

$$(x-\nu)T_n^{(\nu)}(x+1) - (x+n-2\nu)T_n^{(\nu)}(x) - \nu T_n^{(\nu)}(x-1) = 0 \tag{32}$$

$$T_{n+1}^{(\nu)}(x) - (x+n)T_n^{(\nu)}(x) + n\nu T_{n-1}^{(\nu)}(x) = 0 \tag{33}$$

$$T_{n+1}^{(\nu)\,\prime}(x) - T_{n+1}^{(\nu)\,\prime}(x-1) = (n+1)\,T_n^{(\nu)\prime}(x)\,. \tag{34}$$

Proof. Relations (30)–(32) derive immediately from representations (21)–(23) and definitions (14). Similarly, relation (33) derives from (29). Finally, to obtain relation (34), just notice that, from (24), we have $\nabla A = D$. □

Remark 1. *Since $J = \nabla$, we have*

$$\nabla^k T_n^{(\nu)}(x) = n^{\underline{k}}\, T_{n-k}^{(\nu)}(x) \tag{35}$$

where $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$. Hence, we have the identity

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^i T_n^{(\nu)}(x-i) = n^{\underline{k}}\, T_{n-k}^{(\nu)}(x)$$

which generalizes identity (30). In particular, for $k = n$ (and replacing the index i by k), we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k T_n^{(\nu)}(x-k) = n!\,.$$

Recall that the Bernoulli numbers B_n and the harmonic numbers H_n have generating series

$$B(t) = \sum_{n\geq 0} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}\,, \qquad H(t) = \sum_{n\geq 0} H_n t^n = \frac{1}{1-t} \log\frac{1}{1-t}\,.$$

Then, we have the following further identities.

Theorem 3. *The Tricomi continuants satisfy the relations*

$$T_n^{(\nu)\prime}(x) = \sum_{k=1}^{n} \binom{n}{k} (k-1)!\, T_{n-k}^{(\nu)}(x) \tag{36}$$

$$T_{n+1}^{(\nu)\,\prime}(x) = (n+1) \sum_{k=0}^{n} (-1)^k\, B_k \sum_{i=k}^{n} \binom{n}{i} \begin{bmatrix} i \\ k \end{bmatrix} T_{n-i}^{(\nu)}(x) \tag{37}$$

$$T_{n+1}^{(\nu)\,\prime}(x) = \sum_{k=0}^{n} \binom{n+1}{k+1} k!\, T_{n-k}^{(\nu)}(x) \tag{38}$$

$$T_{n+1}^{(\nu)\,\prime}(x+1) - T_{n+1}^{(\nu)\,\prime}(x) = \sum_{k=0}^{n} \binom{n+1}{k+1} (k+1)!\, H_k\, T_{n-k}^{(\nu)}(x) \tag{39}$$

$$T_n^{(\nu)\prime}(x+1) = \sum_{k=0}^{n} \binom{n}{k} k!\, H_k\, T_{n-k}^{(\nu)}(x) \tag{40}$$

where the coefficients B_n are the Bernoulli numbers and the coefficients H_n are the harmonic numbers.

Proof. Identity (36) derives from (19) and (35). Then, from (24), we have $A = B(-D)$ and from this representation we have

$$T_{n+1}^{(\nu)\prime}(x) = (n+1) \sum_{k=0}^{n} \frac{(-1)^k}{k!} B_k\, D^k T_n^{(\nu)}(x)\,.$$

Then, by (20) and (35), we have relation (37). Similarly, from representation (28), we have relation (38). To obtain the fourth relation, notice that

$$\frac{\nabla}{1-\nabla} = \frac{1-\mathrm{e}^{-D}}{\mathrm{e}^{-D}} = \mathrm{e}^{D} - 1 = \Delta\,.$$

Hence

$$\Delta A = \frac{1}{1-\nabla} \log \frac{1}{1-\nabla} = H(\nabla)\,.$$

This representation and (35) imply relation (39). Finally, by (34), relation (39) reduces to (40). □

Remark 2. *Consider the coefficients $T_{n,k}^{(\nu)}$ of the Sheffer matrix (7). Since*

$$T_k^{(\nu)}(t) = \sum_{n\geq k} T_{n,k}^{(\nu)} \frac{t^n}{n!} = G^{(\nu)}(t) \frac{1}{k!} \left(\log \frac{1}{1-t} \right)^k$$

and

$$G^{(\nu)}(t) = \sum_{n\geq 0} G_n^{(\nu)} \frac{t^n}{n!} = (1-t)^{\nu} \mathrm{e}^{\nu t}$$

where $G_n^{(\nu)} = T_{n,0}^{(\nu)}$, then we have

$$T_{n,k}^{(\nu)} = \sum_{i=k}^{n} \binom{n}{i} \begin{bmatrix} i \\ k \end{bmatrix} G_{n-i}^{(\nu)} \tag{41}$$

where the coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are the Stirling numbers of the first kind. Moreover, since

$$(1-t)\, T_{k+1}^{(\nu)\prime}(t) + \nu\, t\, T_{k+1}^{(\nu)}(t) + T_k^{(\nu)}(t) = 0\,,$$

we have the recurrence

$$T_{n+2,k+1}^{(\nu)} = T_{n+1,k}^{(\nu)} + (n+1) T_{n+1,k+1}^{(\nu)} - (n+1)\nu T_{n,k+1}^{(\nu)}\,. \tag{42}$$

Furthermore, since $a(t) = t/\widehat{f}(t) = B(-t)$, recurrence (8) becomes

$$T_{n+1,k+1}^{(\nu)} = \frac{n+1}{k+1} \sum_{i=0}^{n-k} \binom{k+i}{k} (-1)^i B_i T_{n,k+i}^{(\nu)}\,.$$

Finally, notice that from recurrence (42) we have

$$G_{n+2}^{(\nu)} = (n+1) G_{n+1}^{(\nu)} - (n+1)\nu\, G_n^{(\nu)}$$

with $G_0^{(\nu)} = 1$ and $G_1^{(\nu)} = 0$. Thus, if $\nu \le 0$, then $G_n^{(\nu)} \ge 0$ for every $n \in \mathbb{N}$. Consequently, by formula (41), we have $T_{n,k}^{(\nu)} \ge 0$ for every $n, k \in \mathbb{N}$. In conclusion, if $\nu = -m$ with $m \in \mathbb{N}$, then the matrix $\mathcal{T}^{(-m)}$ has non-negative integer entries (as in the examples considered in the Introduction). All these matrices admit a combinatorial interpretation (which we omit here).

4. Turán-like Inequalities

A sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ of real polynomials satisfies the Turán inequalities when $p_{n+1}(x)^2 - p_{n+2}(x)p_n(x) > 0$ for all x belonging to a suitable interval of $\mathbb{R}$. Several classical polynomials satisfy these inequalities, such as the Legendre, Laguerre, Hermite, and ultraspherical polynomials [25,26]. Also, the Tricomi continuants satisfy some inequalities of this kind (but with the direction of the inequality reversed). To obtain such inequalities, we will use the following formula.

