J. Math. Anal. Appl. 473 (2019) 215–257

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Scattering from local deformations of a semitransparent plane

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A R T I C L E   I N F O

Article history:
Received 12 September 2018
Available online 17 December 2018
Submitted by P. Exner

Keywords:
Scattering theory
Point interactions supported by unbounded hypersurfaces
Kreĭn's resolvent formulae

A B S T R A C T

We study scattering for the couple (A_F, A_0) of Schrödinger operators in $L^2(\mathbb{R}^3)$ formally defined as $A_0 = -\Delta + \alpha\,\delta_{\pi_0}$ and $A_F = -\Delta + \alpha\,\delta_{\pi_F}$, $\alpha > 0$, where δ_{π_F} is the Dirac δ-distribution supported on the deformed plane given by the graph of the compactly supported, Lipschitz continuous function $F : \mathbb{R}^2 \to \mathbb{R}$ and π_0 is the undeformed plane corresponding to the choice $F \equiv 0$. We provide a Limiting Absorption Principle, show asymptotic completeness of the wave operators and give a representation formula for the corresponding Scattering Matrix $S_F(\lambda)$. Moreover we show that, as $F \to 0$, $\|S_F(\lambda) - 1\|^2_{\mathfrak{B}(L^2(\mathbb{S}^2))} = \mathcal{O}\left(\int_{\mathbb{R}^2} d\mathbf{x}|F(\mathbf{x})|^\gamma\right)$, $0 < \gamma < 1$.

1. Introduction

In this paper we are concerned with self-adjoint Schrödinger operators in $L^2(\mathbb{R}^3)$ formally defined as

$$A_F = -\Delta + \alpha\,\delta_{\pi_F}\,, \qquad \alpha > 0\,, \tag{1.1}$$

where δ_{π_F} is the Dirac δ-distribution supported on the surface $\pi_F := \{\mathbf{x} \equiv (x^1, \mathbf{x}_\parallel) \in \mathbb{R}^3 \,|\, x^1 = F(\mathbf{x}_\parallel)\}$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is such that:

$F \in C_0^{0,1}(\mathbb{R}^2)$, i.e., F has compact support and is Lipschitz continuous.

For the rigorous definitions of these self-adjoint operators we refer to Section 3 below; here we only notice that the functions in their self-adjointness domain have to satisfy semitransparent boundary conditions of the kind $[\partial_{\mathbf{n}} f]_{\pi_F} = \alpha\, f\restriction_{\pi_F}$, where $[\partial_{\mathbf{n}} f]_{\pi_F}$ denotes the jump of the normal derivative across the surface π_F.

The spectral properties of these operators, mainly presence and estimates on the number of their bound states below the essential spectrum (whenever $\alpha < 0$), have been studied in many papers (see, e.g., [12,14]

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https://doi.org/10.1016/j.jmaa.2018.12.045
0022-247X/

and references therein for the 2D case, where the support of the perturbation is a deformed line; [17] for the 3D case; and the very recent survey [13]). Much less is known regarding their scattering theory: the 2D case has been studied in the paper [15], where existence and completeness of the wave operators have been provided (see also [16] for similar results in 3D in a setting in which the singular potential is supported on a curve) and the scattering matrix was studied (whenever $\alpha < 0$) for the negative part $(-\alpha^2/4, 0)$ of the spectrum; in this case the scattering problem is essentially one-dimensional in the sense that it is described by a 2×2 matrix of reflection and transmission amplitudes. Here we are instead concerned with the more involved 3D case and we provide a comprehensive scattering analysis for the case of repulsive interaction $\alpha > 0$, from a Limiting Absorption Principle, through existence and asymptotic completeness of the wave operators, to a representation formula for the scattering matrix on the half line $(0, +\infty)$ minus the (possibly empty) discrete subset of embedded eigenvalues. Since the singular potential $\alpha\,\delta_{\pi_F}$ does not vanish at infinity, one cannot expect existence of the wave operators for the scattering couple $(A_F, A_\varnothing)$ (here and below $A_\varnothing$ denotes the free Schrödinger operator) and so, as for the 2D case in [15], we consider here the scattering couple (A_F, A_0), where A_0 formally corresponds to $A_0 = -\Delta + \alpha\,\delta_{\pi_0}$ and δ_{π_0} is the Dirac δ-distribution supported on the plane $\pi_0 := \{\mathbf{x} \equiv (x^1, \mathbf{x}_\|) \in \mathbb{R}^3 \,|\, x^1 = 0\}$, i.e. A_0 corresponds to A_F with $F \equiv 0$. Whereas we follow the same strategy as in [24] and [22], due to $A_0 \neq A_\varnothing$ and to the unboundedness of the obstacles, here, with respect to the results provided in [24, Subsection 6.4] and [22, Subsection 5.4], more work is needed and the proofs are, for the most, different and more elaborate.

After supplying some preliminary material in Section 2, we introduce in Section 3 the rigorous definition of the self-adjoint operators A_F by providing their resolvent through a Kreĭn's type formula expressed in terms of the free resolvent (i.e. the resolvent of $A_\varnothing$) (see Theorem 3.4). Let us remark here that the operator A_F could be equivalently defined by quadratic form methods (see, e.g., [4,9,18] and references therein); however, in order to study the Limiting Absorption Principle (LAP for short) and the Scattering Matrix, one needs a convenient resolvent formula. One more important remark about our use of resolvent formulae is the following: proceeding as in [15], one could try to provide a resolvent formula of A_F expressed directly in terms of A_0; however this would imply the use of trace (evaluation) operators in the operator domain of A_0, and these are less well-behaved than in the Sobolev space $H^2(\mathbb{R}^3)$, the self-adjointness domain of $A_\varnothing$ (in particular is not clear what should be the correct trace space). Moreover, such an approach can lead, even in the case of a smooth deformation F, to singular perturbations of A_0 supported on not Lipschitz sets. Therefore we prefer to work with the difference of the two Kreĭn's formulae, one for A_F and the other one for A_0, both expressed in terms of the free resolvent. This suffices, in Section 4, for the proof of LAP for A_F (see Theorem 4.13), obtained after providing LAP for the operator A_0 (see Proposition 4.2) and by building on some abstract results by Renger (see [30] and [31]).

Then, in Section 5, by a careful analysis of the difference of the two resolvent for A_F and A_0, we obtain a Kreĭn's type formula for the resolvent of A_F which contains only the resolvent of A_0 and trace maps on the (compact) support of the deformation F (see Theorem 5.1); such resolvent formula resembles the one used in [15].

LAP proved in Section 4 and the formula for the resolvents difference from Section 5 are the starting point of our analysis of the scattering theory. In Section 6, using the same approach as in Section 5, we prove existence and asymptotic completeness of the wave operators associated to the scattering couple (A_F, A_0) (see Theorem 6.1). In Section 7, following the same kind of reasoning as in [22, Sec. 4] (see also [23, Rem. 5.7]), using this latter resolvent formula, Birman–Yafaev stationary scattering theory and the Kato–Birman invariance principle, we obtain a representation formula for the Scattering Matrix $S_F(\lambda)$ of the scattering couple (A_F, A_0) for any energy $\lambda \in (0, +\infty)\backslash\sigma_p^+(A_F)$, where $\sigma_p^+(A_F)$ is the (possibly empty) discrete set of embedded eigenvalues (see Theorem 7.1 and Corollary 7.2). Finally, see Theorem 7.7, using such a representation formula, we provide an estimate, in operator norm, on the difference $S_F(\lambda) - 1$; in particular, as the deformation $F \to 0$, one gets

$$\|S_F(\lambda) - 1\|^2_{\mathfrak{B}(L^2(\mathbb{S}^2))} = \mathcal{O}\left(\int_{\mathbb{R}^2} d\mathbf{x}_\parallel \, |F(\mathbf{x}_\parallel)|^\gamma \right) , \qquad 0 < \gamma < 1 .$$

The paper is concluded with a technical appendix containing the proof of the LAP for the self-adjoint operator corresponding to the Laplacian with a δ-interaction in one dimension.

2. Notation and preliminaries

• We denote by Δ the distributional Laplacian on $\mathcal{D}'(\mathbb{R}^3)$.
• We denote by $A_\varnothing$ the *free (positive) Laplacian* on $\mathbb{R}^3$; this is the self-adjoint operator

$$A_\varnothing : H^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) , \qquad A_\varnothing f := -\Delta f , \tag{2.1}$$

with purely absolutely continuous spectrum $\sigma(A_\varnothing) = \sigma_{ac}(A_\varnothing) = [0, +\infty)$. The corresponding *free resolvent* operator is

$$R_\varnothing(z) := (A_\varnothing - z)^{-1} : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3) \qquad z \in \mathbb{C}\backslash[0, +\infty) . \tag{2.2}$$

• For any pair X, Y of Banach spaces, we denote the Banach space of bounded linear operators from X to Y with $\mathfrak{B}(X, Y)$ and set $\mathfrak{B}(X) = \mathfrak{B}(X, X)$; the two-sided ideal of compact operators is $\mathfrak{S}_\infty(X, Y)$ ($\mathfrak{S}_\infty(X) = \mathfrak{S}_\infty(X, X)$).
• For any complex number $z \in \mathbb{C}\backslash[0, +\infty)$ we define its square root with the branch cut such that Im $\sqrt{z} > 0$.
• In order to avoid the appearance of cumbersome expressions we shall use the following short-hand notation, for any pair of real numbers $B_1, B_2 \in \mathbb{R}$:

$$B_1 \leqslant_c B_2 \quad \Leftrightarrow \quad B_1 \leqslant \text{const.}\, B_2 , \quad \text{for some finite constant const.} > 0 .$$

2.1. Test functions and distributions

For $k \in \{1, 2, 3, ...\}$, we consider the vector space $\mathcal{D}(\mathbb{R}^k)$ of smooth, compactly supported functions $f : \mathbb{R}^k \to \mathbb{C}$ and equip this space with the well known inductive limit topology. The topological dual space $\mathcal{D}'(\mathbb{R}^k)$ is the space of Schwartz distributions on $\mathbb{R}^k$. Unless otherwise stated, all derivatives considered in the sequel are to be understood in the sense of distributions.

2.2. Sobolev spaces

Let us now introduce Sobolev spaces on $\mathbb{R}^k$ of L^2 type, of both integer and fractional order.
The Sobolev space of positive integer order $n \in \mathbb{N}$ is the Hilbert space

$$H^n(\mathbb{R}^k) := \{f \in \mathcal{D}'(\mathbb{R}^k) \mid \partial^\alpha f \in L^2(\mathbb{R}^k) \text{ for all } \alpha \in \mathbb{N}^k,\, |\alpha| \leqslant n\} ,$$

endowed with the standard inner product

$$\langle f|g\rangle_{H^n(\mathbb{R}^k)} := \sum_{\alpha \in \mathbb{N}^k,\, |\alpha| \leqslant n} \langle \partial^\alpha f|\partial^\alpha g\rangle_{L^2(\mathbb{R}^k)}$$

which induces the norm $\|f\|_{H^n(\mathbb{R}^k)} := \sqrt{\langle f|f\rangle_{H^n(\mathbb{R}^k)}}$. Of course, $H^0(\mathbb{R}^k) = L^2(\mathbb{R}^k)$.

For any $r \in [0,+\infty)\backslash\mathbb{N}$, let us denote with $[r]$ its integer part and put $\rho := r - [r]$. The Sobolev space of fractional order r is

$$H^r(\mathbb{R}^k) := \left\{ f \in \mathcal{D}'(\mathbb{R}^k) \;\middle|\; \partial^\alpha f \in L^2(\mathbb{R}^k) \text{ for } \alpha \in \mathbb{N}^k,\, |\alpha| \leqslant [r]\,, \text{ and } \int\limits_{\mathbb{R}^k\times\mathbb{R}^k} d\mathbf{x}\, d\mathbf{y}\, \frac{|\partial^\alpha f(\mathbf{x}) - \partial^\alpha f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{k+2\rho}} < +\infty \text{ for } \alpha \in \mathbb{N}^k,\, |\alpha| = [r] \right\};$$

this is also a complex Hilbert space with the inner product

$$\langle f|g\rangle_{H^r(\mathbb{R}^k)} := \langle f|g\rangle_{H^{[r]}(\mathbb{R}^k)} + \sum_{\alpha\in\mathbb{N}^k, |\alpha|=[r]} \int\limits_{\mathbb{R}^k\times\mathbb{R}^k} d\mathbf{x}\, d\mathbf{y}\, \frac{\overline{(\partial^\alpha f(\mathbf{x}) - \partial^\alpha f(\mathbf{y}))}\,(\partial^\alpha g(\mathbf{x}) - \partial^\alpha g(\mathbf{y}))}{|\mathbf{x}-\mathbf{y}|^{k+2\rho}}\,, \tag{2.3}$$

inducing the norm $\|f\|_{H^r(\mathbb{R}^k)} := \sqrt{\langle f|f\rangle_{H^r(\mathbb{R}^k)}}$.

As well known, for any $r \in [0,+\infty)$, the space $H^r(\mathbb{R}^k)$ coincides with the Banach space obtained by taking the closure of $\mathcal{D}(\mathbb{R}^k)$ with respect to the norm $\|\cdot\|_{H^r(\mathbb{R}^k)}$. We denote with $H^{-r}(\mathbb{R}^k)$ its topological dual; so, each $f \in H^{-r}(\mathbb{R}^k)$ is a continuous linear form on $H^r(\mathbb{R}^k)$ and, by restriction to $\mathcal{D}(\mathbb{R}^k)$, it can be identified with a distribution on $\mathbb{R}^k$.

In some of the forthcoming proofs, we will also use the well-known equivalent norm on the Sobolev spaces $H^r(\mathbb{R}^k)$ ($r \in \mathbb{R}$) defined by standard functional calculus in terms of real powers of the operator $(\mathbf{1} - \Delta_k)$ (Δ_k is the free Laplacian on $\mathbb{R}^k$) as follows:

$$|\!|\!| f |\!|\!|_{H^r(\mathbb{R}^k)} := \|(\mathbf{1} - \Delta_k)^{r/2} f\|_{L^2(\mathbb{R}^k)}\,. \tag{2.4}$$

Of course, using the distributional Fourier transform $\mathfrak{F} : \mathcal{D}'(\mathbb{R}^k) \to \mathcal{D}'(\mathbb{R}^k)$, normalized so as to be unitary with respect to the inner product of $L^2(\mathbb{R}^k)$, the above norm can be re-expressed as

$$|\!|\!| f |\!|\!|_{H^r(\mathbb{R}^k)} = \|(1 + |\mathbf{k}|^2)^{r/2}\, \mathfrak{F} f\|_{L^2(\mathbb{R}^k)}\,. \tag{2.5}$$

It is a well-known fact that the norm (2.4) is equivalent to the norm introduced previously on $H^r(\mathbb{R}^k)$ for any $r \in \mathbb{R}$. More explicitly, this means that for any $f \in H^r(\mathbb{R}^k)$ there holds true the following chain of inequalities

$$|\!|\!| f |\!|\!|_{H^r(\mathbb{R}^k)} \leqslant_c \|f\|_{H^r(\mathbb{R}^k)} \leqslant_c |\!|\!| f |\!|\!|_{H^r(\mathbb{R}^k)}\,. \tag{2.6}$$

2.3. Sobolev spaces on domains

Whenever $\Omega \subseteq \mathbb{R}^k$ is an open set with a Lipschitz boundary, we define

$$H^s(\Omega) := \{f \in \mathcal{D}'(\Omega) \mid \exists \tilde{f} \in H^s(\mathbb{R}^k) \text{ such that } \tilde{f} \upharpoonright \Omega = f\}$$

and

$$H^s_{\overline{\Omega}}(\mathbb{R}^k) := \{f \in H^s(\mathbb{R}^k) \mid \mathrm{supp} f \subseteq \overline{\Omega}\}\,.$$

2.4. Basic results on Sobolev spaces

Remark 2.1. In what follows we shall often use without further notice the continuous Sobolev embedding $H^s(\mathbb{R}^k) \hookrightarrow H^r(\mathbb{R}^k)$ for $s \geq r$; in particular to infer that given an operator $O : H^r(\mathbb{R}^k) \to H^{r'}(\mathbb{R}^{k'})$ then:

i) If $O \in \mathfrak{B}(H^r(\mathbb{R}^k), H^{r'}(\mathbb{R}^{k'}))$ then $O \in \mathfrak{B}(H^s(\mathbb{R}^k), H^{s'}(\mathbb{R}^{k'}))$ for all $r \leq s$, and $s' \leq r'$;
ii) If $O \in \mathfrak{S}_p(H^r(\mathbb{R}^k), H^{r'}(\mathbb{R}^{k'}))$ then $O \in \mathfrak{S}_p(H^s(\mathbb{R}^k), H^{s'}(\mathbb{R}^{k'}))$ for all $r \leq s$, $s' \leq r'$, and $1 \leq p \leq +\infty$.

A similar remark holds true when dealing with the embedding between weighted Sobolev spaces, see Section 2.6.

Lemma 2.2. *Let $h, k \in \{1, 2, 3, ...\}$ and consider the vector spaces $\mathcal{D}(\mathbb{R}^h) \otimes \mathcal{D}(\mathbb{R}^k)$ (where $\otimes$ indicates the algebraic tensor product), $\mathcal{D}(\mathbb{R}^h \times \mathbb{R}^k)$ and $H^r(\mathbb{R}^h \times \mathbb{R}^k)$. Then, the following inclusions hold true for all $r \in \mathbb{R}$:*

$$\mathcal{D}(\mathbb{R}^h) \otimes \mathcal{D}(\mathbb{R}^k) \subset \mathcal{D}(\mathbb{R}^h \times \mathbb{R}^k) \subset H^r(\mathbb{R}^h \times \mathbb{R}^k)\,.$$

Moreover, the sets on the l.h.s. of the above inclusions are dense subspaces of the topological vector spaces on the r.h.s., assuming the latter are endowed with their natural topologies.

Proof. The thesis is a just restatement of known results. For the proof of the first inclusion, see, e.g., [8, p. 74, Prop. 6.1] or [35, p. 409, Th. 39.2]; for the proof of the second inclusion, see [8, p. 182, Th. 13.2] or [34, p. 107, Prop. 13.1]. □

Lemma 2.3. *Let $h, k \in \{1, 2, 3, ...\}$ and $r \geqslant 0$. Then, there hold the following continuous embeddings:*

$$H^r(\mathbb{R}^{h+k}) \hookrightarrow H^{r_h}(\mathbb{R}^h) \otimes H^{r_k}(\mathbb{R}^k) \qquad \textit{for all } r_k + r_h \leqslant r\,; \tag{2.7}$$

$$H^{r_h}(\mathbb{R}^h) \otimes H^{r_k}(\mathbb{R}^k) \hookrightarrow H^{\min(r_h,r_k)}(\mathbb{R}^{h+k}) \qquad \textit{for all } r_k, r_h \geqslant 0\,. \tag{2.8}$$

Proof. Consider the norms introduced in Eq. (2.4). To prove both statements (2.7) and (2.8) it suffices to recall that these norms are equivalent to the standard norms on $H^r(\mathbb{R}^k)$ and to notice that they fulfil the elementary inequalities reported hereafter for any factorized function of the form $f = u_h \otimes u_k \in \mathcal{D}(\mathbb{R}^h) \otimes \mathcal{D}(\mathbb{R}^k)$.

On the one hand, noting that $(1+|\mathbf{q}_h|^2+|\mathbf{q}_k|^2)^r \geqslant (1+|\mathbf{q}_h|^2+|\mathbf{q}_k|^2)^{r_h}(1+|\mathbf{q}_h|^2+|\mathbf{q}_k|^2)^{r_k} \geqslant (1+|\mathbf{q}_h|^2)^{r_h} \times (1+|\mathbf{q}_k|^2)^{r_k}$ for $r \geqslant 0$ and $r_h + r_k \leqslant r$, we have

$$\begin{aligned} \|f\|^2_{H^r(\mathbb{R}^{h+k})} &= \int_{\mathbb{R}^h \times \mathbb{R}^k} d\mathbf{q}_h d\mathbf{q}_k\, (1+|\mathbf{q}_h|^2+|\mathbf{q}_k|^2)^r\, |(\mathfrak{F}u_h)(\mathbf{q}_h)|^2\, |(\mathfrak{F}u_k)(\mathbf{q}_k)|^2 \\ &\geqslant \int_{\mathbb{R}^h \times \mathbb{R}^k} d\mathbf{q}_h\, d\mathbf{q}_k\, (1+|\mathbf{q}_h|^2)^{r_h}(1+|\mathbf{q}_k|^2)^{r_k}\, |(\mathfrak{F}u_h)(\mathbf{q}_h)|^2\, |(\mathfrak{F}u_k)(\mathbf{q}_k)|^2 \\ &= \|u_h\|^2_{H^{r_h}(\mathbb{R}^h)}\, \|u_k\|^2_{H^{r_k}(\mathbb{R}^k)} = \|u_h \otimes u_k\|^2_{H^{r_h}(\mathbb{R}^h)\otimes H^{r_k}(\mathbb{R}^k)}\,. \end{aligned}$$

On the other hand, for any $r, r_h, r_k \geqslant 0$ one can use the trivial estimates $(1+|\mathbf{q}_h|^2+|\mathbf{q}_k|^2) \leqslant (1+|\mathbf{q}_h|^2)(1+|\mathbf{q}_k|^2)$ and $\min(r_h, r_k) \leqslant r_h$, $\min(r_h, r_k) \leqslant r_k$ to obtain the following chain of inequalities:

$$\begin{aligned}\|f\|^2_{H^{\min(r_h,r_k)}(\mathbb{R}^{h+k})} &= \int\limits_{\mathbb{R}^h\times\mathbb{R}^k} d\mathbf{q}_h\, d\mathbf{q}_k\, (1+|\mathbf{q}_h|^2+|\mathbf{q}_k|^2)^{\min(r_h,r_k)}\, |(\mathfrak{F}u_h)(\mathbf{q}_h)|^2\, |(\mathfrak{F}u_k)(\mathbf{q}_k)|^2 \\ &\leqslant \int\limits_{\mathbb{R}^h\times\mathbb{R}^k} d\mathbf{q}_h\, d\mathbf{q}_k\, (1+|\mathbf{q}_h|^2)^{r_h}(1+|\mathbf{q}_k|^2)^{r_k}\, |(\mathfrak{F}u_h)(\mathbf{q}_h)|^2\, |(\mathfrak{F}u_k)(\mathbf{q}_k)|^2 \\ &= \|u_h\|^2_{H^{r_h}(\mathbb{R}^h)}\, \|u_k\|^2_{H^{r_k}(\mathbb{R}^k)} = \|u_h\otimes u_k\|^2_{H^{r_h}(\mathbb{R}^h)\otimes H^{r_k}(\mathbb{R}^k)}\,.\end{aligned}$$

The above bounds suffice to infer the claims stated in Eqs. (2.7) and (2.8), by standard density arguments (see also the previous Lemma 2.2). □

2.5. Sobolev spaces on the boundary and trace operators

Let us consider the space $\mathbb{R}^3$ and indicate with boldface letters $\mathbf{x} = (x^1, x^2, x^3)$ a set of Cartesian coordinates on it. We consider the plane

$$\pi_0 := \{\mathbf{x}\in\mathbb{R}^3 \,|\, x^1 = 0\}\ ,$$

and refer to it as the "*flat*" or the "*non-deformed*" plane. We write $\mathbf{x}_\parallel = (x^2, x^3)$ for the coordinates induced by the set of coordinates $\mathbf{x}$ on π_0, which allow to naturally identify the latter with $\mathbb{R}^2$.

For $F \in C^{0,1}_0(\mathbb{R}^2)$, as in our assumptions, we write supp$F$ for the support of F and consider the surface

$$\pi_F := \{\mathbf{x}\in\mathbb{R}^3 \,|\, x^1 = F(\mathbf{x}_\parallel)\}\ ,$$

which is referred to as the "deformed" plane. Needless to say, the two planes π_0 and π_F coincide when $F = 0$. Besides, let us remark that, similarly to π_0, the plane π_F can also be identified with $\mathbb{R}^2$ considering the change of coordinates

$$(x^1,\mathbf{x}_\parallel) \mapsto (y^1,\mathbf{y}_\parallel) := (x^1 - F(\mathbf{x}_\parallel), \mathbf{x}_\parallel)\ .$$

Correspondingly, one has an isomorphism I_F of $H^r(\mathbb{R}^3)$ into itself for any order $r\in\mathbb{R}$ such that $|r|\leqslant 1$ (see [19, Sec. 1.3.3] or [25, Ch. 3]), defined by $I_F f(x^1,\mathbf{x}_\parallel) = f(x^1 + F(\mathbf{x}_\parallel),\mathbf{x}_\parallel)$.

To proceed, let us consider the map $\tau_0 : \mathcal{D}(\mathbb{R}^3)\to\mathcal{D}(\mathbb{R}^2)$ defined by

$$(\tau_0 f)(\mathbf{x}_\parallel) := f(0,\mathbf{x}_\parallel)\ ,$$

i.e. the evaluation of smooth functions on the plane π_0. As well known [20], this map can be uniquely extended to a surjective, continuous operator $\tau_0 : H^{r+1/2}(\mathbb{R}^3)\to H^r(\mathbb{R}^2)$ for all $r > 0$.

We shall also use the map $\tau_F : \mathcal{D}(\mathbb{R}^3)\to\mathcal{D}(\mathbb{R}^2)$ defined by

$$(\tau_F f)(\mathbf{x}_\parallel) := f(F(\mathbf{x}_\parallel),\mathbf{x}_\parallel)\ ,$$

i.e. the evaluation of smooth functions on the surface π_F.

Remark 2.4. Noticing that $(\tau_F f)(\mathbf{x}_\parallel) = (\tau_0 I_F f)(\mathbf{x}_\parallel)$, we infer that this map can be uniquely extended to a surjective, continuous operator

$$\tau_F \in \mathfrak{B}(H^{r+1/2}(\mathbb{R}^3), H^r(\mathbb{R}^2)) \qquad \text{for all } r\in(0,1/2]\ ;$$

see also, e.g., [25, Th. 3.37].

