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A hybrid redesign for robust stabilization without unit input

Riccardo Ballaben, Uros Sutulovic, Davide Invernizzi, and Luca Zaccarian

***Abstract*— We propose a redesign strategy to avoid an arbitrary value of the input signal. With a hybrid architecture based on a switching logic with two modes we obtain robust global asymptotic stability and ensure that the input never takes the unwanted value, while preserving the nominal closed-loop behaviour in a neighbourhood of the origin. In the case where the minimization problem is too computationally expensive, we provide a simpler, albeit more conservative, way to determine the scaling factor. We then present a nonlinear generalization to a fairly general class of input-affine nonlinear systems. Numerical examples are used to illustrate the theoretical results.**

***Index Terms*— Hybrid systems, aerospace, Lyapunov methods**

I. Introduction

CONTROLLING linear systems in the presence of input saturation has been extensively studied in the literature (see, *e.g.*, [10], [13] and references therein). Less attention has been devoted to the robust stabilization problem with reverse input saturation constraints, such as avoiding a specific input value. Among the few existing works, [1] addresses the case of avoiding an unwanted value of an m-dimensional control input. Recently, [2], [14] considered reverse polytopic input constraints for linear systems based on hybrid switching.

A relevant case study where unwanted input values should be avoided is the stabilization problem for underactuated UAVs where the attitude dynamics is fully actuated but the position dynamics is not, and the control force can be applied only along certain directions of the airframe. Changing the attitude of the platform to align the delivered force along the direction needed for position control requires specifying a reference attitude that becomes singular in free-falling conditions [6], [7], when the commanded thrust is zero. Existing designs either impose conservative saturation bounds on the control force [6] or modify the reference attitude in the neighborhood of the singularity [9]. The former solution imposes performance limitations since aggressive maneuvers, such as flips to push the UAV downward with a large thrust, are discarded by design. Instead, the latter solution precludes achieving a global stability result. Another potential application example is spacecraft stabilization by means of control moment gyros, where the gimbal-locking condition is problematic. Avoiding certain input values could resolve the associated singularity issue instead of modifying the gimbal rates allocation algorithm when passing through singularities, which locally perturbs the torque commanded by the attitude controller [12].

An important challenge is to obtain robust stabilization, despite the inevitably discontinuous action associated with this goal. In this work we focus on single input control systems that we hybridly redesign preserving robust closed-loop stability. A more general problem is solved with hybrid tools in [2], [14], which deal with multivariable linear systems and where the set of input values to be avoided is given by the union of a finite number of closed polytopes. While we address the simpler problem considered in [1] for single input systems, our redesign allows for a natural extension covering a large class of input-affine nonlinear systems. The proposed schemes guarantee uniform semiglobal dwell-time of the switching instants and robustness to small perturbations.

This work is organized as follows. In Section II we solve the problem for the linear case, and Section III applies our solution to a class of input-affine nonlinear systems.

Notation. $\mathbb{R}$ ($\mathbb{R}_{>0}, \mathbb{R}_{\geq 0}$) denotes the set of (positive, nonnegative) real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices. I denotes the identity matrix. Given $A \in \mathbb{R}^{m\times n}$, $||A|| : \mathbb{R}^{m\times n} \to \mathbb{R}$ is the matrix norm induced by the vector norm $|\cdot|$. $\text{He}(S) = S^\top + S$, and $\lambda_M(S)$ denotes the maximum eigenvalue of S. Given a set $\mathcal{A} \subset \mathbb{R}^n$, $|x|_\mathcal{A} = \inf\{|x-y| : y \in \mathcal{A}\}$ is the distance of a point x from a set $\mathcal{A}$.

II. Linear Redesign

A. Problem Formulation

Consider the strictly proper linear plant

$$\begin{cases} \dot{x}_p = A_p x_p + B_p u_n \\ y = C_p x_p, \end{cases} \tag{1}$$

with $x_p \in \mathbb{R}^{n_p}$, $u_n \in \mathbb{R}$, $y \in \mathbb{R}^{m_p}$, in feedback interconnection with a linear dynamic controller

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u_n = C_c x_c + D_c y, \end{cases} \tag{2}$$

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with $x_c \in \mathbb{R}^{n_c}$. Defining the full state $x := [x_p^\top \; x_c^\top]^\top \in \mathbb{R}^n$, with $n = n_p + n_c$, the output u_n can be written as

$$u_n = D_c C_p x_p + C_c x_c = K \begin{bmatrix} x_p \\ x_c \end{bmatrix} \tag{3}$$

and the linear feedback interconnection becomes

$$\dot{x} = Ax + Bu_n = (A + BK)x, \tag{4}$$

with $A = \begin{bmatrix} A_p & 0 \\ B_c C_p & A_c \end{bmatrix}, B = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, K = \begin{bmatrix} D_c C_p & C_c \end{bmatrix}$.

Any linear stabilizing control design (such as H_∞ or linear quadratic control) yields a controller of the form (2), thus making our redesign technique broadly applicable.

