

THE MGT-FOURIER MODEL IN THE SUPERCRITICAL CASE

MONICA CONTI, LORENZO LIVERANI, VITTORINO PATA

ABSTRACT. We address the energy transfer in the differential system

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u = -\eta \Delta \theta \\ \theta_t - \kappa \Delta \theta = \eta \Delta u_{tt} + \alpha \eta \Delta u_t \end{cases}$$

made by a Moore-Gibson-Thompson equation in the supercritical regime, hence antidissipative, coupled with the classical heat equation. The asymptotic properties of the related solution semigroup depend on the strength of the coupling, ruling the competition between the Fourier damping and the MGT antidamping. Exponential stability will be shown always to occur, provided that the coupling constant is sufficiently large with respect to the other structural parameters. A fact of general interest will be also discussed, namely, the impossibility of attaining the optimal exponential decay rate of a given dissipative system via energy estimates.

1. PREAMBLE: THE MGT EQUATION

The Moore-Gibson-Thompson (MGT) equation is the third-order in time PDE

$$(1.1) \quad \partial_{ttt}u + \alpha \partial_{tt}u - \beta \Delta \partial_t u - \gamma \Delta u = 0,$$

ruling the evolution of the unknown variable $u = u(\mathbf{x}, t) : \Omega \times [0, \infty) \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. Here, $-\Delta$ is the Laplace-Dirichlet operator, while $\alpha, \beta, \gamma > 0$ are fixed structural parameters.

Equation (1.1) originally arises in the modeling of wave propagation in viscous thermally relaxing fluids [36, 39]. Nonetheless, its first appearance goes back to a very old paper of Stokes [38]. Later on, several authors understood that the MGT equation may pop up in the description of a large variety of physical phenomena, ranging from viscoelasticity to thermal conduction. In particular, we want to highlight the interpretation of (1.1) as a model for the vibrations in a standard linear viscoelastic solid [14, 21]. Such a model can also be obtained as a particular case of the equation of linear viscoelasticity deduced within a rheological framework using a linear combination of springs and dashpots (see e.g. [19])

$$\partial_{tt}u(t) - g(0)\Delta u(t) - \int_0^\infty g'(s)\Delta u(t-s) ds = 0,$$

upon choosing the exponential kernel

$$g(s) = \varkappa e^{-\alpha s} + \frac{\gamma}{\alpha},$$

2010 *Mathematics Subject Classification.* 35B40, 35Q74, 74F05.

Key words and phrases. MGT equation, Fourier law, thermoviscoelasticity, critical and supercritical regime, solution semigroup, exponential stability.

where the constant

$$(1.2) \quad \varkappa = \beta - \frac{\gamma}{\alpha}$$

must be strictly positive (see [17]).

As far as the mathematical analysis of (1.1) is concerned, there is nowadays a vast literature (see, e.g., [7, 8, 17, 24, 25, 26, 34] and references therein). Let us briefly subsume the main results obtained so far. For every choice of the parameters $\alpha, \beta, \gamma > 0$, the MGT equation turns out to generate a strongly continuous semigroup of solutions on the natural weak energy space

$$H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

Here, with standard notation, L^2 is the Lebesgue space of square summable functions, while H_0^1 is the Sobolev space of square summable functions along with their first derivatives, with null trace on the boundary $\partial\Omega$. However, the asymptotic behavior of the solutions dramatically depends on the constant \varkappa defined in (1.2), which in the MGT context is usually referred to as the *stability number*. Indeed, depending on the sign of \varkappa , the equation may exhibit dissipative or antidissipative features. More precisely, we have the following picture:

- ◊ If $\varkappa > 0$ the solutions decay exponentially fast.
- ◊ If $\varkappa = 0$ the (nontrivial) solutions are bounded but do not decay.
- ◊ If $\varkappa < 0$ there are solutions with an exponential blow up.

For this reason, the regimes $\varkappa > 0$, $\varkappa = 0$ and $\varkappa < 0$ are usually referred to in the literature as *subcritical*, *critical* and *supercritical*, respectively. In particular, when $\varkappa = 0$ there is the conservation of an appropriate energy, equivalent to the square norm in the phase space. In view of our previous discussion, we may conclude that the MGT equation is a model of viscoelasticity *only* in the subcritical regime, whereas when $\varkappa \leq 0$ the viscoelastic interpretation is completely lost. Nonetheless the critical and supercritical regimes remain very interesting, both from the physical and the mathematical viewpoint. In the critical regime, one might also ask whether or not a further damping mechanism (e.g., of memory type) could induce the uniform decay of the energy, hence driving the regime into subcritical. In general, this is false: an example in this direction has been given in [15]. In fact, each damping mechanism is peculiar, and it may require a sharp dedicated analysis. It is then not a coincidence that almost all the results available in the literature are set in the subcritical regime.

2. INTRODUCTION

2.1. The MGT-Fourier model. The object of our analysis is the system obtained by coupling the MGT equation (1.1) with the classical Fourier heat equation

$$(2.1) \quad \begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u = -\eta \Delta \theta, \\ \theta_t - \kappa \Delta \theta = \eta \Delta u_{tt} + \alpha \eta \Delta u_t, \end{cases}$$

supplemented with the Dirichlet boundary conditions

$$u(\mathbf{x}, t) = \theta(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega.$$

Here $\alpha, \beta, \gamma > 0$ are the usual MGT parameters, $\kappa > 0$ is the thermal conductivity, while $\eta \neq 0$ is the coupling constant.

System (2.1) has been first addressed by the authors of [2], within the key assumption that the stability number \varkappa defined in (1.2) be strictly positive. In which case, there is a clear physical interpretation, namely, a thermoviscoelastic model describing the vibrations in a viscoelastic heat conductor obeying the Fourier heat conduction law. The main result of [2], besides the generation of the solution semigroup, is that the total energy decays exponentially fast for every value of the coupling constant η . Let aside the undeniable interest for the model, which also happens to be the first example in the literature of a coupled MGT equation, the predicted exponential decay is not that surprising. Indeed, each single equation of (1.2) gives rise to an exponentially stable semigroup. In addition, the coupling is fairly nice, in the sense that when one performs the basic estimates in the weak energy space, the contributions produced by the η -terms (which in principle could create problems) cancel each other. In fact, the action of the coupling is typically the one of transferring energy, as well as dissipation, between the equations of a given system. But in this case both equations exhibit enough dissipation by themselves from the very beginning. It is also worth mentioning that for the analogous system modeling a thermoviscoelastic plate

$$\begin{cases} u_{ttt} + \alpha u_{tt} + \beta \Delta^2 u_t + \gamma \Delta^2 u = -\eta \Delta \theta, \\ \theta_t - \kappa \Delta \theta = \eta \Delta u_{tt} + \alpha \eta \Delta u_t, \end{cases}$$

where in the MGT equation $-\Delta$ is replaced by the bilaplacian, the solution semigroup is not only exponentially stable, but also analytic (see [10]).

2.2. A general discussion. There are instead interesting situations where a system is made, say, by two equations, one of which dissipates through a damping mechanism, while the other one preserves the energy. Now the coupling becomes essential, since its action allows to transfer dissipation to the undamped equation, in such a way that the whole system becomes globally stable as time goes to infinity. Several examples of this kind can be found in the literature. For instance, we address the reader to the works [1, 3, 4, 5, 6, 11, 12, 16, 18, 20, 22, 27, 28, 29, 30, 31, 32, 33, 37, 40], just to name a few. In some of these contributions, the dissipation is not mechanical, but only thermal through a heat equation of Fourier type. On the other hand, it is well known that the latter possesses highly regularizing properties, due to the fact that the Fourier damping mechanism is particularly strong. As a byproduct, even a very small coupling (e.g., with a coupling constant $|\eta| \ll 1$) is enough to get stability. But let now push our discussion a little bit further. We may ask what happens if we couple a *dissipative* equation with an *antidissipative* one. The picture now is more intriguing, and the strength of the coupling comes into play. To hope for stability, it is necessary that the equations share their energies to a certain extent. This translates into the fact that the coupling cannot be too weak. But even if the coupling is strong enough to bypass a certain critical threshold, the system may remain unstable if the action of the antidamping is more effective than the one of the damping. To the best of our knowledge, the first analysis of this kind has been made in the very recent paper [9], where a simple (yet not so simple) system of ODEs is

considered, that is,

$$\begin{cases} \ddot{u} + u + a\dot{u} = \eta\dot{v}, \\ \ddot{v} + v - b\dot{v} = -\eta\dot{u}, \end{cases}$$

$a, b > 0$ being a damping and an antidamping parameter, respectively. Here, stability occurs only if $a > b$ and $|\eta|$ is sufficiently large. Loosely speaking, we need not only more damping than antidamping, but also a fairly good communication between the two equations. There is also another interesting issue in connection with the case $a > b$, namely, to find the value of η ensuring the best decay. The answer, in contrast to what one might think, is not $|\eta|$ as large as possible; on the contrary, there is an optimal finite value of $|\eta|$, depending on the parameters a and b . However, this is not the general rule. For instance, one can construct a similar system of two oscillators, but with a different coupling involving u and v rather than their derivatives, where the best decay rate is reached asymptotically when $|\eta| \rightarrow \infty$.