The operators τ_0, τ_F are commonly referred to as *traces* on the planes π_0, π_F. Considering then the $C^{0,1}$ open domains

$$\Omega_F^{\pm} := \{\mathbf{x} \in \mathbb{R}^3 \,|\, x^1 > \pm F(\mathbf{x}_\parallel)\}\,,$$

the trace operator τ_F can be extended to the larger spaces

$$H^r(\mathbb{R}^3 \backslash \pi_F) := H^r(\Omega_F^-) \oplus H^r(\Omega_F^+) \qquad \text{for } r \in (0,1)$$

by setting

$$\tau_F : H^{r+1/2}(\Omega_F^-) \oplus H^{r+1/2}(\Omega_F^+) \to H^r(\mathbb{R}^2)\,, \quad \tau_F(f_- \oplus f_+) := \frac{1}{2}\,(\tau_F^- f_- + \tau_F^+ f_+)\,,$$

where

$$\tau_F^{\pm} \in \mathfrak{B}(H^{r+1/2}(\Omega_F^{\pm}), H^r(\mathbb{R}^2)) \qquad \text{for } r \in (0,1)$$

are the lateral traces defined as the unique bounded extensions of the evaluation maps

$$\tau_F^{\pm} f(\mathbf{x}_\parallel) := \lim_{\varepsilon \downarrow 0} f(F(\mathbf{x}_\parallel) \pm \varepsilon, \mathbf{x}_\parallel)\,, \qquad f \in H^{r+1/2}(\Omega_F^{\pm}) \cap C(\overline{\Omega_F^{\pm}})\,.$$

In terms of the above lateral traces, for later convenience we introduce the notations

$$f \restriction_{\pi_F^{\pm}} := \tau_F^{\pm} f\,, \qquad f \restriction_{\pi_F} := \tau_F f\,, \qquad [f\,]_{\pi_F} := \tau_F^+ f - \tau_F^- f\,, \tag{2.9}$$

2.6. Weighted Sobolev spaces

For any $k \in \mathbb{N}$ and $w \in L^1_{loc}(\mathbb{R}^k, [0, +\infty))$, we consider the weighted Sobolev-type spaces of integer order $n \in \{0, 1, 2, ...\}$ defined as

$$H^n_w(\mathbb{R}^k) := \left\{ f \in \mathcal{D}'(\mathbb{R}^k) \;\middle|\; \|f\|^2_{H^n_w(\mathbb{R}^k)} < +\infty \right\}\,, \qquad \|f\|^2_{H^n_w(\mathbb{R}^k)} := \sum_{|\alpha| \leqslant n} \int_{\mathbb{R}^k} d\mathbf{x}\; w(\mathbf{x})\, \big|(\partial^\alpha f)(\mathbf{x})\big|^2\,; \tag{2.10}$$

in particular, for $n = 0$ we set

$$H^0_w(\mathbb{R}^k) \equiv L^2_w(\mathbb{R}^k) := \left\{ f \in \mathcal{D}'(\mathbb{R}^k) \;\middle|\; \|f\|^2_{L^2_w(\mathbb{R}^k)} < +\infty \right\}\,, \qquad \|f\|^2_{L^2_w(\mathbb{R}^k)} := \int_{\mathbb{R}^k} d\mathbf{x}\; w(\mathbf{x})\, |f(\mathbf{x})|^2\,.$$

For any $\theta \in (0, 1)$, we define the analogous fractional order spaces by complex interpolation putting

$$H^{n+\theta}_w(\mathbb{R}^k) := [H^n_w(\mathbb{R}^k), H^{n+1}_w(\mathbb{R}^k)]_\theta\;. \tag{2.11}$$

In passing, let us remark that the non-weighted fractional Sobolev spaces introduced in subsection 2.2 could be equivalently characterized by complex interpolation, via a relation analogous to (2.11). To be more precise, there holds $H^{n+\theta}(\mathbb{R}^k) = [H^n(\mathbb{R}^k), H^{n+1}(\mathbb{R}^k)]_\theta$; of course, this identity must be understood in the sense that the usual topology on $H^{n+\theta}(\mathbb{R}^k)$ descending from the inner product (2.3) is equivalent to (though, different from) the natural interpolation topology on $[H^n(\mathbb{R}^k), H^{n+1}(\mathbb{R}^k)]_\theta$.

Now, let us assume that $1/w \in L^1_{loc}(\mathbb{R}^k, [0, +\infty))$ is an admissible weight as well; then, for any $r \geqslant 0$ we introduce the weighted space of negative order $-r$ in terms of the standard L^2-duality setting

$$H_w^{-r}(\mathbb{R}^k) := \left(H_{1/w}^r(\mathbb{R}^k)\right)' . \tag{2.12}$$

Taking into account the latter position and [7, p. 98, Cor. 4.5.2], it can be readily inferred that a relation analogous to Eq. (2.11) does indeed hold true for any $n \in \mathbb{Z}$ and all $\theta \in (0,1)$.

Our analysis involves, in particular, the weights $w_{s_1}(x^1) := (1+|x^1|^2)^{s_1}$ on $\mathbb{R}$, $w_{s_\parallel}(\mathbf{x}_\parallel) := (1+|\mathbf{x}_\parallel|^2)^{s_\parallel}$ on $\mathbb{R}^2$ and their tensor product $w_{s_1,s_\parallel}(x^1,\mathbf{x}_\parallel) := w_{s_1}(x^1)\, w_{s_\parallel}(\mathbf{x}_\parallel)$ on $\mathbb{R}^3$, for suitable $s_1, s_\parallel \in \mathbb{R}$. For any $r \in \mathbb{R}$, we indicate the corresponding weighted spaces with the short-hand notations

$$H_{s_1}^r(\mathbb{R}) \equiv H_{w_{s_1}}^r(\mathbb{R}) , \qquad H_{s_\parallel}^r(\mathbb{R}^2) \equiv H_{w_{s_\parallel}}^r(\mathbb{R}^2) , \qquad H_{s_1,s_\parallel}^r(\mathbb{R}^3) \equiv H_{w_{s_1,s_\parallel}}^r(\mathbb{R}^3) ;$$

moreover, noting the elementary identity $1/w_{s_1} = w_{-s_1}$ and the related analogues for $w_{s_\parallel}$ and $w_{s_1,s_\parallel}$, from Eq. (2.12) we infer the duality relations

$$H_{-s_1}^{-r}(\mathbb{R}) = \left(H_{s_1}^r(\mathbb{R})\right)' , \qquad H_{-s_\parallel}^{-r}(\mathbb{R}^2) = \left(H_{s_\parallel}^r(\mathbb{R}^2)\right)' , \qquad H_{-s_1,-s_\parallel}^{-r}(\mathbb{R}^3) = \left(H_{s_1,s_\parallel}^r(\mathbb{R}^3)\right)' . \tag{2.13}$$

Before proceeding, let us mention that all the weighted spaces considered above are indeed Hilbert spaces, endowed with the corresponding natural inner products. For any given pair $\mathcal{H}_1, \mathcal{H}_2$ of these spaces, we regard the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ as a Hilbert space itself, equipped with the usual inner product defined on factorized elements and extended to the whole space by linearity.

In the forthcoming Lemma 2.5 we collect a number of results providing a more explicit characterization of the weighted spaces described previously; these results will be employed systematically in the derivation of the subsequent developments, in particular in Section 4 for the proof of the LAP.

Lemma 2.5. *The following statements i)–iv) hold true.*
i) Assume that $s_1, s_\parallel \geqslant 0$; then, for all $r \in \mathbb{R}$ the following embeddings define continuous maps:

$$H_{s_1}^r(\mathbb{R}) \hookrightarrow H^r(\mathbb{R}) , \qquad H_{s_\parallel}^r(\mathbb{R}^2) \hookrightarrow H^r(\mathbb{R}^2) , \qquad H_{s_1,s_\parallel}^r(\mathbb{R}^r) \hookrightarrow H^r(\mathbb{R}^3) .$$

ii) Let $s_1, s_\parallel \in \mathbb{R}$ and consider the multiplication operators

$$\begin{gathered} I_{s_1} : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}) , \ u_1 \mapsto w_{s_1}^{1/2} u_1 , \qquad I_{s_\parallel} : \mathcal{D}'(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2) , \ u_\parallel \mapsto w_{s_\parallel}^{1/2} u_\parallel , \\ I_{s_1,s_\parallel} : \mathcal{D}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3) , \ f \mapsto w_{s_1,s_\parallel}^{1/2} f . \end{gathered} \tag{2.14}$$

For any $r \in \mathbb{R}$, these operators define by restriction the isomorphism of Banach spaces

$$I_{s_1} : H_{s_1}^r(\mathbb{R}) \to H^r(\mathbb{R}), \quad I_{s_\parallel} : H_{s_\parallel}^r(\mathbb{R}^2) \to H^r(\mathbb{R}^2), \quad I_{s_1,s_\parallel} : H_{s_1,s_\parallel}^r(\mathbb{R}^3) \to H^r(\mathbb{R}^3).$$

iii) Let $s_1, s_\parallel \in \mathbb{R}$; then, for all $r, r' \in \mathbb{R}$ with $r \geqslant r'$ the following embeddings define continuous maps:

$$H_{s_1}^r(\mathbb{R}) \hookrightarrow H_{s_1}^{r'}(\mathbb{R}) , \qquad H_{s_\parallel}^r(\mathbb{R}^2) \hookrightarrow H_{s_\parallel}^{r'}(\mathbb{R}^2) , \qquad H_{s_1,s_\parallel}^r(\mathbb{R}^3) \hookrightarrow H_{s_1,s_\parallel}^{r'}(\mathbb{R}^3) .$$

iv) Let $s_1, s_\parallel \in \mathbb{R}$; then, for all $r_1, r_\parallel \geqslant 0$ the following embedding defines a continuous map:

$$H_{s_1}^{r_1}(\mathbb{R}) \otimes H_{s_\parallel}^{r_\parallel}(\mathbb{R}^2) \hookrightarrow H_{s_1,s_\parallel}^{\min(r_1,r_\parallel)}(\mathbb{R}^3) .$$

Proof. In the following we discuss separately the proofs of items i)–iv). Concerning items i)–iii), as examples we derive the corresponding statements involving the space $H_{s_1}^{r_1}(\mathbb{R})$; the analogous claims regarding $H_{s_\parallel}^r(\mathbb{R}^2)$ and $H_{s_1,s_\parallel}^r(\mathbb{R}^3)$ can be inferred by similar arguments.

i) For $s_1 \geqslant 0$ and $r = n \in \{0, 1, 2, ...\}$, it can be easily checked by direct inspection that $\|u_1\|^2_{H^n_{-s_1}(\mathbb{R})} \leqslant \|u_1\|^2_{H^n(\mathbb{R})} \leqslant \|u_1\|^2_{H^n_{s_1}(\mathbb{R})}$, which proves that $H^n_{s_1}(\mathbb{R}) \hookrightarrow H^n(\mathbb{R})$ and $H^n(\mathbb{R}) \hookrightarrow H^n_{-s_1}(\mathbb{R})$. Recalling our position (2.11) and its dual analogue, from here we infer by interpolation that $H^r_{s_1}(\mathbb{R}) \hookrightarrow H^r(\mathbb{R})$ and $H^r(\mathbb{R}) \hookrightarrow H^r_{-s_1}(\mathbb{R})$ for all $r \geqslant 0$. The first of these relations proves the thesis for $r \geqslant 0$. On the other hand, on account of the duality relation in Eq. (2.13), the second relation allow us to infer that $H^r_{s_1} = (H^{-r}_{-s_1}(\mathbb{R}))' \hookrightarrow (H^{-r}(\mathbb{R}))' = H^r(\mathbb{R})$ for $r \leqslant 0$, which completes the proof.
ii) First of all, let us remark that for any $s_1 \in \mathbb{R}$ the map I_{s_1} defined in Eq. (2.14) is invertible, with inverse given by $I^{-1}_{s_1} = I_{-s_1}$.

Next, let $n \in \{0, 1, 2, ...\}$. It can be checked by direct inspection that the weight $w^{1/2}_{s_1} = w_{s_1/2}$ on $\mathbb{R}$ fulfills the conditions of [36, p. 263, Def. 6.1] and that $u_1 \mapsto \|w^{1/2}_{s_1} u_1\|^2_{H^n(\mathbb{R})}$ determines a norm on $H^n_{s_1}(\mathbb{R})$ which is equivalent to that of Eq. (2.10) (compare with [36, p. 267, Eq. (6.18)]); so, from [36, p. 267, Th. 6.9] it follows that $H^n_{s_1}(\mathbb{R}) = F^n_{2,2}(\mathbb{R}, w^{1/2}_{s_1})$, where $F^r_{2,2}(\mathbb{R}, w^{1/2}_{s_1})$ are the weighted Triebel–Lizorkin spaces defined according to [36, p. 264, Def. 6.3]. Due to our position (2.13), the latter identity and [32, p. 244, Eq. (7)] yield in addition the dual relation $H^{-n}_{s_1}(\mathbb{R}) = F^{-n}_{2,2}(\mathbb{R}, w^{1/2}_{s_1})$, again for $n \in \{0, 1, 2, ...\}$. Furthermore, let us recall that $H^r(\mathbb{R}) = F^r_{2,2}(\mathbb{R}, 1) \equiv F^r_{2,2}(\mathbb{R})$ for any $r \in \mathbb{R}$ (see, e.g., [36, p. 3, Eq. (1.8)]). On the other hand, [36, p. 265, Th. 6.5] states that the map $F^r_{2,2}(\mathbb{R}, w^{1/2}_{s_1}) \to F^r_{2,2}(\mathbb{R})$, $u_1 \mapsto w^{1/2}_{s_1} u_1$ is an isomorphism of Banach spaces for all $r \in \mathbb{R}$. In view of the facts mentioned previously, the latter result allows us to infer that the map $\big(I_{s_1} \restriction H^r_{s_1}(\mathbb{R})\big) : H^r_{s_1}(\mathbb{R}) \to H^r(\mathbb{R})$ is a Banach isomorphism for all $r \in \mathbb{Z}$. Then, the analogous statement for arbitrary $r \in \mathbb{R}$ follows by interpolation from [21, p. 46, Th. 2.1.6], recalling our definition (2.11) and its dual counterpart.
iii) The previously proven item ii) implies that $H^r_{s_1}(\mathbb{R})$ is isomorphic to $H^r(\mathbb{R})$ for any $r \in \mathbb{R}$. Then the thesis follows straightforwardly from the standard Sobolev embedding $H^r(\mathbb{R}) \hookrightarrow H^{r'}(\mathbb{R})$, holding true for $r \geqslant r'$.
iv) Again, due to item ii) we know that $H^{r_1}_{s_1}(\mathbb{R})$, $H^{r_\parallel}_{s_\parallel}(\mathbb{R}^2)$ and $H^{\min(r_1,r_\parallel)}_{s_1,s_\parallel}(\mathbb{R}^3)$ are respectively isomorphic to $H^{r_1}(\mathbb{R})$, $H^{r_\parallel}(\mathbb{R}^2)$ and $H^{\min(r_1,r_\parallel)}(\mathbb{R}^3)$. On the other hand, Eq. (2.8) of Lemma 2.3 allows us to infer that $H^{r_1}(\mathbb{R}) \otimes H^{r_\parallel}(\mathbb{R}^2)$ is continuously embedded into $H^{\min(r_1,r_\parallel)}(\mathbb{R}^3)$ for any $r_1, r_\parallel \geqslant 0$. Altogether, the previous arguments yield the thesis. □

The subsequent Lemmata 2.6 and 2.7, characterize the free resolvent operator $R_\varnothing(z)$ and the trace τ_F on the plane π_F as bounded maps on the weighted spaces under analysis.

Lemma 2.6. *Let $s_1, s_\parallel \in \mathbb{R}$ and $r \in \mathbb{R}$. Then, for all $z \in \mathbb{C}\backslash[0, +\infty)$ there holds*

$$R_\varnothing(z) \in \mathfrak{B}(H^r_{s_1,s_\parallel}(\mathbb{R}^3), H^{r+2}_{s_1,s_\parallel}(\mathbb{R}^3)) \,. \tag{2.15}$$

Proof. First of all, let us notice that by [29, p. 170, Lem. 1] we have $R_\varnothing(z) \in \mathfrak{B}(L^2_{s_1,s_\parallel}(\mathbb{R}^3))$ for all $s_1, s_\parallel \in \mathbb{R}$; from here it can be inferred that $R_\varnothing(z) \in \mathfrak{B}(L^2_{s_1,s_\parallel}(\mathbb{R}^3), H^2_{s_1,s_\parallel}(\mathbb{R}^3))$, using arguments similar to those described in the proof of Th. 4.2 in [24]. This yields the thesis for $r = 0$; then, proceeding by induction it can be inferred by similar arguments that Eq. (2.15) holds true as well for all $r = 2n$ with $n \in \{0, 1, 2, ...\}$.

To proceed let us remark that $R_\varnothing(z) = R_\varnothing(\overline{z})^*$, where * indicates the adjoint with respect to the usual L^2-duality; thus, recalling the basic relation (2.13) we obtain $R_\varnothing(z) \in \mathfrak{B}(H^{-2n-2}_{-s_1,-s_\parallel}(\mathbb{R}^3), H^{-2n}_{-s_1,-s_\parallel}(\mathbb{R}^3))$ for all $n \in \{0, 1, 2, ...\}$, which is equivalent to Eq. (2.15) for $r = -2n$ after the obvious replacement $(s_1\,, s_\parallel) \to (-s_1\,, -s_\parallel)$.

Finally, notice that our definition (2.11) and its dual counter part yield the interpolation identity

$$[H^{2n}_{s_1,s_\parallel}(\mathbb{R}^3), H^{2n+2}_{s_1,s_\parallel}(\mathbb{R}^3)]_\theta = H^{2n+2\theta}_{s_1,s_\parallel}(\mathbb{R}^3) \quad \text{for any } \theta \in (0, 1) \text{ and all } n \in \mathbb{Z};$$

then, from [21, p. 46, Th. 2.1.6] we can conclude that the thesis stated in Eq. (2.15) holds true for arbitrary $r \in \mathbb{R}$ by interpolation. □

Lemma 2.7. *Let $s_1, s_\parallel \in \mathbb{R}$, $r \in (0, 1/2]$. Then, the evaluation map $\tau_F : \mathcal{D}(\mathbb{R}^3) \to \mathcal{D}(\mathbb{R}^2)$, $(\tau_F f)(\mathbf{x}_\parallel) := f(F(\mathbf{x}_\parallel), \mathbf{x}_\parallel)$ (see subsection 2.5) can be uniquely extended to a continuous operator*

$$\tau_F \in \mathfrak{B}(H^{r+1/2}_{s_1,s_\parallel}(\mathbb{R}^3), H^r_{s_\parallel}(\mathbb{R}^2)) \,.$$

Proof. In order to avoid misunderstandings, throughout the present proof we shall temporarily indicate with $\hat{\tau}_F$ the usual trace operator introduced in Remark 2.4, mapping $H^{r+1/2}(\mathbb{R}^3)$ into $H^r(\mathbb{R}^2)$ for $r \in (0, 1/2]$. It can be checked by elementary computations that on $\mathcal{D}(\mathbb{R}^3)$ we have

$$\tau_F = I^{-1}_{s_\parallel}\, W^{(F)}_{s_1}\, \hat{\tau}_F\, I_{s_1,s_\parallel} \,, \tag{2.16}$$

where $I_{s_\parallel}, I_{s_1,s_\parallel}$ are the Banach isomorphisms introduced in item ii) of Lemma 2.5 and $W^{(F)}_{s_1}$ indicates the multiplication operator defined by $(W^{(F)}_{s_1} u_\parallel)(\mathbf{x}_\parallel) := w^{-1/2}_{s_1}(F(\mathbf{x}_\parallel))\, u_\parallel(\mathbf{x}_\parallel)$.

Taking into account the identity (2.16) and the fact that $\mathcal{D}(\mathbb{R}^3)$ is a dense subset of $H^{r+1/2}_{s_1,s_\parallel}(\mathbb{R}^3)$, the thesis follows as soon as we can prove that $W^{(F)}_{s_1} \in \mathfrak{B}(H^r(\mathbb{R}^2))$ for $r \in (0, 1/2]$. On the other hand, due to our assumption $F \in C^{0,1}_0(\mathbb{R}^2)$, we have $w^{-1/2}_{s_1}(F(\cdot)) \in C^{0,1}_b(\mathbb{R}^2)$ (i.e. $w^{-1/2}_{s_1}(F(\cdot))$ is Lipschitz continuous and uniformly bounded); by [2, p. 13, Th. 1.9.2], this suffices to infer that $W^{(F)}_{s_1}$ is indeed a bounded multiplier in $H^r(\mathbb{R}^2)$ for all $r \in (0, 1)$ (hence, in particular, for $r \in (0, 1/2]$), which yields the thesis in view of the previous considerations. □

Remark 2.8. The proof of Lemma 2.7 can be easily generalized to cases where F fulfills stronger regularity assumptions. In particular, for $F = 0$ one readily obtains

$$\tau_0 \in \mathfrak{B}(H^{r+1/2}_{s_1,s_\parallel}(\mathbb{R}^3), H^r_{s_\parallel}(\mathbb{R}^2)) \qquad \text{for all } \; r > 0 \,.$$

3. Schrödinger operators A_0 and A_F, and their resolvents

In this section we give a rigorous definition of the operators A_0 and A_F and obtain a Kreĭn's type formula for their resolvents. We remark that a rigorous definition of the operators A_0 and A_F can also be obtained, more directly, starting from the associated quadratic form, see, e.g., [4,9,18]. However, since we shall extensively use Kreĭn's type resolvent formulae, we prefer to use them also to characterize the domain and action of the operators A_0 and A_F.

We consider first the case in which the interaction is supported on the surface π_F, and give several definitions and results. The corresponding definitions for the plane π_0 are simply obtained by setting $F = 0$; obviously, all the results obtained for generic $F \in C^{0,1}_0(\mathbb{R})$ remain true. Indeed, in the flat case, by exploiting the factorized structure of $\mathbb{R}^3$, several quantities can be explicitly computed and some results can be improved. We pursue this goal at the end of the section.

Remark 3.1. It is a well known fact (see also Lemma 2.6) that

$$R_\varnothing(z) \in \mathfrak{B}(H^r(\mathbb{R}^3), H^{r+2}(\mathbb{R}^3)) \qquad \text{for all } r \in \mathbb{R} \,,$$

where $R_\varnothing(z)$ is the resolvent of the free Laplacian (see Eqs. (2.1), (2.2)).

3.1. Resolvent and rigorous definition of A_F

Next, we introduce several families of operators defined by means of the trace operator τ_F and the free resolvent $R_\varnothing(z)$.

Taking into account Remarks 2.4 and 3.1, we define the operator

$$\breve{G}_F(z) := \tau_F R_\varnothing(z) : L^2(\mathbb{R}^3) \to H^{1/2}(\mathbb{R}^2) \qquad \text{with } z \in \mathbb{C}\backslash[0,+\infty)\,, \tag{3.1}$$

which admits a unique continuous extension

$$\breve{G}_F(z) \in \mathfrak{B}(H^{r-3/2}(\mathbb{R}^3), H^r(\mathbb{R}^2)) \qquad \text{for } r \in (0,1/2]\,.$$

The corresponding adjoint (meant in the sense of the Sobolev duality $H^{-r}(\mathbb{R}^k) = (H^r(\mathbb{R}^k))'$) with conjugate spectral parameter is the *single layer operator*

$$G_F(z) := (\tau_F R_\varnothing(\bar{z}))^* : H^{-1/2}(\mathbb{R}^2) \to L^2(\mathbb{R}^3) \qquad \text{with } z \in \mathbb{C}\backslash[0,+\infty)\,,$$

which admits a unique continuous extension

$$G_F(z) \in \mathfrak{B}(H^{-r}(\mathbb{R}^2), H^{3/2-r}(\mathbb{R}^3)) \qquad \text{for } r \in (0,1/2]\,. \tag{3.2}$$

Next, taking into account Eq. (3.2), we consider the trace of the single layer operator

$$M_F(z) := \tau_F G_F(z)\,,$$

so that

$$M_F(z) \in \mathfrak{B}(H^{-1/2}(\mathbb{R}^2), H^{1/2}(\mathbb{R}^2)). \tag{3.3}$$

The Sobolev indices in the latter claim are fixed by the restriction $r \in (0,1/2]$ in Eq. (3.2), and by the fact that it must be $1-r \in (0,1/2]$ as well. Let us notice that with respect to the $H^{-1/2}(\mathbb{R}^2)$-$H^{1/2}(\mathbb{R}^2)$ duality induced by the $L^2(\mathbb{R}^2)$ scalar product, one has

$$M_F(z)^* = (\tau_F R_\varnothing(z)\tau_F^*)^* = M_F(\bar{z})\,. \tag{3.4}$$

Remark 3.2. Notice that the result in Eq. (3.3) can be improved whenever F is smooth: in this case $M_F(z) \in \mathfrak{B}(H^{r-1/2}(\mathbb{R}^2), H^{r+1/2}(\mathbb{R}^2))$ for any $r \in \mathbb{R}$ (see, e.g., [28, Prop. 13]).

For any given $\alpha > 0$, we define the operator

$$\Gamma_F(z) := \big(1 + \alpha\, M_F(z)\big)\,. \tag{3.5}$$

The following lemma guarantees the invertibility of the operator $\Gamma_F(z)$ for z far away from the real positive axis. Indeed, in view of [11, Th. 2.19], one has that $\Gamma_F^{-1}(z) \in \mathfrak{B}(H^r(\mathbb{R}^2))$ for all $|r| < 1/2$ and $z \in \mathbb{C}\backslash[0,+\infty)$, see Theorem 3.4 and Remark 3.5 below.

