Absence of mobility edges in mosaic Wannier-Stark lattices

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Mobility edges, separating localized from extended states, are known to arise in the single-particle energy spectrum of certain one-dimensional models with quasiperiodic disorder. Recently, some works claimed rather unexpectedly that mobility edges can exist even in disorder-free one-dimensional models, suggesting as an example the so-called mosaic Wannier-Stark lattice, where a Stark potential is applied on every M sites of the lattice. Here we present an exact spectral analysis of the mosaic Wannier-Stark Hamiltonian and prove that strictly speaking there are not mobility edges, separating extended and localized states. Specifically, we prove that the energy spectrum is almost pure point, with all the wave functions displaying a higher than exponential localization, with the exception of $(M-1)$ isolated extended states at energies around which a countably infinite number of localized states, with a diverging localization size, condense.

I. INTRODUCTION

Anderson localization[1–4], i.e. the inhibition of wave diffusion in disordered media via destructive interference of multiply scattered waves, is a universal phenomenon observed in a variety of classical and quantum systems, with experimental demonstrations reported in different areas of physics ranging from photonics[5–7], acoustics[8], matter waves[9–13] and quantum matter[14], to mention a few. The kind of disorder and the spatial dimension of the system are pivotal to Anderson localization, since they strongly affect the appearance of localization transitions and the existence of mobility edges[2–4,15–17], i.e. critical energies separating extended and localized states in the spectrum. In low-dimensional systems with uncorrelated disorder, localization transitions and mobility edges are prevented[4,15]. Conversely, quasi-periodic systems, i.e. quasicrystals, can show localization transitions and mobility edges even in one-spatial dimension (see e.g.[18–32] and references therein). Quasiperiodic models displaying mobility edges include special incommensurate potentials displaying a generalized Aubry-André self-duality[20–22], slowly varying incommensurate potentials[18,23–28], flat-band lattices[29], and quasiperiodic mosaic lattices[30], to mention a few. A different form of mobility edges, separating localized and critical (rather than extended) wave functions, has been also predicted and experimentally observed in certain quasiperiodic potentials[33–40].

Recently, some works suggested that mobility edges can exist in models without disorder nor incommensurate potentials, i.e. in disorder-free systems[41–43]. Specifically, they considered the so-called mosaic Wannier-Stark lattice, i.e. a lattice in which a Stark (linear gradient) potential is applied at every M sites in the lattice. For $M = 1$ the Hamiltonian shows a pure point spectrum, the Wannier-Stark ladder energy spectrum, with localized wave functions and a corresponding periodic dynamics in the time domain (the famous Bloch oscillations)[44–47]. However, when $M \geq 2$ mobility edges are claimed to arise, with the coexistence of extended states and Wannier-Stark localized states in the energy spectrum.

In this work we present an exact analytical solution to the mosaic Wannier-Stark Hamiltonian and show that the energy spectrum is almost pure point and thus, strictly speaking, there are not mobility edges. Specifically, we show that all eigenfunctions are localized with a higher than exponential localization, with the exception of $(M-1)$ isolated extended states. The localized eigenfunctions can be classified into two sets: high-energy wave functions, with energies outside the lattice band, and low-energy wave functions, with energies inside the lattice band. The eigenenergies of the low-energy wave functions condensate toward the energies of the isolated extended states, while the energies of the high-energy wave functions are unbounded and approximate the usual Wannier-Stark ladder in the high-energy limit. While the high-energy wave functions are tightly localized in very few sites of the lattice, the low-energy wave functions can extend over a large size w of the lattice, however asymptotically they decay faster than any exponential and thus they are normalizable wave functions belonging to the point spectrum of the Hamiltonian. However, as the energy E of the localized wave function approaches one of the $(M-1)$ isolated energies of thee extended states, the localization size w diverges.