2.3. The supercritical case. In the light of our previous comments, we can now move to the core of the present paper, which can be summarized by the following question:

What happens if we consider the MGT-Fourier system (2.1) when the stability number \varkappa is zero, or even negative?

If $\varkappa \leq 0$, the MGT equation is no longer dissipative, and the only dissipation mechanism is contributed by the heat equation. Even more so, if \varkappa is strictly negative, then we have a competition between the two equations, as the MGT one becomes antidissipative. In this case, we will prove that if the coupling constant is very small in modulus, the two equations are almost unrelated, and the explosive character of the MGT one becomes predominant, pushing the total energy to exponential blow up. Instead, if $|\eta|$ is sufficiently large and \varkappa slightly negative, it is reasonable to expect that the dissipation provided by the Fourier component should prevail. But as $\varkappa \rightarrow -\infty$ the antidamping becomes stronger and stronger. Thus, in principle, it is hard to predict if stability can still be attained via the coupling. Nevertheless, for *any* value of \varkappa , no matter how negative, there exists a critical threshold for $|\eta|$ beyond which exponential stability occurs, meaning that the damping mechanism of the classical heat equation is stronger than any possible structural antidamping of the MGT equation. From the mathematical viewpoint, the main difficulty lies in the fact that in [2] all the estimates are obtained using the functional

$$\mathbf{W} = \int_{\Omega} |u_{tt} + \alpha u_t|^2 d\mathbf{x} + \frac{\gamma}{\alpha} \int_{\Omega} |\nabla u_t + \alpha \nabla u|^2 d\mathbf{x} + \varkappa \int_{\Omega} |\nabla u_t|^2 d\mathbf{x} + \int_{\Omega} |\theta|^2 d\mathbf{x},$$

which, for $\varkappa > 0$, turns out to be equivalent to the energy, and appears in a natural way when performing the basic multiplications dictated by the phase space of the problem. Unfortunately, \mathbf{W} becomes a pseudoenergy when $\varkappa = 0$, so that its decay provides no information on the decay of the energy itself. And \mathbf{W} ceases to be even a pseudoenergy when $\varkappa < 0$, as it may assume negative values. Accordingly, a more refined analysis is required, via further energy-like functionals. Each of them contributes with terms with the right sign, but introducing at the same time terms playing against dissipation. A delicate balance is needed, in order to convey the whole system towards stability.

2.4. The results. We consider system (2.1), where the MGT equation lies either in the critical or in the supercritical regime. Hence, defining

$$\mu = \gamma - \alpha\beta = -\alpha\kappa,$$

we restrict our analysis to the novel case $\mu \geq 0$. After proving the existence of the solution semigroup $S(t)$ on the natural weak energy space

$$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega),$$

we will show that, for every value $\mu \geq 0$, the exponential decay of the energy takes place provided that the coupling constant η is sufficiently large in modulus. More precisely, there exists a structural threshold $\mathfrak{t} > 0$, independent of η , such that exponential stability occurs whenever

$$\eta^2 > \mathfrak{t}\mu.$$

This tells in particular that in the critical regime $\mu = 0$ we have exponential stability for *every* $\eta \neq 0$. Conversely, in the supercritical regime $\mu > 0$, exponentially growing trajectories always arise if $|\eta|$ is small.

Aiming to deepen our comprehension of the phenomenon in the supercritical regime $\mu > 0$, a further question is to be addressed:

How does the thermal conductivity κ influence the stability threshold \mathfrak{t} ?

We will see that $\mathfrak{t} \rightarrow \infty$ when $\kappa \rightarrow 0$, which is highly expected: if the heat equation is weakly damped, a large coupling constant is needed in order to transfer enough dissipation to the mechanical part. Much more surprising, at first glance, is that $\mathfrak{t} \rightarrow \infty$ when $\kappa \rightarrow \infty$ either, as one might reckon that a stronger dissipation in the Fourier law would transmit a stronger dissipation to the whole system. If so, one should obtain the uniform decay of the energy by keeping η fixed, upon arbitrarily increasing κ . On the contrary, we propose the following physical interpretation, complying with our findings: If κ is very large, then θ tends to decay in a very short time, and the effect is that the mechanical part, which would blow up in absence of heat interaction, does not see the coupling, unless $|\eta|$ is sufficiently large compared with κ .

A further issue concerns with the decay rate in dependence of η , once all the other quantities are fixed. What happens is that the optimal decay occurs for a certain value $\pm\eta_*$ of the coupling constant, depending on the structural parameters, that clearly satisfies the condition $\eta_*^2 > \mathfrak{t}\mu$, but at the same time is relatively small in modulus. So there exists the most efficient coupling in terms of energy transmission. And when $|\eta| \rightarrow \infty$, the decay rate eventually deteriorates, becoming zero in the limit. But finding the *exact* value of the best decay rate is quite another story. Indeed, in the final Appendix we will show that, in general and in particular in our case, establishing the best exponential decay rate via energy estimates is a hopeless task.

3. FUNCTIONAL SETTING AND NOTATION

Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be a real Hilbert space, and let $A : H \rightarrow H$ be a strictly positive selfadjoint operator with domain $\mathfrak{D}(A) \subset H$, where the embedding is not necessarily compact. For $\sigma \in \mathbb{R}$, we define the hierarchy of continuously nested Hilbert spaces

$$H^\sigma = \mathfrak{D}(A^{\sigma/2}),$$

endowed with the scalar products and norms (σ will be always omitted whenever zero)

$$\langle u, v \rangle_\sigma = \langle A^{\sigma/2}u, A^{\sigma/2}v \rangle \quad \text{and} \quad \|u\|_\sigma = \|A^{\sigma/2}u\|.$$

For $\sigma > 0$, it is understood that $H^{-\sigma}$ denotes the completion of the domain, so that $H^{-\sigma}$ is the dual space of H^σ . The symbol $\langle \cdot, \cdot \rangle$ will also denote the duality pairing between $H^{-\sigma}$ and H^σ . We recall the Poincaré inequality

$$\|u\|_{\sigma-1} \leq \frac{1}{\sqrt{\lambda_1}} \|u\|_\sigma, \quad \forall u \in H^\sigma,$$

where $\lambda_1 > 0$ is the minimum of the spectrum of A (the first eigenvalue if the spectrum is discrete). Finally, we introduce the phase space of our problem, namely, the product Hilbert space

$$\mathcal{H} = H^1 \times H^1 \times H \times H,$$

endowed with the norm

$$\|(u, v, w, \theta)\|_{\mathcal{H}}^2 = \|v + \alpha u\|_1^2 + \|w + \alpha v\|^2 + \|v\|_1^2 + \|\theta\|^2.$$

Such a norm, which comes from the MGT structure, is well known to be equivalent to the standard product norm in \mathcal{H} (see, e.g., [17]).

4. THE SOLUTION SEMIGROUP

In greater generality, we consider the abstract problem

$$(4.1) \quad \begin{cases} u_{ttt} + \alpha u_{tt} + \beta A u_t + \gamma A u = \eta A \theta, \\ \theta_t + \kappa A \theta = -\eta A u_{tt} - \alpha \eta A u_t, \end{cases}$$

in the MGT critical or supercritical regime, i.e., with

$$\mu = \gamma - \alpha\beta \geq 0.$$

The system is subject to the initial conditions

$$(4.2) \quad \begin{cases} u(0) = u_0, \\ u_t(0) = v_0, \\ u_{tt}(0) = w_0, \\ \theta(0) = \theta_0, \end{cases}$$

where $\mathbf{u}_0 = (u_0, v_0, w_0, \theta_0) \in \mathcal{H}$ is an arbitrarily given initial datum.

Remark 4.1. For the concrete system (2.1) of the Introduction, A is the Laplace-Dirichlet operator $-\Delta$ acting on the Hilbert space $H = L^2(\Omega)$, while $H^1 = H_0^1(\Omega)$.

The first result addresses the well-posedness of the problem.