Lemma 3.3. *Let $d_z := \inf_{\lambda\in[0,+\infty)} |\lambda - z|$. Then there exists $z_0 \in \mathbb{C}\backslash[0,+\infty)$ such that*

$$\Gamma_F^{-1}(z) \in \mathfrak{B}(H^r(\mathbb{R}^2)) \qquad \textit{for all } |r| < 1/2 \textit{ and } z \in \mathbb{C} \textit{ such that } d_z > d_{z_0}\,. \tag{3.6}$$

Proof. We start with the proof of the following statement: for any $\varepsilon \in (0, 1/2)$, and $z \in \mathbb{C}\backslash[0, +\infty)$, on has

$$\|M_F(z)\|_{\mathfrak{B}(H^{-1/2+\varepsilon}(\mathbb{R}^2),H^{1/2-\varepsilon}(\mathbb{R}^2))} \leqslant_c d_z^{-\varepsilon} \qquad \left(d_z := \inf_{\lambda\in[0,+\infty)} |\lambda - z|\right). \tag{3.7}$$

Recall that $\tau_F \in \mathfrak{B}(H^{r+1/2}(\mathbb{R}^3), H^r(\mathbb{R}^2))$ for all $r \in (0, 1/2]$, which suffices to infer $\tau_F^* \in \mathfrak{B}(H^{-r}(\mathbb{R}^2), H^{-r-1/2}(\mathbb{R}^3))$ as well. From the identity $M_F(z) = \tau_F R_\varnothing(z)\tau_F^*$ we obtain the bound

$$\begin{aligned}&\|M_F(z)\|_{\mathfrak{B}(H^{-r+\varepsilon}(\mathbb{R}^2),H^{1-r-\varepsilon}(\mathbb{R}^2))}\\ &\leq \|\tau_F\|_{\mathfrak{B}(H^{3/2-r-\varepsilon}(\mathbb{R}^3),H^{1-r-\varepsilon}(\mathbb{R}^2))}\|R_\varnothing(z)\|_{\mathfrak{B}(H^{-r-1/2+\varepsilon}(\mathbb{R}^3),H^{3/2-r-\varepsilon}(\mathbb{R}^3))}\|\tau_F^*\|_{\mathfrak{B}(H^{-r+\varepsilon}(\mathbb{R}^2),H^{-r-1/2+\varepsilon}(\mathbb{R}^3))}\,.\end{aligned}$$

Where we used $1 - r - \varepsilon \in (0, 1/2]$ and $r - \varepsilon \in (0, 1/2]$. Then, the thesis follows as soon as we can show the norm bound

$$\|R_\varnothing(z)\|_{\mathfrak{B}(H^{-r-1/2+\varepsilon}(\mathbb{R}^3),H^{3/2-r-\varepsilon}(\mathbb{R}^3))} \leqslant_c \frac{1}{d_z^{\varepsilon}}\,. \tag{3.8}$$

Let us fix arbitrarily $f \in H^{-r-1/2+\varepsilon}(\mathbb{R}^3)$ and notice that, using the equivalent Sobolev norm (2.4) (see also Eq. (2.5)), we have

$$\|R_\varnothing(z)f\|^2_{H^{3/2-r-\varepsilon}(\mathbb{R}^3)} \leqslant_c \|(1-\Delta)^{(3/2-r-\varepsilon)/2}R_\varnothing(z)f\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} d\mathbf{k}\, \frac{(1+|\mathbf{k}|^2)^{3/2-r-\varepsilon}}{\left||\mathbf{k}|^2 - z\right|^2}\, |(\mathfrak{F}f)(\mathbf{k})|^2\,.$$

Then, using the elementary estimates $||\mathbf{k}|^2 - z| \geqslant d_z$ and

$$\sup_{\mathbf{k}\in\mathbb{R}^3}\left(\frac{(1+|\mathbf{k}|^2)^{3/2-r-\varepsilon}(1+|\mathbf{k}|^2)^{r+1/2-\varepsilon}}{\left||\mathbf{k}|^2-z\right|^2}\right) \leqslant \sup_{\mathbf{k}\in\mathbb{R}^3}\left(\frac{(1+|\mathbf{k}|^2)^{2-2\varepsilon}}{\left||\mathbf{k}|^2-z\right|^{2-2\varepsilon}d_z^{2\varepsilon}}\right) \leqslant_c \frac{1}{d_z^{2\varepsilon}}$$

and making reference to the relations contained in Eqs. (2.4)–(2.6), we obtain

$$\|R_\varnothing(z)f\|^2_{H^{3/2-r-\varepsilon}(\mathbb{R}^3)} \leqslant_c \frac{1}{d_z^{2\varepsilon}}\int_{\mathbb{R}^3} d\mathbf{k}\,(1+|\mathbf{k}|^2)^{-r-1/2+\varepsilon}\,|(\mathfrak{F}f)(\mathbf{k})|^2 \leqslant_c \frac{1}{d_z^{2\varepsilon}}\,\|f\|^2_{H^{-r-1/2+\varepsilon}(\mathbb{R}^3)}\,;$$

in view of the arbitrariness of $f \in H^{-r-1/2+\varepsilon}(\mathbb{R}^3)$, the above bound proves the statement (3.8), whence (3.7).

To prove claim (3.6), we note first that, by continuous Sobolev embedding,

$$\|M_F(z)\|_{\mathfrak{B}(H^r(\mathbb{R}^2))} \leqslant_c \|M_F(z)\|_{\mathfrak{B}(H^{-1/2+\varepsilon}(\mathbb{R}^2),H^{1/2-\varepsilon}(\mathbb{R}^2))} \leqslant_c \frac{1}{d_z^{\varepsilon}}$$

for any $r \in (-1/2, 1/2)$, and $\varepsilon \in (0, 1/2)$ such that $-1/2+\varepsilon \leq r \leq 1/2-\varepsilon$. Hence, $\|M_F(z)\|_{\mathfrak{B}(H^r(\mathbb{R}^2))} \leqslant_c 1/d_z^{\varepsilon}$ for all $r \in (-1/2, 1/2)$ and $\varepsilon \in (0, 1/2 - |r|)$. As a consequence, for $r \in (-1/2, 1/2)$, $\Gamma_F(z) \in \mathfrak{B}(H^r(\mathbb{R}^2))$ has bounded inverse for any z such that $d_z > \alpha^{1/\varepsilon}$. □

We note that

$$(u, \Gamma_F(z)\, v)_{H^{-1/2}(\mathbb{R}^2),H^{1/2}(\mathbb{R}^2)} = \overline{(v, \Gamma_F(\bar z)\, u)_{H^{-1/2}(\mathbb{R}^2),H^{1/2}(\mathbb{R}^2)}}\,,$$

with $u, v \in H^{-1/2}(\mathbb{R}^2)$, and where $(\cdot,\cdot)_{X',X}$ denotes the duality product between X and its dual X', which follows immediately from the identity $M_F(z)^* = M_F(\bar z)$. Moreover, for all $z, w \in \mathbb{C}\backslash[0, +\infty)$, on has

$$\alpha^{-1}\Gamma_F(z) - \alpha^{-1}\Gamma_F(w) = (z-w)\,\breve{G}_F(w)\,G_F(z)\,. \tag{3.9}$$

To prove the latter identity, note that

$$\breve{G}_F(z) - \breve{G}_F(w) = \tau_F\big(R_\varnothing(z) - R_\varnothing(w)\big) = (z-w)\,\tau_F R_\varnothing(z)R_\varnothing(w) = (z-w)\,\breve{G}_F(z)R_\varnothing(w)\,.$$

By taking the adjoint (in $\bar{z}$ and $\bar{w}$) it follows that $G_F(z) - G_F(w) = (z-w)R_\varnothing(w)G_F(z)$. Hence,

$$M_F(z) - M_F(w) = \tau_F\big(G_F(z) - G_F(w)\big) = (z-w)\,\breve{G}_F(w)\,G_F(z)\,,$$

from which Identity (3.9) readily follows.

By Lemma 3.3, [27, Th. 2.1] and [11, Th. 2.19], we have the following

Theorem 3.4. *There holds*

$$\Gamma_F^{-1}(z) \in \mathfrak{B}(H^r(\mathbb{R}^2)) \qquad \text{for all } |r| < 1/2 \text{ and } z \in \mathbb{C}\backslash[0,+\infty)\,, \tag{3.10}$$

and the bounded linear operator

$$R_F(z) := R_\varnothing(z) - \alpha\, G_F(z)\,\Gamma_F^{-1}(z)\,\breve{G}_F(z) \qquad \text{with } z \in \mathbb{C}\backslash[0,+\infty) \tag{3.11}$$

is the resolvent of the self-adjoint operator A_F which coincides with $A_\varnothing$ on $H^2(\mathbb{R}^3\backslash\pi_F)$ and which is defined by

$$\begin{gathered} Dom(A_F) := \big\{f \in L^2(\mathbb{R}^3) \;\big|\; f = f_z - \alpha\, G_F(z)\,\Gamma_F^{-1}(z)\,\tau_F f_z\,,\ f_z \in H^2(\mathbb{R}^3)\big\} \\ (A_F - z)f := (A_\varnothing - z)f_z\,. \end{gathered}$$

Remark 3.5. Few comments on the proof of Theorem 3.4 are in order. By Lemma 3.3 and [27, Th. 2.1] one has that $R_F(z)$, given in Eq. (3.11), is the resolvent of the operator A_F for d_z large enough. Hence, [11, Th. 2.19], together with Eq. (3.9) and the fact that $\Gamma_F(z)^* = \Gamma_F(\bar{z})$ (see Eq. (3.4) and (3.5)) implies that $\Gamma_F(z)$ is invertible for all $z \in \rho(A_\varnothing)\cap\rho(A_F)$, where $\rho(A_\varnothing)$ and $\rho(A_F)$ denote the resolvent set of $A_\varnothing$ and A_F respectively. That $\rho(A_\varnothing)\cap\rho(A_F) \subseteq \mathbb{C}\backslash[0,+\infty)$ is a consequence of Remark 3.6 below. Indeed, we shall prove that $\sigma(A_F) = \sigma(A_\varnothing) = [0,+\infty)$, hence $\rho(A_\varnothing)\cap\rho(A_F) = \mathbb{C}\backslash[0,+\infty)$, see Remark 6.2.

With f given as in $\mathrm{Dom}(A_F)$ one has

$$\tau_F f = \tau_F f_z - \alpha\, M_F(z)\Gamma_F^{-1}(z)\tau_F f_z = \alpha\,\Gamma_F^{-1}(z)\tau_F f_z\,.$$

Moreover, indicating with $\mathbf{n}$ the unit vector normal to the surface π_F pointing to the right and with $\partial_\mathbf{n}$ the derivative in the direction normal to π_F, using the notations introduced in Eq. (2.9) one has

$$[\partial_\mathbf{n} f]_{\pi_F} = -\,\alpha\,[\partial_\mathbf{n} G_F(z)\,\Gamma_F^{-1}(z)\,\tau_F f_z]_{\pi_F}\,.$$

Since $[\partial_\mathbf{n} G_F(z)\,u]_{\pi_F} = -\,u$ (see, e.g., [25]), the operator A_F can be also characterized as

$$\begin{gathered} \mathrm{Dom}(A_F) = \big\{f \in H^2(\mathbb{R}^3\backslash\pi_F) \;\big|\; f\restriction_{\pi_F^+} = f\restriction_{\pi_F^-} = f\restriction_{\pi_F}\,,\ [\partial_\mathbf{n} f]_{\pi_F} = \alpha\, f\restriction_{\pi_F}\,\big\}\,, \\ A_F f = -\Delta f \qquad \forall\,\mathbf{x} \in \mathbb{R}^3\backslash\pi_F \quad \text{and} \quad f \in \mathrm{Dom}(A_F)\,. \end{gathered}$$

The operator A_F corresponds to the singular perturbation of the free Laplacian formally written as in Eq. (1.1), and given by a delta-type potential of strength α supported on the surface π_F.

Remark 3.6. By the definition of $M_F(z)$ one has that, for all $\lambda > 0$

$$(u, M_F(-\lambda)\, u)_{L^2(\mathbb{R}^2)} = (\tau_F^* u, R_\varnothing(-\lambda)\, \tau_F^* u)_{H^{-1}(\mathbb{R}^3), H^1(\mathbb{R}^3)} \geqslant 0,$$

because $R_\varnothing(-\lambda)$ is a positive definite operator. Hence,

$$\|u\|_{L^2(\mathbb{R}^2)} \|(1 + \alpha M_F(-\lambda))u\|_{L^2(\mathbb{R}^2)} \geqslant \big(u, (1 + \alpha M_F(-\lambda))u\big)_{L^2(\mathbb{R}^2)} \geqslant \|u\|^2_{L^2(\mathbb{R}^2)},$$

which in turn implies that the inverse $\Gamma_F^{-1}(-\lambda) = (1+\alpha M_F(-\lambda))^{-1}$ is a well-defined and bounded operator in $L^2(\mathbb{R}^2)$ for all $\lambda > 0$. This argument, together with the fact that for $\lambda > 0$ the operators $R_\varnothing(-\lambda)$, $G_F(-\lambda)$, and $\breve{G}_F(-\lambda)$ are bounded in $L^2(\mathbb{R}^k)$ $(k = 2, 3)$, allows us to infer that $R_F(-\lambda) \in \mathfrak{B}(L^2(\mathbb{R}^3))$ and $\sigma(A_F) \subseteq [0, +\infty)$.

3.2. Resolvent and rigorous definition of A_0

Obviously, all the results stated in the previous section hold true for the case $F = 0$ as well. In particular, by Theorem 3.4, the operator

$$\Gamma_0(z) = 1 + \alpha\, M_0(z)$$

is certainly invertible in $H^r(\mathbb{R}^2)$ for all $|r| \leqslant 1/2$ and $z \in \mathbb{C}\backslash[0, +\infty)$ (indeed it admits a bounded inverse for all $r \in \mathbb{R}$, see Lemma 3.8 below). Hence, the operator $R_0(z)$ is well defined and it is the resolvent of the self-adjoint operator

$$\mathrm{Dom}(A_0) = \left\{ f \in H^2(\mathbb{R}^3\backslash\pi_0) \;\middle|\; f\restriction_{\pi_0^+} = f\restriction_{\pi_0^-} = f\restriction_{\pi_0},\ [\partial_{\mathbf{n}} f]_{\pi_0} = \alpha\, f\restriction_{\pi_0} \right\},$$

where we used the notations of Eq. (2.9), $\partial_{\mathbf{n}}$ denotes the derivative in the direction normal to the plane π_0, $(\partial_{\mathbf{n}} f)\restriction_{\pi_0^\pm} \equiv (\partial_{x^1} f)\restriction_{x^1 = 0^\pm}$;

$$A_0 f = -\Delta f \qquad \forall\, \mathbf{x} \in \mathbb{R}^3\backslash\pi_0 \quad \text{and} \quad f \in \mathrm{Dom}(A_0)\,.$$

Remark 3.7. Explicitly $R_0(z)$ reads

$$R_0(z) := R_\varnothing(z) - \alpha\, G_0(z)\, \Gamma_0^{-1}(z)\, \breve{G}_0(z)\,, \tag{3.12}$$

and it is a bounded operator for all $z \in \mathbb{C}\backslash[0, +\infty)$.

Before proceeding further let us point out some special properties of the operators $\breve{G}_0(z)$, $G_0(z)$, and $M_0(z)$. For all $r > 0$, since $\tau_0 : H^{r+1/2}(\mathbb{R}^3) \to H^r(\mathbb{R}^2)$, we have

$$\breve{G}_0(z) \in \mathfrak{B}(H^{r-3/2}(\mathbb{R}^3), H^r(\mathbb{R}^2))\,, \qquad G_0(z) \in \mathfrak{B}(H^{-r}(\mathbb{R}^2), H^{3/2-r}(\mathbb{R}^3))\,.$$

Moreover let us recall a fact which was proven in [10]; namely that there holds true the identity

$$M_0(z) = \frac{i}{2}\,(z + \Delta_\parallel)^{-1/2} \in \mathfrak{B}(L^2(\mathbb{R}^2)) \qquad \text{for any } z \in \mathbb{C}\backslash[0, +\infty)\,.$$

In view of the above result, it appears that $M_0(z)$ sends $H^r(\mathbb{R}^2)$ continuously into $H^{r+1}(\mathbb{R}^2)$ for any $r \in \mathbb{R}$; in turn, this allows to infer a stronger version of the statement in Eq. (3.10) in the particular case $F = 0$:

Remark 3.8. For any $z \in \mathbb{C}\backslash[0,+\infty)$ and any $r \in \mathbb{R}$, $\Gamma_0^{-1}(z) \in \mathfrak{B}(H^r(\mathbb{R}^2))$. To see that this is indeed the case, let $u \in H^r(\mathbb{R}^2)$; then, using the equivalent Fourier norm on $H^r(\mathbb{R}^2)$, one obtains

$$\|\Gamma_0^{-1}(z)\,u\|^2_{H^r(\mathbb{R}^2)} \leqslant_c \int_{\mathbb{R}^2} d\mathbf{k}_\parallel\;(1+|\mathbf{k}_\parallel|^2)^r \left|\frac{\sqrt{|\mathbf{k}_\parallel|^2 - z}}{\frac{\alpha}{2} + \sqrt{|\mathbf{k}_\parallel|^2 - z}}\right|^2 |(\mathfrak{F}\,u)(\mathbf{k}_\parallel)|^2 \,.$$

Note that for all $\alpha > 0$ and all $z \in \mathbb{C}\backslash[0,+\infty)$, the map

$$(0,+\infty) \ni k \mapsto \left|\frac{\sqrt{k^2 - z}}{\frac{\alpha}{2} + \sqrt{k^2 - z}}\right|^2$$

is bounded. This and the previous bound suffice to infer the claim.

Remark 3.9. The operator A_0 can be equivalently represented as

$$A_0 = A_0^{(1)} \otimes 1_\parallel + 1_1 \otimes A_0^{(\parallel)}\,, \tag{3.13}$$

where 1_1, $1_\parallel$ indicate, respectively, the identity operators on $L^2(\mathbb{R})$, $L^2(\mathbb{R}^2)$, and

$$A_0^{(1)} := -\,\partial_{x^1x^1} : \mathrm{Dom}A_0^{(1)} \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})\,, \tag{3.14}$$

$$\mathrm{Dom}A_0^{(1)} := \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) \,|\, u'(0^+) - u'(0^-) = \alpha\, u(0)\}\,, \tag{3.15}$$

$$A_0^{(\parallel)} := -\,\Delta_\parallel : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\,.$$

In other words, $A_0^{(1)}$ denotes the self-adjoint operator on $L^2(\mathbb{R})$ corresponding to the formal expression "$-\,\partial_{x^1x^1} + \alpha\,\delta_{\{x^1=0\}}$" (see [3, Th. 3.1.1]) and $A_0^{(\parallel)}$ is the free Laplacian on the plane π_0. We recall that $\sigma(A_0^{(\parallel)}) = \sigma_{ac}(A_0^{(\parallel)}) = [0,+\infty)$ and that, for $\alpha > 0$, one has $\sigma(A_0^{(1)}) = \sigma_{ac}(A_0^{(1)}) = [0,+\infty)$ (see [3, Th. 3.1.4]). Hence,

$$\sigma(A_0) = \sigma_{ac}(A_0) = [0,+\infty)\,.$$

3.3. A formula for the difference of the resolvents

We conclude this section by pointing out the following basic identity:

$$\begin{aligned} R_F(z) - R_0(z) = -\,\alpha\Big[&G_0(z)\,\Gamma_0^{-1}(z)\left(\breve{G}_F(z) - \breve{G}_0(z)\right) \\ &+ \left(G_F(z) - G_0(z)\right)\Gamma_F^{-1}(z)\,\breve{G}_F(z) + G_0(z)\left(\Gamma_F^{-1}(z) - \Gamma_0^{-1}(z)\right)\breve{G}_F(z)\Big]\,, \end{aligned} \tag{3.16}$$

which can be easily derived from Eqs. (3.11) and (3.12) by addition and subtraction of identical terms.

4. The limiting absorption principle

In the present section we derive Limiting Absorption Principles (LAPs) for the resolvent operators $R_0(z)$ and $R_F(z)$; more precisely, we show that, for any $\lambda \in (0,+\infty)$, the limits $\varepsilon \downarrow 0$ of $R_0(\lambda \pm i\varepsilon)$ and $R_F(\lambda \pm i\varepsilon)$ determine bounded operators on suitable functional spaces (namely, on weighted Sobolev spaces, see Section 2.6). The results obtained here will be employed in the forthcoming Sections 6 and 7, concerning respectively the existence and completeness of the wave operators and the scattering matrix associated to the couple (A_F, A_0).

Our approach mainly consists of the following two steps: first, we derive LAP for $R_0(z)$, starting from simpler lower dimensional operators and employing a result of Ben-Artzi and Devinatz [5] (see also [6]) about sums of tensor products; then, we determine an analogous result for $R_F(z)$, using some abstract perturbation techniques of Renger [30].

4.1. The limiting absorption principle for $R_0(z)$

Let us consider the unperturbed resolvent $R_0(z)$ and the corresponding self-adjoint operator A_0.

We recall that the operator A_0 is factorized as $A_0 = A_0^{(1)} \otimes 1_\parallel + 1_1 \otimes A_0^{(\parallel)}$, see Remark 3.9. For any $z \in \mathbb{C}\backslash[0,+\infty)$ we consider the resolvent operators

$$R_0^{(1)}(z) := (A_0^{(1)} - z)^{-1} : L^2(\mathbb{R}) \to \mathrm{Dom}A_0^{(1)}$$

and

$$R_0^{(\parallel)}(z) := (A_0^{(\parallel)} - z)^{-1} : L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)\ .$$

The result of Ben-Artzi and Devinatz we refer to is Th. 3.8 in [5] which, in our setting, grants the validity of LAP for the operator A_0 in the factorized form (3.13) mentioned above. The cited theorem of [5] is here employed by setting $H_1 := A^{(\parallel)}$, $H_2 := A^{(1)}$, $\Lambda := (-\infty, 0]$, and $U = (0,+\infty)$. The corresponding hypotheses on H_1 and H_2 appear to be fulfilled, respectively, in consequence of the results reported in [5, Example 2.2] and of Theorem 4.1 below.

Theorem 4.1. *Assume that $\theta \in (0, 1/2)$ and let $s_1 > 1/2$. Then, for any $\lambda \in (0,+\infty)$, the limits*

$$R_0^{(1),\pm}(\lambda) := \lim_{\varepsilon \downarrow 0} R_0^{(1)}(\lambda \pm i\varepsilon)$$

exist in $\mathfrak{B}(L^2_{s_1}(\mathbb{R}), H^{1+\theta}_{-s_1}(\mathbb{R}))$ and the convergence is uniform in any compact subset $K \subset (0,+\infty)$.

The proof of the latter theorem is based on a series of explicit estimates and it is rather lengthy; for this reason we postpone it to Appendix A.

Using the previous Theorem 4.1 and some known results of Agmon [1], and of Ben-Artzi and Devinatz [5] we can infer the following Proposition.

Proposition 4.2 *(LAP for A_0). Assume that $\theta \in (0, 1/2)$ and let $\sigma_1, \sigma_\parallel > 1/2$. Then, for any $\lambda \in (0,+\infty)$, the limits*

$$R_0^{\pm}(\lambda) := \lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon) \tag{4.1}$$

exist in $\mathfrak{B}(L^2_{\sigma_1,\sigma_\parallel}(\mathbb{R}^3), H^{1+\theta}_{-\sigma_1,-\sigma_\parallel}(\mathbb{R}^3))$ and the convergence is uniform in any compact subset $K \subset (0,+\infty)$; in particular,

$$R_0^{\pm}(\lambda) \in \mathfrak{B}(L^2_{\sigma_1,\sigma_\parallel}(\mathbb{R}^3), L^2_{-\sigma_1,-\sigma_\parallel}(\mathbb{R}^3))\ .$$

Proof. Let us consider the representation (3.13) of A_0 as a sum of tensor products involving the reduced operators $A_0^{(1)}$ and $A_0^{(\parallel)}$. Recall that Theorem 4.1 gives LAP for the resolvent operator $R_0^{(1)}(z)$. Moreover, let us mention the following analogous result for the resolvent $R_0^{(\|)}(z)$ (see [1, Sec. 4]; see also [5, Prop. 5.1]): for all $\sigma_\| > 1/2$ and for any $\lambda \in (0, +\infty)$, the limits $R_0^{(\|),\pm}(\lambda) := \lim_{\varepsilon\downarrow 0} R_0^{(\|)}(\lambda \pm i\varepsilon)$ exist in $\mathfrak{B}(L^2_{\sigma_\|}(\mathbb{R}^2), H^2_{-\sigma_\|}(\mathbb{R}^2))$. Then, noting the elementary identity $L^2_{\sigma_1}(\mathbb{R}) \otimes L^2_{\sigma_\|}(\mathbb{R}^2) = L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3)$ and that the tensor product of Hilbert spaces $H^{1+\theta}_{-\sigma_1}(\mathbb{R}) \otimes H^2_{-\sigma_\|}(\mathbb{R})$ is continuously embedded into $H^{1+\theta}_{-\sigma_1,-\sigma_\|}(\mathbb{R})$ on account of item iv) of Lemma 2.5, the thesis follows straightforwardly from [5, Th. 3.8] (recall the remarks at the beginning of this section). As we remarked in the introduction to this section, the cited theorem of [5] is here employed setting $H_1 := A^{(\|)}$, $H_2 := A^{(1)}$, $\Lambda := (-\infty, 0]$, and $U = (0, +\infty)$; the corresponding hypotheses on H_1 and H_2 are fulfilled, respectively, in consequence of the results in [5, Example 2.2] and in Theorem 4.1. □

4.2. The limiting absorption principle for $R_F(z)$

As previously anticipated, we now proceed to derive LAP for the resolvent $R_F(z)$; to this purpose, we start from the analogous result for $R_0(z)$ determined in the previous subsection and use an abstract perturbation method of Renger [30]. More precisely, we want to use Th. 7 of [30] (see also Prop. 10 of the same paper). To this aim we must check that the operators A_0 and A_F satisfy Hypotheses 1, 8, and 9 of [30]; adapted to our setting those read:

- Hypothesis 1 of [30]. The operators A_0 and A_F are self-adjoint and semibounded (which is certainly true). There exists a constant $c_R \in \mathbb{R}$ such that for all $z \in \mathbb{C}$ with Re $z < c_R$, there holds $R_0(z) \in \mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3))$ and $R_F(z) \in \mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3))$.
- Hypothesis 8 of [30]. For all $\lambda \in (0, +\infty)$, $R_0^\pm(\lambda) := \lim_{\varepsilon\downarrow 0} R_0(\lambda \pm i\varepsilon)$ exist and are continuous in $\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3), L^2_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))$. Moreover, for each compact subset K of $(0, +\infty)$ there exists a constant $c_K > 0$ such that for all $\lambda \in K$ and all $f \in L^2_{2\sigma_1,2\sigma_\|}(\mathbb{R}^3)$ with $R_0^+(\lambda)f = R_0^-(\lambda)f$, there holds $\|R_0^\pm(\lambda)\, f\|_{L^2(\mathbb{R}^3)} \leqslant_c \|f\|_{L^2_{2\sigma_1,2\sigma_\|}(\mathbb{R}^3)}$.
- Hypothesis 9 of [30]. There exists a constant $c_E \in \mathbb{R}$ such that for all $\mu < c_E$ and for some $\gamma > 0$, there holds

$$R_F(\mu) - R_0(\mu) \in \mathfrak{S}_\infty(L^2(\mathbb{R}^3), L^2_{2\sigma_1+\gamma,2\sigma_\|+\gamma}(\mathbb{R}^3)) \; .$$

Here σ_1 and $\sigma_\|$ are some suitable indices for the weights in the Sobolev spaces.