II. MOSAIC WANNIER-STARK HAMILTONIAN: MODEL AND ENERGY SPECTRUM

The spectral properties of the disorder-free mosaic Wannier-Stark lattice[41–43] are defined by the eigenvalue equation

$$E\psi_n = \kappa(\psi_{n+1} + \psi_{n-1}) + V_n\psi_n \equiv \mathcal{H}\psi_n, \qquad (1)$$

where κ is the hopping amplitude between adjacent sites in the lattice and V_n is the Stark potential, applied at

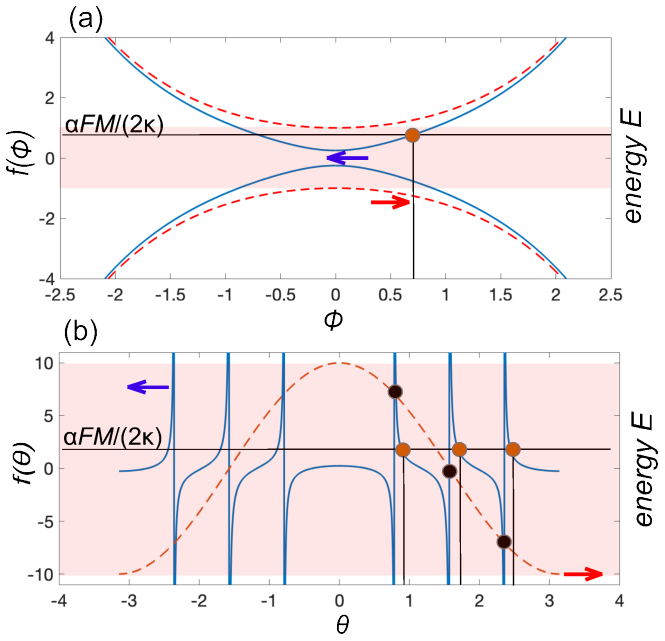

FIG. 1: Geometric solution to Eqs.(7) and (9) for the calculation of the energy spectra $E_\alpha^{(I)}$ and $E_\alpha^{(II)}$. (a) Behavior of the function $f(\phi) = \pm \sinh\phi \cosh(M\phi)/\sinh(M\phi)$ (solid blue curves) for $M = 4$ and energies $E = \pm 2\kappa\cosh\phi$ (dashed red curves). The horizontal solid line yields the quantized value $\alpha F/(2\kappa)$, with α integer. The bold circle corresponds to the simple positive root ϕ to Eq.(7). (b) Same as (a), but with $f(\theta) = \sin\theta\cos(M\theta)/\sin(M\theta)$. The three bold red circles correspond to the $(M-1) = 3$ positive roots θ to Eq.(9) in the range $(0,\pi)$, which are denoted by the additional index $\rho = 1, 2, ..., M-1$. The solid black circles in (b), at which $f(\theta)$ is singular, correspond to the $(M-1)$ energies E_σ of extended states. The shaded light areas in (a) and (b) denote the energy interval $(-2\kappa, 2\kappa)$ of the tight-binding lattice in the absence of the potential. Note that in (a) any energy belonging to $E_\alpha^{(I)}$ falls outside the shaded area, whereas in (b) any energy belonging to $E_\alpha^{(II)}$ falls inside the shaded area.

every M sites of the lattice, i.e.

$$V_n = \begin{cases} Fn & n = 0, \pm M, \pm 2M, \pm 3M, ... \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and F is the force.

For $M = 1$ one recovers the famous Wannier-Stark problem of a quantum particle hopping on a one-dimensional lattice subjected to a dc force F. This problem is exactly solvable and the Hamiltonian $\mathcal{H}$ displays a pure point energy spectrum with equally-spaced energies, $E_\alpha = \alpha F$ ($\alpha = 0, \pm 1, \pm 2, \pm 3, ...$) forming the Wannier-Stark ladder and yielding in the time domain a periodic motion of the wave packet (Bloch oscillations[44–47]). The corresponding wave functions are the well-known Wannier-Stark states, which are given in terms of Bessel functions of first kind,

$$\psi_n^{(\alpha)} = (-1)^{n-\alpha} J_{n-\alpha}(2\kappa/F) \tag{3}$$

and show a higher than exponential localization, i.e. $\lim_{n\to\pm\infty} \psi_n^{(\alpha)} \exp(R|n|) = 0$ for any $R \geq 0$.