Theorem 4.2. *For every $\mathbf{u}_0 \in \mathcal{H}$, problem (4.1)-(4.2) admits a unique weak solution*

$$t \mapsto \mathbf{u}(t) = (u(t), u_t(t), u_{tt}(t), \theta(t)) \in \mathcal{C}([0, \infty), \mathcal{H}),$$

satisfying, for every fixed $t \geq 0$, the further continuity property

$$\mathbf{u}_0 \mapsto \mathbf{u}(t) \in \mathcal{C}(\mathcal{H}, \mathcal{H}).$$

Accordingly, system (4.1) generates a strongly continuous semigroup [13, 35]

$$S(t) : \mathcal{H} \rightarrow \mathcal{H},$$

acting by the rule

$$S(t)\mathbf{u}_0 = \mathbf{u}(t),$$

whose corresponding energy at time t , for the initial datum \mathbf{u}_0 , reads

$$\mathbf{E}(t) = \frac{1}{2} \|S(t)\mathbf{u}_0\|_{\mathcal{H}}^2 = \frac{1}{2} \left[\|u_t(t) + \alpha u(t)\|_1^2 + \|u_{tt}(t) + \alpha u_t(t)\|^2 + \|u_t(t)\|_1^2 + \|\theta(t)\|^2 \right].$$

Proof of Theorem 4.2. The conclusion follows by showing that the energy $\mathbf{E}(t)$ of any Galerkin approximate solution is uniformly bounded on every time-interval $[0, T]$, with a bound of the form $\mathbf{E}(0)h(T)$, for some positive increasing function h . Indeed, from the one side this produces the required uniform bound of the solution. From the other side, since the equation is linear, the same bound holds for the difference of two Galerkin approximants, yielding the convergence of the entire approximating sequence to its (unique) limit in the topology of $\mathcal{C}([0, T], \mathcal{H})$. By the same token, the energy of the difference of two solutions satisfies the same estimate, yielding the continuous dependence.

To this end, we set

$$\alpha_m = \alpha + m,$$

where we choose $m > 0$ large enough that

$$\varkappa_m = \beta - \frac{\gamma}{\alpha_m} = \frac{m\beta - \mu}{\alpha + m} > 0.$$

This trick, first devised in [17], allows to rewrite (4.1) in the form

$$\begin{cases} u_{ttt} + \alpha_m u_{tt} + \beta A u_t + \gamma A u = \eta A \theta + m u_{tt}, \\ \theta_t + \kappa A \theta = -\eta A u_{tt} - \alpha_m \eta A u_t + m \eta A u_t. \end{cases}$$

Note that now the first equation is a subcritical MGT one, plus some lower order terms, which can be multiplied by the standard MGT multiplier $u_{tt} + \alpha_m u_t$, to get

$$\begin{aligned} & \frac{d}{dt} \left[\frac{\gamma}{\alpha_m} \|u_t + \alpha_m u\|_1^2 + \|u_{tt} + \alpha_m u_t\|^2 + \varkappa_m \|u_t\|_1^2 \right] + 2\alpha_m \varkappa_m \|u_t\|_1^2 \\ & = 2\eta \langle A \theta, u_{tt} + \alpha_m u_t \rangle + 2m \langle u_{tt}, u_{tt} + \alpha_m u_t \rangle. \end{aligned}$$

Multiplying instead the second equation of the new system by θ , we obtain

$$\frac{d}{dt} \|\theta\|^2 + 2\kappa \|\theta\|_1^2 = -2\eta \langle A u_{tt} + \alpha_m A u_t, \theta \rangle + 2m \eta \langle A u_t, \theta \rangle.$$

Defining the energy functional

$$\mathbf{W}_m(t) = \frac{\gamma}{\alpha_m} \|u_t(t) + \alpha_m u(t)\|_1^2 + \|u_{tt}(t) + \alpha_m u_t(t)\|^2 + \varkappa_m \|u_t(t)\|_1^2 + \|\theta(t)\|^2,$$

and adding the two differential identities, we end up with

$$(4.3) \quad \frac{d}{dt} \mathbf{W}_m + 2\alpha_m \varkappa_m \|u_t\|_1^2 + 2\kappa \|\theta\|_1^2 = 2m \langle u_{tt}, u_{tt} + \alpha_m u_t \rangle + 2m \eta \langle A u_t, \theta \rangle.$$

Note that W_m is equivalent to the original energy E (and we write $W_m \sim E$), that is,

$$\frac{1}{c} E \leq W_m \leq c E,$$

for some $c > 1$. This is true because of the strict positivity of the parameter \varkappa_m . Making use of the Young and Poincaré inequalities, along with the equivalence above, the right-hand side of (4.3) is immediately controlled by

$$2\alpha_m \varkappa_m \|u_t\|_1^2 + 2\kappa \|\theta\|_1^2 + kW_m,$$

for some $k > 0$. This leads to the differential inequality

$$\frac{d}{dt} W_m \leq kW_m,$$

and a final application of the Gronwall lemma yields

$$W_m(t) \leq W_m(0)e^{kt},$$

providing the sought uniform bound. \square

Remark 4.3. Clearly, the proof remains valid in the subcritical regime $\mu < 0$, where one can simply take $m = 0$. In which case, (4.3) actually tells that $S(t)$ is a contraction semigroup with respect to the equivalent norm in \mathcal{H} dictated by W_0 .

Once the well-posedness result is established, the *quasienergy*

$$W(t) = \frac{\gamma}{\alpha} \|u_t(t) + \alpha u(t)\|_1^2 + \|u_{tt}(t) + \alpha u_t(t)\|^2 - \frac{\mu}{\alpha} \|u_t(t)\|_1^2 + \|\theta(t)\|^2,$$

corresponding to a solution $\mathbf{u}(t)$, is easily seen to fulfill the equality

$$(4.4) \quad \frac{d}{dt} W(t) + 2\kappa \|\theta(t)\|_1^2 = 2\mu \|u_t(t)\|_1^2,$$

for all sufficiently regular initial data. Indeed, W is nothing but the functional W_0 of the previous proof, hence (4.4) is merely obtained by setting $m = 0$ in (4.3). In particular, all the terms containing the coupling constant η disappear. The reason why we refer to W as a quasienergy is that it may assume negative values when $\mu > 0$, whereas it is actually a *pseudoenergy* in the critical regime $\mu = 0$. Although W , contrary to what happens in the subcritical case analyzed in [2], cannot possibly be equivalent to the energy, the identity (4.4) will turn out to be crucial in order to prove the exponential stability of the semigroup.

5. EXPONENTIAL BLOW UP

In the MGT supercritical regime $\mu > 0$, we show that system (4.1) exhibits solutions whose energy blows up exponentially fast, whenever the coupling constant η is small in modulus. We suppose for simplicity that the operator A has at least one eigenvalue $\lambda > 0$. Choosing then a corresponding eigenvector w , we look for solutions to (4.1) of the form

$$u(t) = \phi(t)w \quad \text{and} \quad \theta(t) = \psi(t)w,$$

for some $\phi \in \mathcal{C}^3([0, \infty))$ and $\psi \in \mathcal{C}^1([0, \infty))$. The functions ϕ and ψ are easily seen to fulfill the linear system of ODEs

$$(5.1) \quad \begin{cases} \phi''' + \alpha\phi'' + \beta\lambda\phi' + \gamma\lambda\phi - \eta\lambda\psi = 0, \\ \psi' + \kappa\lambda\psi + \eta\lambda\phi'' + \alpha\eta\lambda\phi' = 0. \end{cases}$$

From the classical ODE theory [23], it is well known that the asymptotic properties of the solutions to (5.1) are completely determined by the (complex) roots of the characteristic polynomial

$$p_\eta(z) = z^4 + (\alpha + \kappa\lambda)z^3 + (\beta\lambda + \alpha\kappa\lambda + \eta^2\lambda^2)z^2 + (\gamma\lambda + \beta\kappa\lambda^2 + \alpha\eta^2\lambda^2)z + \gamma\kappa\lambda^2.$$

In particular, if p_η has a root with strictly positive real part, then there are exponentially blowing up solutions. Accordingly, our claim follows from the next proposition.

Proposition 5.1. *Let $\mu > 0$. Then, for every $\eta \in \mathbb{R}$ with $|\eta|$ small enough, p_η possesses a complex root z_η with $\Re z_\eta > 0$.*

Proof. We first examine the case $\eta = 0$, corresponding to the uncoupled system, whose characteristic polynomial simplifies into

$$p_0(z) = (z + \kappa\lambda)q_0(z),$$

where

$$q_0(z) = z^3 + \alpha z^2 + \beta\lambda z + \gamma\lambda.$$

The roots of q_0 have been carefully analyzed in [17], where it is proved that when $\mu > 0$ there are always two complex conjugate solutions with positive real part, for every fixed $\lambda > 0$. Therefore, p_0 has a root z_0 with $\Re z_0 > 0$. At this point, we use the continuous dependence of the roots of a polynomial on its coefficients (see [41]). Indeed, it is apparent that the coefficients of the polynomial p_η converge to those of p_0 as $\eta \rightarrow 0$. This implies that, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: the polynomial p_η has a root z_η falling within an ε -neighborhood of z_0 , provided that $|\eta| < \delta$. Choosing $\varepsilon < \Re z_0$, we are done. \square

Remark 5.2. The request that A has at least one eigenvalue is not really necessary to prove the result, although it greatly simplifies the analysis. If A has no eigenvalues, the same conclusion can be drawn by analyzing the spectrum of the generator of the semigroup $S(t)$, which leads to the study of the equation $p_\eta(z) = 0$, for values of λ in the spectrum of A . In which case, the existence of a root with positive real part provides a lower bound for the growth rate of the semigroup (see [17]).