Problem 1: Under the assumption that $A + BK$ is Hurwitz, define a hybrid controller modifying the closed-loop (1)-(2) such that the following properties hold:

(i) GAS: the origin is robustly globally asymptotically stable.
(ii) Unit input avoidance: the redesigned input u never takes the value 1 along solutions.
(iii) Local preservation: the redesigned controller preserves the nominal linear closed-loop dynamics (4) in a neighborhood of the origin.
(iv) Output feedback: the redesigned controller only uses the knowledge of x_c and y, namely, only quantities available to the linear controller.

Remark 1: Item (ii) of Problem 1 can be generalized to any nonzero value by properly rescaling u_n and B_p.

In item (i) and the rest of the paper, we characterize robustness of asymptotic stability in the sense of [5, Def 7.15.].

B. Redesigned controller architecture

We propose an architecture based on the hybrid dynamical systems formalism in [5]. A logic variable $q \in \{-1, 1\}$ is introduced to model a switch between the nominal feedback interconnection (1)-(2) ($q = 1$), and a modified feedback interconnection obtained by rescaling the output of the control system ($q = -1$). The redesigned output selection induced by q is given by

$$u = \frac{1 - 2q\bar{\varepsilon}}{1 - 2\bar{\varepsilon}} u_n = \begin{cases} u_n, & \text{if } q = 1 \\ \dfrac{1 + 2\bar{\varepsilon}}{1 - 2\bar{\varepsilon}} u_n, & \text{if } q = -1 \end{cases}, \tag{5}$$

where $\bar{\varepsilon} \in (0, \frac{1}{2})$ is a tuning parameter determining the trade-off between robustness to measurement noise and preservation of the nominal closed-loop performance. The resulting feedback interconnection is shown in Figure 1. The logic defining the value of q is implemented using the function

$$\phi(u, q) = (q + 2\bar{\varepsilon})u - q - \bar{\varepsilon}, \tag{6}$$

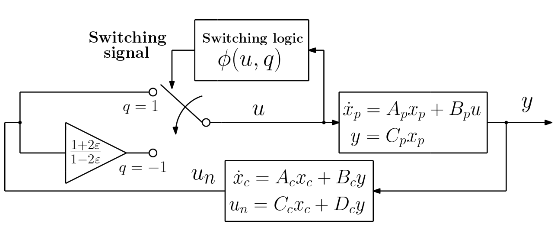

Fig. 1. Redesigned plant/controller feedback interconnection.

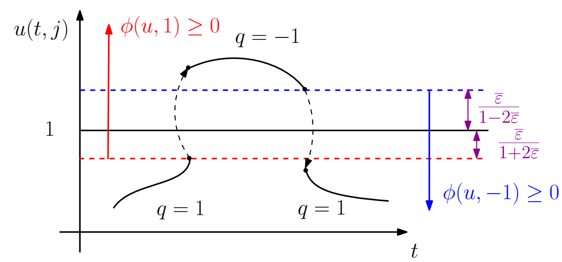

Fig. 2. Example of input evolution generated by the hybrid architecture with jump sets generated by the function $\boldsymbol{\phi(u,q)}$.

which induces the flow and jump sets

$$\begin{aligned} \mathcal{C} &:= \{(x, q) \in \mathbb{R}^n \times \{-1, 1\} \mid \phi(u, q) \leq 0\} \\ \mathcal{D} &:= \{(x, q) \in \mathbb{R}^n \times \{-1, 1\} \mid \phi(u, q) \geq 0\}. \end{aligned} \tag{7}$$

The jump set $\mathcal{D}$ generated by the inequality $\phi(u, q) \geq 0$, shown in Figure 2, can be expressed as

$$\phi(u, q) \geq 0 \Leftrightarrow \begin{cases} u \geq 1 - \dfrac{\bar{\varepsilon}}{1 + 2\bar{\varepsilon}} & \text{for} \quad q = 1, \\ u \leq 1 + \dfrac{\bar{\varepsilon}}{1 - 2\bar{\varepsilon}} & \text{for} \quad q = -1. \end{cases} \tag{8}$$

Defining the augmented state $z = (x, q) \in \mathbb{R}^n \times \{-1, 1\}$ and the jump map toggling the value of q and leaving x unchanged, the hybrid system describing the dynamics of z can be written as

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ 0 \end{bmatrix}, & z \in \mathcal{C} \\ z^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} x \\ -q \end{bmatrix}, & z \in \mathcal{D}. \end{cases} \tag{9}$$

The closed-loop hybrid dynamical system is then given by (5), (6), (7), (9). Exploiting expression (8), Figure 2 illustrates a possible trajectory of the input and how the switching logic ensures that it never crosses the line $u = 1$.

C. Main results and tuning method

In accordance with Figure 2, to ensure $\frac{\bar{\varepsilon}}{1-2\bar{\varepsilon}} > 0$ and $\frac{\bar{\varepsilon}}{1+2\bar{\varepsilon}} > 0$, we constrain $\bar{\varepsilon} \in (0, \frac{1}{2})$, which also guarantees that $u = 0, q = 1$ is not in the jump set and that the scaling factor in equation (5) is not unbounded. More specifically, we show next that $\exists \varepsilon^* \in (0, \frac{1}{2})$ such that, for any $\bar{\varepsilon} \in (0, \varepsilon^*)$, hybrid system (5), (6), (7), (9) solves Problem 1. We also provide a computationally attractive way to estimate ε^*.