The forthcoming Proposition 4.5 (see also Remark 4.6) proves Hypothesis 1; Propositions 4.2 and 4.12 prove Hypothesis 8, and Proposition 4.9 proves Hypothesis 9. They are later employed, together with Th. 7 of [30], in the proof of Theorem 4.13 (were the allowed values of the indices σ_1 and $\sigma_\|$ are explicitly given).

Lemma 4.3. *Let $s_1, s_\| \in \mathbb{R}$, $r \in (0, 1/2]$. Then, for all $z \in \mathbb{C}\backslash[0, +\infty)$, there hold:*

$$\breve{G}_F(z) \in \mathfrak{B}(H^{r-3/2}_{s_1,s_\|}(\mathbb{R}^3), H^r_{s_\|}(\mathbb{R}^2)) \; ; \tag{4.2}$$

$$G_F(z) \in \mathfrak{B}(H^{-r}_{s_\|}(\mathbb{R}^2), H^{3/2-r}_{s_1,s_\|}(\mathbb{R}^3)) \; . \tag{4.3}$$

Proof. Recall that $\breve{G}_F(z) = \tau_F R_\varnothing(z)$ by definition (see Eq. (3.1)); then, the statement in Eq. (4.2) follows readily from Lemmata 2.6 and 2.7. The analogous claim in Eq. (4.3) can be derived by simple duality arguments recalling Eq. (2.12) and the basic relation $G_F(z) = \breve{G}_F(\overline{z})^*$. □

The following lemma adapts the result of Lemma 3.3 (in particular of Eq. (3.6)) to weighted Sobolev spaces.

Lemma 4.4. *Let $s_\parallel \in \mathbb{R}$ and $|r| < 1/2$. Then there exists $z_0 \in \mathbb{C}\backslash[0, +\infty)$ such that*

$$\Gamma_F^{-1}(z) \in \mathfrak{B}(H^r_{s_\parallel}(\mathbb{R}^2)) \qquad \textit{for all } z \in \mathbb{C} \textit{ such that } d_z > d_{z_0}\ . \tag{4.4}$$

Proof. Let us first recall that for $|r| < 1/2$ and z away from the positive real axis ($d_z > d_{z_0}$) the map $\Gamma_F(z) : H^r(\mathbb{R}^2) \to H^r(\mathbb{R}^2)$ is bounded, invertible and can be expressed as $\Gamma_F(z) = 1 + \alpha\, M_F(z)$ in terms of the operator $M_F(z) = \tau_F\, G_F(z)$ (see Lemma 3.3).

Keeping in mind the definition of $M_F(z)$, by Lemmata 2.7 and 4.3 we get $M_F(z) \in \mathfrak{B}(H^{-1/2}_{s_\parallel}(\mathbb{R}^2), H^{1/2}_{s_\parallel}(\mathbb{R}^2))$ for all $s_\parallel \in \mathbb{R}$; so, in particular, by item iii) of Lemma 2.5 (see also Remark 2.1) we have $M_F(z) \in \mathfrak{B}(H^r_{s_\parallel}(\mathbb{R}^2))$ for all $|r| \leq 1/2$. This implies that $\Gamma_F(z) \in \mathfrak{B}(H^r_{s_\parallel}(\mathbb{R}^2))$, under the same assumptions on r and $s_\parallel$.

Next, let $s_\parallel \geqslant 0$ and $|r| < 1/2$ and notice that $H^r_{s_\parallel}(\mathbb{R}^2) \subset H^r(\mathbb{R}^2)$ by item i) of Lemma 2.5; then, the previous considerations and the bounded inverse theorem yield the thesis (4.4).

On the other hand, for $s_\parallel < 0$ the analogous statement can be derived by the usual duality arguments noting that $\Gamma_F(z) = \Gamma_F(\overline{z})^*$. □

Proposition 4.5 *(Check of Hypothesis 1 of [30]). Assume that $\eta, \eta' > 0$ and let $s_1, s_\parallel \in \mathbb{R}$. Then there exists $z_0 \in \mathbb{C}\backslash[0, +\infty)$ such that $R_F(z) \in \mathfrak{B}\big(H^{\eta-3/2}_{s_1,s_\parallel}(\mathbb{R}^3), H^{1/2+\eta}_{s_1,s_\parallel}(\mathbb{R}^3) \cap H^{3/2-\eta'}_{s_1,s_\parallel}(\mathbb{R}^3)\big)$ for all $z \in \mathbb{C}$ such that $d_z > d_{z_0}$. In particular there holds*

$$R_F(z) \in \mathfrak{B}(L^2_{s_1,s_\parallel}(\mathbb{R}^3)) \qquad \textit{for all } z \in \mathbb{C} \textit{ such that } d_z > d_{z_0}\ .$$

Proof. Consider the Kreĭn's type relation (3.11) for $R_F(z)$; then, the thesis follows from a straightforward application of Lemmata 2.5, 2.6, 4.3 and 4.4. □

Remark 4.6. In view of the facts pointed out in Remark 2.8, it appears that the results stated in Lemmata 4.3 and 4.4 can be easily generalized under stronger hypotheses on F. In particular, for $F = 0$ we have

$$\breve{G}_0(z) \in \mathfrak{B}(H^{r-3/2}_{s_1,s_\parallel}(\mathbb{R}^3), H^r_{s_\parallel}(\mathbb{R}^2)) \quad \text{and} \quad G_0(z) \in \mathfrak{B}(H^{-r}_{s_\parallel}(\mathbb{R}^2), H^{3/2-r}_{s_1,s_\parallel}(\mathbb{R}^3))$$

for all $r > 0$,

$$\Gamma_0^{-1}(z) \in \mathfrak{B}(H^r_{s_\parallel}(\mathbb{R}^2)) \qquad \text{for all } r \in \mathbb{R} \text{ and } z \in \mathbb{C}\backslash[0, +\infty)\ .$$

Of course, Proposition 4.5 continues to hold true if $F = 0$.

Let us now proceed to characterize the resolvent difference $R_F(z) - R_0(z)$ as a compact operator between suitable, weighted Sobolev spaces. Our main result in this direction is stated in Proposition 4.9, whose proof relies on the forthcoming Lemmata 4.7 and 4.8.

Lemma 4.7. *Let $s_1, s_\parallel \in \mathbb{R}$. Then, for all $r \in (0, 1/2)$, $\varepsilon \in (0, r]$ and all $t_\parallel \in \mathbb{R}$ there holds*

$$\tau_F - \tau_0 \in \mathfrak{S}_\infty(H^{r+1/2}_{s_1,s_\parallel}(\mathbb{R}^3), H^{r-\varepsilon}_{t_\parallel}(\mathbb{R}^2))\ .$$

Proof. First of all, let us remark that by Lemma 2.7 (see also the related Remark 2.8) we have $\tau_F - \tau_0 \in \mathfrak{B}(H^{r+1/2}_{s_1,s_\parallel}(\mathbb{R}^3), H^r_{s_\parallel}(\mathbb{R}^2))$ for any $s_1, s_\parallel \in \mathbb{R}$ and for all $r \in (0, 1/2)$.

It can be easily checked that the range of $\tau_F - \tau_0$ fulfills

$$\operatorname{ran}(\tau_F - \tau_0) \subseteq \{u \in H^r_{s_\parallel}(\mathbb{R}^2) \,|\, \operatorname{supp} u \subset \operatorname{supp} F\} = \{u \in H^r_{t_\parallel}(\mathbb{R}^2) \,|\, \operatorname{supp} u \subset \operatorname{supp} F\}$$

for all $r \in (0, 1/2)$ and all $t_\parallel \in \mathbb{R}$. The r.h.s. of the above equation is a closed subset of $H^r(\mathbb{R}^2)$ which, for any bounded, open subset $B_F \subset \mathbb{R}^2$ such that $\mathrm{supp}F \subset B_F$, can be isomorphically identified (as a Hilbert space) with $\{u \in H_0^r(B_F) \,|\, \mathrm{supp}u \subset \mathrm{supp}F\} \subset H_0^r(B_F)$.

Since B_F is bounded, it follows that $H_0^r(B_F)$ can be compactly embedded into $H_0^{r-\varepsilon}(B_F)$ (see [20, p. 99, Th. 16.1]). Since $r-\varepsilon \in [0, 1/2)$, one can extend elements of $H_0^{r-\varepsilon}(B_F)$ to elements of $H_{t_\parallel}^{r-\varepsilon}(\mathbb{R}^2)$ which vanish a.e. outside B_F; as well known this procedure gives a continuous map sending $H_0^{r-\varepsilon}(B_F)$ into $H_{t_\parallel}^{r-\varepsilon}(\mathbb{R}^2)$ (see [20, p. 60, Th. 11.4], the weight $w_{t_\parallel}$ is inessential since elements are extended by setting them to zero outside B_F). The above considerations show that the map $\tau_F - \tau_0 : H_{s_1,s_\parallel}^{r+1/2}(\mathbb{R}^3) \to H_{t_\parallel}^{r-\varepsilon}(\mathbb{R}^2)$ is compact, since it can be obtained by composition of a compact operator with continuous ones. □

Lemma 4.8. *Let $s_1, s_\parallel, t_\parallel \in \mathbb{R}$. Then, for all $r \in (0, 1/2)$, $\varepsilon \in (0, r]$ and $z \in \mathbb{C}\backslash[0, +\infty)$ there hold:*

$$\breve{G}_F(z) - \breve{G}_0(z) \in \mathfrak{S}_\infty(H_{s_1,s_\parallel}^{r-3/2}(\mathbb{R}^3), H_{t_\parallel}^{r-\varepsilon}(\mathbb{R}^2))\,, \tag{4.5}$$

$$G_F(z) - G_0(z) \in \mathfrak{S}_\infty(H_{t_\parallel}^{\varepsilon-r}(\mathbb{R}^2), H_{s_1,s_\parallel}^{3/2-r}(\mathbb{R}^3))\,. \tag{4.6}$$

Proof. First of all, let us recall that $\breve{G}_F(z) - \breve{G}_0(z) = (\tau_F - \tau_0)R_\varnothing(z)$ by definition (see Eq. (3.1)); then, the claim in Eq. (4.5) follows easily from Lemmata 2.6 and 4.7. Taking this into account, we are able to derive the analogous statement in Eq. (4.6) by the usual duality arguments. □

Proposition 4.9 *(Check of Hypothesis 9 of [30]). Let $s_1, s_\parallel, t_1, t_\parallel \in \mathbb{R}$. Then there exists $z_0 \in \mathbb{C}\backslash[0, +\infty)$ such that $R_F(z) - R_0(z) \in \mathfrak{S}_\infty(L_{s_1,s_\parallel}^2(\mathbb{R}^3), L_{t_1,t_\parallel}^2(\mathbb{R}^3))$ for all $z \in \mathbb{C}$ such that $d_z > d_{z_0}$. In particular, there holds*

$$R_F(z) - R_0(z) \in \mathfrak{S}_\infty(L^2(\mathbb{R}^3), L_{t_1,t_\parallel}^2(\mathbb{R}^3)) \quad \textit{for all } z \in \mathbb{C} \textit{ such that } d_z > d_{z_0}.$$

Proof. The main argument employed in the proof is the fact that compact operators are a two-sided ideal of bounded operators.

We proceed to prove separately the compactness of each of the three addenda appearing on the r.h.s. of Eq. (3.16). By Lemmata 4.7 and 4.8, for all $r \in (0, 1/2)$ we infer

$$\tau_F - \tau_0 \in \mathfrak{S}_\infty(H_{s_1,s_\parallel}^{r+1/2}(\mathbb{R}^3), L_{t_\parallel}^2(\mathbb{R}^2))\,; \tag{4.7}$$

$$\breve{G}_F(z) - \breve{G}_0(z) \in \mathfrak{S}_\infty(H_{s_1,s_\parallel}^{r-3/2}(\mathbb{R}^3), L_{t_\parallel}^2(\mathbb{R}^2))\,; \tag{4.8}$$

$$G_F(z) - G_0(z) \in \mathfrak{S}_\infty(L_{t_\parallel}^2(\mathbb{R}^2), H_{s_1,s_\parallel}^{3/2-r}(\mathbb{R}^3)) \tag{4.9}$$

for all $s_1, s_\parallel, t_\parallel \in \mathbb{R}$. Next we analyze the three addenda on the r.h.s. of Eq. (3.16) one by one.

First term: There holds

$$G_0(z)\,\Gamma_0^{-1}(z)\left(\breve{G}_F(z) - \breve{G}_0(z)\right) \in \mathfrak{S}_\infty(L_{s_1,s_\parallel}^2(\mathbb{R}^3), L_{t_1,t_\parallel}^2(\mathbb{R}^3))\,.$$

The claim follows by noticing that by Eq. (4.8) one has $\breve{G}_F(z) - \breve{G}_0(z) \in \mathfrak{S}_\infty(L_{s_1,s_\parallel}^2(\mathbb{R}^3), L_{t_\parallel}^2(\mathbb{R}^2))$, in addition to $\Gamma_0^{-1}(z) \in \mathfrak{B}(L_{t_\parallel}^2(\mathbb{R}^2))$ and $G_0(z) \in \mathfrak{B}(L_{t_\parallel}^2(\mathbb{R}^2),\, L_{t_1,t_\parallel}^2(\mathbb{R}^3))$, see Remark 4.6.

Second term: There holds

$$\left(G_F(z) - G_0(z)\right)\Gamma_F^{-1}(z)\,\breve{G}_F(z) \in \mathfrak{S}_\infty(L_{s_1,s_\parallel}^2(\mathbb{R}^3), L_{t_1,t_\parallel}^2(\mathbb{R}^3))\,.$$

The claim follows by noticing that by Eq. (4.9) one has $G_F(z) - G_0(z) \in \mathfrak{S}_\infty(L_{s_\parallel}^2(\mathbb{R}^2), L_{t_1,t_\parallel}^2(\mathbb{R}^3))$, in addition to $\breve{G}_F(z) \in \mathfrak{B}(L^2(\mathbb{R}^3)_{s_1,s_\parallel},\, L_{s_\parallel}^2(\mathbb{R}^2))$ and $\Gamma_F^{-1}(z) \in \mathfrak{B}(L_{s_\parallel}^2(\mathbb{R}^2))$, see Lemmata 4.3 and 4.4.

Third term: There holds

$$G_0(z)\left(\Gamma_F^{-1}(z)-\Gamma_0^{-1}(z)\right)\breve{G}_F(z)\in\mathfrak{S}_\infty(L^2_{s_1,s_\parallel}(\mathbb{R}^3),L^2_{t_1,t_\parallel}(\mathbb{R}^3))\,.$$

Let us recall that $M_F(z)=\tau_F G_F(z)$ (and similarly for $F=0$). We point out the following basic identity (we are using essentially the identity $(A+B)^{-1}-A^{-1}=-A^{-1}B(A+B)^{-1}$).

$$\begin{aligned}\Gamma_F^{-1}(z)-\Gamma_0^{-1}(z)=&\Big(1+\alpha M_0(z)+\alpha\big(M_F(z)-M_0(z)\big)\Big)^{-1}-\Big(1+\alpha M_0(z)\Big)^{-1}\\=&-\,\alpha\,\Gamma_0^{-1}(z)\left(\tau_F G_F(z)-\tau_0 G_0(z)\right)\Gamma_F^{-1}(z)\,.\end{aligned}$$

Since

$$\tau_F G_F(z)-\tau_0 G_0(z)=(\tau_F-\tau_0)G_F(z)+\tau_0\big(G_F(z)-G_0(z)\big)\,,$$

we infer

$$\Gamma_F^{-1}(z)-\Gamma_0^{-1}(z)=-\alpha\,\Gamma_0^{-1}(z)\Big((\tau_F-\tau_0)G_F(z)+\tau_0\big(G_F(z)-G_0(z)\big)\Big)\,\Gamma_F^{-1}(z)\,.$$

To proceed, let us recall that:
i) $\breve{G}_F(z)\in\mathfrak{B}(L^2_{s_1,s_\parallel}(\mathbb{R}^3),L^2_{s_\parallel}(\mathbb{R}^2))$ and $G_0(z)\in\mathfrak{B}(L^2_{t_\parallel}(\mathbb{R}^2),\,L^2_{t_1,t_\parallel}(\mathbb{R}^3))$.
ii) $\Gamma_F^{-1}(z)\in\mathfrak{B}(L^2_{s_\parallel}(\mathbb{R}^2))$ and $\Gamma_0^{-1}(z)\in\mathfrak{B}(L^2_{t_\parallel}(\mathbb{R}^2))$.
iii) $G_F(z)\in\mathfrak{B}(L^2_{s_\parallel}(\mathbb{R}^2),H^1_{s_1,s_\parallel}(\mathbb{R}^3))$ and $\tau_0\in\mathfrak{B}(H^1_{t_1,t_\parallel}(\mathbb{R}^3),L^2_{t_\parallel}(\mathbb{R}^2))$.
In view of these facts, the thesis follows from $\tau_F-\tau_0\in\mathfrak{S}_\infty(H^1_{s_1,s_\parallel}(\mathbb{R}^3),L^2_{t_\parallel}(\mathbb{R}^2))$ and $G_F(z)-G_0(z)\in\mathfrak{S}_\infty(L^2_{s_\parallel}(\mathbb{R}^2),H^1_{t_1,t_\parallel}(\mathbb{R}^3))$, which are in turn a consequence of Eqs. (4.7) (4.9). □

Remark 4.10. The latter proposition implies in particular that $R_F(z)-R_0(z)$ is a compact operator in $L^2(\mathbb{R}^3)$ for z far away from the real axis, hence for any $z\in\rho(A_F)\cap\rho(A_0)$. From which we infer $\sigma_{ess}(A_F)=\sigma_{ess}(A_0)=[0,+\infty)$.

Finally we check the validity of Hypothesis 8 of [30]. Let us consider the following set of functions $\{\varphi_{\mathbf{k}}\}$, indexed by the labels $\mathbf{k}=(k_1,\mathbf{k}_\parallel)\in\mathbb{R}\times\mathbb{R}^2$ (see [3, p. 85, Eq. (3.4.1)]):

$$\varphi_{\mathbf{k}}:=\varphi^{(1)}_{k_1}\otimes\varphi^{(\parallel)}_{\mathbf{k}_\parallel}\,, \tag{4.10}$$

with

$$\varphi^{(1)}_{k_1}(x^1):=\frac{e^{ik_1x^1}}{\sqrt{2\pi}}-\frac{i\,\alpha}{2|k_1|+i\,\alpha}\,\frac{e^{i\,|k_1|\,|x^1|}}{\sqrt{2\pi}}\,,$$

and

$$\varphi^{(\parallel)}_{\mathbf{k}_\parallel}(\mathbf{x}_\parallel):=\frac{e^{i\mathbf{k}_\parallel\cdot\mathbf{x}_\parallel}}{2\pi}\,.$$

These form a complete set of generalized eigenfunctions for A_0 with respect to a suitable rigging of $L^2(\mathbb{R}^3)$ (see [3] and [26, Ch. VI, Sec. 21]). More precisely, it can be checked by direct inspection that $\varphi_{\mathbf{k}}\in L^2_{-s_1,-s_\parallel}(\mathbb{R}^3)$ for any $\mathbf{k}\in\mathbb{R}^3$ and for all $s_1>1/2$, $s_\parallel>1$; furthermore, denoting with $\langle\ |\ \rangle$ the $(L^2_{-s_1,-s_\parallel},L^2_{s_1,s_\parallel})$-duality pairing induced by the L^2-inner product, we have $\langle\varphi_{\mathbf{k}}|(A_0-|\mathbf{k}|^2)f\rangle=0$ for all $f\in\mathrm{Dom}A_0\cap L^2_{s_1,s_\parallel}(\mathbb{R}^3)$ with $A_0f\in L^2_{s_1,s_\parallel}(\mathbb{R}^3)$.

Keeping in mind the facts mentioned above, in the forthcoming Lemma 4.11 we proceed to point out some notable features of the functions $\mathbb{R}^3 \ni \mathbf{k} \mapsto \langle \varphi_{\mathbf{k}} | f \rangle$, for $f \in L^2_{s_1,s_\parallel}(\mathbb{R}^3)$. We will later employ these features in the proof of Proposition 4.12.

Lemma 4.11. *Assume that $s_1 > 1/2$, $s_\parallel > 1$ and let $f \in L^2_{s_1,s_\parallel}(\mathbb{R}^3)$. Then, the map $\mathbb{R}^3 \ni \boldsymbol{k} \mapsto \langle \varphi_{\mathbf{k}} | f \rangle$ enjoys the following properties:*
i) For all $\eta \in (0,1)$ such that $\eta \leqslant \min(s_1 - 1/2, s_\parallel - 1)$, the map $\boldsymbol{k} \mapsto \langle \varphi_{\mathbf{k}} | f \rangle$ belongs to $C^{0,\eta}(\overline{\mathbb{R}^3})$; moreover,

$$\|\boldsymbol{k} \mapsto \langle \varphi_{\mathbf{k}} | f \rangle\|_{C^{0,\eta}(\overline{\mathbb{R}^3})} \leqslant_c \|f\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)} \,. \tag{4.11}$$

ii) Let $K \subset (0,+\infty)$ be any compact subset and consider the operators $R_0^\pm(\lambda)$ of Eq. (4.1) for $\lambda \in K$. If $R_0^+(\lambda) f = R_0^-(\lambda) f$ for all $\lambda \in K$, then

$$\langle \varphi_{\mathbf{k}} | f \rangle = 0 \quad \textit{for all } \boldsymbol{k} \in \mathbb{R}^3 \textit{ such that } |\boldsymbol{k}|^2 \in K \,.$$

Proof. We discuss separately the proofs of items i) and ii).
i) The thesis follows by obvious density arguments as soon as we can infer the relation stated in Eq. (4.11) for any factorized function of the form $f = u_1 \otimes u_\parallel$, with $u_1 \in L^2_{s_1}(\mathbb{R})$ and $u_\parallel \in L^2_{s_\parallel}(\mathbb{R}^2)$. To this purpose, let us first notice that the Fourier transforms on $\mathbb{R}$ and $\mathbb{R}^2$ induce, respectively, the isomorphisms of Hilbert spaces $\mathfrak{F}_1 : H^{s_1}(\mathbb{R}) \to L^2_{s_1}(\mathbb{R})$ and $\mathfrak{F}_\parallel : H^{s_\parallel}(\mathbb{R}^2) \to L^2_{s_\parallel}(\mathbb{R})$ for any $s_1, s_\parallel \in \mathbb{R}$. Besides, indicating with Θ the Heaviside step function and using the explicit expression (4.10) for $\varphi_{\mathbf{k}}$, by direct computations we get

$$\langle \varphi_{\mathbf{k}} | u_1 \otimes u_\parallel \rangle = \left[(\mathfrak{F}_1^{-1} u_1)(k_1) + \frac{i\,\alpha}{2|k_1| - i\,\alpha} \Big(\big(\mathfrak{F}_1^{-1}(\Theta u_1)\big)(|k_1|) + \big(\mathfrak{F}_1^{-1}((1-\Theta)u_1)\big)(-|k_1|) \Big) \right] (\mathfrak{F}_\parallel^{-1} u_\parallel)(\mathbf{k}_\parallel) \,.$$

Since $\Theta \in L^\infty(\mathbb{R})$, in view of the previously mentioned facts we have $\mathfrak{F}_1^{-1} u_1,\, \mathfrak{F}_1^{-1}(\Theta u_1), \mathfrak{F}_1^{-1}((1-\Theta)u_1) \in H^{s_1}(\mathbb{R})$ and $\mathfrak{F}_\parallel^{-1} u_\parallel \in H^{s_\parallel}(\mathbb{R}^2)$; so, by standard Sobolev embeddings, $\mathfrak{F}_1^{-1} u_1, \mathfrak{F}_1^{-1}(\Theta u_1), \mathfrak{F}_1^{-1}((1-\Theta)u_1) \in C^{0,\eta_1}(\overline{\mathbb{R}})$ for all $\eta_1 \in (0,1)$ such that $\eta_1 < s_1 - 1/2$ and $\mathfrak{F}_\parallel^{-1} u_\parallel \in C^{0,\eta_\parallel}(\overline{\mathbb{R}^2})$ for all $\eta_\parallel \in (0,1)$ such that $\eta_\parallel < s_\parallel - 1$. Let us also remark that the absolute value $k_1 \mapsto |k_1|$ is uniformly Lipschitz-continuous on $\mathbb{R}$ (in fact, $|\,|k_1| - |h_1|\,| \leqslant |k_1 - h_1|$ for all $k_1, h_1 \in \mathbb{R}$). Summing up, for any $\eta \in (0,1)$ fulfilling $\eta \leqslant \min(\eta_1, \eta_\parallel)$ by elementary computations we obtain

$$\sup_{\mathbf{k} \in \mathbb{R}^3} \big| \langle \varphi_{\mathbf{k}} | u_1 \otimes u_\parallel \rangle \big| \leqslant_c \|u_1 \otimes u_\parallel\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)} \,, \tag{4.12}$$

$$\big| \langle \varphi_{\mathbf{k}} | u_1 \otimes u_\parallel \rangle - \langle \varphi_{\mathbf{h}} | u_1 \otimes u_\parallel \rangle \big| \leqslant_c \|u_1 \otimes u_\parallel\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)} |\mathbf{k} - \mathbf{h}|^\eta \,. \tag{4.13}$$