6. EXPONENTIAL STABILITY

We now come to the core of our work, regarding the uniform exponential decay of the solutions to (4.1). Let us first recall two definitions.

Definition 6.1. System (4.1), or more precisely its related semigroup $S(t)$, is said to be *exponentially stable* if

$$(6.1) \quad \mathbf{E}(t) \leq M\mathbf{E}(0)e^{-\omega t},$$

for some constants $\omega > 0$ and $M \geq 1$, both independent of the initial data of the problem.

Definition 6.2. The *exponential decay rate* of the energy E is the best (in the sense of largest) $\omega > 0$ for which exponential stability holds; namely, it is the number

$$\omega_* = \sup \left\{ \omega > 0 : E(t) \leq M E(0) e^{-\omega t} \text{ for some } M \geq 1 \right\}.$$

In order to state a quantitatively precise result, we define the *stability threshold* \mathfrak{t} depending on κ , besides the other structural quantities of the problem but η , as

$$(6.2) \quad \mathfrak{t}(\kappa) = \begin{cases} \frac{2\kappa}{\alpha^2} \left(1 - \frac{\alpha}{\kappa\lambda_1} + \frac{\alpha^2}{\kappa^2\lambda_1^2} \right) & \text{if } \kappa < \alpha/\lambda_1, \\ \frac{2\kappa}{\alpha^2} & \text{if } \kappa \geq \alpha/\lambda_1, \end{cases}$$

which attains its global minimum at $\kappa = \alpha/\lambda_1$. Recall that $\lambda_1 > 0$ is the minimum of the spectrum of A .

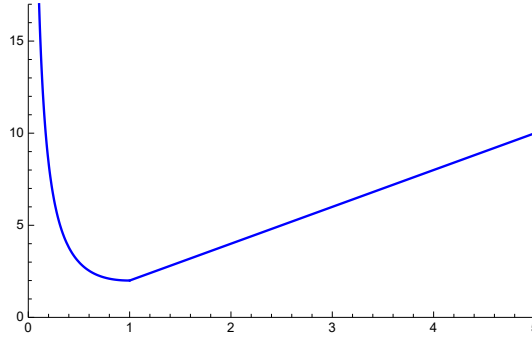


fig. 1 Plot of $\mathfrak{t}(\kappa)$ for $\alpha = \lambda_1 = 1$

Theorem 6.3. Let $\alpha, \beta, \gamma > 0$ be fixed and such that $\mu = \gamma - \alpha\beta \geq 0$. Assume that

$$(6.3) \quad \eta^2 > \mathfrak{t}(\kappa)\mu.$$

Then system (4.1) is exponentially stable.

Theorem 6.3 tells that in the critical regime $\mu = 0$ we have exponential stability for every $\eta \neq 0$. On the other hand, in the supercritical regime $\mu > 0$, the exponential decay of the energy depends on the interplay between the coupling constant η and the Fourier coefficient κ . Observe also that at the global minimum we have

$$\mathfrak{t}(\alpha/\lambda_1) = \frac{2}{\alpha\lambda_1}.$$

Thus the value $\kappa = \alpha/\lambda_1$ is “optimal”, being the one requiring the weakest coupling to drive the system to stability. For such a κ condition (6.3) reads

$$\eta^2 > \frac{2\mu}{\alpha\lambda_1}.$$

The proof, carried out in the next section, requires several steps.

7. PROOF OF THEOREM 6.3

As customary, we will work with sufficiently regular solutions, which stand the forthcoming calculations. The final conclusions follow by density.

7.1. Auxiliary functionals. We introduce the auxiliary functionals

$$\begin{aligned} \mathbf{F}(t) &= \eta \langle \theta(t), u_t(t) \rangle + \frac{\eta^2}{2} \|u_t(t)\|_1^2 + \frac{1}{2} \|\theta(t)\|_{-1}^2, \\ \mathbf{G}(t) &= -\langle u_t(t) - \alpha u(t), u_{tt}(t) + \alpha u_t(t) \rangle. \end{aligned}$$

Lemma 7.1. *The following inequality holds:*

$$\frac{d}{dt} \mathbf{F} + \frac{\alpha \eta^2}{2} \|u_t(t)\|_1^2 \leq \frac{\kappa^2}{\alpha} \|\theta(t)\|_1^2 + \left(\frac{\alpha}{\lambda_1} - \kappa \right) \|\theta(t)\|^2.$$

Proof. Via direct computations,

$$\frac{d}{dt} \mathbf{F} + \alpha \eta^2 \|u_t\|_1^2 = -\kappa \eta \langle \theta, u_t \rangle_1 - \kappa \|\theta\|^2 - \alpha \eta \langle \theta, u_t \rangle.$$

Using the Young and Poincaré inequalities, we estimate the terms in the right-hand side as

$$-\alpha \eta \langle \theta, u_t \rangle \leq \frac{\alpha}{\lambda_1} \|\theta\|^2 + \frac{\alpha \eta^2}{4} \|u_t\|_1^2,$$

and

$$-\kappa \eta \langle \theta, u_t \rangle_1 \leq \frac{\kappa^2}{\alpha} \|\theta\|_1^2 + \frac{\alpha \eta^2}{4} \|u_t\|_1^2.$$

This will do. \square

Remark 7.2. In the Young inequalities above we used the weights $(1/2, 1/2)$. One might argue that the more general weights $(\nu/2, 1/2\nu)$ should be used instead, looking eventually for $\nu > 0$ producing the best estimate. However, doing so, one would realize *a posteriori* that the optimal value is indeed $\nu = 1/2$.

Lemma 7.3. *The following inequality holds:*

$$\frac{d}{dt} \mathbf{G} + \frac{\gamma}{2\alpha} \|u_t + \alpha u\|_1^2 + \frac{1}{2} \|u_{tt} + \alpha u_t\|^2 \leq \ell \|u_t\|_1^2 + \eta \langle \theta, u_t + \alpha u \rangle_1 - 2\eta \langle \theta, u_t \rangle_1,$$

where $\ell = \frac{4\gamma^2 + \mu^2}{2\alpha\gamma} + \frac{2\alpha^2}{\lambda_1}$.

Proof. Via direct computations,

$$\begin{aligned} & \frac{d}{dt} \mathbf{G} + \frac{\gamma}{\alpha} \|u_t + \alpha u\|_1^2 + \|u_{tt} + \alpha u_t\|^2 \\ &= \frac{2\gamma + \mu}{\alpha} \langle u_t, u_t + \alpha u \rangle_1 + \eta \langle \theta, u_t + \alpha u \rangle_1 \\ & \quad - 2\eta \langle \theta, u_t \rangle_1 - \frac{2\mu}{\alpha} \|u_t\|_1^2 + 2\alpha \langle u_t, u_{tt} + \alpha u_t \rangle. \end{aligned}$$

By the Young inequality,

$$\frac{2\gamma + \mu}{\alpha} \langle u_t, u_t + \alpha u \rangle_1 \leq \frac{\gamma}{2\alpha} \|u_t + \alpha u\|_1^2 + \frac{(2\gamma + \mu)^2}{2\alpha\gamma} \|u_t\|_1^2,$$

$$2\alpha \langle u_t, u_{tt} + \alpha u_t \rangle \leq \frac{1}{2} \|u_{tt} + \alpha u_t\|^2 + \frac{2\alpha^2}{\lambda_1} \|u_t\|_1^2,$$

yielding the claim. \square

We are now ready to define our energy functional

$$\mathbf{L}(t) = \mathbf{W}(t) + \rho \mathbf{F}(t) + \varepsilon^2 \mathbf{G}(t),$$

for some $\rho > 0$ and $\varepsilon > 0$ to be chosen later. Combining Lemma 7.1 and Lemma 7.3 together with (4.4), we get

$$\begin{aligned} & \frac{d}{dt} \mathbf{L} + \frac{\varepsilon^2 \gamma}{2\alpha} \|u_t + \alpha u\|_1^2 + \frac{\varepsilon^2}{2} \|u_{tt} + \alpha u_t\|^2 \\ & + \left(\frac{\rho \alpha \eta^2}{2} - 2\mu - \varepsilon^2 \ell \right) \|u_t\|_1^2 + \rho \left(\kappa - \frac{\alpha}{\lambda_1} \right) \|\theta\|^2 + \left(2\kappa - \frac{\rho \kappa^2}{\alpha} \right) \|\theta\|_1^2 \\ & \leq \varepsilon^2 \eta \langle \theta, u_t + \alpha u \rangle_1 - 2\varepsilon^2 \eta \langle \theta, u_t \rangle_1. \end{aligned}$$