Here we provide exact analytical results on the energy spectrum and corresponding eigenfunctions for the mosaic Wannier-Stark Hamiltonian when $M \geq 2$. According to the Simon-Spencer theorem[48], since the potential V_n is unbounded at infinity the absolutely continuous part of the energy spectrum of $\mathcal{H}$ is empty, i.e. the energy spectrum comprises pure point and/or singular continuous parts. The Simon-Spencer theorem basically excludes the existence of bands of extended states for the mosaic Wannier-Stark Hamiltonian, and the corresponding eigenfunctions therefore should be either normalizable (localized) states or critical states. However, the Simon-Spencer theorem does not exclude the existence of isolated extended states, with zero spectral measure of the corresponding energies. This means that, if mobility edges would exist, they should separate localized and critical states. The main result of the present work is that the energy spectrum is almost pure point and there are not mobility edges. The results are summarized by the following *theorem*:

1) The energy spectrum of $\mathcal{H}$ on the infinitely-extended lattice and for $M \geq 2$ is pure point, with corresponding eigenfunctions displaying a higher than exponential localization, with the exception of $(M-1)$ isolated energies, given by

$$E_\sigma = 2\kappa\cos\left(\frac{\pi\sigma}{M}\right) \tag{4}$$

($\sigma = 1, 2, .., M-1$), at which the corresponding eigenfunctions are the following non-normalizable (improper) extended states

$$\psi_n^{(\sigma)} = \sin(n\pi\sigma/M). \tag{5}$$

2) The eigenenergies of the localized wave functions can be grouped into two sets: the high-energy wave functions (set I) and the low-energy wave functions (set II), with corresponding eigenenergies E which fall outside and inside the range $(-2\kappa, 2\kappa)$, respectively. Note that this range defines the energy band of the tight-binding lattice in the absence of the Stark potential.

3) The spectrum $E_\alpha^{(I)}$ of the high-energy wave functions is given by

$$E_\alpha^{(I)} = \pm 2\kappa\cosh\phi_\alpha \tag{6}$$

where $\phi_\alpha > 0$ is the root of the trascendental equation [Fig.1(a)]

$$\frac{\sinh\phi_\alpha\cosh(\phi_\alpha M)}{\sinh(M\phi_\alpha)} = \pm\frac{\alpha F M}{2\kappa} \tag{7}$$

and $\alpha = 0, \pm 1, \pm 2, \pm 3, ...$ is an arbitrary integer.

4) The spectrum $E_{\alpha,\rho}^{(II)}$ of the low-energy wave functions is given by

$$E_{\alpha,\rho}^{(II)} = 2\kappa\cos\theta_{\alpha,\rho} \tag{8}$$

where $0 \leq \theta_{\alpha,\rho} \leq \pi$ are the $(M-1)$ roots of the trascendental equation [Fig.1(b)]

$$\frac{\sin\theta_{\alpha,\rho}\cos(\theta_{\alpha,\rho}M)}{\sin(M\theta_{\alpha,\rho})} = \frac{\alpha FM}{2\kappa}, \tag{9}$$

$\alpha = 0, \pm1, \pm2, \pm3, ...$ is an arbitrary integer and $\rho = 1, 2, .., M-1$ labels the root number of Eq.(9) for a given value of α [see Fig.1(b)].
5) At the lattice sites $nM = 0, \pm M, \pm 2M, \pm 3M, ...$, i.e. where the potential is non-vanishing, the localized wave functions, for both low-energy and high-energy branches, are given in terms of Bessel functions of first kind, namely

$$\psi^{(\alpha)}_{nM} = (-1)^{n-\alpha} J_{n-\alpha}(\Gamma) \tag{10}$$

where we have set

$$\Gamma = \Gamma_\alpha = \frac{2\kappa}{FM}\frac{\sinh\phi_\alpha}{\sinh(M\phi_\alpha)} \tag{11}$$

for the high-energy wave functions, and

$$\Gamma = \Gamma_{\alpha,\rho} = \frac{2\kappa}{FM}\frac{\sin\theta_{\alpha,\rho}}{\sin(M\theta_{\alpha,\rho})} \tag{12}$$

for the low-energy wave functions ($\alpha = 0, \pm1, \pm2, ...$, $\rho = 1, 2, ..., M-1$). The wave functions at the sites $(nM+1)$, i.e. where the potential does not vanish, are given by

$$\psi^{(\alpha)}_{nM+1} = (-1)^{n-\alpha}\frac{\sin[(M-1)\omega]J_{n-\alpha}(\Gamma) - \sin\omega J_{n+1-\alpha}(\Gamma)}{\sin(M\omega)} \tag{13}$$

with $\omega = i\phi_\alpha, i\phi_\alpha + \pi$ for the high-energy wave functions and $\omega = \theta_{\alpha,\rho}$ for the low-energy wave functions.