Theorem 4. *The Tricomi continuants satisfy the relation*

$$T_{n+2}^{(\nu)}(x)T_n^{(\nu)}(x) - T_{n+1}^{(\nu)}(x)^2 = (x-\nu)\sum_{k=0}^{n}\binom{n}{k}(n-k)!\nu^{n-k}T_k^{(\nu)}(x)T_k^{(\nu)}(x+1)\,. \tag{43}$$

Proof. Consider the determinants

$$y_n = \begin{vmatrix} T_{n+2}^{(\nu)}(x) & T_{n+1}^{(\nu)}(x) \\ T_{n+1}^{(\nu)}(x) & T_n^{(\nu)}(x) \end{vmatrix}.$$

By recurrence (2) and by the properties of the determinants, we have

$$y_{n+1} = (n+1)\nu y_n + T_{n+1}^{(\nu)}(x)(T_{n+2}^{(\nu)}(x) - \nu T_{n+1}^{(\nu)}(x))\,.$$

By relation (31), this identity becomes

$$y_{n+1} = (n+1)\nu y_n + (x-\nu)T_{n+1}^{(\nu)}(x)T_{n+1}^{(\nu)}(x+1)\,.$$

This is a linear recurrence of the first order in the form $y_{n+1} = a_{n+1}y_n + b_{n+1}$, where $a_n = n\nu$ and $b_n = (x-\nu)T_n^{(\nu)}(x)T_n^{(\nu)}(x+1)$. The general solution of this recurrence is

$$y_n = a_n^* y_0 + \sum_{k=1}^{n}\frac{a_n^*}{a_k^*}b_k$$

where $y_0 = x - \nu$ and $a_n^* = a_n a_{n-1}\cdots a_1 = n!\nu^n$. This implies formula (43). □

We can now proof the following inequalities.

Theorem 5. *If $\nu > 0$ and $x > \nu$, then*

$$T_{n+2}^{(\nu)}(x)T_n^{(\nu)}(x) - T_{n+1}^{(\nu)}(x)^2 > 0\,. \tag{44}$$

Proof. If $\nu > 0$ and $x > \nu$, then (6) implies $T_n^{(\nu)}(x) > 0$, and consequently $T_n^{(\nu)}(x+1) > 0$. Hence, by formula (43), we have inequality (44). □

5. Congruences

In this section, we will obtain some congruences for the polynomials $T_n^{(\nu)}(x)$ and $T_n^{(\nu)\prime}(x)$, which by formula (6) are polynomials with integer coefficients (considering ν as an arbitrary parameter). First, recall that, given two polynomials $p(x), q(x) \in \mathbb{Z}[x]$, we have $p(x) \equiv q(x) \pmod{p\mathbb{Z}_p[x]}$ when the corresponding coefficients of $p(x)$ and $q(x)$ are congruent modulo p.

First of all, we have the following simple result.

Lemma 1. *Let p be a prime. Then $T_p^{(\nu)}(x) \equiv x^p - x + \nu \pmod{p\mathbb{Z}_p[x]}$.*

Proof. If p is prime, by formula (6), we have (working in $\mathbb{Z}_p[x]$)

$$T_p^{(\nu)}(x) = \sum_{k=0}^{p} \binom{p}{k} \nu^{p-k} (x-\nu)_k = \nu^p + (x-\nu)_p \,.$$

Since $(x)_p = -x + x^p$, we have $(x-\nu)_p = -x + \nu + (x-\nu)^p = -x + \nu + x^p - \nu^p$. This implies our congruence. □

More generally, we have the following theorem.

Theorem 6. *Let p be a prime. Then, for every $n \in \mathbb{N}$, we have the congruence*

$$T_{p+n}^{(\nu)}(x) \equiv (x^p - x + \nu) T_n^{(\nu)}(x) \pmod{p\mathbb{Z}_p[x]} \,. \tag{45}$$

Proof. It is well-known that $\binom{p+n}{k} \equiv \binom{n}{k} + \binom{n}{k-p} \pmod{p}$ for p prime and $n \in \mathbb{N}$. Thus, by formula (6), we have (working in $\mathbb{Z}_p[x]$)

$$\begin{aligned}
T_{p+n}^{(\nu)}(x) &= \sum_{k=0}^{p+n} \binom{p+n}{k} \nu^{p+n-k} (x-\nu)_k \\
&= \sum_{k=0}^{p+n} \left(\binom{n}{k} + \binom{n}{k-p} \right) \nu^{p+n-k} (x-\nu)_k \\
&= \sum_{k=0}^{n} \binom{n}{k} \nu^{p+n-k} (x-\nu)_k + \sum_{k=p}^{p+n} \binom{n}{k-p} \nu^{p+n-k} (x-\nu)_k \\
&= \nu^p \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} (x-\nu)_k + \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} (x-\nu)_{p+k} \\
&= \nu^p \, T_n^{(\nu)}(x) + \sum_{k=0}^{n} \binom{n}{k} \nu^{n-k} (x-\nu)_p (x-\nu)_k \\
&= (\nu^p + (x-\nu)_p) \, T_n^{(\nu)}(x) \,.
\end{aligned}$$

As we observed in the proof of Lemma 1, we have $T_p^{(\nu)}(x) = \nu^p + (x-\nu)_p$. Thus, we have $T_{p+n}^{(\nu)}(x) = T_p^{(\nu)}(x) T_n^{(\nu)}(x)$, and by Lemma 1 we have congruence (45). □

Theorem 6 can be generalized as follows.

Theorem 7. *Let p be a prime. Then, for every $m, n, s \in \mathbb{N}$, $s \geq 1$, we have the congruences*

$$T_{mp+n}^{(\nu)}(x) \equiv (x^p - x + \nu)^m T_n^{(\nu)}(x) \pmod{p\mathbb{Z}_p[x]} \tag{46}$$

$$T_{mp^s+n}^{(\nu)}(x) \equiv (x^{p^s} - x^{p^{s-1}} + \nu^{p^{s-1}})^m T_n^{(\nu)}(x) \pmod{p\mathbb{Z}_p[x]} \,. \tag{47}$$

Proof. Congruence (46) can be proved by induction on m, using congruence (45). Then, congruence (47) can be proved by induction on s, using congruence (46). □

Furthermore, we also have the following congruences.

Theorem 8. *Let p be a prime. Then, for every $n \in \mathbb{N}$, we have the congruence*

$$T_{p+n}^{(\nu)\,\prime}(x) \equiv (x^p - x + \nu) T_n^{(\nu)\prime}(x) - T_n^{(\nu)}(x) \pmod{p\mathbb{Z}_p[x]} \,. \tag{48}$$

Proof. By formula (36), we have (always working in $\mathbb{Z}_p[x]$)

$$\begin{aligned} T_{p+n}^{(\nu)\,\prime}(x) &= \sum_{k=1}^{p+n} \binom{p+n}{k} (k-1)!\, T_{p+n-k}^{(\nu)}(x) \\ &= \sum_{k=1}^{p+n} \left(\binom{n}{k} + \binom{n}{k-p} \right) (k-1)!\, T_{p+n-k}^{(\nu)}(x) \\ &= \sum_{k=1}^{n} \binom{n}{k} (k-1)!\, T_{p+n-k}^{(\nu)}(x) + \sum_{k=p}^{p+n} \binom{n}{k-p} (k-1)!\, T_{p+n-k}^{(\nu)}(x) \\ &= (x^p - x + \nu) \sum_{k=1}^{n} \binom{n}{k} (k-1)!\, T_{n-k}^{(\nu)}(x) + \sum_{k=0}^{n} \binom{n}{k} (p+k-1)!\, T_{n-k}^{(\nu)}(x) \\ &= (x^p - x + \nu)\, T_n^{(\nu)\prime}(x) + (p-1)!\, T_n^{(\nu)}(x) \\ &= (x^p - x + \nu)\, T_n^{(\nu)\prime}(x+1) - T_n^{(\nu)}(x+1) \end{aligned}$$

where we have used congruence (6) and Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$. This proves congruence (48). □

Finally, Theorem 8 can be easily extended in the following way.