We control the right-hand side as

$$\begin{aligned} \varepsilon^2 \eta \langle \theta, u_t + \alpha u \rangle_1 & \leq \frac{\varepsilon \eta}{2} \|\theta\|_1^2 + \frac{\varepsilon^3 \eta}{2} \|u_t + \alpha u\|_1^2, \\ -2\varepsilon^2 \eta \langle \theta, u_t \rangle_1 & \leq \varepsilon \eta \|\theta\|_1^2 + \varepsilon^3 \eta \|u_t\|_1^2. \end{aligned}$$

Hence, we end up with

$$(7.1) \quad \begin{aligned} & \frac{d}{dt} \mathbf{L} + \frac{\varepsilon^2}{2} \left(\frac{\gamma}{\alpha} - \varepsilon \eta \right) \|u_t + \alpha u\|_1^2 + \frac{\varepsilon^2}{2} \|u_{tt} + \alpha u_t\|^2 \\ & + \left(\frac{\rho \alpha \eta^2}{2} - 2\mu - \varepsilon^2 \ell - \varepsilon^3 \eta \right) \|u_t\|_1^2 \\ & + \rho \left(\kappa - \frac{\alpha}{\lambda_1} \right) \|\theta\|^2 + \left(2\kappa - \frac{\rho \kappa^2}{\alpha} - \frac{3\varepsilon \eta}{2} \right) \|\theta\|_1^2 \leq 0. \end{aligned}$$

At this point, we show that we can choose ρ and ε in such a way to obtain a satisfactory differential inequality. Here, assumption (6.3) plays a crucial role.

Lemma 7.4. *Let (6.3) hold. Then, there exists $\rho > 0$ with the following property: for every $\varepsilon > 0$ sufficiently small, there is $\omega = \omega(\varepsilon) > 0$ such that*

$$\frac{d}{dt} \mathbf{L} + \omega \mathbf{E} \leq 0.$$

Proof. Recalling the definition of $\mathfrak{t}(\kappa)$ given in (6.2), we fix the value of $\rho > 0$ to be

$$(7.2) \quad \rho = \frac{4(\mu + \kappa)}{\alpha \eta^2 + \alpha \kappa \mathfrak{t}(\kappa)},$$

and we set

$$\sigma = \frac{2\kappa(\eta^2 - \mathfrak{t}(\kappa)\mu)}{\eta^2 + \kappa \mathfrak{t}(\kappa)}.$$

Observe that (6.3) ensures that $\sigma > 0$. It is convenient to consider two cases.

(i) If $\kappa \leq \alpha/\lambda_1$ the coefficient of $\|\theta\|^2$ in (7.1) is negative. So, we apply the Poincaré inequality to get

$$\rho \left(\kappa - \frac{\alpha}{\lambda_1} \right) \|\theta\|^2 + \left(2\kappa - \frac{\rho \kappa^2}{\alpha} - \frac{3\varepsilon \eta}{2} \right) \|\theta\|_1^2 \geq \left(2\kappa - \frac{\rho \kappa^2}{\alpha} + \frac{\rho \kappa}{\lambda_1} - \frac{\rho \alpha}{\lambda_1^2} - \frac{3\varepsilon \eta}{2} \right) \|\theta\|_1^2.$$

By the choice of ρ and the definition of σ , we see that

$$\frac{\rho\alpha\eta^2}{2} - 2\mu = 2\kappa - \frac{\rho\kappa^2}{\alpha} + \frac{\rho\kappa}{\lambda_1} - \frac{\rho\alpha}{\lambda_1^2} = \sigma.$$

Accordingly, (7.1) becomes

$$(7.3) \quad \begin{aligned} \frac{d}{dt}\mathbf{L} + \frac{\varepsilon^2}{2} \left(\frac{\gamma}{\alpha} - \varepsilon\eta \right) \|u_t + \alpha u\|_1^2 + \frac{\varepsilon^2}{2} \|u_{tt} + \alpha u_t\|^2 \\ + (\sigma - \varepsilon^2\ell - \varepsilon^3\eta) \|u_t\|_1^2 + \frac{1}{2}(2\sigma - 3\varepsilon\eta) \|\theta\|_1^2 \leq 0. \end{aligned}$$

It is then apparent that when ε is sufficiently small, all the coefficients become positive. Hence, calling

$$(7.4) \quad \omega = \min \left\{ \varepsilon^2 \left(\frac{\gamma}{\alpha} - \varepsilon\eta \right), \varepsilon^2, 2(\sigma - \varepsilon^2\ell - \varepsilon^3\eta), \lambda_1(2\sigma - 3\varepsilon\eta) \right\},$$

and making a further use of the Poincaré inequality, we arrive at the desired conclusion.

(ii) If $\kappa > \alpha/\lambda_1$ the coefficient of $\|\theta\|^2$ in (7.1) is positive and can be neglected. With ρ and σ as above, we now have

$$\frac{\rho\alpha\eta^2}{2} - 2\mu = 2\kappa - \frac{\rho\kappa^2}{\alpha} = \sigma.$$

So we end up with (7.3), and we conclude exactly as in the previous step. \square

7.2. Equivalence of the energy. The next step is showing that the functional \mathbf{L} is actually equivalent to the energy \mathbf{E} .

Lemma 7.5. *Assume that*

$$(7.5) \quad \eta^2 > \frac{\mu}{\alpha\lambda_1} \quad \text{and} \quad \rho > \frac{2\mu\lambda_1}{\alpha\eta^2\lambda_1 - \mu}.$$

Then, there exists $c > 1$ such that

$$\frac{1}{c} \mathbf{E} \leq \mathbf{L} \leq c \mathbf{E},$$

for every $\varepsilon > 0$ small.

Proof. Throughout this proof, $c > 1$ will denote a generic constant, depending only on ρ and the structural parameters of the problem. The inequality

$$\mathbf{L} \leq c \mathbf{E}$$

is a straightforward consequence of the Young and Poincaré inequalities (recall that ε is small). By the same token, it is also clear that

$$|\mathbf{G}| \leq c \mathbf{E}.$$

Therefore, we are left to prove the remaining inequality

$$\frac{1}{c} \mathbf{E} \leq \mathbf{W} + \rho \mathbf{F}.$$

Since

$$\mathbf{W} + \rho\mathbf{F} = \|u_{tt} + \alpha u_t\|^2 + \frac{\gamma}{\alpha} \|u_t + \alpha u\|_1^2 + \left(\frac{\rho\eta^2}{2} - \frac{\mu}{\alpha}\right) \|u_t\|_1^2 + \rho\eta \langle \theta, u_t \rangle + \|\theta\|^2 + \frac{\rho}{2} \|\theta\|_{-1}^2,$$

we only need to show that

$$\left(\frac{\rho\eta^2}{2} - \frac{\mu}{\alpha}\right) \|u_t\|_1^2 + \rho\eta \langle \theta, u_t \rangle + \|\theta\|^2 + \frac{\rho}{2} \|\theta\|_{-1}^2 \geq c \left[\|u_t\|_1^2 + \|\theta\|^2 \right].$$

Indeed,

$$\langle \theta, u_t \rangle \geq -\frac{\eta\nu}{2} \|u_t\|_1^2 - \frac{1}{2\eta\nu} \|\theta\|_{-1}^2,$$

for every $\nu \in (0, 1)$. Therefore,

$$\begin{aligned} & \left(\frac{\rho\eta^2}{2} - \frac{\mu}{\alpha}\right) \|u_t\|_1^2 + \rho\eta \langle \theta, u_t \rangle + \|\theta\|^2 + \frac{\rho}{2} \|\theta\|_{-1}^2 \\ & \geq \left(\frac{\rho\eta^2}{2}(1-\nu) - \frac{\mu}{\alpha}\right) \|u_t\|_1^2 + \|\theta\|^2 - \frac{\rho}{2\nu}(1-\nu) \|\theta\|_{-1}^2 \\ & \geq \left(\frac{\rho\eta^2}{2}(1-\nu) - \frac{\mu}{\alpha}\right) \|u_t\|_1^2 + \left(1 - \frac{\rho}{2\nu\lambda_1}(1-\nu)\right) \|\theta\|^2. \end{aligned}$$

The conclusion follows if we find $\nu \in (0, 1)$ for which the two coefficients in the right-hand side are positive, which amounts to requiring, besides $\nu > 0$,

$$1 - \frac{2\lambda_1}{2\lambda_1 + \rho} < \nu < 1 - \frac{2\mu}{\rho\alpha\eta^2}.$$

This is possible since

$$\frac{\mu}{\rho\alpha\eta^2} < \frac{\lambda_1}{2\lambda_1 + \rho} < \frac{1}{2},$$

and the first inequality is nothing but (7.5). \square

7.3. Conclusion of the proof. We choose ρ as in (7.2), so that Lemma 7.4 applies. Besides, we know from (6.3) that

$$\rho > \frac{4\mu}{\alpha\eta^2},$$

and

$$\eta^2 > \min_{\kappa > 0} \mathfrak{t}(\kappa)\mu = \frac{2\mu}{\lambda_1\alpha}.$$

Combining the two inequalities, we obtain

$$\rho(\alpha\eta^2\lambda_1 - \mu) > 4\mu\lambda_1 - \frac{4\mu^2}{\alpha\eta^2} \geq 2\mu\lambda_1,$$

meaning that (7.5) holds true. Therefore, Lemma 7.5 applies as well. Accordingly, fixing $\varepsilon > 0$ small enough, and redefining ω up to the multiplicative constant c , we are led to

$$\frac{d}{dt} \mathbf{L} + \omega \mathbf{L} \leq 0.$$

The Gronwall lemma then gives

$$\mathbf{L}(t) \leq \mathbf{L}(0)e^{-\omega t}.$$

Since $L \sim E$, we finally arrive at the sought energy inequality (6.1). \square

8. OPTIMAL DECAY RATE

Once the exponential stability of the system is established, another relevant issue concerns with the exponential decay rate ω_* of the energy E , introduced in Definition 6.2. Indeed, one might want to look for the value of η that maximizes ω_* , once all the other parameters are fixed. We already know from Section 5 that when $\eta \rightarrow 0$ exponential stability is lost. Let us see what happens when $|\eta| \rightarrow \infty$, where exponential stability certainly occurs in view of Theorem 6.3.