First note that $A + BK$ being Hurwitz implies that there exists $\alpha > 0$ and $P = P^\top > 0$ such that

$$\text{He}(P(A + BK)) \leq -2\alpha P, \tag{10}$$

where α is any value less than or equal to the spectral abscissa of $A + BK$. To simplify the notation in the following equations, we introduce the additional variable $\eta = \frac{1-2\bar{\varepsilon}}{2\bar{\varepsilon}} > 0$, which maps $(0, \frac{1}{2})$ to $(0, \infty)$ with $\bar{\varepsilon} = (2(1+\eta))^{-1}$. It is then possible to select $\varepsilon^* = (2(1+\eta^*))^{-1}$, where $\eta^* \in [0, +\infty)$ is any number satisfying

$$\text{He}(PBK) \leq \eta^* \alpha P. \tag{11}$$

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We emphasize that a large enough η^* solving (11) always exists, which will lead to a small enough $\varepsilon^* = (2(1+\eta^*))^{-1}$. We observe that the smaller the value of ε^*, the closer the nominal controller will get to the unit value before jumping (see Figure 2), which reduces the margin around the value $u = 1$. For this reason, it is of interest to solve the following quasi-convex generalized eigenvalue problem:

$$\eta^* = \min_{\eta, P} \quad \eta \quad \text{subject to } (10), (11). \tag{12}$$

Remark 2: Given the relations $\eta = \frac{1-2\bar{\varepsilon}}{2\bar{\varepsilon}} \in (0, +\infty)$, since $\bar{\varepsilon} \in (0, \frac{1}{2})$, and $\bar{\varepsilon} = \frac{1}{2(1+\eta)}$, we have that maximizing $\bar{\varepsilon}$ is equivalent to minimizing η. Thus the best selection for η is the solution of the optimization problem in (12).

Theorem 1: For any plant-controller interconnection (1)-(2) such that $A+BK$ is Hurwitz, pick any α in (10) smaller than or equal to the spectral abscissa of $A+BK$, and select η^* as in (12) and $\varepsilon^* = (2(1+\eta^*))^{-1}$. For any $\bar{\varepsilon} \in (0, \varepsilon^*)$, hybrid system (5), (6), (7), (9) solves Problem 1.

Remark 3: Following the steps of [2, Theorem 2] it is possible to prove uniform global exponential stability in the t-direction for (5), (6), (7), (9), thus showing a guaranteed t-exponential decrease induced by the redesign law.

The following corollary gives an alternative way to compute a suitable matrix P for the tuning of $\bar{\varepsilon}$, which is more conservative than the solution of (12) but can be used when the minimization problem is too computationally expensive.

Corollary 1: Consider an arbitrary plant-controller interconnection (1)-(2) such that $A + BK$ is Hurwitz. Given the matrix $P = P^\top > 0$ resulting from solving the Lyapunov equation $\text{He}(P(A+BK)) = -I$, select $\eta^* = 2\lambda_M(PBK + (PBK)^\top)$ and $\varepsilon^* = (2(1+\eta^*))^{-1}$. Then, the hybrid system (5), (6), (7), (9) solves Problem 1 for any $\bar{\varepsilon} \in (0, \varepsilon^*)$.

Under mild controllability conditions, the next corollary shows that using linear quadratic control to tune the gain matrix K guarantees that the hybrid system (5), (6), (7), (9) solves Problem 1 for any $\bar{\varepsilon}$ in the admissible range.

Corollary 2: Given any $R = R^\top > 0$ and $Q = Q^\top \geq 0$ and assuming (A, B) stabilizable and $(A, Q^{1/2})$ detectable, when selecting $K = -R^{-1}B^\top P$ with $P = P^\top > 0$ being the unique solution to the algebraic Riccati equation $\text{He}(PA) + Q - PBR^{-1}B^\top P = 0$, then the hybrid system (5), (6), (7), (9) solves Problem 1 for any $\bar{\varepsilon} \in (0, \frac{1}{2})$.
The proof follows immediately by noting that linear quadratic control guarantees a gain margin in the range $[-6\text{dB}, +\infty)$, which implies that the corresponding closed-loop system is robustly GAS for any $\bar{\varepsilon} \in (0, \frac{1}{2})$ such that $\frac{1+2\bar{\varepsilon}}{1-2\bar{\varepsilon}} \geq \frac{1}{2}$, which holds $\forall \bar{\varepsilon} \in (0, \frac{1}{2})$.

Remark 4: The control scheme in Figure 1 can be interpreted as a feedback interconnection of an open loop plant with input u and output u_n connected to a memoryless time-varying gain in the sector $[1, \frac{1+2\bar{\varepsilon}}{1-2\bar{\varepsilon}}]$. Input-Output stability arguments based on the circle criterion can then be exploited to develop generalized dynamical output feedback schemes.

To rule out rapid repeated jumps, we also prove *uniform semiglobal dwell-time*, namely for each $r > 0$ there exists $\tau > 0$ such that, for any solution z satisfying $|z(0,0)| \leq r$, $t_{j+1} - t_j \geq \tau$ for all consecutive jump times t_j, t_{j+1} with $j \geq 1$, and $(t, j) \in \text{dom } z$.

Proposition 1: System (5), (6), (7), (9) enjoys a uniform semiglobal dwell-time property.