Theorem 9. *Let p be a prime. Then, for every $m, n, s \in \mathbb{N}$, $m, s \geq 1$, we have the congruences*

$$\begin{aligned} T_{mp+n}^{(\nu)\,\prime}(x) &\equiv (x^p - x + \nu)^m T_n^{(\nu)\prime}(x) - m(x^p - x + \nu)^{m-1} T_n^{(\nu)}(x) \pmod{p\mathbb{Z}_p[x]} \\ T_{mp^s+n}^{(\nu)\,\prime}(x) &\equiv (x^{p^s} - x^{p^{s-1}} + \nu^{p^{s-1}})^m T_n^{(\nu)\prime}(x) \pmod{p\mathbb{Z}_p[x]}\,. \end{aligned}$$

6. A Binomial Identity

Consider the polynomials $T_n^{(\nu+\mu n)}(x+\mu n)$, where μ is an arbitrary parameter. To obtain their exponential generating series $Q(x;t) = \sum_{n\geq 0} T_n^{(\nu+\mu n)}(x+\mu n)\, \frac{t^n}{n!}$, consider the bivariate generating series

$$\sum_{n,k\geq 0} T_n^{(\nu+\mu k)}(x+\mu k)\, \frac{t^n}{n!} u^k = \sum_{k\geq 0} \frac{e^{(\nu+\mu k)t}}{(1-t)^{x-\nu}}\, u^k = \frac{T^{(\nu)}(x;t)}{1 - e^{\mu t} u}\,.$$

Since $Q(x;t)$ is the diagonal of the above bivariate series, then by Cauchy's integral theorem (see [27], p. 42, [28], or [29], p. 182), we have

$$Q(x;t) = \frac{1}{2\pi i} \oint \frac{T^{(\nu)}(x;z)}{1 - e^{\mu z}\, t/z}\, \frac{dz}{z} = \frac{1}{2\pi i} \oint \frac{T^{(\nu)}(x;z)}{z - t\, e^{\mu z}}\, dz\,.$$

There is only one pole (of the first order) tending to 0 as $t \to 0$, given by the unique (invertible) formal series $\psi(t)$, such that

$$\psi(t) = t\, e^{\mu\psi(t)} \qquad \text{or} \qquad \widehat{\psi}(t) = t\, e^{-\mu t}\,. \tag{49}$$

Hence, by the residue theorem, we have

$$Q(x;t) = \lim_{z\to\psi(t)} \frac{(z-\psi(t))\, T^{(\nu)}(x;z)}{z - t\, e^{\mu z}} = \lim_{z\to\psi(t)} \frac{T^{(\nu)}(x;z)}{1 - \mu\, t\, e^{\mu z}} = \frac{T^{(\nu)}(x;\psi(t))}{1 - \mu\, t\, e^{\mu\psi(t)}}\,.$$

From (49), we have

$$\psi'(t) = \frac{e^{\mu\psi(t)}}{1 - \mu\, t\, e^{\mu\psi(t)}} = \frac{\psi(t)/t}{1 - \mu\, t\, e^{\mu\psi(t)}}\,.$$

Therefore, in conclusion, we have

$$\sum_{n\geq 0} T_n^{(\nu+\mu n)}(x+\mu n)\,\frac{t^n}{n!} = \frac{t\psi'(t)}{\psi(t)}\,T^{(\nu)}(x;\psi(t)) = \Psi\,T^{(\nu)}(x;t) \tag{50}$$

where Ψ is the Sheffer matrix

$$\Psi = \left(\frac{t\psi'(t)}{\psi(t)}, \psi(t)\right) = \left[\binom{n}{k}(\mu n)^{n-k}\right]_{n,k\geq 0}.$$

Notice also that the polynomials $T_n^{(\nu+\mu n)}(x+\mu n)$ form a Sheffer sequence, with spectrum

$$\left(\frac{t\psi'(t)}{\psi(t)}, \psi(t)\right)\left((1-t)^{\nu}\mathrm{e}^{\nu t}, \log\frac{1}{1-t}\right) = \left(\frac{t\psi'(t)}{\psi(t)}(1-\psi(t))^{\nu}\mathrm{e}^{\nu\psi(t)}, \log\frac{1}{1-\psi(t)}\right).$$

Moreover, from series (50), we have the binomial identity

$$\sum_{k=0}^{n}\binom{n}{k}(\mu n)^{n-k}\,T_k^{(\nu)}(x) = T_n^{(\nu+\mu n)}(x+\mu n)\,.$$

Remark 3. *Using a similar approach, we can prove that*

$$\sum_{n\geq 0} T_n^{(\nu)}(x-\mu n)\,\frac{t^n}{n!} = \frac{t\varphi'(t)}{\varphi(t)}\,T^{(\nu)}(x;\varphi(t)) = \Phi\,T^{(\nu)}(x;t)$$

where $\varphi(t)$ is the unique formal series such that $\varphi(t) = t(1-\varphi(t))^{\mu}$ and Φ is the Sheffer matrix

$$\Phi = \left(\frac{t\varphi'(t)}{\varphi(t)}, \varphi(t)\right) = \left[(-1)^{n-k}\binom{\mu n}{n-k}\frac{n!}{k!}\right]_{n,k\geq 0} = \left[\binom{n}{k}(\mu n)_{n-k}\right]_{n,k\geq 0}.$$

Also, in this case the polynomials $T_n^{(\nu)}(x-\mu n)$ form a Sheffer sequence, with spectrum

$$\left(\frac{t\varphi'(t)}{\varphi(t)}, \varphi(t)\right)\left((1-t)^{\nu}\mathrm{e}^{\nu t}, \log\frac{1}{1-t}\right) = \left(\frac{t\varphi'(t)}{\varphi(t)}(1-\varphi(t))^{\nu}\mathrm{e}^{\nu\varphi(t)}, \log\frac{1}{1-\varphi(t)}\right).$$

7. A Carlitz-like Identity

In 1971, Carlitz obtained [30] the following two-parameter binomial identity

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}B_{m+k} = (-1)^m\sum_{k=0}^{m}\binom{m}{k}B_{n+k}$$

for the Bernoulli numbers B_n. This identity has been generalized in various ways to many other numerical and polynomial sequences. For instance, in [31] there is a general theorem proved by the umbral calculus. Similarly, in [32] there is a generalization to the Appell polynomials using a slightly different umbral approach. Here, we will use a similar approach to find a Carlitz-like identity for the Tricomi continuants.