Comments.
(i) The infinitely countable set of energies $E^{(II)}_{\alpha,\rho}$ of the low-energy wave functions are embedded in the interval $(-2\kappa, 2\kappa)$ and condense toward the isolated points E_σ of extended states [Eq.(4)] as $\alpha \to \pm\infty$, bacause $\sin(M\theta_{\alpha,\rho}) \sim 0$. Therefore, set of energies $E^{(II)}_{\alpha,\rho}$ form $(M-1)$ narrow 'bands' centered at the around the energies E_σ.
(ii) As $E^{(II)}_{\alpha,\rho}$ approaches the isolated points E_σ of extended states, from Eqs.(10) and (13) one has $|\psi^{(\alpha)}_{Mn}/\psi^{(\alpha)}_{nM+1}| \ll 1$, i.e. the wave functions with energies close to the isolated points of extended states have negligible excitation at the lattices sites $0, \pm M, \pm 2M, \pm 3M, ...$ where the potential is nonvanishing.
(iii) Owing to the properties of Bessel functions, the spatial size w of the wave function can be estimated from the relation $w \sim 2\Gamma M$, and thus from Eq.(12) it follows that w diverges for the low-energy wave functions as $\alpha \to \pm\infty$, i.e. when the eigenergy $E^{(II)}_{\alpha,\rho}$ approaches one of the points E_σ of the energy spectrum.
(iv) The energies $E^{(I)}_\alpha$ of the high-energy wave functions fall outside the range $(-2\kappa, 2\kappa)$, and in the limit $\alpha \to \pm\infty$ one has $E^{(I)}_\alpha \simeq \alpha FM$ and the corresponding wave function is tightly confined at the lattice site $n = \alpha$. This means that the set of energies $E^{(I)}_\alpha$ forms an almost equally-spaced Wannier-Stark ladder.
(v) A corollary of the theorem is that the mosaic Wannier-Stark Hamiltonian does not show strictly speaking any mobility edge, though the low-energy localized eigenstates become more and more extended in space as their energy approaches the accumulation points E_σ defined by Eq.(4).

Proof. To prove the main theorem, let us write Eq.(1) in the dynamical system form

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} \frac{E-V_n}{\kappa} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \tag{14}$$

from which by iteration one obtains

$$\begin{pmatrix} \psi_{nM+1} \\ \psi_{nM} \end{pmatrix} = \begin{pmatrix} \frac{E-nMF}{\kappa} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{E}{\kappa} & -1 \\ 1 & 0 \end{pmatrix}^{M-1} \times \begin{pmatrix} \psi_{(n-1)M+1} \\ \psi_{(n-1)M} \end{pmatrix}. \tag{15}$$

Unlike previous works[41,43], we do not use Lyapunov exponent analysis and Avila's global theory to determine the localization properties of the wave functions. In fact, while Lyapunov exponent analysis can be safely applied to quasiperiodic or disordered models displaying Anderson-like localization, where the existence of Lyapunov exponent $L(E) \geq 0$ can be proven for any eigenstate (either extended, critical or localized with an exponential localization), it cannot be applied to wave functions with a higher than exponential localization, which is the case of the Wannier-Stark localization. Rather, we provide exact analytical solution to the dynamical system Eq.(15).
For the properties of 2×2 unimodular matrices, one can write

$$\begin{pmatrix} \frac{E}{\kappa} & -1 \\ 1 & 0 \end{pmatrix}^{M-1} = \begin{pmatrix} \frac{\sin(M\omega)}{\sin\omega} & -\frac{\sin[(M-1)\omega)]}{\sin\omega} \\ \frac{\sin[(M-1)\omega)]}{\sin\omega} & -\frac{\sin[(M-2)\omega)]}{\sin\omega} \end{pmatrix} \tag{16}$$