Proposition 8.1. *Let $\alpha, \beta, \gamma, \kappa$ be fixed. Then the exponential decay rate of the energy goes to zero as $|\eta| \rightarrow \infty$.*

Proof. As in Proposition 5.1, we assume that A has at least one eigenvalue $\lambda > 0$, but again this simplifying assumption is not really necessary. Looking for single-mode solutions to (4.1), we boil down once more to system (5.1). Accordingly, all we need to show is that the characteristic polynomial p_η of Section 5 has at least a root whose real part tends to zero as $|\eta| \rightarrow \infty$. Note that

$$p_\eta(0) = \gamma\kappa\lambda^2 > 0.$$

Besides, choosing

$$\varepsilon_\eta = \frac{\gamma\kappa\lambda^2 + 1}{\alpha\lambda^2} \frac{1}{\eta^2},$$

it is readily seen that

$$\lim_{|\eta| \rightarrow \infty} p_\eta(-\varepsilon_\eta) = -1.$$

Thus, for $|\eta|$ large enough, $p_\eta(-\varepsilon_\eta)$ becomes negative, implying that p_η has a (negative) real root $-x_\eta \in (-\varepsilon_\eta, 0)$, where $\varepsilon_\eta \rightarrow 0$. This means that the exponential decay rate is less than or equal to $2x_\eta \rightarrow 0$. \square

Remark 8.2. We should observe that the conclusions of Proposition 8.1 hold no matter if μ is positive or not. Thus, when $|\eta| \rightarrow \infty$ the decay rate of the energy deteriorates also in the subcritical case $\mu < 0$ studied in [2].

Accordingly, the exponential decay rate, which is easily seen to be continuous in η , attains its maximum when $\eta = \pm\eta_*$, for some $\eta_* > 0$, being clear that the picture depends only on the modulus of η . This complies with our calculations: when $\alpha, \beta, \gamma, \kappa$ are fixed, the best decay rate $\omega_b = \omega_b(\eta)$ predicted by Theorem 6.3 is obtained, up to a multiplicative constant, by maximizing (7.4) with respect to $\varepsilon > 0$. In turn, the function $\eta \mapsto \omega_b(\eta)$ reaches its maximum for some $\eta = \pm\eta_b$, with

$$\sqrt{\mathfrak{t}(\kappa)\mu} < \eta_b < \infty,$$

although it is not easy at all to compute such an η_b explicitly. Nevertheless, this does not solve the problem of finding the actual value ω_* of the exponential decay rate (for any fixed η large enough). First, because our estimates provide only sufficient conditions. But, more importantly, because it is generally impossible to establish the exponential decay rate of a linear system via energy estimates. This fact will be discussed in detail in the final Appendix.

Actually, there is a last interesting question to be addressed, namely, to see what happens in the limit situations when the thermal conductivity κ becomes either very small or very large. More precisely, for fixed α, β, γ , let us consider the exponential decay rate

$$\omega_\star = \omega_\star(\kappa, \eta).$$

This function is defined for every $\kappa > 0$ and every η with $|\eta|$ large enough. Then, for every $\kappa > 0$, we set

$$\omega_\kappa = \max_{\eta} \omega_\star(\kappa, \eta).$$

In other words, ω_κ is the best possible decay rate for a fixed κ , which is obtained by suitably modulating the coupling constant η . It is reasonable to expect that $\omega_\kappa \rightarrow 0$ as $\kappa \rightarrow 0$. This is indeed true, but not so impressive from the physical viewpoint (and we omit the proof). On the contrary, it would seem reasonable to bet on $\omega_\kappa \rightarrow \infty$ as $\kappa \rightarrow \infty$. The next proposition tells that you would lose.

Proposition 8.3. *Let α, β, γ be fixed. Then there exists $\xi = \xi(\alpha, \beta, \gamma) > 0$ such that*

$$\limsup_{\kappa \rightarrow \infty} \omega_\kappa \leq 2\xi.$$

Proof. We parallel the proof of Proposition 8.1, considering the characteristic polynomial of Section 5, that here we simply call p , which fulfills $p(0) = \gamma\kappa\lambda^2 > 0$. We will establish the claim by showing that there exists $0 < \zeta \leq \xi$ such that

$$p(-\zeta) < 0,$$

for every κ large enough. This means that p has a real root in $(-\xi, 0)$, implying that $\omega_\kappa \leq 2\xi$. To this aim, we introduce the positive number (as $\mu \geq 0$)

$$r = \frac{\alpha^2\lambda}{\alpha^3 - 4\alpha\beta\lambda + 8\gamma\lambda},$$

and we call ξ the unique real solution to the equation

$$-\xi^3 + \left(\alpha + \frac{\lambda}{r}\right)\xi^2 + \gamma\lambda = 0.$$

It is straightforward to check that $\xi > \alpha$. Then, we split p into the sum

$$p(z) = f(z) + g(z),$$

where

$$\begin{aligned} f(z) &= \kappa\lambda z^3 + (\alpha\kappa\lambda + \eta^2\lambda^2)z^2 + (\beta\kappa\lambda^2 + \alpha\eta^2\lambda^2)z + \gamma\kappa\lambda^2, \\ g(z) &= z^4 + \alpha z^3 + \beta\lambda z^2 + \gamma\lambda z. \end{aligned}$$

Note that g is independent of κ and η . If $\kappa/\eta^2 \leq r$, we set $\zeta = \alpha/2$ to obtain

$$f(-\zeta) = \frac{\alpha^2\lambda^2\eta^2}{8} \left(-2 + \frac{\kappa}{\eta^2 r}\right) \leq -\frac{\alpha^2\lambda^2\eta^2}{8} \leq -\frac{\alpha^2\lambda^2\kappa}{8r}.$$

Instead, if $\kappa/\eta^2 > r$, we set $\zeta = \xi$ and, from the very definition of ξ , we get

$$f(-\zeta) = -\kappa\lambda^2 \left(\beta\xi + \frac{\kappa - r\eta^2}{r\kappa}\xi^2 + \frac{\alpha\eta^2}{\kappa}\xi\right) \leq -\beta\xi\lambda^2\kappa.$$

In both cases, we conclude that $p(-\zeta) = f(-\zeta) + g(-\zeta) \rightarrow -\infty$ as $\kappa \rightarrow \infty$. \square

Remark 8.4. Actually, the same is true also in the subcritical regime $\mu < 0$. Just note that $p(-\alpha) = \lambda\mu(\kappa\lambda - \alpha) \rightarrow -\infty$ as $\kappa \rightarrow \infty$.

9. COMPARISON WITH NUMERICAL RESULTS

The aim of this section is to verify to what extent our theoretical findings are consistent with the numerical simulations. This is of some importance, since our main Theorem 6.3 establishes only sufficient conditions on the exponential decay. As a model, we consider the one-dimensional version of (4.1), i.e., where the underlying Hilbert space H is simply \mathbb{R} and $A = 1$, implying in turn $\lambda_1 = 1$. As far as the MGT parameters are concerned, we take

$$\alpha = 2, \quad \beta = 1, \quad \gamma = 3,$$

meaning that we sit in the supercritical regime with $\mu = 1$. But, in fact, nothing really changes in the forthcoming analysis by selecting different parameters complying with $\mu > 0$. Accordingly, our system reads

$$\begin{cases} u''' + 2u'' + u' + 3u = \eta\theta, \\ \theta' + \kappa\theta = -\eta u'' - 2\eta u', \end{cases}$$

where the *prime* stands time-derivative, and whose characteristic polynomial is

$$p(z) = z^4 + (2 + \kappa)z^3 + (1 + 2\kappa + \eta^2)z^2 + (3 + \kappa + 2\eta^2)z + 3\kappa.$$

9.1. The stability threshold. The first task is comparing the theoretical stability threshold $\mathfrak{t}(\kappa)$ provided by (6.2) with the actual stability threshold $\mathfrak{t}_*(\kappa)$, which certainly exists in the light of Proposition 5.1. With the aid of Mathematica™ we compute $\mathfrak{t}_*(\kappa)$ as

$$\mathfrak{t}_*(\kappa) = \sup \{ \eta^2 : p \text{ has a root with positive real part} \},$$

and we compare the two thresholds in the following figure.