First, we prove the following simple, but important, identity.

Lemma 2. *The binomial transform of the Tricomi continuants is given by*

$$\sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}T_k^{(\nu)}(x) = T_n^{(\nu+\mu)}(x+\mu)\,. \tag{51}$$

Proof. By following series (3), we have $\mathrm{e}^{\mu t}\,T^{(\nu)}(x;t) = T^{(\nu+\mu)}(x+\mu;t)$. This is equivalent to identity (51). □

Now, we can prove the following theorem.

Theorem 10. *For every $m, n \in \mathbb{N}$, the Tricomi continuants satisfy the Carlitz-like identity*

$$\sum_{k=0}^{n} \binom{n}{k} (2\mu)^{n-k} T_{m+k}^{(\nu-\mu)}(x-\mu) = \sum_{k=0}^{m} \binom{m}{k} (-2\mu)^{m-k} T_{n+k}^{(\nu+\mu)}(x+\mu) . \tag{52}$$

Proof. Consider the umbral map defined by the linear isomorphism $\varphi : \mathbb{R}[x] \to \mathbb{R}[x]$ where $\varphi(x^n) = T_n^{(\nu)}(x)$, for all $n \in \mathbb{N}$. Then, by identity (51) proved in Lemma 2, we have

$$\varphi((x+\mu)^n) = T_n^{(\nu+\mu)}(x+\mu) .$$

Since

$$(x-\mu)^m (x+\mu)^n = (x-\mu)^m (x-\mu+2\mu)^n = \sum_{k=0}^{n} \binom{n}{k} (2\mu)^{n-k} (x-\mu)^{m+k}$$

$$(x-\mu)^m (x+\mu)^n = (x+\mu-2\mu)^m (x+\mu)^n = \sum_{k=0}^{m} \binom{m}{k} (-2\mu)^{m-k} (x+\mu)^{n+k} ,$$

we have the umbral identity

$$\sum_{k=0}^{n} \binom{n}{k} (2\mu)^{n-k} (x-\mu)^{m+k} = \sum_{k=0}^{m} \binom{m}{k} (-2\mu)^{m-k} (x+\mu)^{n+k} .$$

Now, by applying φ to both members of this equation, we obtain identity (52). □

Using the same approach, we can also prove the following further identity.

Theorem 11. *The Tricomi continuants satisfy the identity*

$$\sum_{k=0}^{n} \binom{n}{k} (-\mu^2)^{n-k} T_{2k}^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-2\mu)^{n-k} T_{n+k}^{(\nu+\mu)}(x+\mu) . \tag{53}$$

Proof. Consider again the umbral map φ defined in the proof of Theorem 10. Since

$$(x-\mu)^n (x+\mu)^n = (x^2-\mu^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-\mu^2)^{n-k} x^{2k}$$

$$(x-\mu)^n (x+\mu)^n = (x+\mu-2\mu)^n (x+\mu)^n = \sum_{k=0}^{n} \binom{n}{k} (-2\mu)^{n-k} (x+\mu)^{n+k} ,$$

we have the umbral identity

$$\sum_{k=0}^{n} \binom{n}{k} (-\mu^2)^{n-k} x^{2k} = \sum_{k=0}^{n} \binom{n}{k} (-2\mu)^{n-k} (x+\mu)^{n+k} . \tag{54}$$

By applying the umbral map φ to both members of this equation, we obtain identity (53). □

Remark 4. *From identity (53), by the binomial inversion theorem (11), we have*

$$T_{2n}^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} \binom{k}{i} (-2)^{k-i} \mu^{2n-k-i} T_{k+i}^{(\nu+\mu)}(x+\mu) .$$

Remark 5. *For $\mu = 1$, identity (54) becomes*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{2k} = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} (x+1)^{n+k} . \tag{55}$$

This identity can be used to obtain several other identities of the same kind. For instance, if we consider the umbral map $\varphi : \mathbb{R}[x] \to \mathbb{R}$ defined by $\varphi(x^n) = b_n$, where the b_n are the Bell numbers [33], p. 493, then we have $\varphi((x+1)^n) = b_{n+1}$. Thus, by applying φ to both members of (55), we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_{2k} = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} b_{n+k+1} .$$

Similarly, if we consider the harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$, then we have the identities

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} = H_n \qquad \textit{and} \qquad \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} H_k = H_n^{(2)} \qquad (n \geq 1) .$$

Hence, by defining the umbral maps $\varphi_i : \mathbb{R}[x] \to \mathbb{R}$ by setting $\varphi_1(1) = 0$ and $\varphi_1(x^n) = \frac{(-1)^{n-1}}{n}$, for $n \geq 1$, and $\varphi_2(1) = 0$ and $\varphi_2(x^n) = \frac{(-1)^{n-1}}{n} H_n$, for $n \geq 1$, then we have $\varphi_1((x+1)^n) = H_n$ and $\varphi_2((x+1)^n) = H_n^{(2)}$. Thus, by applying φ_1 and φ_2 to (55), we have the identities

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{n-k-1}}{2k} = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} H_{n+k}$$
$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{n-k-1}}{2k} H_{2k} = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} H_{n+k}^{(2)} .$$

Remark 6. *The umbral method we just used to prove the previous identities has an extremely broad scope. Starting from an umbral identity (i.e., a polynomial identity), it is possible to obtain several other identities by applying a suitable umbral map [32,34]. Consider, for instance, Simons identity [32,35]*

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x^k = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} (-1)^{n-k} (x+1)^k .$$

Using once again the umbral map φ defined in the proof of theorem 10, we obtain the following Simons-like identity for the Tricomi continuants

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} T_k^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} (-1)^{n-k} T_k^{(\nu+1)}(x+1) .$$

8. Cayley Continuants

The Cayley continuants are defined by

$$U_n^{(\nu)}(x) = \begin{vmatrix} x & 1 & & & & & \\ \nu & x & 2 & & & & \\ & \nu-1 & x & 3 & & & \\ & & \nu-2 & x & 4 & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & \nu-n+3 & x & n-1 \\ & & & & & \nu-n+2 & x \end{vmatrix}_{n \times n}$$

and were considered by Cayley [36] in relation to the Sylvester continuants (see [6], Volume 2, pp. 425–426, [37])

$$H_n(\lambda) = \begin{vmatrix} \lambda & 1 & & & & & \\ n-1 & \lambda & 2 & & & & \\ & n-2 & \lambda & 3 & & & \\ & & n-3 & \lambda & 4 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 2 & \lambda & n-1 \\ & & & & & 1 & \lambda \end{vmatrix}_{n\times n}$$

with the intention of proving the following surprising identity

$$U_n^{(\nu)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}\binom{n-k}{n-2k}\left(-\frac{1}{2}\right)^k k!\, p_k(\nu-n)\, H_{n-2k}(x)$$

where $p_k(x) = (x+1)(x+3)\cdots(x+2k-1)$.