D. Linear Example study

Consider the linearized vertical dynamics of a quadrotor UAV with mass m

$$\dot{z} = v, \quad m\dot{v} = mg - T_c + d \tag{13}$$

for which we consider the goal of asymptotically stabilizing a desired altitude setpoint z_d in the presence of an unknown constant disturbance d and with the input constraint $T_c \neq 0$, which is needed in hierarchical control strategies to avoid singularity conditions [7]. To solve this task, select $T_c = mg(1-u)$, where u is the output of the following PID control law with feedforward gravity compensation

$$\begin{aligned} \dot{x}_i &= e, \quad \dot{x}_d = -\tfrac{1}{\tau_d}(x_d + e) \\ u &= -\bar{k}_p e - \bar{k}_d x_d - k_i x_i \end{aligned} \tag{14}$$

In (14) $e := z - z_d$ represents the stabilization error and $\bar{k}_p = k_p + \bar{k}_d$ and $\bar{k}_d = \frac{k_d}{\tau_d}$. Defining $x_p := [e \ v]^\top$, $x_c := [x_i - \frac{d}{mgk_i} \ x_d]^\top$, the closed-loop error system associated with (13)-(14) has the same form as (4) with $A_p = [\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}]$, $B_p = [0 \ g]^\top$, $C_p = [1 \ 0]$ and $D_c = -\bar{k}_p$, with the constraint $u \neq 1$.

We consider a quadcopter with mass $m = 0.5$ kg that has to reach a desired altitude of $z_d = -0.1$ m and is subject to a constant disturbance force $d = -2.8$ N (which pushes the drone upwards against the gravity). Assigning the gains $k_p = 1.05$, $k_d = 0.75$ and $k_i = 0.45$ and the time constant as $\tau_d = \frac{k_d}{10k_p}$, we get the successful simulation results in Figure 3. To tune the value of $\bar{\varepsilon}$ we used the approach of Theorem 1, which results in $\bar{\varepsilon} = 0.0439$.

E. Proofs

We begin with some preliminary results. The following Lemma, which will be used in the proofs of Theorem 1 and Proposition 1, ensures that after a jump from $\mathcal{D}$ solutions can not immediately jump again but must first flow.

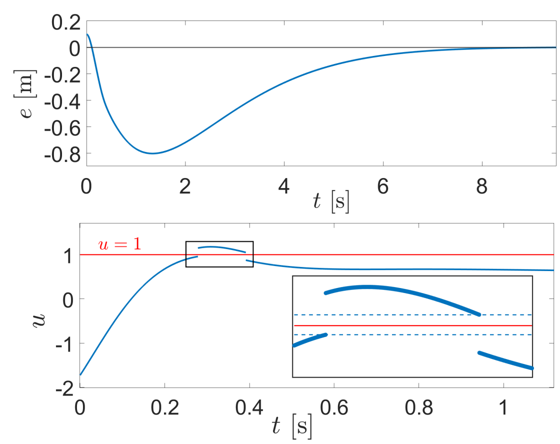

Fig. 3. Closed-loop solution of (13)-(14). Altitude error time evolution (Top). PID output with zoom around $u = 1$, showing the boundary of $\mathcal{D}$ with dashed lines (Bottom).

Lemma 1: Across jumps of (6)-(9) we have $\phi(u^+, q^+) - \phi(u, q) \leq -2\bar{\varepsilon}$.

Proof: Assume $z \in \mathcal{D}$, which implies $\phi(u, q) \geq 0$. From (7) we have $\phi(u, q) = (q + 2\bar{\varepsilon})u - q - \bar{\varepsilon} \geq 0$, namely

$$q + \bar{\varepsilon} \leq (q + 2\bar{\varepsilon})u. \tag{15}$$

Substituting (15) in the expression of $\phi(u^+, q^+)$ gives

$$\begin{aligned}
&\phi(u^+, q^+)\\
&= (-q + 2\bar{\varepsilon})u^+ + q - \bar{\varepsilon} = (-q + 2\bar{\varepsilon})u^+ + q - 2\bar{\varepsilon} + \bar{\varepsilon}\\
&\leq (-q + 2\bar{\varepsilon})u^+ + (q + 2\bar{\varepsilon})u - 2\bar{\varepsilon}\\
&= (-q + 2\bar{\varepsilon})\frac{1 + 2q\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}u_n + (q + 2\bar{\varepsilon})\frac{1 - 2q\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}u_n - 2\bar{\varepsilon}\\
&= -2\bar{\varepsilon},
\end{aligned}$$

which concludes the proof. ■

Let us now proceed with the proof of Theorem 1.

Proof of Theorem 1. To show GAS in item (i) of Problem 1, accounting for the new logical state q, we prove asymptotic stability of the compact set $\mathcal{A}_0 = \{x = 0, q = 1\}$ for the hybrid system (5), (6), (7), (9).