where the complex angle ω is defined by the relation

$$\cos\omega = \frac{E}{2\kappa}. \tag{17}$$

After letting

$$\varphi_n \equiv \psi_{nM}\ , \quad \xi_n \equiv \psi_{nM+1} \tag{18}$$

from Eqs.(15) and (16) one obtains

$$\xi_n = S_{11}\xi_{n-1} + S_{12}\varphi_{n-1} \tag{19}$$
$$\varphi_n = S_{21}\xi_{n-1} + S_{22}\varphi_{n-1} \tag{20}$$

where we have set

$$\begin{aligned} S_{11} &= \frac{\sin(M\omega)}{\sin\omega}\left(\frac{E}{\kappa}-\frac{FM}{\kappa}n\right)-\frac{\sin[(M-1)\omega]}{\sin\omega} \\ S_{12} &= -\frac{\sin[(M-1)\omega]}{\sin\omega}\left(\frac{E}{\kappa}-\frac{FM}{\kappa}n\right)+\frac{\sin[(M-2)\omega]}{\sin\omega} \\ S_{21} &= \frac{\sin(M\omega)}{\sin\omega} \\ S_{22} &= -\frac{\sin[(M-1)\omega]}{\sin\omega}. \end{aligned} \tag{21}$$

Taking into account that $\det S = S_{11}S_{22} - S_{12}S_{21} = 1$, eliminating form Eqs.(19) and (20) the variables ξ_n one obtains the following second-order difference equation for the amplitudes φ_n

$$\varphi_{n+1} + \varphi_{n-1} = (A - Bn)\varphi_n \tag{22}$$

where we have set

$$A \equiv 2\cos(M\omega) \tag{23}$$

$$B \equiv \frac{FM}{\kappa}\frac{\sin(M\omega)}{\sin\omega}. \tag{24}$$

The spectral problem on the infinite lattice defined by Eq.(22) is the usual Wannier-Stark problem on a tight-binding lattice but with energy-dependent dc force, which is solved in terms of J_n Bessel functions of first kind. Here we give a direct solution to the spectral problem exploiting a recursive identity of Bessel functions J_n; a different approach based on a spectral method could be also used.

Let us first assume $B \neq 0$, i.e. $\sin(M\omega)/\sin\omega \neq 0$. Using the recursive relation of $J_n(x)$ Bessel functions

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x) \tag{25}$$

the set of solutions to Eq.(22), which do diverge as $n \to \pm\infty$, is given by

$$\varphi_n^{(\alpha)} = (-1)^{n-\alpha}J_{n-\alpha}(\Gamma) \tag{26}$$

where $\alpha = 0, \pm1, \pm2, \pm3 ...$ is an arbitrary integer number,

$$\Gamma = \frac{2\kappa}{FM}\frac{\sin\omega}{\sin(M\omega)}, \tag{27}$$

and ω is any root of the trascendental equation

$$\sin\omega\frac{\cos(M\omega)}{\sin(M\omega)} = \alpha\frac{FM}{2\kappa}. \tag{28}$$

Since the energy E is real and it is related to the complex angle ω by the relation $E = 2\kappa\cos\omega$ [Eq.(17)], Eq.(28) can be satisfied by letting either $\omega = i\phi$, $\omega = i\phi + \pi$, or $\omega = \theta$, with θ and ϕ real numbers. The first two cases, $\omega = i\phi$ or $\omega = i\phi + \pi$, yield the high-energy wave functions with energies $E_\alpha^{(I)}$ defined by Eqs.(6) and (7), with the + and - signs in the equations corresponding to $\omega = i\phi$ and $\omega = i\phi + \pi$, respectively. On the other hand, the last case $\omega = \theta$ yields the low-energy wave functions with energies $E_\alpha^{(II)}$ defined by Eqs.(8) and (9). In all cases, the corresponding wave functions are given in terms of Bessel functions as in Eqs.(10-12). Finally, using Eq.(20) the wave functions $\psi_{nM+1}^{(\alpha)}$ at sites $nM+1$ are obtained from the relation

$$\psi_{nM+1}^{(\alpha)} = \xi_n = \frac{\varphi_{n+1} - S_{22}\varphi_n}{S_{21}}, \tag{29}$$

which yields Eq.(13). For $M \geq 3$, the wave function amplitudes $\psi_n^{(\alpha)}$ at the other lattice sites $Mn + l$ with $l = 2, 3, ..M-1$, if needed, can be obtained by the recursive relation (14).