Also, the Cayley continuants form a Sheffer (and a cross-) sequence [38,39], having exponential generating series

$$U^{(\nu)}(x;t) = \sum_{n\geq 0} U_n^{(\nu)}(x)\,\frac{t^n}{n!} = \frac{(1+t)^{\frac{x+\nu}{2}}}{(1-t)^{\frac{x-\nu}{2}}}\,. \tag{56}$$

Moreover, they can be expressed as

$$U_n^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x+\nu}{2}\right)^{\underline{k}}\left(\frac{x-\nu}{2}\right)_{n-k}$$

where the polynomials $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ are the falling factorials.

Cayley and Tricomi continuants are related to each other by the following formulas

$$U_n^{(\nu)}(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\, T_k^{(\nu/2)}(-x/2)\, T_{n-k}^{(\nu/2)}(x/2) \tag{57}$$

$$T_n^{(\nu)}(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\, T_k^{(-\nu)}(x)\, U_{n-k}^{(2\nu)}(2x)\,. \tag{58}$$

Indeed, from series (3) and (56), we have

$$T^{(\nu/2)}(-x/2;-t)T^{(\nu/2)}(x/2;t) = U^{(\nu)}(x;t)$$
$$T^{(-\nu)}(x;-t)U^{(2\nu)}(2x;t) = T^{(\nu)}(x;t)$$

and these identities are equivalent to identities (57) and (58), respectively.

By identity (57), the Cayley continuants are expressed in terms of the Tricomi continuants. Now, to express the Tricomi continuants in terms of the Cayley continuants, consider the Sheffer matrix

$$\left((1-t/2)^{\nu}, \frac{t/2}{1-t/2}\right) = \left[\binom{\nu-k}{n-k}\frac{n!}{k!}\frac{(-1)^{n-k}}{2^n}\right]_{n,k\geq 0}.$$

Then, we have

$$\left((1-t/2)^{\nu}, \frac{t/2}{1-t/2}\right)U^{(\nu)}(x;t) = \frac{1}{(1-t)^{\frac{x-\nu}{2}}}$$

or, equivalently,

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{\nu-k}{n-k}\frac{n!}{k!}\,(-1)^{n-k}\,U_k^{(\nu)}(x)=\left(\frac{x-\nu}{2}\right)_n .$$

Thus, replacing x and ν by $2x$ and 2ν, we have

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{2\nu-k}{n-k}\frac{n!}{k!}\,(-1)^{n-k}\,U_k^{(2\nu)}(2x)=(x-\nu)_n .$$

Therefore, by formula (6), we have

$$T_n^{(\nu)}(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{\nu^{n-k}}{2^k}\sum_{i=0}^{k}\binom{2\nu-i}{k-i}\frac{k!}{i!}\,(-1)^{k-i}\,U_i^{(2\nu)}(2x)$$

or, equivalently,

$$T_n^{(\nu)}(x)=\sum_{k=0}^{n}\left(\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}\frac{(-1)^{i-k}}{2^i}\,\nu^{n-i}(2\nu-k)^{\underline{i-k}}\right)U_k^{(2\nu)}(2x) .$$

9. Hankel Matrices

In this section, we will consider the infinite Hankel matrices

$$\mathcal{H}^{(\nu;s)}(x)=\left[\,T_{i+j+s}^{(\nu)}(x)\,\right]_{i,j\geq 0}=\begin{pmatrix} T_s^{(\nu)}(x) & T_{s+1}^{(\nu)}(x) & T_{s+2}^{(\nu)}(x) & \cdots \\ T_{s+1}^{(\nu)}(x) & T_{s+2}^{(\nu)}(x) & T_{s+3}^{(\nu)}(x) & \cdots \\ T_{s+2}^{(\nu)}(x) & T_{s+3}^{(\nu)}(x) & T_{s+4}^{(\nu)}(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the partial $n\times n$ Hankel matrices

$$\mathcal{H}_n^{(\nu;s)}(x)=\left[\,T_{i+j+s}^{(\nu)}(x)\,\right]_{i,j=0}^{n-1}=\begin{pmatrix} T_s^{(\nu)}(x) & T_{s+1}^{(\nu)}(x) & \cdots & T_{s+n-1}^{(\nu)}(x) \\ T_{s+1}^{(\nu)}(x) & T_{s+2}^{(\nu)}(x) & \cdots & T_{s+n}^{(\nu)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ T_{s+n-1}^{(\nu)}(x) & T_{s+n}^{(\nu)}(x) & \cdots & T_{s+2n-2}^{(\nu)}(x) \end{pmatrix}$$

generated by the Tricomi continuants. We will prove that the matrix $\mathcal{H}^{(\nu)}(x)=\mathcal{H}^{(\nu;0)}(x)$ admits a Sheffer LDU-factorization, while the matrix $\mathcal{H}^{(\nu;1)}(x)$ admits a Sheffer LTU-factorization. More precisely, an infinite matrix A has a Sheffer LDU-factorization [40] when there exist two Sheffer matrices S_1 and S_2 with main diagonal 1 and a diagonal matrix D such that $A=S_1DS_2^T$. Similarly, an infinite matrix A has a Sheffer LTU-factorization [40] when there exist two Sheffer matrices S_1 and S_2 with main diagonal 1 and a tridiagonal matrix T such that $A=S_1TS_2^T$.

To obtain these factorizations, we will use the exponential generating series of the infinite matrices $\mathcal{H}^{(\nu)}(x)$ and $\mathcal{H}^{(\nu;1)}(x)$. By Taylor's formula, they are given by

$$h(t,u)=\sum_{m,n\geq 0}T_{m+n}^{(\nu)}(x)\,\frac{t^m}{m!}\frac{u^n}{n!}=T^{(\nu)}(x;t+u)$$

$$h'(t,u)=\sum_{m,n\geq 0}T_{m+n+1}^{(\nu)}(x)\,\frac{t^m}{m!}\frac{u^n}{n!}=\frac{\partial}{\partial t}\,T^{(\nu)}(x;t+u) .$$

Moreover, we will use the following lemmas proved in [40].

Lemma 3. *Consider two Sheffer matrices*

$$R = [r_{n,k}]_{n,k\geq 0} = (g_1(t), f_1(t)) \qquad \text{and} \qquad S = [s_{n,k}]_{n,k\geq 0} = (g_2(t), f_2(t))\,,$$

a diagonal matrix $D = [\, k!\, h_k\, \delta_{n,k}\,]_{n,k\geq 0}$, and the exponential generating series $h(t) = \sum_{n\geq 0} h_n \frac{t^n}{n!}$. Then, the exponential generating series of the matrix RDS^T is given by

$$\sum_{i,j\geq 0} \left(\sum_{k=0}^{i\wedge j} r_{i,k} s_{j,k} h_k k! \right) \frac{t^i}{i!} \frac{u^j}{j!} = g_1(t) g_2(u)\, h(f_1(t) f_2(u)).$$

Lemma 4. *Consider two Sheffer matrices*

$$R = [r_{n,k}]_{n,k\geq 0} = (g_1(t), f_1(t)) \qquad \text{and} \qquad S = [s_{n,k}]_{n,k\geq 0} = (g_2(t), f_2(t))\,,$$

and a tridiagonal matrix $T = [\, h!\, t_{h,k}\,]_{h,k\geq 0}$, where

$$t_{h,k} = b_k \delta_{h,k+1} + a_k \delta_{h,k} + k c_k \delta_{h,k-1}\,.$$

Consider the exponential generating series $a(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!}$, $b(t) = \sum_{n\geq 0} b_n \frac{t^n}{n!}$, and $c(t) = \sum_{n\geq 0} c_n \frac{t^n}{n!}$. Then, the exponential generating series of the matrix RTS^T is given by

$$\sum_{i,j\geq 0} \left(\sum_{h,k\geq 0} r_{i,k} s_{j,k} h! t_{h,k} \right) \frac{t^i}{i!} \frac{u^j}{j!} = g_1(t) g_2(u)\, F(t,u)$$

where

$$F(t,u) = a(f_1(t) f_2(u)) + f_1(t) b(f_1(t) f_2(u)) + f_2(u) c'(f_1(t) f_2(u))\,.$$

Now, we can obtain our first factorization.