Consider the output signal defined in (5). For $q = 1$, the control law returns the nominal state feedback $u = Kx$. Since η^* is selected according to (12), then the closed-loop flow dynamics $\dot{x} = (A + BK)x$ satisfies (10), (11) for a suitable $\alpha > 0$ and $P = P^\top > 0$. Consider now $q = -1$. The control law (5) reads

$$u = \frac{1 + 2\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}Kx = \frac{1 - 2\bar{\varepsilon} + 4\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}Kx = Kx + \frac{4\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}Kx, \tag{16}$$

and the resulting closed-loop flow dynamics reads

$$\dot{x} = \left(A + BK + \frac{4\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}BK\right)x. \tag{17}$$

Given $P = P^\top > 0$ and η^* solving (12), consider an arbitrary $\bar{\varepsilon} \in (0, \varepsilon^*)$ and define $\eta = \frac{1-2\bar{\varepsilon}}{2\bar{\varepsilon}}$. For dynamics (17) we obtain, using the bounds (10), (11)

$$\begin{aligned}
&\mathrm{He}(P(A + BK + \frac{4\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}BK))\\
&= \mathrm{He}(P(A + BK)) + \frac{2}{\eta}\mathrm{He}(PBK)\\
&\leq \mathrm{He}(P(A + BK)) + \frac{2}{\eta}\eta^*\alpha P\\
&\leq -2\alpha P + 2\frac{\eta^*}{\eta}\alpha P = 2\alpha P(\frac{\eta^*}{\eta} - 1) < 0
\end{aligned} \tag{18}$$

where we use the fact that, for $\bar{\varepsilon} \in (0, \varepsilon^*)$, we have $\eta^* = \frac{1-2\varepsilon^*}{2\varepsilon^*} < \frac{1-2\bar{\varepsilon}}{2\bar{\varepsilon}} = \eta$, which implies $\frac{\eta^*}{\eta} < 1$. Thus, we have that P defines a common quadratic Lyapunov function for the nominal and modified closed-loop dynamics.

Consider now the candidate Lyapunov function $V = \frac{1}{2}x^\top Px$, which is positive definite with respect to $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 = \{x = 0, q = 1\} \cup \{x = 0, q = -1\}$. Due to (10) and (18), via (17) we have that the Lyapunov function decreases when the solution flows, i.e., $\dot{V}(z) < 0,\ \forall z \in \mathcal{C}\backslash\mathcal{A}$. It is immediate to see from (9) that V is also constant across jumps, since $x^+ = x$. Moreover, Lemma 1 ensures that after any jump the solution must flow so that V must decrease, which implies global asymptotic stability of $\mathcal{A}$ by the Invariance Principle for hybrid dynamical systems. More specifically, since (5), (6), (7), (9) satisfies the Hybrid Basic Conditions in [5, Assumption 6.5.], then we can apply the result in [4, Theorem S13].

Since $\mathcal{A}$ is the union of the two points $\mathcal{A}_0, \mathcal{A}_1$, then solutions either converge to $\mathcal{A}_0$ or to $\mathcal{A}_1$. Assume now that there exists a solution $z(t, j)$ that asymptotically converges to $\mathcal{A}_1$. Uniform global attractivity of $\mathcal{A}$ implies that for any $\delta > 0$ there exist $T > 0$ such that $|z(t, j)|_{\mathcal{A}_1} \leq \delta,\ \forall(t, j) \in \mathrm{dom}\ z$ s.t. $t + j \geq T$. Pick a small enough δ such that $|z|_{\mathcal{A}_1} \leq \delta$ implies $|u| \leq \left|\frac{1+2\bar{\varepsilon}}{1-2\bar{\varepsilon}}Kx\right| < 1 - \frac{\bar{\varepsilon}}{1+2\bar{\varepsilon}}$. Such a δ always exists since $u = \frac{1-2q\bar{\varepsilon}}{1-2\bar{\varepsilon}}Kx = 0$ for $x = 0$ and $|x| \leq |z|_{\mathcal{A}_1}$. Then, $|u| < 1 - \frac{\bar{\varepsilon}}{1+2\bar{\varepsilon}}\ \forall(t, j) \in \mathrm{dom}\ z$ such that $t + j \geq T$. From the expression of $\mathcal{D}$ in (8) it follows that the solution must, at some point, jump to $q^+ = 1$ and can never jump again, which means that no solution can asymptotically converge to $\{x = 0, q = -1\}$. This, together with the fact that solutions from $\mathcal{A}_1 = \{x = 0, q = -1\} \in \mathcal{D}$ jump to $\mathcal{A}_0$, proves global attractivity of $\mathcal{A}_0$. Lyapunov stability of $\mathcal{A}_0$ follows from Lyapunov stability of $\mathcal{A}$ and the fact that $\mathcal{A}_0, \mathcal{A}_1$ are two disjoint points. Finally, robust global asymptotic stability of $\mathcal{A}_0$ follows from global attractivity and Lyapunov stability of $\mathcal{A}_0$ and the fact that system (9) satisfies the Hybrid Basic Conditions [5, Assumption 6.5], which allows using [5, Theorem 7.21].