Let us give a few more details about the derivation of the latter estimates. On the one hand, we have

$$\begin{aligned} \sup_{\mathbf{k} \in \mathbb{R}^3} \big| \langle \varphi_{\mathbf{k}} | u_1 \otimes u_\parallel \rangle \big| &\leqslant_c \Big[\|\mathfrak{F}_1^{-1} u_1\|_{C^0(\overline{\mathbb{R}})} + \|\mathfrak{F}_1^{-1}(\Theta u_1)\|_{C^0(\overline{\mathbb{R}})} + \|\mathfrak{F}_1^{-1}((1-\Theta)u_1)\|_{C^0(\overline{\mathbb{R}})} \Big] \|\mathfrak{F}_\parallel^{-1} u_\parallel\|_{C^0(\overline{\mathbb{R}^2})} \\ &\leqslant_c \Big[\|\mathfrak{F}_1^{-1} u_1\|_{H^{s_1}(\mathbb{R})} + \|\mathfrak{F}_1^{-1}(\Theta u_1)\|_{H^{s_1}(\mathbb{R})} + \|\mathfrak{F}_1^{-1}((1-\Theta)u_1)\|_{H^{s_1}(\mathbb{R})} \Big] \|\mathfrak{F}_\parallel^{-1} u_\parallel\|_{H^{s_\parallel}(\mathbb{R}^2)} \\ &\leqslant_c \|u_1 \otimes u_\parallel\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)} \,. \end{aligned}$$

Taking this into account, for $|\mathbf{k}-\mathbf{h}| \geqslant 1$ we readily get $|\langle \varphi_{\mathbf{k}} | u_1 \otimes u_\parallel \rangle - \langle \varphi_{\mathbf{h}} | u_1 \otimes u_\parallel \rangle| \leqslant_c 2 \sup_{\mathbf{k} \in \mathbb{R}^3} |\langle \varphi_{\mathbf{k}} | u_1 \otimes u_\parallel \rangle| \leqslant_c \|u_1 \otimes u_\parallel\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)} |\mathbf{k} - \mathbf{h}|^\eta$. On the other hand, for $|\mathbf{k} - \mathbf{h}| < 1$ we have

$$
\begin{aligned}
&\big|\langle\varphi_{\mathbf{k}}|u_1\otimes u_\parallel\rangle-\langle\varphi_{\mathbf{h}}|u_1\otimes u_\parallel\rangle\big|\\
&\leqslant_c \Big[\|\mathfrak{F}_1^{-1}u_1\|_{C^{0,\eta_1}(\overline{\mathbb{R}})}+\|\mathfrak{F}_1^{-1}(\Theta u_1)\|_{C^{0,\eta_1}(\overline{\mathbb{R}})}\\
&\qquad+\|\mathfrak{F}_1^{-1}((1-\Theta)u_1)\|_{C^{0,\eta_1}(\overline{\mathbb{R}})}\Big]\|\mathfrak{F}_\parallel^{-1}u_\parallel\|_{C^{0,\eta_\parallel}(\overline{\mathbb{R}^2})}\Big(|k_1-h_1|^{\eta_1}+|\mathbf{k}_\parallel-\mathbf{h}_\parallel|^{\eta_\parallel}\Big)\\
&\leqslant_c \Big[\|\mathfrak{F}_1^{-1}u_1\|_{H^{s_1}(\mathbb{R})}+\|\mathfrak{F}_1^{-1}(\Theta u_1)\|_{H^{s_1}(\mathbb{R})}\\
&\qquad+\|\mathfrak{F}_1^{-1}((1-\Theta)u_1)\|_{H^{s_1}(\mathbb{R})}\Big]\|\mathfrak{F}_\parallel^{-1}u_\parallel\|_{H^{s_\parallel}(\mathbb{R}^2)}\,|\mathbf{k}-\mathbf{h}|^{\min(\eta_1,\eta_\parallel)}\\
&\leqslant_c \|u_1\otimes u_\parallel\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,|\mathbf{k}-\mathbf{h}|^{\eta}\,.
\end{aligned}
$$

Estimates (4.12) and (4.13) allow us to infer the relation (4.11) for $f=u_1\otimes u_\parallel$, which yields the thesis.
ii) First of all, for all $z\in\mathbb{C}\backslash[0,+\infty)$, from [26, p. 121, Cor. 2] we infer that

$$(R_0(z)\,f)(\mathbf{x})=\int_{\mathbb{R}^3}d\mathbf{k}\,\frac{\langle\varphi_{\mathbf{k}}|f\rangle}{|\mathbf{k}|^2-z}\,\varphi_{\mathbf{k}}(\mathbf{x})\qquad(\mathbf{x}\in\mathbb{R}^3)\,.$$

On account of Proposition 4.2, for any $f\in L^2_{s_1,s_\parallel}(\mathbb{R}^3)$ with $s_1>1/2$, $s_\parallel>1$ and for all $\lambda\in K$, the condition $R_0^+(\lambda)f=R_0^-(\lambda)f$ can be rephrased as follows using the above integral kernel identity:

$$0=\lim_{\varepsilon\downarrow 0}\langle\big[R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\big]f\,|\,f\rangle=\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3}d\mathbf{k}\,\frac{2i\varepsilon}{(|\mathbf{k}|^2-\lambda)^2+\varepsilon^2}\,|\langle\varphi_{\mathbf{k}}|f\rangle|^2\,.$$

Recalling that the function $\mathbf{k}\mapsto\langle\varphi_{\mathbf{k}}|f\rangle$ is, in particular, continuous (see item i) of this Lemma) and denoting with $d\sigma_r$ the spherical measure on the 2-sphere $S^2_r:=\{\mathbf{k}\in\mathbb{R}^3\,|\,|\mathbf{k}|^2=r\}$ induced by the usual Lebesgue measure on $\mathbb{R}^3$, the above relation can be rewritten as

$$0=\lim_{\varepsilon\downarrow 0}\int_0^{+\infty}dr\,\frac{2i\varepsilon}{(r-\lambda)^2+\varepsilon^2}\int_{S^2_r}d\sigma_r(\mathbf{k})\,|\langle\varphi_{\mathbf{k}}|f\rangle|^2\,,$$

and yields $0=\int_{S^2_\lambda}d\sigma_\lambda(\mathbf{k})\,|\langle\varphi_{\mathbf{k}}|f\rangle|^2$. This suffices to infer that $|\langle\varphi_{\mathbf{k}}|f\rangle|^2=0$ for almost every $\mathbf{k}\in S^2_\lambda$, which in view of the continuity of $\mathbf{k}\mapsto\langle\varphi_{\mathbf{k}}|f\rangle$ implies the thesis. □

Proposition 4.12 *(Check of Hypothesis 8 of [30]). Assume that $s_1>1$, $s_\parallel>3/2$ and let $K\subset(0,+\infty)$ be any compact subset. Then, for all $\lambda\in K$ and for all $f\in L^2_{s_1,s_\parallel}(\mathbb{R}^3)$ such that $R_0^+(\lambda)f=R_0^-(\lambda)f$, there holds*

$$\|R_0^{\pm}(\lambda)f\|_{L^2(\mathbb{R}^3)}\leqslant_c\|f\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,.$$

Proof. Let us fix arbitrarily $\varepsilon_0>0$ and $K\subset(0,+\infty)$. Of course, on account of the definition (4.1) of the operators $R_0^{\pm}(\lambda)$, the thesis follows as soon as we are able to derive the following uniform bound (4.14) for all $\varepsilon\in(0,\varepsilon_0)$, $\lambda\in K$ and for some constant $c>0$ depending only on ε_0 and K:

$$\|R_0(\lambda\pm i\varepsilon)f\|_{L^2(\mathbb{R}^3)}\,\leqslant\,c\,\|f\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,.\tag{4.14}$$

To this purpose, let us first remark that [26, p. 111, Th. 2] yields

$$\|R_0(\lambda\pm i\varepsilon)f\|^2_{L^2(\mathbb{R}^3)}=\int_{\mathbb{R}^3}d\mathbf{k}\,\frac{|\langle\varphi_{\mathbf{k}}|f\rangle|^2}{\big|\,|\mathbf{k}|^2-(\lambda\pm i\varepsilon)\big|^2}.$$

Let us introduce the notation $\mathbb{R}^3\backslash K$ to denote the set of all $\mathbf{k} \in \mathbb{R}^3$ such that $|\mathbf{k}|^2 \notin K$. Due to the assumption $R_0^+(\lambda)f = R_0^-(\lambda)f$, by item ii) of Lemma 4.11 we have $\langle \varphi_{\mathbf{k}}|f\rangle = 0$ for all $\mathbf{k} \in \mathbb{R}^3$ with $|\mathbf{k}|^2 \in K$. Therefore, we obtain

$$\|R_0(\lambda \pm i\varepsilon)f\|^2_{L^2(\mathbb{R}^3)} = \int\limits_{\mathbb{R}^3\backslash K} d\mathbf{k}\, \frac{|\langle \varphi_{\mathbf{k}}|f\rangle|^2}{(|\mathbf{k}|^2-\lambda)^2+\varepsilon^2}\,. \tag{4.15}$$

Assume that $\lambda \in K$ is an interior point; in this case, we have $\big||\mathbf{k}|^2-\lambda\big| \geqslant \delta$ for some fixed $\delta > 0$ and for all $\mathbf{k} \in \mathbb{R}^3\backslash K$. Thus, taking into account that [26, p. 121, Cor. 2] gives $\|f\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} d\mathbf{k}\, |\langle\varphi_{\mathbf{k}}|f\rangle|^2$, from Eq. (4.15) we infer

$$\|R_0(\lambda \pm i\varepsilon)\, f\|^2_{L^2(\mathbb{R}^3)} \leqslant \frac{1}{\delta^2} \int\limits_{\mathbb{R}^3\backslash K} d\mathbf{k}\, |\langle\varphi_{\mathbf{k}}|f\rangle|^2 \leqslant_c \|f\|^2_{L^2(\mathbb{R}^3)} \leqslant_c \|f\|^2_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,,$$

where the last inequality obviously holds true for all $s_1, s_\parallel \geqslant 0$.

Next, let us consider the case where $\lambda \in K$ is a boundary point; in this situation, it is convenient to re-express Eq. (4.15) as

$$\|R_0(\lambda \pm i\varepsilon)f\|^2_{L^2(\mathbb{R}^3)} = \int\limits_{(\mathbb{R}^3\backslash K)\,\cap\,\{||\mathbf{k}|^2-\lambda|\geqslant 1\}} d\mathbf{k}\, \frac{|\langle\varphi_{\mathbf{k}}|f\rangle|^2}{(|\mathbf{k}|^2-\lambda)^2+\varepsilon^2} + \int\limits_{(\mathbb{R}^3\backslash K)\,\cap\,\{||\mathbf{k}|^2-\lambda|<1\}} d\mathbf{k}\, \frac{|\langle\varphi_{\mathbf{k}}|f\rangle|^2}{(|\mathbf{k}|^2-\lambda)^2+\varepsilon^2}\,.$$

On the one hand, by computations similar to those described before we easily get

$$\int\limits_{(\mathbb{R}^3\backslash K)\,\cap\,\{||\mathbf{k}|^2-\lambda|\geqslant 1\}} d\mathbf{k}\, \frac{|\langle\varphi_{\mathbf{k}}|f\rangle|^2}{(|\mathbf{k}|^2-\lambda)^2+\varepsilon^2} \leqslant \|f\|^2_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,.$$

On the other hand, let us consider the unit vector $\hat{\mathbf{k}} := \mathbf{k}/|\mathbf{k}|$ and notice that item ii) of Lemma 4.11 implies $\langle\varphi_{\sqrt{\lambda}\,\hat{\mathbf{k}}}|f\rangle = 0$ for all $\lambda \in K$; so, for all $s_1 > 1/2$, $s_\parallel > 1$ and $\eta \in (0,1)$ with $\eta \leqslant \min(s_1 - 1/2, s_\parallel - 1)$, by item i) of the same Lemma (see, in particular, Eq. (4.11)) we get $|\langle\varphi_{\mathbf{k}}|f\rangle| = |\langle\varphi_{\mathbf{k}}|f\rangle - \langle\varphi_{\sqrt{\lambda}\,\hat{\mathbf{k}}}|f\rangle| \leqslant_c \big||\mathbf{k}| - \sqrt{\lambda}\big|^\eta \|f\|_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}$. Taking this into account, we infer that

$$\int\limits_{(\mathbb{R}^3\backslash K)\cap\{||\mathbf{k}|^2-\lambda|<1\}} d\mathbf{k}\, \frac{|\langle\varphi_{\mathbf{k}}|f\rangle|^2}{\big((|\mathbf{k}|^2-\lambda)^2+\varepsilon^2\big)} \leqslant_c \left(\int\limits_{(\mathbb{R}^3\backslash K)\,\cap\,\{||\mathbf{k}|^2-\lambda|<1\}} d\mathbf{k}\, \frac{\big|\,|\mathbf{k}|-\sqrt{\lambda}\,\big|^{2\eta}}{(|\mathbf{k}|^2-\lambda)^2} \right) \|f\|^2_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,.$$

Under the stronger hypotheses $s_1 > 1$, $s_\parallel > 3/2$ and $1/2 < \eta \leqslant \min(s_1 - 1/2, s_\parallel - 1)$, the latter relation yields

$$\int\limits_{(\mathbb{R}^3\backslash K)\cap\{||\mathbf{k}|^2-\lambda|<1\}} d\mathbf{k}\, \frac{|\langle\varphi_{\mathbf{k}}|f\rangle|^2}{(|\mathbf{k}|^2-\lambda)^2+\varepsilon^2} \leqslant_c \left(\int\limits_{\{|r^2-\lambda|<1\}} dr\, \frac{1}{|r-\sqrt{\lambda}|^{2-2\eta}} \right) \|f\|^2_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)} \leqslant_c \|f\|^2_{L^2_{s_1,s_\parallel}(\mathbb{R}^3)}\,.$$

Summing up, the arguments described above prove Eq. (4.14) for all $\lambda \in K$ (either in the interior or on boundary), thus implying the thesis. □

We are now ready to state the main result of this subsection.

Theorem 4.13 *(LAP for A_F). Let $\sigma_1 > 1/2$, $\sigma_\| > 3/4$, and denote by $\sigma_p^+(A_F) := (0,+\infty) \cap \sigma_p(A_F)$ the (possibly empty) set of embedded eigenvalues of A_F.*

Then $\sigma_p^+(A_F)$ is a discrete set and for any $\lambda \in (0,+\infty)\backslash\sigma_p^+(A_F)$, the limits

$$R_F^\pm(\lambda) := \lim_{\varepsilon\downarrow 0} R_F(\lambda \pm i\varepsilon)$$

exist in $\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3), L^2_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))$ and the convergence is uniform in any compact subset $K \subset (0,+\infty)\backslash\sigma_p^+(A_F)$.

Proof. As already anticipated, the results derived previously in this work allow us to infer the thesis by a straightforward application of [30, Th. 7]; in the following we give more details about this claim.

Firstly, Proposition 4.5 shows that $R_0(z), R_F(z) \in \mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3))$ for any $\sigma_1, \sigma_\| \in \mathbb{R}$; this and the previously discussed properties of the corresponding operators A_0, A_F prove Hypothesis 1 of [30].

Secondly, for all $\sigma_1, \sigma_\| > 1/2$, Proposition 4.2 grants the existence of the limits $R_0^\pm(\lambda) := \lim_{\varepsilon\downarrow 0} R_0(\lambda \pm i\varepsilon)$ in $\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3), L^2_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))$; furthermore, under the stronger assumptions $\sigma_1 > 1/2$, $\sigma_\| > 3/4$, by Proposition 4.12 we have $\|R_0^\pm(\lambda)f\|_{L^2(\mathbb{R}^3)} \leqslant_c \|f\|_{L^2_{2\sigma_1,2\sigma_\|}(\mathbb{R}^3)}$ for all $u \in L^2_{2\sigma_1,2\sigma_\|}(\mathbb{R}^3)$ such that $R_0^+(\lambda)f = R_0^-(\lambda)f$. The above remarks prove that Hypothesis 8 of [30] holds true.

Thirdly, Proposition 4.9 implies, in particular,

$$R_F(z) - R_0(z) \in \mathfrak{S}_\infty(L^2(\mathbb{R}^3), L^2_{2\sigma_1+\gamma,2\sigma_\|+\gamma}(\mathbb{R}^3))$$

for any $\sigma_1, \sigma_\| \in \mathbb{R}$ and $\gamma > 0$, thus yielding Hypothesis 9 of [30].

Taking into account [30, Prop. 10], the above arguments suffice to infer that all the hypotheses of [30, Th. 7] are fulfilled by the pair A_0, A_F. □

5. An alternative formula for the resolvents difference

The aim of this section is to re-write the resolvent difference $R_F(z) - R_0(z)$ in a way that is convenient to get existence and completeness of the wave operators for the scattering couple (A_F, A_0) and to get a representation of the corresponding scattering matrix. In particular we show that $R_F(z) - R_0(z)$ depends only on the restriction of the traces τ_F and τ_0 to supp(F).

Let us recall the resolvent formulae that we obtained for the two self-adjoint operators A_0 and A_F, respectively (see Eqs. (3.12), (3.11)):

$$R_0(z) := R_\varnothing(z) - \alpha\, G_0(z)\, \Gamma_0^{-1}(z)\, \breve{G}_0(z)\,, \tag{5.1}$$

$$R_F(z) := R_\varnothing(z) - \alpha\, G_F(z)\, \Gamma_F^{-1}(z)\, \breve{G}_F(z)\,, \tag{5.2}$$

where

$$\begin{aligned} G_0(z) &\in \mathfrak{B}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^3))\,, & G_F(z) &\in \mathfrak{B}(L^2(\mathbb{R}^2), L^2(\mathbb{R}^3))\,,\\ \breve{G}_0(z) &\in \mathfrak{B}(L^2(\mathbb{R}^3), L^2(\mathbb{R}^2))\,, & \breve{G}_F(z) &\in \mathfrak{B}(L^2(\mathbb{R}^3), L^2(\mathbb{R}^2))\,,\\ \Gamma_0(z) := \big(1 + \alpha\, \tau_0 G_0(z)\big) &\in \mathfrak{B}(L^2(\mathbb{R}^2))\,, & \Gamma_F(z) := \big(1 + \alpha\, \tau_F G_F(z)\big) &\in \mathfrak{B}(L^2(\mathbb{R}^2))\,,\\ \Gamma_0^{-1}(z) &\in \mathfrak{B}(L^2(\mathbb{R}^2))\,, & \Gamma_F^{-1}(z) &\in \mathfrak{B}(L^2(\mathbb{R}^2))\,. \end{aligned} \tag{5.3}$$

Here and below z is any point in $\mathbb{C}\backslash[0,+\infty)$. The resolvent formula in the following Theorem resembles the one used in [15]:

Theorem 5.1. *The resolvent difference $R_F(z)-R_0(z)$ depends only on the restriction of the traces τ_F and τ_0 to $\Sigma := \mathrm{supp}(F)$:*

$$R_F(z)=R_0(z)+g_\Sigma(z)\Lambda_{F,\Sigma}(z)\breve g_\Sigma(z)\,,\qquad z\in\mathbb{C}\backslash[0,+\infty) \tag{5.4}$$

where

$$\begin{gathered}\breve g_\Sigma(z):L^2(\mathbb{R}^3)\to L^2(\Sigma)\oplus L^2(\Sigma)\,,\quad \breve g_\Sigma(z):=\tau_\Sigma R_0(z)\,,\\ \tau_\Sigma:H^{r+1/2}(\mathbb{R}^3)\to L^2(\Sigma)\oplus L^2(\Sigma)\,,\quad \tau_\Sigma f:=(\tau_F f|\Sigma)\oplus(\tau_0 f|\Sigma)\quad \textit{for } r>0\,,\\ g_\Sigma(z):L^2(\Sigma)\oplus L^2(\Sigma)\to L^2(\mathbb{R}^3)\,,\quad g_\Sigma(z):=(g_\Sigma(\bar z))^*\,,\\ \Lambda_{F,\Sigma}(z):L^2(\Sigma)\oplus L^2(\Sigma)\to L^2(\Sigma)\oplus L^2(\Sigma)\,,\\ \Lambda_{F,\Sigma}(z):=-\alpha\left(1+\alpha J\tau_\Sigma g_\Sigma(z)\right)^{-1}J\,,\quad J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\end{gathered}$$

Proof. Combining (5.1) and (5.2), one gets (here and below we use the matrix block operator notation)

$$R_F(z)-R_0(z)=-\alpha\left(G_F(z),\,G_0(z)\right)\begin{pmatrix}\Gamma_F^{-1}(z)&0\\0&-\Gamma_0^{-1}(z)\end{pmatrix}\begin{pmatrix}\breve G_F(z)\\ \breve G_0(z)\end{pmatrix}. \tag{5.5}$$

Defining

$$\breve g_0(z):=\tau_0R_0(z)\,,\qquad \breve g_F(z):=\tau_FR_0(z)$$

and

$$g_0(z):=\breve g_0(\bar z)^*\,,\qquad g_F(z):=\breve g_F(\bar z)^*\,,$$

one has

$$\breve g_0(z)=\Gamma_0^{-1}(z)\,\breve G_0(z)\,,\qquad g_0(z)=G_0(z)\,\Gamma_0^{-1}(z)$$

and so

$$\breve G_0(z)=\Gamma_0(z)\,\breve g_0(z)\,,\qquad G_0(z)=g_0(z)\,\Gamma_0(z). \tag{5.6}$$

From the latter identity, together with the definition of $\Gamma_0(z)$ given above, one infers that

$$\gamma_0(z):=\left(1-\alpha\,\tau_0g_0(z)\right)\in\mathfrak{B}(L^2(\mathbb{R}^2))$$

has a bounded inverse:

$$\gamma_0^{-1}(z)=\Gamma_0(z)\,. \tag{5.7}$$

Hence, by Eq. (5.1),

$$R_\varnothing(z)=R_0(z)+\alpha\,g_0(z)\,\gamma_0^{-1}(z)\,\breve g_0(z).$$

By applying τ_F to the latter equation it follows that

$$\breve G_F(z)=\tau_FR_\varnothing(z)=\breve g_F(z)+\alpha\,\tau_Fg_0(z)\,\gamma_0^{-1}(z)\,\breve g_0(z) \tag{5.8}$$

and, by taking the adjoint (in $\bar{z}$),

$$G_F(z) = g_F(z) + \alpha\, g_0(z)\, \gamma_0^{-1}(z)\, \tau_0 g_F(z), \tag{5.9}$$

where we used $(\tau_F g_0(\bar{z}))^* = (\breve{g}_F(\bar{z})\tau_0^*)^* = \tau_0 g_F(z)$.