Theorem 12. *The Hankel matrix $\mathcal{H}^{(\nu)}(x)$ admits the Sheffer LDU-factorization*

$$\mathcal{H}^{(\nu)}(x) = \mathcal{S}^{(\nu)}(x)\, \mathcal{D}^{(\nu)}(x)\, \mathcal{S}^{(\nu)}(x)^T \tag{59}$$

where

$$\mathcal{S}^{(\nu)}(x) = \left(\frac{e^{\nu t}}{(1-t)^{x-\nu}}, \frac{t}{1-t} \right) = \left[\binom{n}{k} T_{n-k}^{(\nu)}(x+k) \right]_{n,k\geq 0} \tag{60}$$

and

$$\mathcal{D}^{(\nu)}(x) = \left[\, k!\, (x-\nu)_k\, \delta_{n,k}\, \right]_{n,k\geq 0}.$$

Moreover, we have the Hankel determinants

$$\det \left[\, T_{i+j}^{(\nu)}(x) \,\right]_{i,j=0}^{n-1} = \prod_{k=0}^{n-1} k!\, (x-\nu)_k\,.$$

Proof. The exponential generating series of the Hankel matrix $\mathcal{H}^{(\nu)}(x)$ can be written as

$$\begin{aligned} \sum_{m,n\geq 0} T_{m+n}^{(\nu)}(x) \frac{t^m}{m!} \frac{u^n}{n!} &= T^{(\nu)}(x; t+u) = \frac{e^{\nu(t+u)}}{(1-t-u)^{x-\nu}} \\ &= \frac{e^{\nu t}}{(1-t)^{x-\nu}} \frac{e^{\nu u}}{(1-u)^{x-\nu}} \frac{1}{\left(1 - \frac{t}{1-t}\frac{u}{1-u}\right)^{x-\nu}} = g(t) g(u)\, h(f(t) f(u)) \end{aligned}$$

where

$$g(t) = \frac{e^{\nu t}}{(1-t)^{x-\nu}} = T^{(\nu)}(x;t)\,, \qquad f(t) = \frac{t}{1-t}\,, \qquad h(t) = \frac{1}{(1-t)^{x-\nu}}\,.$$

By Lemma 3, this implies the stated factorization. Notice that such a factorization is inherited by the partial matrices, namely $\mathcal{H}_n^{(\nu)}(x) = \mathcal{S}_n^{(\nu)}(x)\,\mathcal{D}_n^{(\nu)}(x)\,\mathcal{S}_n^{(\nu)}(x)^T$. This implies the stated determinants. □

From Theorem 12, we can obtain the following identities.

Theorem 13. *We have the identity*

$$T_{m+n}^{(\nu)}(x) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k} k!\,(x-\nu)_k\,T_{m-k}^{(\nu)}(x+k)T_{n-k}^{(\nu)}(x+k) \tag{61}$$

and the linearization formula

$$T_m^{(\nu)}(x)T_n^{(\nu)}(x) = \sum_{i,j=0}^{m,n} \binom{m}{i}\binom{n}{j}\left(\sum_{k=0}^{i\wedge j} \left|{i \atop k}\right|\left|{j \atop k}\right|(-1)^k k!\,(x-\nu)^{\underline{k}}\right) T_{m+n-i-j}^{(\nu)}(x)\,. \tag{62}$$

Proof. By Lemma 3 and Theorem 12, we have that identity (61) is equivalent to factorization (59). Now, we can reverse this identity by observing that in the proof of Theorem 12 we obtained the relation

$$T^{(\nu)}(x;t+u) = \frac{T^{(\nu)}(x;t)\,T^{(\nu)}(x;u)}{\left(1-\frac{t}{1-t}\frac{u}{1-u}\right)^{x-\nu}}$$

or, equivalently,

$$T^{(\nu)}(x;t)\,T^{(\nu)}(x;u) = \left(1-\frac{t}{1-t}\frac{u}{1-u}\right)^{x-\nu} T^{(\nu)}(x;t+u)\,. \tag{63}$$

Since

$$\left(1-\frac{t}{1-t}\frac{u}{1-u}\right)^{x-\nu} = g(t)g(u)\,(1-f(t)f(u))^{x-\nu}$$

with

$$g(t) = 1\,, \qquad f(t) = \frac{t}{1-t}\,, \qquad h(t) = (1-t)^{x-\nu}\,,$$

we can apply Lemma 3 where $R = S = (g(t), f(t)) = \left(1, \frac{t}{1-t}\right)$ is the Lah matrix (12) and $D = \left[\,k!(x-\nu)^{\underline{k}}\,\delta_{n,k}\,\right]_{n,k\geq 0}$. This means that we have the expansion

$$\left(1-\frac{t}{1-t}\frac{u}{1-u}\right)^{x-\nu} = \sum_{i,j\geq 0}\left(\sum_{k=0}^{i\wedge j}\left|{i \atop k}\right|\left|{j \atop k}\right|(-1)^k k!\,(x-\nu)^{\underline{k}}\right)\frac{t^i}{i!}\frac{u^j}{j!}\,.$$

In conclusion, by this identity and identity (63), we obtain the linearization formula (62). □

In the next theorem, we obtain our second factorization.

Theorem 14. *The Hankel matrix $\mathcal{H}^{(\nu;1)}(x)$ admits the Sheffer LTU-factorization*

$$\mathcal{H}^{(\nu;1)}(x) = \mathcal{S}^{(\nu)}(x)\,\mathcal{T}^{(\nu)}(x)\,\mathcal{S}^{(\nu)}(x)^T$$

where $\mathcal{S}^{(\nu)}(x)$ is the Sheffer matrix (60) and $\mathcal{T}^{(\nu)}(x) = \left[\,h!\,t_{h,k}\,\right]_{h,k\geq 0}$, where

$$t_{h,k} = (x-\nu)_{k+1}\,\delta_{h,k+1} + (x+2k)\,(x-\nu)_k\,\delta_{h,k} + k\,(x-\nu)_k\,\delta_{h,k-1}\,.$$