Let us now continue with the proof that item (ii) of Problem 1 is also satisfied. Consider a generic solution $(t, j) \to z(t, j)$ to system (9) and assume that $z(0, 0)$ is such that $u(0, 0) \neq 1$. Observe from the definition of $\mathcal{C}$ that the solution can only flow for $\phi(u, q) \leq 0$. This implies that $z(t, j)$ can not flow with the value $u = 1$ while $z \in \mathcal{C}$, due to the definition of $\mathcal{D}$ in (8). What is left to check is that, after a jump of z, u is guaranteed to be not unitary. With a slight abuse of notation, we will express the jump in the control law generated by the jump map as $u^+ = \frac{1+2q^+\bar{\varepsilon}}{1-2q^+\bar{\varepsilon}}u$. For $q = 1$ we have $u^+ = \frac{1+2\bar{\varepsilon}}{1-2\bar{\varepsilon}}u \geq \frac{1+2\bar{\varepsilon}}{1-2\bar{\varepsilon}}(1 - \frac{\bar{\varepsilon}}{1+2\bar{\varepsilon}}) = \frac{1+\bar{\varepsilon}}{1-2\bar{\varepsilon}} > 1$. For $q = -1$ instead we have $u^+ = \frac{1-2\bar{\varepsilon}}{1+2\bar{\varepsilon}}u \leq \frac{1-2\bar{\varepsilon}}{1+2\bar{\varepsilon}}(1 + \frac{\bar{\varepsilon}}{1-2\bar{\varepsilon}}) = \frac{1-\bar{\varepsilon}}{1+2\bar{\varepsilon}} < 1$. Thus, u is guaranteed not to land on the value $u = 1$ after a jump from $\mathcal{D}$.

To prove item (iii) of Problem 1, it is enough to show that the controller never switches to $q = -1$ when $u = 0$. Substituting $u = 0, q = 1$ in (8) and recalling that $\bar{\varepsilon} > 0$ gives $0 \geq 1 - \frac{\bar{\varepsilon}}{1+2\bar{\varepsilon}} \Leftrightarrow \bar{\varepsilon} \geq 1 + 2\bar{\varepsilon}$, which is never true for $\bar{\varepsilon} \in (0, \frac{1}{2})$, thus leading to a contradiction. Then, the hybrid controller never switches to the modified control law in a neighbourhood of the origin. Additionally, the jump dynamics only depends on the knowledge of u and q, and does not require knowledge of the plant state, which proves item (iv) of Problem 1. □

Proof of Corollary 1. Select $P = P^\top > 0$ satisfying the Lyapunov equation $\mathrm{He}(P(A + BK)) = -I$ and $\eta > 2\lambda_M(PBK + (PBK)^\top)$. The same steps as those in the proof of Theorem 1 can be followed, with (18) replaced by

$$\begin{aligned}
&\mathrm{He}(P(A + BK)) + \frac{4\bar{\varepsilon}}{1 - 2\bar{\varepsilon}}\mathrm{He}(PBK)\\
&= -I + \frac{2}{\eta}\mathrm{He}(PBK) < -I + \frac{1}{\lambda_M(\mathrm{He}(PBK))}\mathrm{He}(PBK),
\end{aligned}$$

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where the last term is negative semi-definite. Following the proof of Theorem 1, the four items of Problem 1 follow. □

We now prove a uniform dwell-time result, which is of independent interest and is only based on a *Uniform Global Boundedness* (UGB) assumption (also known as *Lagrange stability*), namely for each $r > 0$, $\exists M_r > 0$ such that all solutions z with $|z(0,0)| \leq r$ satisfy $|z(t,j)| \leq M_r$ for all $(t,j) \in \text{dom} z$. Its proof is inspired by [3, §2].

Lemma 2: For a nominally well-posed hybrid system with hybrid data $(F, G, \mathcal{C}, \mathcal{D})$, if $\mathcal{D} \cap G(\mathcal{D}) = \emptyset$ and solutions are uniformly globally bounded, then solutions enjoy a uniform semiglobal dwell-time property.

Proof: Due to UGB, all solutions satisfying $|\phi(0,0)| \leq r$ are uniformly bounded. Then, applying [11, Lemma 2.7], we obtain [1] that each solution enjoys a dwell-time property. From the stated assumptions, to prove uniformity we proceed by contradiction. Suppose that for some $r > 0$ there exist solutions ϕ_k, $k \in \mathbb{N}$ and indices $i_k \geq 1$ with jump times t_{i_k} and t_{i_k+1} satisfying $t_{i_k+1} - t_{i_k} = \tau_k$ and $\tau_k \to 0$ as $k \to \infty$. Similar to [3, Lemma 1], define shifted solutions $\bar{\phi}_k$ as $\bar{\phi}_k(t,j) = \phi_k(t_{i_k-1} + t, i_k + j - 1)$ (namely, $\bar{\phi}_k$ is the tail of ϕ_k with the shrinking dwell-time between its jump times t_1 and t_2). Due to nominal well-posedness [5, Def. 6.2] there exists a converging subsequence of $\bar{\phi}_k$ whose limit is a solution $\bar{\phi}_\infty$. By construction, the domain of $\bar{\phi}_\infty$ satisfies $t_2 - t_1 = \lim_{k\to\infty} \tau_k = 0$. However, this solution contradicts the dwell-time property established by [11, Lemma 2.7]. ■

Proof of Proposition 1. First observe that system (5), (6), (7), (9) satisfies the hybrid basic conditions of [5, As. 6.5] and due to [5, Thm 6.8] it is nominally well-posed. Moreover, the UGS property proven in Theorem 1 implies uniform global boundedness, and $\mathcal{D} \cap G(\mathcal{D}) = \emptyset$ follows from Lemma 1. Then the result follows from Lemma 2. □