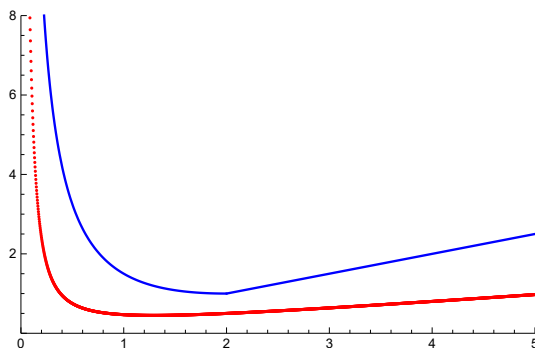
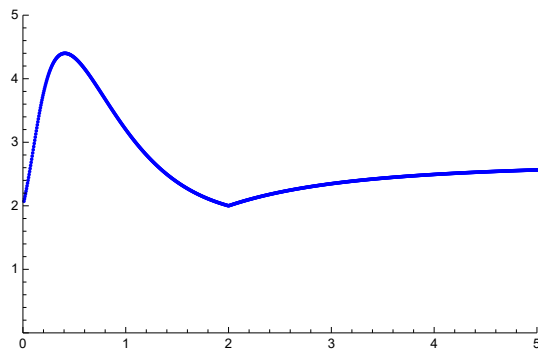


fig.2 Theoretical $\mathfrak{t}(\kappa)$ (blue) vs actual $\mathfrak{t}_*(\kappa)$ computed numerically (red).

As expected, $\mathfrak{t}(\kappa) > \mathfrak{t}_*(\kappa)$ for every κ . Nonetheless, both functions have a linear growth for κ large, and they also exhibit the same behavior when $\kappa \rightarrow 0$. This is quite clear from the graph of the ratio $\mathfrak{t}/\mathfrak{t}_*$.

fig. 3 Ratio $t(\kappa)/t_*(\kappa)$ for $\kappa < 5$

Remark 9.1. Note that $t(\kappa)$ does not really depend on the particular operator A , but only on the minimum λ_1 of its spectrum. However, we might add that, in the one-dimensional case, our theoretical results are in fact better and closer to the numerical simulations, since in the estimates (precisely those of Lemma 7.4) we can lean on the fact that the 0-norm and the 1-norm coincide. Indeed, we actually have

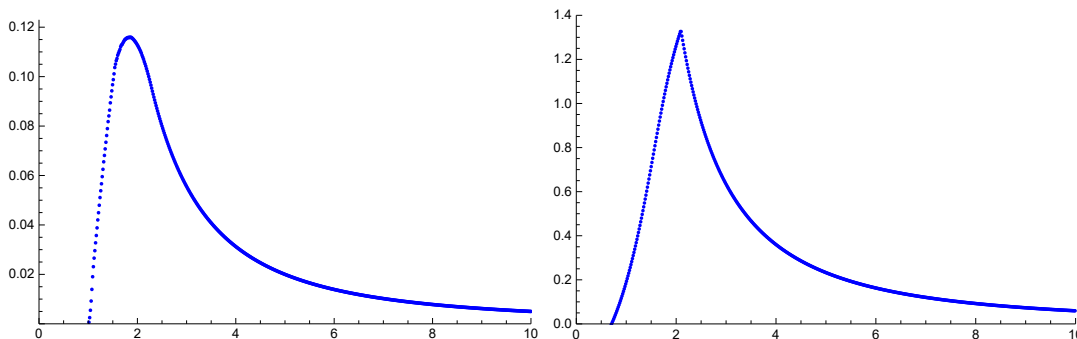
$$t(\kappa) = \frac{2\kappa}{\alpha^2} \left(1 - \frac{\alpha}{\kappa\lambda_1} + \frac{\alpha^2}{\kappa^2\lambda_1^2} \right), \quad \forall \kappa > 0.$$

9.2. Optimal decay. As already mentioned, the value ω_b found in Theorem 6.3, is certainly worse than the actual decay rate ω_* , the latter being easily computed from the roots of p . Nonetheless, their shapes as functions of η are similar, and the maximum points are pretty close. We perform the numerical analysis for $\kappa = 2$, and we recall that, up to a multiplicative constant, ω_b is computed by means of (7.4), which for this particular choice of the parameters reads

$$\omega = \omega(\eta) = \min \left\{ \varepsilon^2 \left(\frac{3}{2} - \varepsilon\eta \right), \varepsilon^2, 2\sigma - \frac{133}{6}\varepsilon^2 - 2\varepsilon^3\eta, 2\sigma - 3\varepsilon\eta \right\}.$$

Accordingly,

$$\omega_b = \omega_b(\eta) = \sup_{\varepsilon > 0} \omega(\eta).$$

fig. 4 Theoretical decay rate $\omega_b(\eta)$ (left) vs actual decay rate $\omega_*(\eta)$ (right) for $\eta < 10$.

Note that $\eta_b \approx 1.86$ whereas $\eta_* \approx 2.15$.

APPENDIX:
EXPONENTIAL DECAY RATE AND ENERGY ESTIMATES

In general, for a given linear differential system, it is impossible to attain the exponential decay rate (in the sense of Definition 6.2) via energy estimates, except in the trivial situation where the energy inequality (6.1) holds for $M = 1$. To show this fact, we look at the simplest possible example, namely, the semigroup generated by the second-order differential equation

$$(A.1) \quad x'' + x' + x = 0,$$

modeling the position $x = x(t) \in \mathbb{R}$ of an oscillator subject to dynamical friction. Setting $y = x'$, the total energy of (A.1), which is the sum of the elastic potential and the kinetic energies, reads

$$\mathbf{e} = \frac{1}{2}[x^2 + y^2].$$

The exponential decay rate of \mathbf{e} is completely determined by the complex roots of the characteristic polynomial

$$p(\lambda) = \lambda^2 + \lambda + 1.$$

Since the (conjugate) roots have both real part equal to $-1/2$, we deduce the exponential decay rate $\omega_* = 1$. Let us see now what happens if we try to attain the exponential decay via energy estimates. For the case under investigation, one multiplies equation (A.1) by $y + \varepsilon x$, for $\varepsilon \in (0, 1)$. Incidentally, such a multiplier is the classical one used in the asymptotic analysis of hyperbolic ODEs and PDEs. Then, straightforward computations entail the differential identity

$$(A.2) \quad \frac{d}{dt}\mathbf{g} + \mathbf{f} = 0,$$

having set

$$\mathbf{g} = (1 + \varepsilon)x^2 + y^2 + 2\varepsilon xy \quad \text{and} \quad \mathbf{f} = 2\varepsilon x^2 + 2(1 - \varepsilon)y^2.$$

Both \mathbf{g} and \mathbf{f} are equivalent to the energy \mathbf{e} , due to the fact that $\varepsilon \in (0, 1)$. At this point, the (only possible) strategy is showing that

$$(A.3) \quad \omega \mathbf{g} \leq \mathbf{f},$$

for some $\omega > 0$. In which case, one infers from (A.2) the differential inequality

$$\frac{d}{dt}\mathbf{g} + \omega \mathbf{g} \leq 0,$$

and an application of the Gronwall lemma yields

$$\mathbf{g}(t) \leq \mathbf{g}(0)e^{-\omega t}.$$

From the equivalence $\mathbf{g} \sim \mathbf{e}$, we conclude that \mathbf{e} decays at an exponential rate ω as well. In fact, we are looking for the best possible ω that can be obtained by means of this procedure, which coincides with the largest ω such that (A.3) holds or, equivalently, with the largest ω such that the inequality

$$(2\varepsilon - \omega - \omega\varepsilon)t^2 - 2\omega\varepsilon t + 2 - 2\varepsilon - \omega \geq 0$$

is verified for all $t \in \mathbb{R}$. This translates into the conditions

$$(A.4) \quad \begin{cases} 2\varepsilon - \omega - \omega\varepsilon > 0, \\ (2\omega\varepsilon)^2 - 4(2\varepsilon - \omega - \omega\varepsilon)(2 - 2\varepsilon - \omega) \leq 0. \end{cases}$$

We have fallen into a simple optimization problem in two variables: maximize the function

$$F(\varepsilon, \omega) = \omega$$

on the admissible region

$$\mathfrak{A} = \{(\varepsilon, \omega) \in (0, 1) \times (0, 1] \text{ such that (A.4) holds}\}.$$

The maximum of the (linear) extension of F on the closure of \mathfrak{A} is attained on the boundary. Since equality in the first constraint of (A.4) leads to a minimum, F assumes its maximum when equality holds in the second constraint, yielding

$$\omega = 1 - \sqrt{\frac{1 - 3\varepsilon + 3\varepsilon^2}{1 + \varepsilon - \varepsilon^2}}.$$

By elementary computations, the latter quantity is maximized when $\varepsilon = 1/2$, where $\omega = \omega_b$ with

$$\omega_b = 1 - \frac{\sqrt{5}}{5}.$$

Since the pair $(1/2, \omega_b) \in \mathfrak{A}$, the problem is solved. Summarizing, the best possible decay obtained through this method is exactly $\omega_b \approx 0.55$, with a sizable gap compared to the actual decay rate $\omega_* = 1$.