Moreover, we have the Hankel determinants

$$\det\Big[\,T^{(\nu)}_{i+j+1}(x)\,\Big]_{i,j=0}^{n-1} = \left(\prod_{k=0}^{n-1} k!\,(x-\nu)_k\right)\tau^{(\nu)}_n(x)$$

where the $\tau^{(\nu)}_n(x)$ *are the polynomials defined by the exponential generating series*

$$\tau^{(\nu)}(x;t) = \sum_{n\geq 0} \tau^{(\nu)}_n(x)\,\frac{t^n}{n!} = \frac{\mathrm{e}^{\frac{\nu t}{1-t}}}{(1-t)^{x-\nu}}\,. \tag{64}$$

Proof. Since

$$\sum_{n\geq 0} T^{(\nu)}_{n+1}(x)\,\frac{t^m}{m!} = \frac{\partial}{\partial t}\,T^{(\nu)}(x;t) = \frac{x-\nu t}{(1-t)^{x-\nu+1}}\,\mathrm{e}^{\nu t}\,,$$

the exponential generating series of the Hankel matrix $\mathcal{H}^{(\nu;1)}(x)$ can be written as

$$\begin{aligned}\sum_{m,n\geq 0} T^{(\nu)}_{m+n+1}(x)\,\frac{t^m}{m!}\frac{u^n}{n!} &= \frac{\partial}{\partial t}\,T^{(\nu)}(x;t+u) = \frac{x-\nu(t+u)}{(1-t-u)^{x-\nu+1}}\,\mathrm{e}^{\nu(t+u)} \\ &= \frac{\mathrm{e}^{\nu t}}{(1-t)^{x-\nu}}\,\frac{\mathrm{e}^{\nu u}}{(1-u)^{x-\nu}}\,\frac{x-\nu(t+u)}{(1-t)(1-u)}\,\frac{1}{\left(1-\dfrac{t}{1-t}\dfrac{u}{1-u}\right)^{x-\nu+1}}\,.\end{aligned}$$

Since

$$\frac{x-\nu t-\nu u}{(1-t)(1-u)} = x + (x-\nu)\frac{t}{1-t} + (x-\nu)\frac{u}{1-u} + (x-2\nu)\frac{t}{1-t}\frac{u}{1-u}\,,$$

we have

$$\sum_{m,n\geq 0} T^{(\nu)}_{m+n+1}(x)\,\frac{t^m}{m!}\frac{u^n}{n!} = g(t)g(u)\,\big(a(f(t)f(u)) + f(t)b(f(t)f(u)) + f(u)c'(f(t)f(u))\big)$$

where

$$g(t) = \frac{\mathrm{e}^{\nu t}}{(1-t)^{x-\nu}} = T^{(\nu)}(x;t)\,, \qquad f(t) = \frac{t}{1-t}\,,$$

$$a(t) = \frac{x+(x-2\nu)t}{(1-t)^{x-\nu+1}}\,, \qquad b(t) = \frac{x-\nu}{(1-t)^{x-\nu}}\,, \qquad c'(t) = \frac{x-\nu}{(1-t)^{x-\nu}}\,.$$

In particular, we have

$$a_n = (x+2n)\,(x-\nu)_n\,, \qquad b_n = (x-\nu)_{n+1}\,, \qquad c_n = b_{n-1} = (x-\nu)_n\,.$$

Therefore, by Lemma 4, we have the stated factorization. Again, such a factorization is inherited by the partial matrices, namely $\mathcal{H}^{(\nu;1)}_n(x) = \mathcal{S}^{(\nu)}_n(x)\,\mathcal{T}^{(\nu)}_n(x)\,\mathcal{S}^{(\nu)}_n(x)^T$. Then, we have

$$\det\mathcal{H}^{(\nu;1)}_n(x) = \det\mathcal{T}^{(\nu)}_n(x)$$

Notice that $\mathcal{T}^{(\nu)}(x) = \big[\,h!\,(x-\nu)_k\,t'_{h,k}\,\big]_{h,k\geq 0}$, where

$$t'_{h,k} = (x-\nu+k)\,\delta_{h,k+1} + (x+2k)\,\delta_{h,k} + k\,\delta_{h,k-1}\,.$$

Then, we have

$$\det\mathcal{T}^{(\nu)}_n(x) = \left(\prod_{k=0}^{n-1} k!\,(x-\nu)_k\right)\tau^{(\nu)}_n(x)$$

where $\tau^{(\nu)}_n(x) = \det\big[\,t'_{h,k}\,\big]_{h,k=0}^{n-1}$, that is

$$\tau_n^{(\nu)}(x) = \begin{vmatrix} x & 1 & & & & & \\ x-\nu & x+2 & 2 & & & & \\ & x-\nu+1 & x+4 & 3 & & & \\ & & x-\nu+2 & x+6 & 4 & & \\ & & \cdots & \cdots & \cdots & & \\ & & & x-\nu+n-3 & x+2(n-2) & n-1 \\ & & & & x-\nu+n-2 & x+2(n-1) \end{vmatrix}_{n\times n}.$$

Therefore, the polynomials $\tau_n^{(\nu)}(x)$ satisfy the recurrence

$$\tau_{n+2}^{(\nu)}(x) = (x+2n+2)\,\tau_{n+1}^{(\nu)}(x) - (n+1)(x-\nu+n)\,\tau_n^{(\nu)}(x)$$

with the initial values $\tau_0^{(\nu)}(x) = 1$ and $\tau_1^{(\nu)}(x) = x$. Now, by the usual techniques of the formal series, it is straightforward to obtain the generating series (64). This concludes the proof. □

Remark 7. *The polynomials $\tau_n^{(\nu)}(x)$ are themselves continuants, and by series (64), they also form a Sheffer (and a cross-) sequence with spectrum*

$$\tau^{(\nu)} = \left((1-t)^{\nu} e^{\frac{\nu t}{1-t}}, \log\frac{1}{1-t}\right). \tag{65}$$

Also, these polynomials can be expressed in terms of the Laguerre polynomials. Indeed, from series (3) and (4), we have

$$\tau_n^{(\nu)}(x) = L_n^{(x-\nu-1)}(-\nu)\,.$$

For this reason, we call these polynomials Tricomi continuants of the second kind. In Section 10, we will give some representations of these continuants.

Theorems 12 and 14 can be generalized as follows.

Theorem 15. *The Hankel matrix*

$$\mathcal{H}^{(\nu)}(\lambda;x) = \left[\,\lambda T_{i+j}^{(\nu)}(x) + T_{i+j+1}^{(\nu)}(x)\,\right]_{i,j\geq 0}$$

admits the Sheffer LTU-factorization

$$\mathcal{H}^{(\nu)}(\lambda;x) = \mathcal{S}^{(\nu)}(x)\,\mathcal{T}^{(\nu+\lambda)}(x+\lambda)\,\mathcal{S}^{(\nu)}(x)^T$$

where $\mathcal{S}^{(\nu)}(x)$ is the Sheffer matrix (60) and $\mathcal{T}^{(\nu)}(x)$ is the tridiagonal matrix defined in Theorem 14. Moreover, we have the Hankel determinants

$$\det\left[\,\lambda T_{i+j}^{(\nu)}(x) + T_{i+j+1}^{(\nu)}(x)\,\right]_{i,j=0}^{n-1} = \left(\prod_{k=0}^{n-1} k!\,(x-\nu)_k\right)\tau_n^{(\nu+\lambda)}(x+\lambda)\,.$$

Proof. By Theorems 12 and 14, we have

$$\mathcal{H}^{(\nu)}(\lambda;x) = \lambda\mathcal{H}^{(\nu)}(x) + \mathcal{H}^{(\nu;1)}(\lambda;x) = \mathcal{S}^{(\nu)}(x)\,(\lambda\mathcal{D}^{(\nu)}(x) + \mathcal{T}^{(\nu)}(x))\,\mathcal{S}^{(\nu)}(x)^T\,.$$

Now, it is straightforward to prove that

$$\lambda\mathcal{D}^{(\nu)}(x) + \mathcal{T}^{(\nu)}(x) = \mathcal{T}^{(\nu+\lambda)}(x+\lambda)\,.$$

This implies the stated factorization. Then, by Theorem 14, we have the stated determinant. □

By Theorems 12 and 14, we can also compute the following Hankel determinants.