III. Nonlinear Redesign

A. Main results

We now extend the proposed hybrid architecture to a class of input-affine nonlinear systems

$$\dot{x} = f(x) + g(x)u, \tag{19}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}$ is the control input. For a compact set $\mathcal{A} \subset \mathbb{R}^n$, assume that f and g are continuous functions of x and that there exists a continuous selection $u = u_n(x)$ and a continuously differentiable Lyapunov function V on $\mathbb{R}^n$, positive definite with respect to $\mathcal{A}$ and radially unbounded, such that, for the closed-loop system $\dot{x} = f(x) + g(x)u_n(x)$, it holds that

$$\begin{aligned}\dot{V}(x) &= \nabla V(x)^\top f(x) + \nabla V(x)^\top g(x)u_n(x) \\ &=: -\psi(x) < 0, \quad \forall x \in \mathbb{R}^n \backslash \mathcal{A}.\end{aligned} \tag{20}$$

Paralleling the reasoning in Figure 2, assume the following.

Assumption 1: There exists $\bar{\varepsilon} \in (0, \frac{1}{2}]$ such that $u(x) < 1 - \frac{\bar{\varepsilon}}{1+2\bar{\varepsilon}}$ for all $x \in \mathcal{A}$.

[1]Note that precompactness (which requires completeness) is required in [11, Lemma 2.7], but only uniform global boundedness is used in its proof.

Following the redesign presented for the linear case, we introduce a logic state $q \in \{-1, 1\}$ and define the full state $z = (x, q) \in \mathbb{R}^n \times \{-1, 1\}$ and a modified control law

$$u = \frac{1 - 2q\varepsilon(x)}{1 - 2\varepsilon(x)} u_n(x), \tag{21}$$

where the constant $\bar{\varepsilon}$ used for the linear case in (5) is replaced by a globally bounded function $x \mapsto \varepsilon(x) : \mathbb{R}^n \to [0, \bar{\varepsilon})$, to be defined, satisfying $|\varepsilon(x)| < \bar{\varepsilon}$ for all $x \in \mathbb{R}^n$, where $\bar{\varepsilon}$ comes from Assumption 1. Mimicking the linear redesign, the flow set $\mathcal{C}$ and jump set $\mathcal{D}$ are induced by the function

$$\phi(x, q) = (q + 2\varepsilon(x))u - q - \varepsilon(x), \tag{22}$$

as in (6), (7). The redesigned system is given by (21) with

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} f(x) + g(x)u \\ 0 \end{bmatrix}, & z \in \mathcal{C} \\ z^+ &= \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} x \\ -q \end{bmatrix}, & z \in \mathcal{D}, \end{aligned} \tag{23}$$

where the function $x \mapsto \varepsilon(x)$ is selected as

$$\varepsilon(x) = \mu\bar{\varepsilon} \frac{\psi(x)}{2|\nabla V(x)^\top g(x) u_n(x)| + \psi(x)}, \tag{24}$$

and $\mu \in (0, 1)$ is a parameter determining the trade-off between avoidance robustness and preservation of the nominal controller. Observe that (24) implies $0 \leq \varepsilon(x) < \bar{\varepsilon} \leq \frac{1}{2}$, and that continuous differentiability of V and continuity of f and g imply that ε is a continuous function of x.

The following theorem is a nonlinear extension of the linear results in Theorem 1 and Proposition 1.

Theorem 2: Under Assumption 1, the set $\mathcal{A} \times \{1\}$ is robustly globally asymptotically stable for (7), (21)–(24). Additionally, system (7), (21)–(24) satisfies items (ii) and (iii) of Problem 1 and enjoys uniform semiglobal dwell-time.

B. Example

Consider the plant from [8, Section 3.3]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x) + g(x)u = \begin{bmatrix} x_2 + x_1^2 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{25}$$

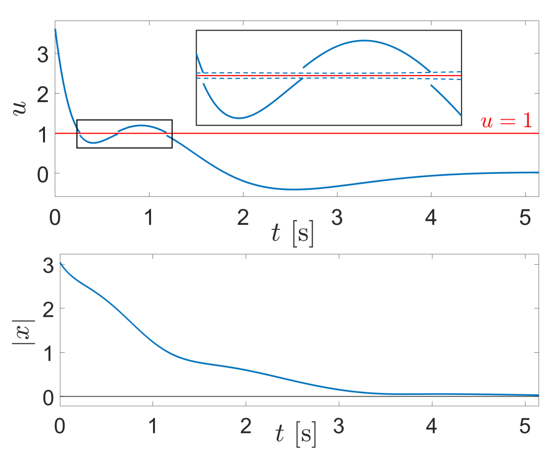

Fig. 4. Closed-loop solution of the hybrid redesigned feedback (7), (21), (22), (23), (24) applied to (25),(26).

where $x = [x_1\ x_2]^\top \in \mathbb{R}^2$ is the state and $u \in \mathbb{R}$ is the control input. For the selection

$$u = u_n = -2x_1 - 2x_2 - 3x_1^2 - 2x_1(x_2 + x_1^2), \tag{26}$$

it is shown in [8] that $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$ is a strong Lyapunov function with respect to the origin, as its negative definite time derivative is

$$\dot{V}(x) = -x_1^4 - 2x_1^3 - 2x_1^2 x_2 - 2x_1^2 - 2x_1 x_2 - x_2^2 =: -\psi(x).$$

Since (26) is zero at the origin, we may select $\overline{\varepsilon} = \frac{1}{2}$ for (25). Additionally, it is immediate to verify that $\nabla V(x)^\top g(x) = x_2 + x_1 + x_1^2$.