By Eqs. (5.6), (5.8) and (5.9) (together with Eq. (5.7)) it follows that

$$\begin{aligned}\begin{pmatrix}\breve{G}_F(z)\\ \breve{G}_0(z)\end{pmatrix} &= \begin{pmatrix}\breve{g}_F(z) + \alpha\, \tau_F g_0(z)\, \gamma_0^{-1}(z)\, \breve{g}_0(z)\\ \gamma_0^{-1}(z)\, \breve{g}_0(z)\end{pmatrix}\\ &= \begin{pmatrix}1 & \alpha\, \tau_F g_0(z)\, \gamma_0^{-1}(z)\\ 0 & \gamma_0^{-1}(z)\end{pmatrix}\begin{pmatrix}\breve{g}_F(z)\\ \breve{g}_0(z)\end{pmatrix}.\end{aligned}$$

Similarly, for $(G_F(z),\, G_0(z))$ one has that

$$\begin{aligned}(G_F(z),\, G_0(z)) &= \big(g_F(z) + \alpha\, g_0(z)\, \gamma_0^{-1}(z)\, \tau_0 g_F(z), g_0(z)\gamma_0^{-1}(z)\big)\\ &= \big(g_F(z), g_0(z)\big)\begin{pmatrix}1 & 0\\ \alpha\, \gamma_0^{-1}(z)\, \tau_0 g_F(z) & \gamma_0^{-1}(z)\end{pmatrix}.\end{aligned}$$

By Eqs. (5.3) and (5.9), it follows that

$$\Gamma_F(z) = 1 + \alpha\, \tau_F g_F(z) + \alpha^2\, \tau_F g_0(z)\, \gamma_0^{-1}(z)\, \tau_0 g_F(z).$$

Now, let us set

$$\begin{gathered}N_1(z) = 1 + \alpha\, \tau_F g_F(z)\,, \qquad N_2(z) = \alpha\, \tau_F g_0(z)\,,\\ N_3(z) = \alpha\, \tau_0 g_F(z)\,, \quad N_4(z) = -\gamma_0(z)\,.\end{gathered}$$

With this notation, one has

$$\Gamma_F(z) = N_1(z) - N_2(z) N_4{}^{-1}(z) N_3(z)\,, \qquad \Gamma_0(z) = -N_4{}^{-1}(z)$$

and (here we temporarily omit the dependence on z)

$$\begin{aligned}&\begin{pmatrix}1 & 0\\ \alpha\, \gamma_0^{-1}\, \tau_0 g_F & \gamma_0^{-1}\end{pmatrix}\begin{pmatrix}\Gamma_F^{-1} & 0\\ 0 & -\Gamma_0^{-1}\end{pmatrix}\begin{pmatrix}1 & \alpha\, \tau_F g_0\, \gamma_0^{-1}\\ 0 & \gamma_0^{-1}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ -N_4{}^{-1} N_3 & -N_4{}^{-1}\end{pmatrix}\begin{pmatrix}\big(N_1 - N_2 N_4{}^{-1} N_3\big)^{-1} & 0\\ 0 & N_4\end{pmatrix}\begin{pmatrix}1 & -N_2 N_4{}^{-1}\\ 0 & -N_4{}^{-1}\end{pmatrix}\\ &= \begin{pmatrix}\big(N_1 - N_2 N_4{}^{-1} N_3\big)^{-1} & -\big(N_1 - N_2 N_4{}^{-1} N_3\big)^{-1} N_2 N_4{}^{-1}\\ -N_4{}^{-1} N_3 \big(N_1 - N_2 N_4{}^{-1} N_3\big)^{-1} & N_4{}^{-1} N_3 \big(N_1 - N_2 N_4{}^{-1} N_3\big)^{-1} N_2 N_4{}^{-1} + N_4{}^{-1}\end{pmatrix}\\ &= \begin{pmatrix}N_1 & N_2\\ N_3 & N_4\end{pmatrix}^{-1} = \begin{pmatrix}1 + \alpha\, \tau_F g_F & \alpha\, \tau_F g_0\\ \alpha\, \tau_0 g_F & -(1 - \alpha\, \tau_0 g_0)\end{pmatrix}^{-1},\end{aligned}$$

where we used the inversion formula for block matrices. Hence we can set

$$\gamma(z) := \frac{J}{\alpha} + \begin{pmatrix}\tau_F g_F(z) & \tau_F g_0(z)\\ \tau_0 g_F(z) & \tau_0 g_0(z)\end{pmatrix}\,; \qquad J = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and we have that

$$\gamma(z) \in \mathfrak{B}\big(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)\big)$$

has a bounded inverse

$$\gamma^{-1}(z) = \alpha \begin{pmatrix} 1 & 0 \\ \alpha\, \gamma_0(z)^{-1}\, \tau_0 g_F(z) & \gamma_0(z)^{-1} \end{pmatrix} \begin{pmatrix} \Gamma_F(z)^{-1} & 0 \\ 0 & -\Gamma_0(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha\, \tau_F g_0(z)\, \gamma_0(z)^{-1} \\ 0 & \gamma_0(z)^{-1} \end{pmatrix}.$$

Going back to the formula for the resolvent difference, see Eq. (5.5), we set

$$g(z) := \big(g_F(z),\, g_0(z)\big)\,, \qquad \breve{g}(z) := \begin{pmatrix} \breve{g}_F(z) \\ \breve{g}_0(z) \end{pmatrix}$$

and we have that

$$R_F(z) - R_0(z) = -\, g(z)\, \gamma^{-1}(z)\, \breve{g}(z)\ .$$

We remark that all the quantities at the r.h.s. of the latter equation are written in terms of the operator $R_0(z)$ and of the traces τ_0 and τ_F.

Since $\gamma(z)$ is a continuous bijection from $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ onto itself, from now on we work in the following setting:

$$\begin{gathered} \tau := \tau_F \oplus \tau_0 \in \mathfrak{B}\big(H^s(\mathbb{R}^3), L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)\big)\,, \qquad s > 1/2\,, \\ \breve{g}(z) := \tau R_0(z) \in \mathfrak{B}\big(L^2(\mathbb{R}^3), L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)\big)\,, \\ g(z) := \breve{g}(\bar{z})^* \in \mathfrak{B}\big(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2), L^2(\mathbb{R}^3)\big)\,, \\ \tau g(z) \in \mathfrak{B}\big(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)\big)\,. \end{gathered}$$

We freely use the identifications

$$\begin{aligned} L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) &\equiv L^2(\Sigma) \oplus L^2(\Sigma^c) \oplus L^2(\Sigma) \oplus L^2(\Sigma^c) \\ &\equiv L^2(\Sigma) \oplus L^2(\Sigma) \oplus L^2(\Sigma^c) \oplus L^2(\Sigma^c)\,. \end{aligned}$$

Let us now introduce, in the component $L^2(\Sigma^c) \oplus L^2(\Sigma^c)$, the orthogonal projections

$$P_0(\phi \oplus \psi) := \frac{\phi + \psi}{2} \oplus \frac{\phi + \psi}{2}\,, \qquad Q_0 := 1 - P_0\,.$$

This induces the further decomposition

$$L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \equiv L^2(\Sigma) \oplus L^2(\Sigma) \oplus \mathrm{Range}(P_0) \oplus \mathrm{Range}(Q_0)\,. \tag{5.10}$$

We then define the orthogonal projectors in $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$

$$P = 1 \oplus P_0\,, \qquad Q = 1 \oplus Q_0\,.$$

From now on a vector $\phi_F \oplus \phi_0 \in L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ will be decomposed as

$$\phi_F \oplus \phi_0 \equiv \phi_\circ \oplus \phi_+ \oplus \phi_-\,,$$

where

$$\phi_\circ = \phi_F^\Sigma \oplus \phi_0^\Sigma\,,$$
$$\phi_+ := P_0(\phi_F^{\Sigma^c} \oplus \phi_0^{\Sigma^c})\,, \quad \phi_- := Q_0(\phi_F^{\Sigma^c} \oplus \phi_0^{\Sigma^c})\,,$$
$$\phi_{F/0}^{\Sigma/\Sigma^c} := \phi_{F/0}|\Sigma/\Sigma^c\,,$$

and we used the notation $\phi|\Sigma$ (resp. $\phi|\Sigma^c$) to denote the restriction of the function ϕ to the set Σ (resp. Σ^c).

We also introduce the decompositions

$$\tau \equiv \tau_\Sigma \oplus \tau_{\Sigma^c} \equiv P\tau \oplus Q_0\tau_{\Sigma^c} \equiv \tau_\Sigma \oplus P_0\tau_{\Sigma^c} \oplus Q_0\tau_{\Sigma^c}$$

where $\tau_{\Sigma/\Sigma^c} f := \tau f|\Sigma/\Sigma^c$. This induces the further decompositions

$$\breve{g}(z) \equiv \breve{g}_\Sigma(z) \oplus \breve{g}_{\Sigma^c}(z) \equiv P\breve{g}(z) \oplus Q_0\breve{g}_{\Sigma^c}(z) \equiv \breve{g}_\Sigma(z) \oplus P_0\breve{g}_{\Sigma^c}(z) \oplus Q_0\breve{g}_{\Sigma^c}(z)\,,$$

where $\breve{g}_{\Sigma/\Sigma^c}(z)f := (\tau R_0(z)f)|\Sigma/\Sigma^c$.

Since $\tau_F f|\Sigma^c = \tau_0 f|\Sigma^c$, one has $Q_0\tau_{\Sigma^c} = Q_0\breve{g}_{\Sigma^c}(z) = 0$ and so

$$\mathrm{Range}(\tau) \subseteq \mathrm{Range}(P)\,, \qquad \mathrm{Range}(\breve{g}(z)) \subseteq \mathrm{Range}(P)\,.$$

Thus $P\breve{g}(z) = \breve{g}(z)$; by duality $g(z) = g(z)P$ and so $g(z)(\mathsf{0} \oplus \mathsf{0} \oplus Q_0) = 0$. Moreover

$$\tau g(z) = P\tau g(z)P\,.$$

Equivalently, using block operator matrix notation with respect to the decomposition (5.10) and setting $g_{\Sigma/\Sigma^c}(z) := \breve{g}_{\Sigma/\Sigma^c}(\bar{z})^*$, one has

$$\tau g(z) = \begin{pmatrix} \tau_\Sigma g_\Sigma(z) & \tau_\Sigma g_{\Sigma^c}(z) & \mathsf{0} \\ \tau_{\Sigma^c} g_\Sigma(z) & \tau_{\Sigma^c} g_{\Sigma^c}(z) & \mathsf{0} \\ \mathsf{0} & \mathsf{0} & \mathsf{0} \end{pmatrix}.$$

Then, since

$$J : \mathrm{Range}(P) \to \mathrm{Range}(Q)\,, \qquad J : \mathrm{Range}(Q) \to \mathrm{Range}(P)\,,$$

one gets

$$\gamma(z) = \begin{pmatrix} J/\alpha + \tau_\Sigma g_\Sigma(z) & \tau_\Sigma g_{\Sigma^c}(z) & \mathsf{0} \\ \tau_{\Sigma^c} g_\Sigma(z) & \tau_{\Sigma^c} g_{\Sigma^c}(z) & J/\alpha \\ \mathsf{0} & J/\alpha & \mathsf{0} \end{pmatrix}. \tag{5.11}$$

Since $\gamma(z)$ is surjective, for any given $\psi_\circ \in L^2(\Sigma) \oplus L^2(\Sigma)$ there exists $\phi_\circ \oplus \phi_+ \oplus \phi_- \equiv \phi_F \oplus \phi_0 \in L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ such that

$$\psi_\circ \oplus 0 \oplus 0 = \gamma(z)(\phi_\circ \oplus \phi_+ \oplus \phi_-)\,.$$

By Eq. (5.11), since J is injective, one gets $\phi_+ = 0$ and $\psi_\circ = (J/\alpha + \tau_\Sigma g_\Sigma(z))\phi_\circ$. Therefore $J/\alpha + \tau_\Sigma g_\Sigma(z)$ is surjective. Since $J/\alpha + \tau_\Sigma g_\Sigma(z) = (J/\alpha + \tau_\Sigma g_\Sigma(\bar{z}))^*$, $\ker(J/\alpha + \tau_\Sigma g_\Sigma(z)) = \mathrm{ran}(J/\alpha + \tau_\Sigma g_\Sigma(\bar{z}))^\perp = \{0\}$. Hence $J/\alpha + \tau_\Sigma g_\Sigma(z)$ is a continuous bijection and so, by the inverse mapping theorem, has a bounded inverse. We introduce the notation

$$\begin{aligned} \Lambda_{F,\Sigma}(z) :=& -\big(J/\alpha + \tau_\Sigma g_\Sigma(z)\big)^{-1} \\ =& -\alpha\big(1 + \alpha J\tau_\Sigma g_\Sigma(z)\big)^{-1} J \in \mathfrak{B}(L^2(\Sigma) \oplus L^2(\Sigma))\,. \end{aligned}$$

Therefore, using again the inversion formula for block operator matrices,

$$\gamma(z)^{-1}=\begin{pmatrix}-\Lambda_{F,\Sigma}(z) & 0 & \alpha\Lambda_{F,\Sigma}(z)\tau_\Sigma g_{\Sigma^c}(z)J\\ 0 & 0 & \alpha J\\ \alpha J\tau_{\Sigma^c}g_\Sigma(z)\Lambda_{F,\Sigma}(z) & \alpha J & -\alpha^2 J\left(\tau_{\Sigma^c}g_\Sigma(z)\Lambda_{F,\Sigma}(z)\tau_\Sigma g_{\Sigma^c}(z)+\tau_{\Sigma^c}g_{\Sigma^c}(z)\right)J\end{pmatrix}.$$

In conclusion,

$$g(z)\,\gamma(z)^{-1}\,\breve{g}(z)=-g_\Sigma(z)\,\Lambda_{F,\Sigma}(z)\,\breve{g}_\Sigma(z)$$

and the proof is concluded. □

6. Existence and asymptotic completeness of the wave operators

The aim of this section is to prove the existence and completeness of the wave operators relative to the couple (A_F, A_0). We make use of the resolvent formula given in Theorem 5.1 together with [22, Th. 2.8] and LAP. Recall that $\sigma(A_0)=\sigma_{ac}(A_0)=[0,+\infty)$.

Theorem 6.1. *Let A_F, A_0 be defined as above. Then, the wave operators*

$$W_\pm(A_F,A_0):=\text{s}-\lim_{t\to\pm\infty}e^{itA_F}e^{-itA_0}\,,$$
$$W_\pm(A_0,A_F):=\text{s}-\lim_{t\to\pm\infty}e^{itA_0}e^{-itA_F}P_{ac}(A_F),$$

where $P_{ac}(A_F)$ denotes the orthogonal projector on the absolutely continuous subspace relative to A_F, exist and are asymptotically complete, i.e., are complete and $\sigma_{sc}(A_F)=\emptyset$.

Proof. Taking into account the resolvent formula (5.4), according to [22, Th. 2.8 and Rem. 2.1], in order to get existence and completeness of the wave operators one needs to show that

$$\sup_{(\lambda,\varepsilon)\in I\times(0,1)}\sqrt{\varepsilon}\,\|g_\Sigma(\lambda\pm i\varepsilon)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2(\mathbb{R}^3))}<+\infty \tag{6.1}$$

and

$$\sup_{(\lambda,\varepsilon)\in I\times(0,1)}\|\Lambda_{F,\Sigma}(\lambda\pm i\varepsilon)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}<+\infty\,, \tag{6.2}$$

for any open and bounded interval I such that $\overline{I}\subset\mathbb{R}\backslash(\{0\}\cup\sigma_p^+(A_F))$. Since $z\mapsto g_\Sigma(z)$ and $z\mapsto\Lambda_{F,\Sigma}(z)$ are continuous on $\rho(A_0)=\mathbb{C}\backslash[0,+\infty)$, it suffices to prove (6.1) and (6.2) whenever $\overline{I}\subset(0,+\infty)\backslash\sigma_p^+(A_F)$.

Let $\mu\in\rho(A_0)\cap\mathbb{R}=(-\infty,0)$; by the inequality (see [22, Eq. (3.16)])

$$\begin{aligned}&\varepsilon\,\|g_\Sigma(\lambda\pm i\varepsilon)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2(\mathbb{R}^3))}\\
&\leqslant\frac{1}{2}\left(|\mu-\lambda|+\varepsilon\right)\|g_\Sigma(\mu)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3))}\times\\
&\quad\times\left(\|\breve{g}_\Sigma(\lambda+i\varepsilon)\|_{\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3),L^2(\Sigma)\oplus L^2(\Sigma))}+\|\breve{g}_\Sigma(\lambda-i\varepsilon)\|_{\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3),L^2(\Sigma)\oplus L^2(\Sigma))}\right)\\
&\leqslant\frac{1}{2}\left(|\mu-\lambda|+\varepsilon\right)\|g_\Sigma(\mu)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3))}\|\tau_\Sigma\|_{\mathfrak{B}(H^1_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3),L^2(\Sigma)\oplus L^2(\Sigma))}\times\\
&\quad\times\left(\|R_0(\lambda+i\varepsilon)\|_{\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3),H^1_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))}+\|R_0(\lambda-i\varepsilon)\|_{\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3),H^1_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))}\right),\end{aligned}$$

inequality (6.1) is consequence of LAP for A_0 (see Proposition 4.2).

By the relation (see [22, Eq. (4.2) and Lem. 4.2], see also [11, Eq. (2.9)])

$$\Lambda_{F,\Sigma}(\lambda \pm i\varepsilon) \ = \Lambda_{F,\Sigma}(\mu)\big[1 + (\lambda - \mu \pm i\varepsilon)\breve{g}_\Sigma(\mu)\big(1 - (\lambda - \mu \pm i\varepsilon)R_F(\lambda \pm i\varepsilon)\big)g_\Sigma(\mu)\Lambda_{F,\Sigma}(\mu)\big]\,,$$

one gets the inequality (here we set $\nu := |\lambda - \mu| + 1$)

$$\begin{aligned}&\|\Lambda_{F,\Sigma}(\lambda \pm i\varepsilon)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))} \leqslant \|\Lambda_{F,\Sigma}(\mu)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}\\ &+ \nu\|\Lambda_{F,\Sigma}(\mu)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}\|\breve{g}_\Sigma(\mu)\|^2_{\mathfrak{B}(L^2(\mathbb{R}^3),L^2(\Sigma)\oplus L^2(\Sigma))}\\ &+ \nu^2\|\Lambda_{F,\Sigma}(\mu)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}\|g_\Sigma(\mu)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))}\times\\ &\quad\times \|R_F(\lambda \pm i\varepsilon)\|_{\mathfrak{B}(L^2_{\sigma_1,\sigma_\|}(\mathbb{R}^3),L^2_{-\sigma_1,-\sigma_\|}(\mathbb{R}^3))}\,.\end{aligned}$$

Hence inequality (6.1) is consequence of LAP for A_F (see Proposition 4.13).

Finally, since LAP implies absence of the singular continuous spectrum (see e.g. [24, Cor. 4.7]), $\sigma_{sc}(A_F)$ is empty. □

Remark 6.2. As immediate consequence of Theorem 6.1, one gets

$$\sigma_{ac}(A_F) = \sigma_{ac}(A_0) = [0, +\infty)\,.$$

We conjecture that there are no embedded eigenvalues for A_F. Hence, if this is the case, $\sigma(A_F) = \sigma_{ac}(A_F)$ and $P_{ac}(A_F)$ is the identity.

7. The scattering matrix

In this section our aim is (following the same strategy as in [22, Sec. 4]) to use the Kato–Birman invariance principle to recover the scattering matrix $S_F(\lambda)$ for the scattering couple (A_F, A_0) from the one for $(R_F(\mu), R_0(\mu))$, $\mu \in \rho(A_0) \cap \rho(A_F)$. Let us denote by

$$S_F := W_+^* W_- : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

the scattering operator corresponding to the wave operators $W_\pm \equiv W_\pm(A_F, A_0)$.

Let

$$\mathcal{F}_0 : L^2(\mathbb{R}^3) \to L^2((0, +\infty); L^2(\mathbb{S}^2))\,,$$

be the unitary which diagonalizes A_0; below we will provide an explicit representation for $\mathcal{F}_0$ in terms of generalized eigenfunctions introduced in Eq. (4.10). Then, define the scattering matrix

$$S_F(\lambda) : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)\,, \qquad \lambda \in (0, +\infty)\backslash\sigma_p^+(A_F)$$

by the relation

$$[\mathcal{F}_0 S_F \mathcal{F}_0^* u](\lambda) = S_F(\lambda)u(\lambda)\,, \qquad u : (0, +\infty)\backslash\sigma_p^+(A_F) \to L^2(\mathbb{S}^2)\,.$$

Let us now denote by $W_\pm^\mu$ the wave operators for the scattering couple $(R_0(\mu), R_F(\mu))$; below we show that they exist and are complete. We denote by S_F^μ the corresponding scattering operator $S_F^\mu := (W_+^\mu)^* W_-^\mu$. As above, we then define the corresponding scattering matrix by

$$S_F^\mu(\lambda) : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)\,, \qquad [\mathcal{F}_0^\mu S_F^\mu (\mathcal{F}_0^\mu)^* u](\lambda) = S_F^\mu(\lambda) u(\lambda)\,,$$

where $[\mathcal{F}_0^\mu f](\lambda) := \frac{1}{\lambda}\,[\mathcal{F}_0 f](\mu + 1/\lambda)$ is the unitary which diagonalizes $R_0(\mu)$.

The forthcoming theorem provides a more explicit representation of $S_F^\mu(\lambda)$.

Theorem 7.1. *The wave operator $W_\pm^\mu$ exist, are complete and the corresponding scattering matrix has the representation*

$$S_F^\mu(\lambda) = 1 - 2\pi i L_F^\mu(\lambda)\Lambda_{F,\Sigma}(\mu)\big[1 - \breve{g}_\Sigma(\mu)\big(R_F(\mu) - (\lambda + i0)\big)^{-1} g_\Sigma(\mu)\Lambda_{F,\Sigma}(\mu)\big] L_F^\mu(\lambda)^*,$$

for all λ such that $\mu + \frac{1}{\lambda} \in (0, +\infty)\backslash\sigma_p^+(A_F)$, and where

$$L_F^\mu(\lambda) : L^2(\Sigma) \oplus L^2(\Sigma) \to L^2(\mathbb{S}^2)\,,$$
$$L_F^\mu(\lambda)(\phi_F^\Sigma \oplus \phi_0^\Sigma) := \frac{1}{\lambda}\left[\mathcal{F}_0 g_\Sigma(\mu)(\phi_F^\Sigma \oplus \phi_0^\Sigma)\right](\mu + 1/\lambda)\,.$$

Proof. At first, let us notice the relations

$$\begin{aligned} \big(R_0(\mu) - z\big)^{-1} &= -\frac{1}{z}\left(\frac{1}{z}\,R_0(\mu + 1/z) + 1\right)\,, \\ \big(R_F(\mu) - z\big)^{-1} &= -\frac{1}{z}\left(\frac{1}{z}\,R_F(\mu + 1/z) + 1\right)\,. \end{aligned} \tag{7.1}$$

Therefore, by LAP for A_0 and A_F (see Theorems 4.1 and 4.13) the limits

$$\big(R_0(\mu) - (\lambda \pm i0)\big)^{-1} := \lim_{\varepsilon \downarrow 0} \big(R_0(\mu) - (\lambda \pm i\varepsilon)\big)^{-1}\,,$$

with $\mu + \frac{1}{\lambda} \in (0, +\infty)$, exists in $\mathfrak{B}(L^2_{\sigma_1,\sigma_\parallel}(\mathbb{R}^3), H^s_{-\sigma_1,-\sigma_\parallel}(\mathbb{R}^3))$ for $\sigma_1, \sigma_\parallel > 1/2$ and $s \in (1, 3/2)$ (see Proposition 4.2), and the limits

$$\big(R_F(\mu) - (\lambda \pm i0)\big)^{-1} := \lim_{\varepsilon \downarrow 0} \big(R_F(\mu) - (\lambda \pm i\varepsilon)\big)^{-1}\,,$$

with $\mu + \frac{1}{\lambda} \in (0, +\infty)\backslash\sigma_p^+(A_F)$, exist in $\mathfrak{B}(L^2_{\sigma_1,\sigma_\parallel}(\mathbb{R}^3), L^2_{-\sigma_1,-\sigma_\parallel}(\mathbb{R}^3))$ for $\sigma_1 > 1/2$ and $\sigma_\parallel > 3/4$. Moreover, by Proposition 4.5, Remark 4.6 and Lemma 2.7 one gets

$$\breve{g}_\Sigma(\mu) \in \mathfrak{B}\big(L^2_{-\sigma_1,-\sigma_\parallel}(\mathbb{R}^3), L^2(\Sigma) \oplus L^2(\Sigma)\big)\,,$$
$$g_\Sigma(\mu) = \breve{g}_\Sigma(\mu)^* \in \mathfrak{B}\big(L^2(\Sigma) \oplus L^2(\Sigma), L^2_{\sigma_1,\sigma_\parallel}(\mathbb{R}^3)\big)$$

for all $\sigma_1, \sigma_\parallel \in \mathbb{R}$. Therefore the following limits exist:

$$\lim_{\varepsilon \downarrow 0} \breve{g}_\Sigma(\mu)\big(R_0(\mu) - (\lambda \pm i\varepsilon)\big)^{-1}\,; \tag{7.2}$$

$$\lim_{\varepsilon \downarrow 0} \breve{g}_\Sigma(\mu)\big(R_F(\mu) - (\lambda \pm i\varepsilon)\big)^{-1}\,;$$

$$\lim_{\varepsilon \downarrow 0} \breve{g}_\Sigma(\mu)\big(R_F(\mu) - (\lambda \pm i\varepsilon)\big)^{-1} g_\Sigma(\mu)\,. \tag{7.3}$$

Let us now show that $\breve{g}_\Sigma(\mu)$ is weakly-$R^0(\mu)$ smooth, i.e. by [37, p. 154, Lem. 2],

$$\sup_{0<\varepsilon<1} \varepsilon\,\|\breve{g}_\Sigma(\mu)(R_0(\mu) - (\lambda \pm i\varepsilon))^{-1}\|^2_{\mathfrak{B}(L^2(\Sigma), L^2(\Sigma)\oplus L^2(\Sigma))} \leqslant c_\lambda < +\infty\,, \quad \text{for a.e. } \lambda\,. \tag{7.4}$$

By (7.1), this is consequence of

$$\sup_{0<\varepsilon<1} \varepsilon \, \|\breve{g}_\Sigma(\mu) R_0(\mu + 1/\lambda \pm i\varepsilon)\|^2_{\mathfrak{B}(L^2(\Sigma), L^2(\Sigma)\oplus L^2(\Sigma))} \leqslant C_\lambda < +\infty \,, \quad \text{for a.e. } \lambda \,.$$

To prove the latter claim let us note that, by

$$\breve{g}_\Sigma(\mu) R_0(z) = \tau_\Sigma R_0(\mu) R_0(z) = \tau_\Sigma R_0(z) R_0(\mu) = \breve{g}_\Sigma(z) R_0(\mu) = (R_0(\mu) g_\Sigma(\bar{z}))^* \,,$$

it is consequence of Eq. (6.1), which has been shown to hold in the proof of Theorem 6.1. Therefore, by the factorization of the difference $R_F(\mu) - R_0(\mu)$ provided in Theorem 5.1, by the existence of the limits (7.2)–(7.3) and by the bound (7.4), the hypotheses in [37, p. 178, Th. 4'] are satisfied, which suffices to infer the thesis. □

By [22, Lem. 4.2] (be aware that there the resolvent of an operator A is defined as $(-A+z)^{-1}$) one has the identity

$$\Lambda_{F,\Sigma}(\mu)\big[1 - \breve{g}_\Sigma(\mu)\big(R_F(\mu) - z\big)^{-1} g_\Sigma(\mu)\Lambda_{F,\Sigma}(\mu)\big] = \Lambda_{F,\Sigma}(\mu + 1/z) \,.$$

This gives the existence of the limit

$$\begin{aligned}\Lambda_{F,\Sigma}(\lambda^+) :=& \lim_{\varepsilon\downarrow 0} \Lambda_{F,\Sigma}(\lambda + i\varepsilon) \\ =& \Lambda_{F,\Sigma}(\mu)\big[1 - \breve{g}_\Sigma(\mu)\big(R_F(\mu) - (1/(\lambda-\mu) + i0)\big)^{-1} g_\Sigma(\mu)\Lambda_{F,\Sigma}(\mu)\big] \,.\end{aligned}$$