Let us finally assume $B = 0$ in Eq.(22), i.e. $\sin(M\omega) = 0$ with $\omega \neq 0, \pi$, which is satisfied by letting $\omega = \sigma\pi/M$ for $\sigma = 1, 2, ..., M-1$. Correspondingly, the eigenenergies are $E_\sigma = 2\kappa\cos(\pi\sigma/M)$. In this case $S_{21} = 0$ while $S_{11}, S_{12} \neq 0$, so that to avoid divergences ss $n \to \pm\infty$ of the solution to Eqs.(19) and (20) one necessarily must have $\varphi_n = 0$, i.e. the wave function ψ_l vanishes identically at the lattice sites $l = nM = 0, \pm M, \pm 2M, \pm 3M, ...$ where the potential V_l is nonvanishing. This means that the wave function ψ_l is also an eigenfunction of the tight-binding lattice without any potential, with eigenenergy E_σ. Such solutions are the well-known Bloch waves given by Eq.(5), which identically vanish at the lattice sites $0, \pm M, \pm 2M, \pm 3M, ...$ This concludes the proof of statements 1)-5) given above. The comments (i-v) are simple corollaries of the main theorem.

III. NUMERICAL RESULTS AND COMMENTS

To illustrate and support the analytical results given in the previous section, we present some numerical results of energy spectra and localization properties of corresponding wave functions. The results are obtained by diagonalization of the matrix Hamiltonian $\mathcal{H}$ assuming a finite lattice of large size L with open boundary conditions. We also comment on the pseudo mobility edges that have been introduced in previous work, where the inverse participation ratio (IPR) was used to discriminate between 'localized' and 'extended' states. For a wave function normalized such that $\sum_{n=1}^{L}|\psi_n^{(\alpha)}|^2 = 1$, the IPR is defined by the relation

$$IPR_\alpha = \sum_{n=1}^{L}\left|\psi_n^{(\alpha)}\right|^4. \tag{30}$$

The IPR of an extended state takes a small value and scales as L^{-1}, hence vanishing in the thermodynamic limit $L \to \infty$, while it remains finite for a localized state. Figure 2(a) shows the numerically-computed energy spectrum for $M = 4$ and for $F/\kappa = 0.5$ in a lattice of size $L = 1000$. The corresponding IPR of the