Applying the redesign to (25), Theorem 2 holds. Choosing $\mu = 0.1$ and initial conditions $x_1(0) = -0.5$, $x_2(0) = -3$ leads to the desirable simulations shown in Figure 4.

C. Proof of Theorem 2

We first show that the modified input (21), (24) ensures Lyapunov decrease along flowing solutions.

Lemma 3: Given plant (19), a stabilizer u_n and a function V satisfying (20), along the solutions of the hybrid system (7), (21), (22), (23), (24) it holds that $\dot{V}(z) < 0$, $\forall z \in (\mathcal{C} \cup \mathcal{D}) \backslash (\mathcal{A} \times \{-1, 1\})$.

Proof: For $q = 1$, and any $z = (x, q)$ with $x \notin \mathcal{A}$, we have $u = u_n(x)$ and (20) implies $\dot{V}(z) = -\psi(x) < 0$. For the non-trivial case $q = -1$, and $z = (x, q)$ with $x \notin \mathcal{A}$, first note that $\mu < 1$ and $\overline{\varepsilon} \leq \frac{1}{2}$, together with (24) and the fact that $\mu\overline{\varepsilon} < \frac{1}{2}$, imply that $4\varepsilon(x)|V(x)^\top g(x) u_n(x)| < \psi(x) - 2\varepsilon(x)\psi(x)$, which can be rearranged to get

$$\frac{4\varepsilon(x)}{1 - 2\varepsilon(x)} |\nabla V(x)^\top g(x) u_n(x)| < \psi(x).$$

Then, proceeding as in (16), we obtain

$$\begin{aligned} \dot{V}(z) &= \nabla V(x)^\top f(x) + \nabla V(x)^\top g(x) u = \nabla V(x)^\top f(x) \\ &+ \nabla V(x)^\top g(x) u_n(x) + \frac{4\varepsilon(x)}{1 - 2\varepsilon(x)} \nabla V(x)^\top g(x) u_n(x) \\ &= -\psi(x) + \frac{4\varepsilon(x)}{1 - 2\varepsilon(x)} \nabla V(x)^\top g(x) u_n(x) < 0, \end{aligned}$$

as to be proven. ■

The following Lemma ensures that after a jump the solution must flow, unless it jumps from $\mathcal{A} \times \{-1\}$. Its proof is the same as the one presented for Lemma 1, so it is omitted.

Lemma 4: Let z be a solution of (7), (21), (22), (23), (24). Then, $\phi(x^+, q^+) - \phi(x, q) \leq -2\varepsilon(x) < 0$ for all $z \in \mathcal{D}$.

Proof of Theorem 2. Mimicking the proof of Theorem 1, we start by showing that $\mathcal{A} \times \{-1, 1\}$ is robustly globally asymptotically stable for (7), (21), (22), (23), (24). For $z \in \mathcal{C}$, Lemma 3 guarantees the decrease of V.

Consider now $z \in \mathcal{D}$. Since the jump map in (23) toggles the value of q and V only depends on x, we have that the Lyapunov function remains constant across jumps. Additionally, Lemma 4 ensures that after a jump the solution must flow. Since the Hybrid Basic Conditions hold, then global asymptotic stability of $\mathcal{A} \times \{-1, 1\}$ follows from the Invariance Principle for hybrid dynamical systems [4, Theorem S13]. Following the remaining steps of the proof of Theorem 1 we may prove robust global asymptotic stability of $\mathcal{A} \times \{1\}$.

Items (ii) and (iii) of Problem 1 can be proven exploiting at the structure of $\mathcal{C}$ and $\mathcal{D}$. Substituting $\varepsilon = \varepsilon(x)$ into (8) and using the fact that $\varepsilon(x)$ is a continuous function of x, only zero for $x \in \mathcal{A}$, we conclude that there exists a neighbourhood of $\mathcal{A} \times \{1\}$ where the nominal controller is preserved. Furthermore, $u = 1$ being in the interior of $\mathcal{D}$ implies that solutions never flow with $u = 1$. Finally, $u^+ \neq 1$ can be proven by following the bounds on u^+ derived at the end of Theorem 1 with $\overline{\varepsilon}$ replaced by $\varepsilon(x)$.

The semiglobal dwell-time property can be proven by following exactly the same steps as those in the proof of Proposition 1, by exploiting Lemma 2. □

IV. Conclusions

Motivated by the nonzero input assumptions required in the controller design for underactuated UAVs [7], we proposed a hybrid redesign, based on a switching logic with two modes, avoiding unitary inputs in linear systems and in a class of input-affine nonlinear systems. Future work includes nonlinear extensions to UAV control and exploiting the degrees of freedom highlighted in Remark 4 for developing generalized dynamic schemes.

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