Taking into account the identities

$$-\Lambda_{F,\Sigma}(\lambda + i\varepsilon)\left(\frac{J}{\alpha} + \tau_\Sigma g_\Sigma(\lambda + i\varepsilon)\right) = 1 = -\left(\frac{J}{\alpha} + \tau_\Sigma g_\Sigma(\lambda + i\varepsilon)\right)\Lambda_{F,\Sigma}(\lambda + i\varepsilon) \,,$$

$$\left(\frac{J}{\alpha} + \tau_\Sigma g_\Sigma(\lambda + i\varepsilon)\right)\frac{J}{\alpha}\left(1 + \alpha J \tau_\Sigma g_\Sigma(\lambda + i\varepsilon)\right)^{-1} = 1 = \left(1 + \alpha J \tau_\Sigma g_\Sigma(\lambda + i\varepsilon)\right)^{-1}\left(\frac{J}{\alpha} + \tau_\Sigma g_\Sigma(\lambda + i\varepsilon)\right)\frac{J}{\alpha}$$

and considering the limit $\varepsilon \downarrow 0$, this also provides the existence of the inverse $(1 + \tau_\Sigma g_\Sigma(\lambda + i0))^{-1}$ and the identity

$$\Lambda_{F,\Sigma}(\lambda^+) = -\alpha\left(1 + \alpha J M_{F,\Sigma}(\lambda^+)\right)^{-1} J \,,$$

where we set

$$M_{F,\Sigma}(\lambda^+) := \tau_\Sigma g_\Sigma(\lambda + i0) = \lim_{\varepsilon\downarrow 0} \tau_\Sigma g_\Sigma(\lambda + i\varepsilon) \,.$$

Since, by the invariance principle, one has the relation (see [37, Ch. 2, Sec. 6, Eq. (14)])

$$S_F(\lambda) = S_F^\mu\big(1/(\lambda - \mu)\big) \,, \tag{7.5}$$

Theorem 7.1 has the following

Corollary 7.2. *For all $\lambda \in (0, +\infty)\backslash\sigma_p^+(A_F)$ the scattering matrix for the scattering couple (A_F, A_0) is given by*

$$S_F(\lambda) = 1 + 2\pi i \alpha L_F(\lambda)\left(1 + \alpha J M_{F,\Sigma}(\lambda^+)\right)^{-1} J L_F(\lambda)^* \,, \tag{7.6}$$

where $L_F(\lambda) : L^2(\Sigma) \oplus L^2(\Sigma) \to L^2(\mathbb{S}^2)$ *is given by*

$$\begin{aligned}&\big(L_F(\lambda)(\phi_F^\Sigma \oplus \phi_0^\Sigma)\big)(\boldsymbol{\xi})\\ &= \frac{\lambda^{1/4}}{2^{1/2}}\frac{1}{(2\pi)^{3/2}}\int_\Sigma d\mathbf{x}_\parallel \left(e^{-i\sqrt{\lambda}\xi_1 F(\mathbf{x}_\parallel)} - \frac{\alpha\, e^{-i\sqrt{\lambda}\,|\xi_1|\,|F(\mathbf{x}_\parallel)|}}{\alpha + 2i\sqrt{\lambda}\,|\xi_1|}\right) e^{-i\sqrt{\lambda}\,\boldsymbol{\xi}_\parallel\cdot\mathbf{x}_\parallel}\,\phi_F^\Sigma(\mathbf{x}_\parallel)\\ &\quad + \frac{\lambda^{1/4}}{(4\pi)^{1/2}}\left(1 - \frac{\alpha}{\alpha + 2i\sqrt{\lambda}\,|\xi_1|}\right)\widehat{\widetilde{\phi_0^\Sigma}}(\sqrt{\lambda}\,\boldsymbol{\xi}_\parallel)\,.\end{aligned}$$

Here $\widehat{\ }$ *denotes the Fourier transform in* $L^2(\mathbb{R}^2)$, $\widetilde{\phi_0^\Sigma}$ *is the extension by zero of* ϕ_0^Σ *to the whole* $\mathbb{R}^2$ *and* $\boldsymbol{\xi} \equiv (\xi_1, \boldsymbol{\xi}_\parallel)$, $|\xi_1|^2 + \|\boldsymbol{\xi}_\parallel\|^2 = 1$.

Proof. By Theorem 7.1 and by Eq. (7.5), one gets Eq. (7.6) with

$$L_F(\lambda)(\phi_F^\Sigma \oplus \phi_0^\Sigma) := (\lambda - \mu)\big(\mathcal{F}_0 g_\Sigma(\mu)(\phi_F^\Sigma \oplus \phi_0^\Sigma)\big)(\lambda)\,.$$

Then by

$$(\lambda - \mu)\big(\mathcal{F}_0 R_0(\mu) f\big)(\lambda) = (\mathcal{F}_0 f)(\lambda)$$

and by

$$\big(g_\Sigma(\mu)(\phi_F^\Sigma \oplus \phi_0^\Sigma)\big)(\lambda) = R_0(\mu)(\tau_{F,\Sigma}^*\phi_F^\Sigma + \tau_{0,\Sigma}^*\phi_0^\Sigma)\,,$$

where $\tau_{F/0,\Sigma} u := \tau_{F/0} u|\Sigma$, one gets

$$L_F(\lambda)(\phi_F^\Sigma \oplus \phi_0^\Sigma) = \mathcal{F}_0(\tau_{F,\Sigma}^*\phi_F^\Sigma + \tau_{0,\Sigma}^*\phi_0^\Sigma)(\lambda)\,.$$

By the structure of the generalized eigenfunctions of A_0 (see Eq. (4.10)), one gets

$$\begin{aligned}&\mathcal{F}_0 : L^2(\mathbb{R}^3) \to L^2\big((0, +\infty); L^2(\mathbb{S}^2)\big)\,,\\ &\big((\mathcal{F}_0 f)(\lambda)\big)(\boldsymbol{\xi}) := \frac{\lambda^{1/4}}{2^{1/2}}\int_{\mathbb{R}^3} \overline{\varphi}_{\sqrt{\lambda}\,\boldsymbol{\xi}}(\boldsymbol{x})\, f(\boldsymbol{x})\, d\boldsymbol{x}\,,\end{aligned}$$

where $\varphi_{\boldsymbol{k}}(\boldsymbol{x})$ was defined in Eq. (4.10), $\boldsymbol{x} \equiv (x^1, \mathbf{x}_\parallel)$, $\boldsymbol{\xi} \equiv (\xi_1, \boldsymbol{\xi}_\parallel)$, $|\xi_1|^2 + |\boldsymbol{\xi}_\parallel|^2 = 1$ and $\int_{\mathbb{R}^3}$ is to be understood as the L^2-limit of $\int_{|\mathbf{x}|\leqslant R}$ for $R \nearrow +\infty$. Therefore

$$\begin{aligned}&\big(L_F(\lambda)(\phi_F^\Sigma \oplus \phi_0^\Sigma)\big)(\boldsymbol{\xi})\\ &= \frac{\lambda^{1/4}}{2^{1/2}}\int_\Sigma d\mathbf{x}_\parallel \left(\overline{\varphi}_{\sqrt{\lambda}\,\boldsymbol{\xi}}(F(\mathbf{x}_\parallel), \mathbf{x}_\parallel)\phi_F^\Sigma(\mathbf{x}_\parallel) + \overline{\varphi}_{\sqrt{\lambda}\,\boldsymbol{\xi}}(0, \mathbf{x}_\parallel)\phi_0^\Sigma(\mathbf{x}_\parallel)\right)\\ &= \frac{\lambda^{1/4}}{2^{1/2}}\frac{1}{(2\pi)^{3/2}}\int_\Sigma d\mathbf{x}_\parallel \left(e^{-i\sqrt{\lambda}\xi_1 F(\mathbf{x}_\parallel)} - \frac{\alpha\, e^{-i\sqrt{\lambda}\,|\xi_1|\,|F(\mathbf{x}_\parallel)|}}{\alpha + 2i\sqrt{\lambda}\,|\xi_1|}\right) e^{-i\sqrt{\lambda}\,\boldsymbol{\xi}_\parallel\cdot\mathbf{x}_\parallel}\,\phi_F^\Sigma(\mathbf{x}_\parallel)\\ &\quad + \frac{\lambda^{1/4}}{(4\pi)^{1/2}}\left(1 - \frac{\alpha}{\alpha + 2i\sqrt{\lambda}\,|\xi_1|}\right)\widehat{\widetilde{\phi_0^\Sigma}}(\sqrt{\lambda}\,\boldsymbol{\xi}_\parallel)\,. \quad \square\end{aligned}$$

Let us now denote by $S_0(\lambda)$, $L_0(\lambda)$, $\Lambda_{0,\Sigma}(\lambda^+)$, $M_{0,\Sigma}(\lambda^+)$ the operators $S_F(\lambda)$, $L_F(\lambda)$, $\Lambda_{F,\Sigma}(\lambda^+)$, $M_{F,\Sigma}(\lambda^+)$ corresponding to the choice $F = 0$; obviously $S_0(\lambda) = 1$. By

$$\begin{aligned} S_F(\lambda) - 1 &= S_F(\lambda) - S_0(\lambda) \\ &= 2\pi i\big(L_F(\lambda) - L_0(\lambda)\big)\Lambda_{F,\Sigma}(\lambda^+)L_F(\lambda)^* \\ &\quad + 2\pi i L_0(\lambda)\Lambda_{F,\Sigma}(\lambda^+)\big(M_{F,\Sigma}(\lambda^+) - M_{0,\Sigma}(\lambda^+)\big)\Lambda_{0,\Sigma}(\lambda^+)L_F(\lambda)^* \\ &\quad + 2\pi i L_0(\lambda)\Lambda_{0,\Sigma}(\lambda^+)\big(L_F(\lambda)^* - L_0(\lambda)^*\big) \end{aligned}$$

one gets

$$\begin{aligned} &\|S_F(\lambda) - 1\|_{\mathfrak{B}(L^2(\mathbb{S}^2))} \\ &\leqslant 2\pi\big(K_{F,\Sigma}(\lambda) + K_{0,\Sigma}(\lambda)\big)\,\|L_F(\lambda) - L_0(\lambda)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2(\mathbb{S}^2))} \\ &\quad + 2\pi K_{F,\Sigma}(\lambda)K_{0,\Sigma}(\lambda)\,\|M_{F,\Sigma}(\lambda^+) - M_{0,\Sigma}(\lambda^+)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}\,, \end{aligned}$$

where

$$K_{F/0,\Sigma}(\lambda) := \|L_{F/0}(\lambda))\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2(\mathbb{S}^2))}\|\Lambda_{F/0,\Sigma}(\lambda^+)\|_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}\,.$$

The next Lemmata provide estimates on the norms of the differences $L_F(\lambda) - L_0(\lambda)$ and $M_{F,\Sigma}(\lambda^+) - M_{0,\Sigma}(\lambda^+)$ and hence on the norm of $S_F(\lambda) - 1$.

Lemma 7.3. *For all $\lambda \in (0,+\infty)\backslash\sigma_p^+(A_F)$, there holds*

$$\|L_F(\lambda) - L_0(\lambda)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma),L^2(\mathbb{S}^2))} \leqslant \frac{2\sqrt{\lambda}}{\pi^2}\int_\Sigma d\mathbf{x}_\parallel\left(1 - \frac{\sin\big(\sqrt{\lambda}\,F(\mathbf{x}_\parallel)\big)}{\sqrt{\lambda}\,F(\mathbf{x}_\parallel)}\right).$$

Proof. Indicating with $d\sigma$ the usual surface element on the sphere $\mathbb{S}^2$, we have

$$\begin{aligned} &\big\|\big(L_F(\lambda) - L_0(\lambda)\big)(\phi_F^\Sigma \oplus \phi_0^\Sigma)\big\|^2_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} d\sigma(\boldsymbol{\xi})\big|\big((L_F(\lambda) - L_0(\lambda))(\phi_F^\Sigma \oplus \phi_0^\Sigma)\big)(\boldsymbol{\xi})\big|^2 \\ &\leqslant \frac{\lambda^{1/2}}{2(2\pi)^3}\int_{\mathbb{S}^2} d\sigma(\boldsymbol{\xi})\left(\int_\Sigma d\mathbf{x}_\parallel\left(\left|e^{-i\sqrt{\lambda}\xi_1 F(\mathbf{x}_\parallel)} - 1\right| + \left|e^{-i\sqrt{\lambda}\,|\xi_1|\,|F(\mathbf{x}_\parallel)|} - 1\right|\right)|\phi_F^\Sigma(\mathbf{x}_\parallel)|\right)^2 \\ &\leqslant \frac{\lambda^{1/2}}{2}\frac{\|\phi_F^\Sigma\|^2_{L^2(\Sigma)}}{(2\pi)^3}\int_\Sigma d\mathbf{x}_\parallel\left(\int_{\mathbb{S}^2} d\sigma(\boldsymbol{\xi})\left(\left|e^{-i\sqrt{\lambda}\xi_1 F(\mathbf{x}_\parallel)} - 1\right| + \left|e^{-i\sqrt{\lambda}\,|\xi_1|\,|F(\mathbf{x}_\parallel)|} - 1\right|\right)^2\right) \\ &\leqslant 4\lambda^{1/2}\frac{\|\phi_F^\Sigma\|^2_{L^2(\Sigma)}}{(2\pi)^3}\int_\Sigma d\mathbf{x}_\parallel\left(\int_{\mathbb{S}^2} d\sigma(\boldsymbol{\xi})\Big(1 - \cos\big(\sqrt{\lambda}\,\xi_1 F(\mathbf{x}_\parallel)\big)\Big)\right) \\ &= 2\lambda^{1/2}\frac{\|\phi_F^\Sigma\|^2_{L^2(\Sigma)}}{\pi^2}\int_\Sigma d\mathbf{x}_\parallel\left(1 - \frac{\sin\sqrt{\lambda}\,F(\mathbf{x}_\parallel)}{\sqrt{\lambda}\,F(\mathbf{x}_\parallel)}\right), \end{aligned}$$

where we used the basic identity $\int_0^\pi dx\cos(z\cos x)\sin x = \frac{2}{z}\sin z$. □

Remark 7.4. By the inequality $0 \le 1 - \frac{\sin x}{x} \leqslant \frac{|x|}{\pi}$ one gets

$$\|L_F(\lambda) - L_0(\lambda)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma), L^2(\mathbb{S}^2))} \leqslant \frac{2\lambda}{\pi^3} \|F\|_{L^1(\Sigma)} .$$

Lemma 7.5. *Let $r = \frac{1}{2} + \gamma$, $0 < \gamma < 1$ and $s > 1$. Then*

$$\|\tau_{F,\Sigma} - \tau_{0,\Sigma}\|^2_{\mathfrak{B}(H^r(I)\otimes H^s(\Omega), L^2(\Sigma))} \leqslant_c \left\||F|^{2\gamma}\right\|_{L^1(\Sigma)} .$$

Proof. Using the Sobolev inequalities

$$|u_1(x_1) - u_1(y_1)| \leqslant_c \|u_1\|_{H^r(I)} |x_1 - y_1|^\gamma ,$$
$$\sup_{\mathbf{x}_\parallel \in \Omega} |u_\parallel(\mathbf{x}_\parallel)| \leqslant_c \|u_\parallel\|_{H^s(\Omega)} ,$$

one obtains

$$\int_\Sigma d\mathbf{x}_\parallel \, |u_1(F(\mathbf{x}_\parallel)) - u_1(0)|^2 \, |u_\parallel(\mathbf{x}_\parallel)|^2 \leqslant_c \|u_1\|^2_{H^r(I)} \|u_\parallel\|^2_{H^s(\Omega)\|} \int_\Sigma d\mathbf{x}_\parallel \, |F(\mathbf{x}_\parallel)|^{2\gamma} ,$$

which proves the thesis. □

Lemma 7.6. *There holds*

$$\begin{aligned}
&\|M_{F,\Sigma}(\lambda^+) - M_{0,\Sigma}(\lambda^+)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma), L^2(\Sigma)\oplus L^2(\Sigma))} \\
&\leqslant_c \left(N^2_{s_1,s_2,\varepsilon}(\lambda^+) N^2_{F,\Sigma,s_1,s_2} + \left(N^2_{s_1,s_2,\varepsilon}(\lambda^+) + N^2_{s_1,s_2,\varepsilon}(\lambda^-)\right) N^2_{0,\Sigma,s_1,s_2}\right) \times \\
&\times \|\tau_{F,\Sigma} - \tau_{0,\Sigma}\|^2_{\mathfrak{B}(H^{-s_1+3/2-\varepsilon}(I)\otimes H^{-s_2+2}(\Omega), L^2(\Sigma))} ,
\end{aligned}$$

where $\varepsilon > 0$, $\frac{1}{2} < s_1 < 1 - \varepsilon$, $\frac{1}{2} < s_2 < \frac{3}{2}$,

$$N_{F/0,\Sigma,s_1,s_2} := \|\tau_{F/0,\Sigma}\|_{\mathfrak{B}(H^{s_1}(I)\otimes H^{s_2}(\Omega), L^2(\Sigma))} ,$$
$$N_{s_1,s_2,\varepsilon}(\lambda^\pm) := \|(r_I \otimes r_\Omega) R_0(\lambda^\pm)(r_I^* \otimes r_\Omega^*)\|_{\mathfrak{B}(H^{-s_1}_{\overline{I}}(\mathbb{R})\otimes H^{-s_2}_{\overline{\Omega}}(\mathbb{R}^2), H^{-s_1+3/2-\varepsilon}(I)\otimes H^{-s_2+2}(\Omega))} ,$$

$\Omega \subset \mathbb{R}^2$ is a open ball containing Σ, $I \subset \mathbb{R}$ is an open bounded interval which contains range(F) *and $r_I : H^{s_1}_{\sigma_1}(\mathbb{R}) \to H^{s_1}(I)$, $r_\Omega : H^{s_2}_{\sigma_\parallel}(\mathbb{R}^2) \to H^{s_2}(\Omega)$ are the restriction operators.*

Proof. One has $H^s(I)' \simeq H^{-s}_{\overline{I}}(\mathbb{R})$ and $H^s(\Omega)' \simeq H^{-s}_{\overline{\Omega}}(\mathbb{R}^2)$ for any $s \in \mathbb{R}$, (see, e.g., [25, Ths. 3.14, 3.29]); we use such identifications in the following. Then, the duals of the restriction operators $r_I^* : H^{-s_1}_{\overline{I}}(\mathbb{R}) \to H^{-s_1}_{-\sigma_1}(\mathbb{R})$ and $r_\Omega^* : H^{-s_2}_{\overline{\Omega}}(\mathbb{R}^2) \to H^{-s_2}_{-\sigma_\parallel}(\mathbb{R}^2)$ are given by the extensions by zero.

We know that

$$R_0^{(1)}(\lambda^\pm) \in \mathfrak{B}\big(L^2_{\sigma_1}(\mathbb{R}), H^{3/2-\varepsilon}_{-\sigma_1}(\mathbb{R})\big) , \quad \text{for } \sigma_1 > 1/2,\ 0 < \varepsilon < 1/2 ,$$

and

$$R_0^{(\parallel)}(\lambda^\pm) \in \mathfrak{B}\big(L^2_{\sigma_\parallel}(\mathbb{R}^2), H^2_{-\sigma_\parallel}(\mathbb{R}^2)\big) , \quad \sigma_\parallel > 1/2 .$$

Thus

$$r_I R_0^{(1)}(\lambda^\pm) r_I^* \in \mathfrak{B}\big(L^2_{\overline{I}}(\mathbb{R}), H^{3/2-\varepsilon}(I)\big) \tag{7.7}$$

and

$$r_\Omega R_0^{(\|)}(\lambda^\pm)r_\Omega^* \in \mathfrak{B}\big(L^2_{\overline{\Omega}}(\mathbb{R}^2), H^2(\Omega)\big)\,. \tag{7.8}$$

Hence, by duality,

$$r_I R_0^{(1)}(\lambda^\pm)r_I^* \in \mathfrak{B}\big(H^{-3/2+\varepsilon}_{\overline{I}}(\mathbb{R}), L^2(I)\big) \tag{7.9}$$

and

$$r_\Omega R_0^{(\|)}(\lambda^\pm)r_\Omega^* \in \mathfrak{B}\big(H^{-2}_{\overline{\Omega}}(\mathbb{R}^2), L^2(\Omega)\big)\,. \tag{7.10}$$

By Eqs. (7.7)–(7.10) and by interpolation ([1]) one gets

$$r_I R_0^{(1)}(\lambda^\pm)r_I^* \in \mathfrak{B}(H^{-s_1}_{\overline{I}}(\mathbb{R}), H^{-s_1+3/2-\varepsilon}(I))\,, \quad \text{for } 0 \leqslant s_1 \leqslant 3/2-\varepsilon\,,$$

and

$$r_\Omega R_0^{(\|)}(\lambda^\pm)r_\Omega^* \in \mathfrak{B}(H^{-s_2}_{\overline{\Omega}}(\mathbb{R}^2), H^{-s_2+2}(\Omega))\,, \quad \text{for } 0 \leqslant s_2 \leqslant 2\,.$$

Therefore, since $A_0 = A_0^{(1)} \otimes 1 + 1 \otimes A_0^{(\|)}$, by [5] one has

$$(r_I \otimes r_\Omega)R_0(\lambda^\pm)(r_I^* \otimes r_\Omega^*) \in \mathfrak{B}(H^{-s_1}_{\overline{I}}(\mathbb{R}) \otimes H^{-s_2}_{\overline{\Omega}}(\mathbb{R}^2), H^{-s_1+3/2-\varepsilon}(I) \otimes H^{-s_2+2}(\Omega))\,.$$

Since

$$(\tau_{F,\Sigma} - \tau_{0,\Sigma})R_0(\lambda^\pm)\tau^*_{F/0,\Sigma} = (\tau_{F,\Sigma} - \tau_{0,\Sigma})(r_I \otimes r_\Omega)R_0(\lambda^\pm)(r_I^* \otimes r_\Omega^*)\tau^*_{F/0,\Sigma}\,,$$

one has, for any $\frac{1}{2} < s_1 < 1-\varepsilon$ and $\frac{1}{2} < s_2 < \frac{3}{2}$,

$$\begin{aligned}
&\|(\tau_{F,\Sigma} - \tau_{0,\Sigma})R_0(\lambda^\pm)\tau^*_{F/0,\Sigma}\|_{\mathfrak{B}(L^2(\Sigma))}\\
&\leqslant N_{s_1,s_2,\varepsilon}(\lambda^\pm)\, N_{F/0,\Sigma,s_1,s_2}\, \|\tau_{F,\Sigma} - \tau_{0,\Sigma}\|_{\mathfrak{B}(H^{-s_1+3/2-\varepsilon}(I)\otimes H^{-s_2+2}(\Omega), L^2(\Sigma))}\,.
\end{aligned}$$

Using the block operator matrix notation one has

$$\begin{aligned}
&M_{F,\Sigma}(\lambda^+) - M_{0,\Sigma}(\lambda^+)\\
&= \begin{pmatrix} \tau_{F,\Sigma}(\tau_{F,\Sigma}R_0(\lambda^-))^* - \tau_{0,\Sigma}(\tau_{0,\Sigma}R_0(\lambda^-))^* & (\tau_{F,\Sigma} - \tau_{0,\Sigma})(\tau_{0,\Sigma}R_0(\lambda^-))^* \\ \tau_{0,\Sigma}((\tau_{F,\Sigma} - \tau_{0,\Sigma})R_0(\lambda^-))^* & 0 \end{pmatrix}
\end{aligned}$$

and so (for brevity here we omitted some norm indexes)

$$\begin{aligned}
&\|M_{F,\Sigma}(\lambda^+) - M_{0,\Sigma}(\lambda^+)\|^2_{\mathfrak{B}(L^2(\Sigma)\oplus L^2(\Sigma))}\\
&\leqslant_c \|(\tau_{F,\Sigma} - \tau_{0\Sigma})(\tau_{F,\Sigma}R_0(\lambda^-))^*\|^2 + \|\tau_{0,\Sigma}((\tau_{F,\Sigma} - \tau_{0\Sigma})R_0(\lambda^-))^*\|^2\\
&\quad + \|(\tau_{F,\Sigma} - \tau_{0\Sigma})(\tau_{0,\Sigma}R_0(\lambda^+))^*\|^2\\
&\leqslant_c \|(\tau_{F,\Sigma} - \tau_{0\Sigma})R_0(\lambda^+)\tau^*_{F,\Sigma}\|^2 + \|(\tau_{F,\Sigma} - \tau_{0\Sigma})R_0(\lambda^-)\tau^*_{0,\Sigma}\|^2
\end{aligned}$$

[1] One should keep in mind the previously mentioned isomorphisms $H^s(I)' \simeq H^{-s}_{\overline{I}}(\mathbb{R})$, $H^s(\Omega)' \simeq H^{-s}_{\overline{\Omega}}(\mathbb{R}^2)$ (for $s \in \mathbb{R}$), recall the duality interpolation theorem stated in [7, Cor. 4.5.2] and notice that the spaces $H^s(I)$, $H^s(\Omega)$ enjoy the interpolation property (see, e.g., [33]).

$$
\begin{aligned}
&+ \|(\tau_{F,\Sigma} - \tau_{0\Sigma})R_0(\lambda^+)\tau_{0,\Sigma}^*\|^2 \\
&\leqslant_c \left(N^2_{s_1,s_2,\varepsilon}(\lambda^+)N^2_{F,\Sigma,s_1,s_2} + \left(N^2_{s_1,s_2,\varepsilon}(\lambda^+) + N^2_{s_1,s_2,\varepsilon}(\lambda^-)\right) N^2_{0,\Sigma,s_1,s_2}\right) \times \\
&\times \|\tau_{F,\Sigma} - \tau_{0,\Sigma}\|^2_{\mathfrak{B}(H^{-s_1+3/2-\varepsilon}(I)\otimes H^{-s_2+2}(\Omega),L^2(\Sigma))} . \quad \square
\end{aligned}
$$

Summing up, using the previous Lemma 7.6 with ε replaced by $\varepsilon/4$ $(0 < \varepsilon < 1)$ and $s_1 = 1/2 + \varepsilon/4$, $s_2 = 1 - \varepsilon/4$, one gets

Theorem 7.7. *For all $\lambda \in (0, +\infty)\backslash\sigma_p^+(A_F)$ and $0 < \varepsilon < 1$, the following estimate holds:*

$$
\|S_F(\lambda) - 1\|^2_{\mathfrak{B}(L^2(\mathbb{S}^2))} \leqslant c_{F,\Sigma}(\lambda)\, \|F\|_{L^1(\mathbb{R}^2)} + c_{F,\Sigma,\varepsilon}(\lambda)\, \|\, |F|^{1-\varepsilon}\|_{L^1(\mathbb{R}^2)} , \tag{7.11}
$$

where

$$
\begin{gathered}
c_{F,\Sigma}(\lambda) := c\left(K_{F,\Sigma}(\lambda) + K_{0,\Sigma}(\lambda)\right)^2 , \\
c_{F,\Sigma,\varepsilon}(\lambda) := c\, K^2_{F,\Sigma}(\lambda)K^2_{0,\Sigma}(\lambda)\, \left(N^2_\varepsilon(\lambda^+)N^2_{F,\Sigma,\varepsilon} + \left(N^2_\varepsilon(\lambda^+) + N^2_\varepsilon(\lambda^-)\right) N^2_{0,\Sigma,\varepsilon}\right) , \\
N_{F/0,\Sigma,\varepsilon} := N_{F/0,\Sigma,\frac{1}{2}+\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}} , \qquad N_\varepsilon(\lambda^\pm) := N_{\frac{1}{2}+\frac{\varepsilon}{4},1-\frac{\varepsilon}{4},\frac{\varepsilon}{4}}(\lambda^\pm) .
\end{gathered}
$$

Remark 7.8. Since the constants $c_{F,\Sigma}(\lambda)$ and $c_{F,\Sigma,\varepsilon}(\lambda)$ are bounded and away from zero as $F \to 0$, by (7.11) one gets, for any $0 < \gamma < 1$,

$$
\|S_F(\lambda) - 1\|^2_{\mathfrak{B}(L^2(\mathbb{S}^2))} = \mathcal{O}\left(\int_{\mathbb{R}^2} d\mathbf{x}_\| |F(\mathbf{x}_\|)|^\gamma\right) .
$$

Acknowledgments

We thank the anonymous referee for the stimulating remarks.