Remark A.1. As we mentioned above, for a linear semigroup $S(t)$ acting on a Hilbert space \mathcal{H} , the exponential decay rate is attained via energy estimates when (6.1) is satisfied with $M = 1$. This occurs if and only the inequality

$$\langle \mathbb{A}u, u \rangle_{\mathcal{H}} \leq -\frac{\omega}{2} \|u\|_{\mathcal{H}}^2$$

holds true for all vectors u in the domain of \mathbb{A} , being \mathbb{A} the infinitesimal generator of $S(t)$. Certainly this is not the case for the system (4.1) of this paper, and not even for the much simpler equation (A.1). It is however true for the semigroup generated by the linear parabolic equation

$$u_t + Au = 0,$$

with A as in Section 3.

Acknowledgments. We thank the anonymous Referee for useful suggestions and comments.

REFERENCES

- [1] F. Alabau-Boussouira, Z. Wang, L. Yu, *A one-step optimal energy decay formula for indirectly nonlinearly damped hyperbolic systems coupled by velocities*, ESAIM Control Optim. Calc. Var. **23** (2017), 721–749.
- [2] M.S. Alves, C. Buriol, M.V. Ferreira, J. E. Muñoz Rivera, M. Sepúlveda, O. Vera, *Asymptotic behaviour for the vibrations modeled by the standard linear solid model with a thermal effect*, J. Math. Anal. Appl. **399** (2013), 472–479.
- [3] K. Ammari, S. Nicaise, *Stabilization of a transmission wave/plate equation*, J. Differential Equations **249** (2010), 707–727.
- [4] G. Avalos, I. Lasiecka, *Exponential stability of a thermoelastic system without mechanical dissipation*, Rend. Istit. Mat. Univ. Trieste **28** (1997), 1–28.
- [5] G. Avalos, I. Lasiecka, *The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system*, Semigroup Forum **57** (1998), 278–292.
- [6] G. Avalos, I. Lasiecka, R. Triggiani, *Heat-wave interaction in 2-3 dimensions: optimal rational decay rate*, J. Math. Anal. Appl. **437** (2016), 782–815.
- [7] F. Bucci, I. Lasiecka, *Feedback control of the acoustic pressure in ultrasonic wave propagation*, Optimization **68** (2019), 1811–1854.
- [8] F. Bucci, L. Pandolfi, *On the regularity of solutions to the Moore-Gibson-Thompson equation: a perspective via wave equations with memory*, J. Evol. Equ. **20** (2020), 837–867.
- [9] M. Conti, L. Liverani, V. Pata, *A note on the energy transfer in coupled differential systems*, Commun. Pure Appl. Anal., to appear.
- [10] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, *On the analyticity of the MGT-viscoelastic plate with heat conduction*, J. Differential Equations **269** (2020), 7862–7880.
- [11] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, *A new approach to MGT-thermoviscoelasticity*, Discrete Contin. Dyn. Syst., to appear.
- [12] M. Conti, V. Pata, R. Quintanilla, *Thermoelasticity of Moore-Gibson-Thompson type with history dependence in the temperature*, Asymptot. Anal. **120** (2020), 1–21.
- [13] R.F. Curtain, H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag, New York, 1995.
- [14] B. D’Acunto, A. D’Anna, P. Renno, *On the motion of a viscoelastic solid in presence of a rigid wall*, Z. Angew. Math. Phys. **34** (1983), 421–438.
- [15] F. Dell’Oro, I. Lasiecka, V. Pata, *The Moore-Gibson-Thompson equation with memory in the critical case*, J. Differential Equations **261** (2016), 4188–4222.
- [16] F. Dell’Oro, V. Pata, *On the stability of Timoshenko systems with Gurtin-Pipkin thermal law*, J. Differential Equations **257** (2014), 523–548.
- [17] F. Dell’Oro, V. Pata, *On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity*, Appl. Math. Optim. **76** (2017), 641–655.
- [18] R. Denk, F. Kammerlander, *Exponential stability for a coupled system of damped-undamped plate equations*, IMA J. Appl. Math. **83** (2018), 302–322.
- [19] A.D. Drozdov, V.B. Kolmanovskii, *Stability in viscoelasticity*, North-Holland, Amsterdam, 1994.
- [20] H.D. Fernández Sare, R. Racke, *On the stability of damped Timoshenko systems: Cattaneo versus Fourier law*, Arch. Rational Mech. Anal. **194** (2009), 221–251.
- [21] G.C. Gorain, S.K. Bose, *Exact controllability and boundary stabilization of torsional vibrations of an internally damped flexible space structure*, J. Optim. Theory Appl. **99** (1998), 423–442.
- [22] J. Hao, Z. Liu, *Stability of an abstract system of coupled hyperbolic and parabolic equations*, Z. Angew. Math. Phys. **64** (2013), 1145–1159.
- [23] M.W. Hirsch, S. Smale, *Differential equations, dynamical systems and linear algebra*, Academic Press, New York, 1974.
- [24] B. Kaltenbacher, I. Lasiecka, R. Marchand, *Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Control Cybernet. **40** (2011), 971–988.

- [25] B. Kaltenbacher, I. Lasiecka, M.K. Pospieszalska, *Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Math. Models Methods Appl. Sci. **22** (2012), n.11, 1250035.
- [26] B. Kaltenbacher, V. Nikolić, *The Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time*, Math. Models Methods Appl. Sci. **29** (2019), 2523–2556.
- [27] J.U. Kim, *On the energy decay of a linear thermoelastic bar and plate*, SIAM J. Math. Anal. **23** (1992), 889–899.
- [28] Z. Liu, S. Zheng, *Exponential stability of the Kirchhoff plate with thermal or viscoelastic damping*, Quarterly Appl. Math. **55** (1997), 551–564.
- [29] Z. Liu, S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC, Boca Raton, 1999.
- [30] J.E. Muñoz Rivera, R. Racke, *Smoothing properties, decay, and global existence of solutions to nonlinear coupled systems of thermoelastic type*, SIAM J. Math. Anal. **26** (1995), 1547–1563.
- [31] J.E. Muñoz Rivera, R. Racke, *Large solutions and smoothing properties for nonlinear thermoelastic systems*, J. Differential Equations **127** (1996), 454–483.
- [32] J.E. Muñoz Rivera, R. Racke, *Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability*, J. Math. Anal. Appl. **276** (2002), 248–278.
- [33] J.E. Muñoz Rivera, R. Racke, *Transmission problems in (thermo)viscoelasticity with Kelvin-Voigt damping: nonexponential, strong, and polynomial stability*, SIAM J. Math. Anal. **49** (2017), 3741–3765.
- [34] R. Marchand, T. McDevitt, R. Triggiani, *An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability*, Math. Methods Appl. Sci. **35** (2012), 1896–1929.
- [35] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.
- [36] F.K. Moore, W.E. Gibson, *Propagation of weak disturbances in a gas subject to relaxation effects*, J. Aero/Space Sci. **27** (1960), 117–127.
- [37] M.L. Santos, D.S. Almeida Júnior, J.E. Muñoz Rivera, *The stability number of the Timoshenko system with second sound*, J. Differential Equations **253** (2012), 2715–2733.
- [38] Professor Stokes, *An examination of the possible effect of the radiation of heat on the propagation of sound*, Philos. Mag. Series 4 **1** (1851), 305–317.
- [39] P.A. Thompson, *Compressible-fluid dynamics*, McGraw-Hill, New York, 1972.
- [40] R. Triggiani, *Heat-viscoelastic plate interaction: analyticity, spectral analysis, exponential decay*, Evol. Equ. Control Theory **7** (2018), 153–182.
- [41] D.J. Uherka, A.M. Sergott, *On the continuous dependence of the roots of a polynomial on its coefficients* Amer. Math. Monthly **84** (1977), 368–370.

POLITECNICO DI MILANO - DIPARTIMENTO DI MATEMATICA
 VIA BONARDI 9, 20133 MILANO, ITALY
 Email address: monica.conti@polimi.it (M. Conti)
 Email address: lorenzo.liverani@polimi.it (L. Liverani)
 Email address: vittorino.pata@polimi.it (V. Pata)