Theorem 16. *The Hankel determinants of order* 2 *are given by*

$$\det\left[\,T^{(\nu)}_{i+j+2}(x)\,\right]_{i,j=0}^{n-1} = \left(\prod_{k=0}^{n} k!\,(x-\nu)_k\right)\sum_{k=0}^{n}\frac{\tau^{(\nu)}_k(x)^2}{k!\,(x-\nu)_k}\,.$$

Proof. Let $h_n = \det\left[\,T^{(\nu)}_{i+j}(x)\,\right]_{i,j=0}^{n-1}$, $h'_n = \det\left[\,T^{(\nu)}_{i+j+1}(x)\,\right]_{i,j=0}^{n-1}$ and $h''_n = \det\left[\,T^{(\nu)}_{i+j+2}(x)\,\right]_{i,j=0}^{n-1}$. Then, by Dodgson's formula [41] (or Jacobi identity [42], p. 303), we have the following linear recurrence of the first order

$$h_{n+1}h''_{n+1} = h_{n+2}h''_n + \left(h'_{n+1}\right)^2$$

with the initial value $h''_0 = 1$. By Theorems 12 and 14, we have $h'_n = h_n\tau^{(\nu)}_n(x)$ and

$$h''_{n+1} = (n+1)(x-\nu)_{n+1}h''_n + h_{n+1}\tau^{(\nu)}_{n+1}(x)^2\,.$$

Now, it is straightforward to show that the solution of this recurrence is given by the stated formula. □

10. Representations

In this final section, we will find some identities for the Tricomi continuants $T^{(\nu)}_n(x)$ and $\tau^{(\nu)}_n(x)$ involving the generalized Stirling numbers and, in particular, we will show that such continuants can be expressed in terms of each other.

Consider the generalized Stirling numbers defined by the Sheffer matrices (9) and the exponential polynomials $S_n(x)$ defined by the exponential generating series

$$\sum_{n\geq 0} S_n(x)\frac{t^n}{n!} = \mathrm{e}^{x(\mathrm{e}^t-1)}\,.$$

Since

$$(\mathrm{e}^{-\mu t}, 1-\mathrm{e}^{-t})\,T^{(\nu)}(x;t) = \mathrm{e}^{(x-\nu-\mu)t}\mathrm{e}^{-\nu(\mathrm{e}^{-t}-1)}$$

we have the identity

$$\sum_{k=0}^{n}(-1)^{n-k}S^{(\mu)}_{n,k}\,T^{(\nu)}_k(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}S_{n-k}(-\nu)\,(x-\nu-\mu)^k\,.$$

Then, by the Stirling inversion theorem (10), we have the representation

$$T^{(\nu)}_n(x) = \sum_{k=0}^{n} s^{(\mu)}_{n,k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_{k-i}(-\nu)\,(x-\nu-\mu)^i\,.$$

In particular, for $\mu = 0, 1$, we have the ordinary Stirling numbers of the first kind [33], namely $s^{(0)}_{n,k} = \left[{n \atop k}\right]$ and $s^{(1)}_{n,k} = \left[{n+1 \atop k+1}\right]$. Therefore,

$$T^{(\nu)}_n(x) = \sum_{k=0}^{n}\left[{n \atop k}\right]\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_{k-i}(-\nu)\,(x-\nu)^i$$

$$T^{(\nu)}_n(x) = \sum_{k=0}^{n}\left[{n+1 \atop k+1}\right]\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}S_{k-i}(-\nu)\,(x-\nu-1)^i\,.$$

Similarly, we have the identity

$$(\mathrm{e}^{-\mu t}, 1-\mathrm{e}^{-t})\,\tau^{(\nu)}(x;t) = \mathrm{e}^{(x-\nu-\mu)t}\mathrm{e}^{\nu(\mathrm{e}^{t}-1)}\,,$$

and consequently

$$\sum_{k=0}^{n}(-1)^{n-k}S_{n,k}^{(\mu)}\,\tau_k^{(\nu)}(x)=\sum_{k=0}^{n}\binom{n}{k}S_{n-k}(\nu)\,(x-\nu-\mu)^k\,.$$

Again, by the Stirling inversion theorem (10), we have the representation

$$\tau_n^{(\nu)}(x)=\sum_{k=0}^{n}s_{n,k}^{(\mu)}\sum_{i=0}^{k}\binom{k}{i}S_{k-i}(\nu)\,(x-\nu-\mu)^i$$

and, in particular,

$$\tau_n^{(\nu)}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\sum_{i=0}^{k}\binom{k}{i}S_{k-i}(\nu)\,(x-\nu)^i$$

$$\tau_n^{(\nu)}(x)=\sum_{k=0}^{n}\begin{bmatrix}n+1\\k+1\end{bmatrix}\sum_{i=0}^{k}\binom{k}{i}S_{k-i}(\nu)\,(x-\nu-1)^i\,.$$

Consider now the Lah matrix (12). Since

$$\left(1,\frac{t}{1+t}\right)T^{(\nu)}(x;t)=\tau^{(-\nu)}(-x;-t)\,,$$

we have the relation

$$\sum_{k=0}^{n}\left|\begin{matrix}n\\k\end{matrix}\right|(-1)^{n-k}T_k^{(\nu)}(x)=(-1)^n\tau_n^{(-\nu)}(-x)\,,$$

or

$$\tau_n^{(\nu)}(x)=\sum_{k=0}^{n}\left|\begin{matrix}n\\k\end{matrix}\right|(-1)^kT_k^{(-\nu)}(-x)\,.$$

Finally, by the Lah inversion theorem (13), we also have the inverse representation

$$T_n^{(\nu)}(x)=\sum_{k=0}^{n}\left|\begin{matrix}n\\k\end{matrix}\right|(-1)^k\tau_k^{(-\nu)}(-x)\,.$$

These last relations are equivalent to the following matrix identities

$$\overline{L}\,T^{(\nu)}=\overline{\tau^{(-\nu)}}\,,\qquad \tau^{(\nu)}=L\,\overline{T^{(-\nu)}}\qquad\text{and}\qquad T^{(\nu)}=L\,\overline{\tau^{(-\nu)}}$$

where $T^{(\nu)}$ and $\tau^{(\nu)}$ are the Sheffer matrices (7) and (65).

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