This work was supported by: INdAM, Gruppo Nazionale per la Fisica Matematica; "Progetto Giovani GNFM 2017 – Dinamica quasi classica per il modello di polarone" fostered by GNFM-INdAM; INFN, Istituto Nazionale di Fisica Nucleare.

Appendix A. LAP for the Laplacian plus a δ-interaction in dimension one

In this section we prove Theorem 4.1. This result implies Limiting Absorption Principle (LAP) for the operator $A_0^{(1)}$, which is formally written as the Laplacian plus a δ-interaction of strength α in dimension one, and is rigorously defined in Eqs. (3.14)–(3.15).

Proof of Theorem 4.1. Unless otherwise stated, throughout all the proof we implicitly understand the assumptions $\theta \in (0, 1/2)$ and $s_1 > 1/2$; moreover, let us arbitrarily fix $\varepsilon_0 > 0$ and a compact subset $K \subset (0, +\infty)$. Then, on account of item ii) in Lemma 2.5, the thesis is proved as soon as we are able to infer the following uniform bound for all $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in K$ and for some constant $c > 0$ (depending on ε_0 and K, but not on ε and λ):

$$
\|I_{-s_1}R_0^{(1)}(\lambda \pm i\varepsilon)\, u\|^2_{H^{1+\theta}(\mathbb{R})} \leqslant c\, \|u\|_{L^2_{s_1}(\mathbb{R})} . \tag{A.1}
$$

As an example, in the sequel we proceed to evaluate the expression $\|I_{-s_1}R_0^{(1)}(\lambda + i\varepsilon)\, u\|^2_{H^{1+\theta}(\mathbb{R})}$; altogether, the forthcoming uniform bounds (A.4), (A.5) and (A.11) imply the corresponding version of Eq. (A.1). Of course, similar results can be derived also for $\|I_{-s_1}R_0^{(1)}(\lambda - i\varepsilon)\, u\|^2_{H^{1+\theta}(\mathbb{R})}$, ultimately yielding the thesis.

The starting point of our analysis is the following integral kernel identity, holding true for any given $u \in L^2_{s_1}(\mathbb{R}) \subset L^2(\mathbb{R})$ and for all $z \in \mathbb{C}\backslash[0,+\infty)$ (see [3, p. 77, Th. 3.1.2]):

$$\big(R_0^{(1)}(z)\,u\big)(x) = \int_{\mathbb{R}} dy \left[\frac{i}{2\sqrt{z}}\; e^{i\sqrt{z}\,|x-y|} - \frac{i\alpha}{2\sqrt{z}\,(\alpha - 2i\sqrt{z})}\; e^{i\sqrt{z}\,(|x|+|y|)}\right] u(y)\,. \tag{A.2}$$

Differentiating the above identity, by the dominated convergence theorem we obtain

$$\big(R_0^{(1)}(z)\,u\big)'(x) = -\frac{1}{2}\int_{\mathbb{R}} dy \left[\mathrm{sgn}(x-y)\; e^{i\sqrt{z}\,|x-y|} - \frac{\alpha\;\mathrm{sgn}x}{\alpha - 2i\sqrt{z}}\; e^{i\sqrt{z}\,(|x|+|y|)}\right] u(y)\,. \tag{A.3}$$

Let us also remark that most of our arguments rely on the use of the elementary identity $\sqrt{\lambda + i\varepsilon} = (1/\sqrt{2})\big[\sqrt{\sqrt{\lambda^2+\varepsilon^2}+\lambda} + i\,\sqrt{\sqrt{\lambda^2+\varepsilon^2}-\lambda}\,\big]$, along with some related simple estimates.

Firstly, performing few elementary manipulations, from Eq. (A.2) we obtain

$$\begin{aligned}
&\|I_{-s_1} R_0^{(1)}(\lambda+i\varepsilon)\,u\|^2_{L^2(\mathbb{R})}\\
&= \int_{\mathbb{R}} dx\; w_{-s_1}(x) \left|\int_{\mathbb{R}} dy \left[\frac{e^{i\sqrt{\lambda+i\varepsilon}\,|x-y|}}{\alpha - 2i\sqrt{\lambda+i\varepsilon}} + \frac{i\alpha\big(e^{i\sqrt{\lambda+i\varepsilon}\,|x-y|} - e^{i\sqrt{\lambda+i\varepsilon}\,(|x|+|y|)}\big)}{2\sqrt{\lambda+i\varepsilon}\,(\alpha - 2i\sqrt{\lambda+i\varepsilon})}\right] u(y)\right|^2 .
\end{aligned}$$

By means of the Cauchy–Schwarz inequality and of the bounds $\big|e^{i\sqrt{\lambda+i\varepsilon}\,|x-y|}/(\alpha - 2i\sqrt{\lambda+i\varepsilon})\big| \leqslant 1/\sqrt{\alpha^2+4\lambda}$, $\big|i\alpha\big(e^{i\sqrt{\lambda+i\varepsilon}\,|x-y|} - e^{i\sqrt{\lambda+i\varepsilon}\,(|x|+|y|)}\big)/(2\sqrt{\lambda+i\varepsilon}(\alpha - 2i\sqrt{\lambda+i\varepsilon}))\big| \leqslant \alpha/\sqrt{\lambda\,(\alpha^2+4\lambda)}$, the above identity implies the following relations ([2]):

$$\begin{aligned}
&\|I_{-s_1} R_0^{(1)}(\lambda+i\varepsilon)\,u\|^2_{L^2(\mathbb{R})}\\
&\leqslant \frac{2\,(\alpha^2+\lambda)}{\lambda\,(\alpha^2+4\lambda)} \left(\int_{\mathbb{R}} dx\; w_{-s_1}(x)\right)^2 \left(\int_{\mathbb{R}} dy\; w_{s_1}(y)\,|u(y)|^2\right) \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}\,.
\end{aligned} \tag{A.4}$$

Next, let us consider the expression (A.3) for $(R_0^{(1)}(z)\,u)'(x)$; bearing in mind the previous estimate (A.4) and using again the Cauchy–Schwarz inequality, along with the elementary relations $|e^{i\sqrt{\lambda+i\varepsilon}|x-y|}| \leqslant 1$,

$$\big|\alpha\, e^{i\sqrt{\lambda+i\varepsilon}\,(|x|+|y|)}/(\alpha - 2i\sqrt{\lambda+i\varepsilon})\big| \leqslant \alpha/\sqrt{\alpha^2+4\lambda}$$

and $|(w^{1/2}_{-s_1})'(x)| \leqslant (s_1/2)\, w^{1/2}_{-s_1}(x)$, by similar computations we get

$$\begin{aligned}
&\big\|\big(I_{-s_1} R_0^{(1)}(\lambda+i\varepsilon)u\big)'\big\|^2_{L^2(\mathbb{R})}\\
&\leqslant \frac{\alpha^2+2\lambda}{\alpha^2+4\lambda}\left[1 + \frac{{s_1}^2(\alpha^2+\lambda)}{2\lambda(\alpha^2+2\lambda)}\right] \left(\int_{\mathbb{R}} dx\, w_{-s_1}(x)\right)^2 \left(\int_{\mathbb{R}} dy\, w_{s_1}(y)\,|u(y)|^2\right) \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}.
\end{aligned} \tag{A.5}$$

Let us now pass to the evaluation of the following seminorm, corresponding to the inner product defined as in Eq. (2.3):

[2] The first estimate in Eq. (A.3) gives an indication of the fact that the arguments employed here cannot be extended to include the boundary case $\lambda = 0$; this is the underlying reason behind our assumption $\lambda \in (0,+\infty)$.

$$\big|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)'\big|_\theta^2 := \mathcal{I}^{(<)}_{\theta,s_1} + \mathcal{I}^{(>)}_{\theta,s_1}\,, \tag{A.6}$$

$$\mathcal{I}^{(\lessgtr)}_{\theta,s_1} := \int\limits_{\{|x-y|\,\lessgtr\,1\}} dx\,dy\; \frac{\big|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)'(x) - (I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)'(y)\big|^2}{|x-y|^{1+2\theta}}\,.$$

On the one hand, taking into account Eq. (A.5) we readily obtain

$$\begin{aligned} \mathcal{I}^{(>)}_{\theta,s_1} &\leqslant_c \int\limits_{\mathbb{R}} dx\, \big|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)'(x)\big|^2 \int\limits_{\{|x-y|\,>\,1\}} dy\; \frac{1}{|x-y|^{1+2\theta}} \\ &\leqslant_c \|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)'\|^2_{L^2(\mathbb{R})} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}\,. \end{aligned} \tag{A.7}$$

On the other hand, the derivation of a uniform bound for $\mathcal{I}^{(<)}_{\theta,s_1}$ is less straightforward. To attain such a bound let us first point out the following relation, which can be readily inferred by addition and subtraction of identical terms and by elementary symmetry arguments:

$$\mathcal{I}^{(<)}_{\theta,s_1} \leqslant_c \mathcal{J}^{(1)}_{\theta,s_1} + \mathcal{J}^{(2)}_{\theta,s_1} + \mathcal{J}^{(3)}_{\theta,s_1}\,; \tag{A.8}$$

$$\begin{aligned} \mathcal{J}^{(1)}_{\theta,s_1} &:= \int\limits_{\{|x-y|<1\}} dx\,dy\, w_{s_1}(y)\, \frac{\big|w^{1/2}_{-s_1}(x) - w^{1/2}_{-s_1}(y)\big|^2}{|x-y|^{1+2\theta}}\, \big|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)'(x)\big|^2 \\ &+ \int\limits_{\{|x-y|<1\}} dx\,dy \Bigg(w_{s_1}(x)\, \frac{\big|(w^{1/2}_{-s_1})'(x) - (w^{1/2}_{-s_1})'(y)\big|^2}{|x-y|^{1+2\theta}} \\ &+ \big(w_{s_1}(x) + w_{s_1}(y)\big)\, w_{s_1}(x)\, |(w^{1/2}_{-s_1})'(x)|^2\, \frac{\big|w^{1/2}_{-s_1}(x) - w^{1/2}_{-s_1}(y)\big|^2}{|x-y|^{1+2\theta}} \Bigg) \big|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)(x)\big|^2 \end{aligned}$$

$$\begin{aligned} \mathcal{J}^{(2)}_{\theta,s_1} &:= \int\limits_{\{|x-y|<1\}} dx\,dy\; w_{s_1}(x)\, |(w^{1/2}_{-s_1})'(y)|^2 \times \\ &\quad \times \frac{\big|(I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)(x) - (I_{-s_1} R_0^{(1)}(\lambda + i\varepsilon)\, u)(y)\big|^2}{|x-y|^{1+2\theta}}\,, \end{aligned}$$

$$\mathcal{J}^{(3)}_{\theta,s_1} := \int\limits_{\{|x-y|<1\}} dx\,dy\; w_{-s_1}(x)\, \frac{\big|(R_0^{(1)}(\lambda + i\varepsilon)\, u)'(x) - (R_0^{(1)}(\lambda + i\varepsilon)\, u)'(y)\big|^2}{|x-y|^{1+2\theta}}\,.$$

Concerning the term $\mathcal{J}^{(1)}_{\theta,s_1}$, it can be checked by direct inspection that the integrals over $y \in (x-1, x+1)$ of the expressions involving the weights $w_{\pm s_1}$ (and their derivatives) are uniformly bounded for $x \in \mathbb{R}$.

Let us give a few more details about this statement. First of all, it should be noticed that $w_{s_1}(x)\, |(w^{1/2}_{-s_1})'(x)|^2 = {s_1}^2 x^2/(1+x^2)^2 \leqslant {s_1}^2/4$ for all $x \in \mathbb{R}$. Furthermore, starting from the elementary identity $w_{s_1}(y) - w_{s_1}(x) = 2s_1 \int_0^{y-x} dt\,(x+t)\,(1+(x+t)^2)^{s_1-1}$, for any given $s_1 \in \mathbb{R}$ and for all $x, y \in \mathbb{R}$ with $|x-y| < 1$ one gets

$$\begin{aligned} &|w_{s_1}(y) - w_{s_1}(x)| \\ &\leqslant 2s_1 \left(\sup_{x\in\mathbb{R},\, y\in(x-1,x+1)} \left| \int\limits_0^{y-x} dt\,(x+t)\,(1+(x+t)^2)^{s_1-1}/(1+x^2)^{s_1} \right| \right) w_{s_1}(x) \\ &\leqslant_c w_{s_1}(x)\, |x-y|; \end{aligned}$$

for $|x-y|<1$, the latter relation yields in particular $w_{s_1}(y) \leqslant w_{s_1}(x) + |w_{s_1}(y) - w_{s_1}(x)| \leqslant_c w_{s_1}(x)$ and $|w^{1/2}_{-s_1}(x) - w^{1/2}_{-s_1}(y)| = |w_{-s_1/2}(x) - w_{-s_1/2}(y)| \leqslant_c w^{1/2}_{-s_1}(x)\, |x-y|$. In a similar way, it can even be shown that $|(w^{1/2}_{-s_1})'(x) - (w^{1/2}_{-s_1})'(y)| \leqslant_c w^{1/2}_{-s_1}(x)\, |x-y|$. Taking into account the previously mentioned facts, it is easy to infer that

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|<1\}} dy\, w_{s_1}(y)\, \frac{\big|w^{1/2}_{-s_1}(x) - w^{1/2}_{-s_1}(y)\big|^2}{|x-y|^{1+2\theta}} < +\infty\,,$$

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|<1\}} dy\, w_{s_1}(x)\, \frac{\big|(w^{1/2}_{-s_1})'(x) - (w^{1/2}_{-s_1})'(y)\big|^2}{|x-y|^{1+2\theta}} < +\infty\,,$$

and

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|<1\}} dy \big(w_{s_1}(x) + w_{s_1}(y)\big)\, w_{s_1}(x)\, |(w^{1/2}_{-s_1})'(x)|^2\, \frac{\big|w^{1/2}_{-s_1}(x) - w^{1/2}_{-s_1}(y)\big|^2}{|x-y|^{1+2\theta}} < +\infty.$$

Then, recalling the bounds derived previously in Eqs. (A.4) and (A.5), we readily obtain $\mathcal{J}^{(1)}_{\theta,s_1} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}$.

As for the expression $\mathcal{J}^{(2)}_{\theta,s_1}$, computations similar to those described above grant that $w_{s_1}(x)\, |(w^{1/2}_{-s_1})'(y)|$ is uniformly bounded for all $x, y \in \mathbb{R}$ with $|x-y|<1$; in addition, by the Cauchy–Schwarz inequality we have

$$\begin{aligned}
&\big|(I_{-s_1} R^{(1)}_0(\lambda + i\varepsilon)\, u)(x) - (I_{-s_1} R^{(1)}_0(\lambda + i\varepsilon)\, u)(y)\big| \\
&= \left| \int_y^x dt\, (I_{-s_1} R^{(1)}_0(\lambda + i\varepsilon)\, u)'(t) \right| \\
&\leqslant \big\|\big(I_{-s_1} R^{(1)}_0(\lambda + i\varepsilon)\, u\big)'\big\|^2_{L^2(\mathbb{R})}\, |x-y|\,.
\end{aligned}$$

The above arguments, along with the estimate (A.5), allows us to infer that $\mathcal{J}^{(2)}_{\theta,s_1} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}$.

In order to derive a uniform bound for $\mathcal{J}^{(3)}_{\theta,s_1}$, let us indicate with Θ the Heaviside step function; this is such that $\Theta(t) = 1$ for $t \geqslant 0$, $\Theta(t) = 0$ for $t < 0$ and $\Theta(t) + \Theta(-t) = 1$ almost everywhere on $\mathbb{R}$. Then, using the explicit expression (A.3) for $(R^{(1)}_0(\lambda + i\varepsilon)\, u)'$, by a carefully devised procedure of addition and subtraction of identical terms and by triangular inequality (plus a number of elementary manipulations) we obtain

$$\mathcal{J}^{(3)}_{\theta,s_1} \leqslant_c J_1 + J_2 + J_3\,; \tag{A.9}$$

$$\begin{aligned}
J_1 := \int_{\{|x-y|<1\}} dx\, dy\, \frac{w_{-s_1}(x)\, \Theta(x-y)}{|x-y|^{1+2\theta}} \Bigg[&\left| \int_y^x dt\, e^{i\sqrt{\lambda + i\varepsilon}\,(x-t)} u(t) \right|^2 \\
&+ \left| \big(e^{i\sqrt{\lambda+i\varepsilon}\,(x-y)} - 1\big) \int_{-\infty}^y dt\, e^{i\sqrt{\lambda+i\varepsilon}\,(y-t)} u(t) \right|^2 \Bigg],
\end{aligned}$$

$$J_2 := \int\limits_{\{|x-y|<1\}} dx\,dy\, \frac{w_{-s_1}(x)\,\Theta(x-y)}{|x-y|^{1+2\theta}} \Bigg[\bigg| \int\limits_y^x dt\; e^{i\sqrt{\lambda+i\varepsilon}\,(t-y)} u(t) \bigg|^2 + \bigg| \Big(e^{i\sqrt{\lambda+i\varepsilon}\,(x-y)} - 1\Big) \int\limits_x^{+\infty} dt\; e^{i\sqrt{\lambda+i\varepsilon}\,(t-x)} u(t) \bigg|^2 \Bigg],$$

$$J_3 := \int\limits_{\{|x-y|<1\}} dx\,dy\, \frac{w_{-s_1}(x)}{|x-y|^{1+2\theta}} \left| \frac{\alpha}{\alpha - 2i\sqrt{\lambda+i\varepsilon}} \right|^2 \bigg| \int\limits_{\mathbb{R}} dt\; e^{i\sqrt{\lambda+i\varepsilon}\,|t|}\, u(t) \bigg|^2 \times \times \Big| (\mathrm{sgn} x)\, e^{i\sqrt{\lambda+i\varepsilon}\,|x|} - (\mathrm{sgn} y)\, e^{i\sqrt{\lambda+i\varepsilon}\,|y|} \Big|^2 .$$

The terms J_1, J_2 can be treated similarly; as an example, let us consider J_1. Since $|e^{i\sqrt{\lambda+i\varepsilon}\,(x-t)}| \leqslant 1$ for $y < t < x$, by the Cauchy–Schwarz inequality we have $\big| \int_y^x dt\, e^{i\sqrt{\lambda+i\varepsilon}\,(x-t)} u(t) \big|^2 \leqslant |x-y|\, \|u\|^2_{L^2_{s_1}(\mathbb{R})}$. Again by the Cauchy–Schwarz inequality, we get

$$\bigg| \int\limits_{-\infty}^{y} dt\, e^{i\sqrt{\lambda+i\varepsilon}\,(y-t)} u(t) \bigg|^2 \leqslant \int\limits_{-\infty}^{y} dt\, w_{-s_1}(t)\, e^{-\sqrt{2}\,(y-t)\sqrt{\sqrt{\lambda^2+\varepsilon^2}-\lambda}} \|u\|^2_{L^2_{s_1}(\mathbb{R})} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})} \,;$$

moreover, we have $|e^{i\sqrt{\lambda+i\varepsilon}\,(x-y)} - 1| \leqslant_c |x-y|$. The above remarks allow us to infer that

$$J_1 \leqslant_c \left(\int\limits_{\mathbb{R}} dx\; w_{-s_1}(x) \right) \left(\int\limits_{\{|x-y|<1\}} dy\; \frac{1}{|x-y|^{2\theta}} \right) \|u\|^2_{L^2_{s_1}(\mathbb{R})} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})} . \tag{A.10}$$

Finally, let us pass to the term J_3. Estimates similar to those employed before yield $|\alpha/(\alpha - 2i\sqrt{\lambda+i\varepsilon}\,)| \leqslant \alpha/\sqrt{\alpha^2+4\lambda}$ and $\big| \int_{\mathbb{R}} dt\, e^{i\sqrt{\lambda+i\varepsilon}\,|t|} u(t) \big|^2 \leqslant \big(\int_{\mathbb{R}} dt\, w_{-s_1}(t)\, e^{-\sqrt{2}\,|t|\sqrt{\sqrt{\lambda^2+\varepsilon^2}-\lambda}} \big) \|u\|^2_{L^2_{s_1}(\mathbb{R})} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}$. Moreover, indicating again with Θ the Heaviside step function, by simple symmetry considerations (and related changes of integration variables) we have

$$\int\limits_{\{|x-y|<1\}} dx\,dy\; w_{-s_1}(x)\, \frac{\big| (\mathrm{sgn} x)\, e^{i\sqrt{\lambda+i\varepsilon}\,|x|} - (\mathrm{sgn} y)\, e^{i\sqrt{\lambda+i\varepsilon}\,|y|} \big|^2}{|x-y|^{1+2\theta}}$$

$$= 2 \int\limits_{\{|x-y|<1\}} dx\,dy\; w_{-s_1}(x)\, \Theta(x)\, \Theta(y)\, \frac{\big| e^{i\,x\sqrt{\lambda+i\varepsilon}} - e^{i\,y\,\sqrt{\lambda+i\varepsilon}} \big|^2}{|x-y|^{1+2\theta}} + 2 \int\limits_{\{x+y<1\}} dx\,dy\; w_{-s_1}(x)\, \Theta(x)\, \Theta(y)\, \frac{\big| e^{i\,x\,\sqrt{\lambda+i\varepsilon}} + e^{i\,y\,\sqrt{\lambda+i\varepsilon}} \big|^2}{|x+y|^{1+2\theta}} .$$

The first addendum in the second line of the above equation can be easily proved to be finite noting that $|e^{i\,x\sqrt{\lambda+i\varepsilon}} - e^{i\,y\,\sqrt{\lambda+i\varepsilon}}| \leqslant_c |x-y|$; on the other hand, the finiteness of the second addendum can be inferred noting that $|e^{i\,x\sqrt{\lambda+i\varepsilon}} + e^{i\,y\,\sqrt{\lambda+i\varepsilon}}| \leqslant 2$ (for all $x, y > 0$) and introducing a system of polar coordinates in the quadrant $x, y > 0$.

More precisely, let us put $x = \rho\cos\theta$, $y = \rho\sin\theta$ for $\rho \in (0, +\infty)$, $\theta \in (0, \pi/2)$. Then, noting that $\cos\theta + \sin\theta \geqslant 1$ for $\theta \in (0, \pi/2)$, we obtain the following for all $s_1 > 1/2$ and $\theta \in (0, 1/2)$:

$$\int_{\{x+y<1\}} dx\,dy\, \frac{w_{-s_1}(x)\,\Theta(x)\,\Theta(y)}{|x+y|^{1+2\eta}}$$
$$= \int_0^{\pi/2} d\theta \int_{\{\rho(\cos\theta+\sin\theta)<1\}} d\rho\, \frac{1}{\big(1+\rho^2+\rho^4\sin^2\theta\cos^2\theta\big)^{s_1}\rho^{2\eta}\,|\cos\theta+\sin\theta|^{1+2\eta}}$$
$$\leqslant \frac{\pi}{2}\int_0^1 d\rho\,\frac{1}{\big(1+\rho^2\big)^{s_1}\rho^{2\eta}} < +\infty\,.$$

The above arguments show that $J_3 \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}$, which along with Eqs. (A.9) and (A.10) yields $\mathcal{J}^{(3)}_{\theta,s_1} \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}$.

In view of the previously described results, Eqs. (A.6), (A.7) and (A.8) allow us to infer that

$$\big|\big(I_{-s_1}R^{(1)}_0(\lambda+i\varepsilon)\,u\big)'\big|^2_\theta \leqslant_c \|u\|^2_{L^2_{s_1}(\mathbb{R})}\,. \tag{A.11}$$

Summing up, Eqs. (A.4), (A.5) and (A.11) (together with the definition of the norm on $H^{1+\eta}(\mathbb{R})$ descending from Eq. (2.3)) imply the claim (A.1) stated at the beginning of the present proof; as already mentioned, this suffices to prove the thesis. □

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