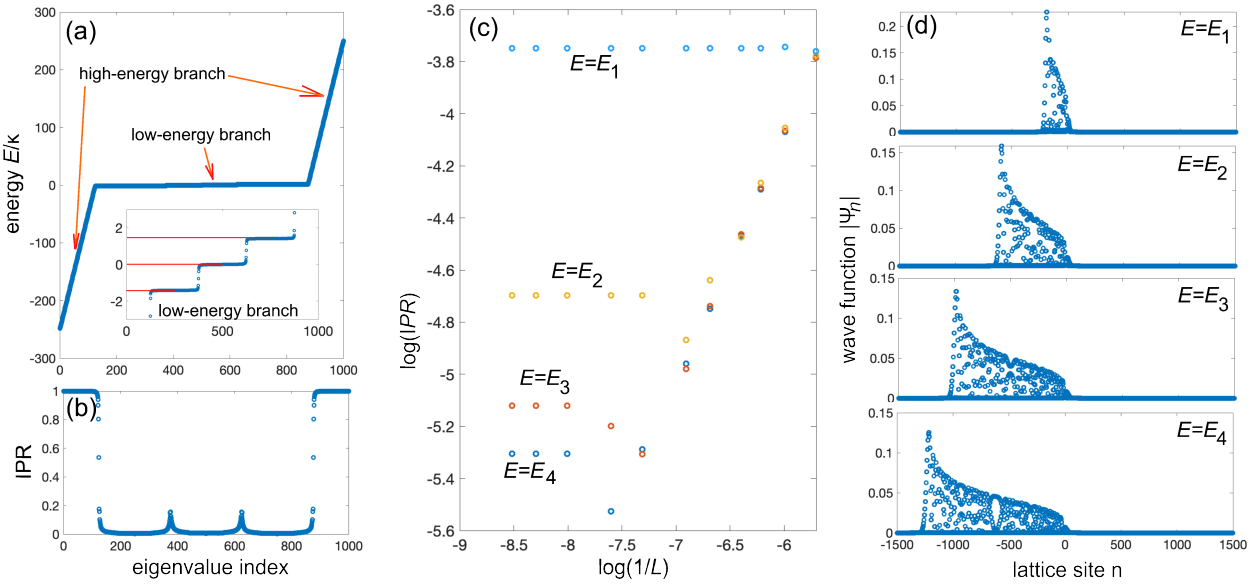

FIG. 2: (a) Numerically-computed energy in a lattice of size $L = 1000$ with open boundary conditions for parameter values $M = 4$ and $F/\kappa = 0.5$. The inset shows an enlargement of the energy spectrum corresponding to the low-energy branch $E_\alpha^{(II)}$, clearly showing that the eigenenergies condensate toward the $(M-1) = 3$ values $0, \pm\sqrt{2}\kappa$, corresponding to the energies of the isolated extended states (horizontal red curves). (b) The IPR of the corresponding wave functions. Note that the IPR of the high-energy wave functions is very close to one, indicating strong localization, whereas the IPR of the low-energy wave functions is small, indicating a low degree of localization. (c) Behavior of $\log(IPR)$ versus $\log(1/L)$ for four low-energy wave functions with energies $E_{1,2,3,4}$ that approach the zero energy of the extended state ($E_1 = 0.020\kappa$, $E_2 = 0.0067\kappa$, $E_3 = 0.0040\kappa$, and $E_4 = 0.0032\kappa$). The slopes of the curves at $L \to \infty$ give the fractal dimension β of the wave functions, with $\beta = 1$ for extended states, $0 < \beta < 1$ for critical states and $\beta = 0$ for localized states. Note that all wave functions are localized, even though the IPR is very small. A weakly-localized state differs from a critical or a fully extended state because the IPR settles down to a constant (albeit small) value as $L \to \infty$, and the fractal dimension β correspondingly vanishes. (d) Shape of the wave function amplitudes $|\psi_n|$ corresponding to the four energies in a lattice of size $L = 3000$. Note that the wave functions are very weakly localized, extending over several hundreds of lattice sites.

wave functions is shown in Fig.2(b). The energy spectrum in Fig.2(a) clearly shows that, besides high-energy eigenstates, a large fraction of the wave functions, namely $\sim L(1-1/M)$ wave functions, have their energy in three narrow regions [see the insets in Fig.2(a)] that condensate toward the $(M-1) = 3$ energies $E_\sigma = 0, \pm\sqrt{2}\kappa$ of extended states. As shown in Fig.2(b), the IPR of the high-energy wave functions is very close to one, indicating a tight localization of the wave functions. On the other hand, the IPR of the low-energy wave functions is small, reaching a value down to ~ 0.005 close to the three energies E_σ. However, a small value of the IPR alone does not necessarily mean that the wave function is an extended or critical state, it just tells us that the excitation spreads over several sites of the lattice. The nature of the function $\psi_n^{(\alpha)}$ is at best captured by looking at its fractal dimension β_α, which is defined by (see e.g.[35,36,49])

$$\beta_\alpha = \lim_{L\to\infty} \frac{\ln IPR_\alpha}{\ln(1/L)} \tag{31}$$

For a localized wave function one has $\beta_\alpha = 0$, for an extended wave function one has $\beta_\alpha = 1$, whereas for a critical wave function one has $0 < \beta_\alpha < 1$. In our example, the energy spectrum contains $(M-1) = 3$ fully extended states, at the energies $E_\sigma = 0, \pm\sqrt{2}\kappa$, and for such states one clearly has $\beta = 1$. In Fig.2(c) we show the numerically-computed behavior of $\log(IPR)$ versus $\log(1/L)$ for four wave functions with energies $E_1 = 0.020\kappa$, $E_2 = 0.0067\kappa$, $E_3 = 0.0040\kappa$, $E_4 = 0.0032\kappa$ that approach the zero energy value of one of the three extended states. The figure clearly shows that, for a fixed energy (albeit very close to zero) the behavior of $\log(IPR)$ becomes independent of $\log(1/L)$ for large enough system size L, indicating that the slope β vanishes and the wave function is not strictly an extended state, although the excitation can spread over many sites of the lattice. As an example, in Fig.2(d) we plot the wave function amplitudes for the four energies in a lattice of size $L = 3000$, clearly showing that, even though the excitation spreads over several hundreds of sites in the lattice, with a very small IPR, the wave function asymptotically decays toward zero and thus belongs to the point spectrum of the Hamiltonian in the $L \to \infty$ limit.

In systems with a finite size L, a relevant number of eigen-

states, with energies very close to the $(M-1)$ values E_σ, are nevertheless extended over the entire lattice and can be thus considered as 'extended' states in a broad sense. The mobility edges, i.e. the energies separating such wave functions extended over the entire size L of the lattice from localized wave functions, clearly shrink toward the $(M-1)$ energies E_σ in a system of large size L, indicating that the spectral extent (but not the number of wave functions) of such 'extended' states shrink to zero in the $L \to \infty$ limit, according to the Simon-Spencer theorem[48]. This result is clearly at odd with the results presented in Ref.[41]. However, one can retrieve the results of Ref.[41], and in particular the form of pseudo mobility edges, introducing the notion of an 'extended state' in a weaker sense, by classifying a wave function as an 'extended' state whenever its IPR is smaller than an assigned (small) number ϵ, and a 'localized' state when its IPR is larger than ϵ. Using the property of Bessel functions that $J_n(\Gamma)$ extends over $\sim 2|\Gamma|$ sites of the lattice, we can roughly estimate the size w of a narrow-energy wave function using Eq.(27), i.e.

$$w \sim 2M|\Gamma| = \left| \frac{4\kappa}{F} \frac{\sin\theta}{\sin(M\theta)} \right|, \tag{32}$$

where the angle θ is related to the energy E via the relation $E = 2\kappa\cos\theta$. For a wave function with excitation uniformly distributed over w sites of the lattice, the IPR is clearly estimated by the relation

$$IPR \sim 1/w \tag{33}$$

and thus from Eqs.(32) and (33) one obtains

$$IPR \sim \left| \frac{F}{4\kappa} \frac{\sin(M\theta)}{\sin\theta} \right|. \tag{34}$$

The pseudo mobility edges are thus obtained from the relation $IPR = \epsilon$, i.e.

$$\left| \frac{F}{4\kappa} \frac{\sin(M\theta)}{\sin\theta} \right| = \epsilon. \tag{35}$$

If we assume $\epsilon = 1/2$ and let $a_M \equiv \sin(M\theta)/\sin\theta$, the pseudo-mobility edges are thus defined by the relation

$$\left| \frac{F}{\kappa} a_M \right| = 2 \tag{36}$$

which is precisely the result obtained in Ref.[41] for the mobility edges [see Eq.(11) of this reference].

IV. CONCLUSIONS

In summary, we reported on the exact analytical solution of the spectral problem of the mosaic Wannier-Stark Hamiltonian, a tight-binding model which has been introduced in recent works[41–43] as an example of a disorder-free system displaying mobility edges, separating localized and extended states. This result looks quite surprising since so far all known one-dimensional models displaying mobility edges require some kind of (incommensurate) disorder. Our results indicate that for the mosaic Wannier-Stark Hamiltonian strictly speaking there are not mobility edges, separating extended and localized states. Specifically, we proved that the energy spectrum is almost pure point, with all the wave functions displaying a higher than exponential localization, typical of Wannier-Stark localization, with the exception of $(M-1)$ isolated extended states. The energy spectrum comprises two sets of countably infinite number of localized states, the low-energy and high-energy wave functions. While the high-energy wave functions are tightly localized, the low-energy wave functions are weakly localized, and they become more and more extended as their energies approach the energies of the isolated extended states.

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