



Substructuring-based characterization of periodic straight beams from finite-element unit cells

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ARTICLE INFO

Keywords:

Periodic beams
Computational homogenization
Beam characterization
Finite elements
Thin-walled structures
Shell elements
Equivalent beam properties
Reduced-order modeling

ABSTRACT

Heterogeneous beam-like structures are commonly represented by equivalent one-dimensional models to allow efficient structural analyses and reduced-order simulations. When the heterogeneity is distributed periodically along the beam axis, however, the determination of consistent effective properties remains challenging, especially for complex geometries and thin-walled configurations modeled with either three-dimensional solid or shell elements. This work presents a finite-element framework for the characterization of periodic straight beams based on a representative axial cell. The approach provides the full effective beam stiffness matrix and the associated inertial quantities, while remaining solver-agnostic and directly applicable to standard finite-element models.

The method is assessed against benchmark periodic structures from the literature, including sandwich beams with periodic cores, thin-walled box beams with and without transverse ribs, and an unsymmetrical corrugated-strip-core sandwich beam providing direct comparisons of transverse-shear and torsional stiffness characteristics. For the thin-walled box-beam configurations, both three-dimensional and shell-based characterizations are considered. The method is assessed against benchmark periodic structures from the literature, including sandwich beams with periodic cores and thin-walled box beams with and without transverse ribs. For the latter class, both three-dimensional and shell-based characterizations are considered.

The framework is then applied to a spar-like embossed C-section representative of lightweight aerostructural layouts. Effective properties obtained from a single periodic cell are used to define an equivalent one-dimensional beam model, and the resulting response is compared against detailed finite-element simulations. A rectangular perforated periodic solid beam provides a complementary free-free modal assessment. The results show that the proposed framework captures the dominant constitutive terms and reproduces the global static response and low-frequency beam-like dynamics of the reference structures, while preserving the flexibility needed for complex periodic geometries.

1. Introduction

Beam-like structures with periodic heterogeneity along the span are encountered in many engineering applications, including sandwich members with repeated cores, ribbed and stiffened thin-walled components, corrugated structural parts, and lightweight aerostructures with recurring internal details. Although these systems are inherently three-dimensional, their global response is often

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<https://doi.org/10.1016/j.finel.2026.104653>

Received 5 June 2026; Received in revised form 23 September 2026; Accepted 23 September 2026

Available online 2 October 2026

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governed by one-dimensional kinematics. This motivates the construction of equivalent beam models that retain the effect of the periodic architecture while allowing efficient structural analyses, design studies, and reduced-order simulations.

The main difficulty is that, for periodic straight beams, the representative entity is no longer an isolated cross section. Geometry, thickness, and internal reinforcement vary along the beam axis, so the characterization must be performed on an axial cell rather than on an isolated section. As a consequence, classical section-only procedures are not directly sufficient, and the reduced constitutive description must account for the three-dimensional nature of the periodic cell from which the beam properties are inferred.

A substantial body of literature has addressed this problem through asymptotic homogenization and computational unit-cell procedures for periodic beam-like structures. The asymptotic foundations for spanwise-periodic beams were established in particular by Buannic and Cartraud [1,2]. Later computational implementations demonstrated that these ideas can be effectively transferred to representative-cell finite-element (FE) workflows [3–5].

Recent developments have established three-dimensional axial-cell formulations capable of identifying fully coupled shear-deformable beam constitutive laws for structures with spanwise-periodic heterogeneity [6–8]. The present work focuses on a matrix-based unit-cell reduction workflow that operates directly on assembled finite-element stiffness and mass matrices, treats solid and conventional shell discretizations within the same handle-based formulation, and transfers both constitutive and actual unit-cell inertial properties to same-length beam elements used in static and low-frequency modal models.

The approach builds on a previously developed matrix-based characterization strategy for beam-like structures [9] and its shell-oriented extension [10], but it is presented here in a self-contained form specialized to the periodic setting. The algebraic ingredients inherited from the previous developments are the matrix-based substructuring and condensation procedure, the handle representation of beam kinematics, and the energy-equivalent identification of stiffness and inertial properties. The related shell extension provides the corresponding treatment of interface translational and rotational degrees of freedom. The specific contribution of the present work is the combined formulation and assessment of these ingredients in the axial-periodic setting. The resulting workflow operates directly on assembled finite-element stiffness and mass matrices, provides a common handle-based treatment of solid and conventional shell interfaces, and transfers both constitutive and actual inertial properties to equivalent beam models for static and low-frequency dynamic analyses. In this setting, the characterization domain is no longer an isolated cross section or a prismatic beam segment, but a finite three-dimensional cell bounded by two end interfaces.

Within this scope, the contributions of the paper are: (i) a two-end, handle-based formulation for extracting the full equivalent beam constitutive matrix from an axially periodic finite-element cell; (ii) a common characterization workflow for solid and shell cell models based only on their assembled stiffness and mass matrices and interface definitions; (iii) the evaluation of the resulting constitutive and inertial properties against literature benchmarks and genuinely spanwise-heterogeneous solid and thin-walled configurations; and (iv) the predictive use of these properties in equivalent one-dimensional static and low-frequency modal models. The numerical assessment also examines the sensitivity of the identified properties to alternative cell cuts and to the number of repeated periods included in the characterization domain.

The scope of the present study is deliberately focused on response regimes that can be described consistently by equivalent one-dimensional beam properties, including quasi-static loading conditions and low-frequency dynamics governed by the first global deformation modes. Other lines of research address high-frequency wave propagation [11], dispersive effects in periodic structures [12], and dynamic homogenization based on frequency-dependent formulations [13,14]; these topics are not considered in the present framework.

Likewise, the issue of cut dependence is discussed from a practical perspective through alternative periodic-cell selections, with emphasis on the observation that the dominant constitutive terms remain close while the more sensitive differences are associated with comparatively small coupling terms.

The remainder of the paper is organized as follows. Section 2 reviews the main lines of research on periodic beam characterization and clarifies the positioning of the present contribution. Section 3 presents the FE characterization framework for periodic straight beams and summarizes the main implementation ingredients inherited from the previously developed substructuring and shell-based formulations. Section 4 introduces the benchmark configurations and the numerical setup. Section 5 discusses validation results against the literature, including shell-versus-solid comparisons. Section 6 presents the spar-like periodic structure and the corresponding effective properties. Section 7 compares the resulting equivalent one-dimensional model with full three-dimensional simulations. Final remarks and perspectives are collected in Section 8.

2. Literature review on periodic beam characterization

The literature on periodic beam-like structures can be organized around three main lines: formal asymptotic homogenization of spanwise-periodic media, computational procedures that turn those ideas into FE workflows on a representative cell, and adjacent reduced-order beam methodologies that help delimit the range of applicability of equivalent one-dimensional models. A first line of work is the asymptotic homogenization of periodic heterogeneous beams. Earlier asymptotic developments for periodic beam-like media can be traced back to Kolpakov [15]. The formulations developed by Buannic and Cartraud [1,2] provide a clear decomposition of the original three-dimensional elastic problem into a microscopic cell problem and a macroscopic beam problem, obtained through separation of scales.

These contributions are foundational because they establish a rigorous route from a spanwise-periodic three-dimensional structure to an equivalent beam constitutive law. They also clarify the role of the periodic cell and of the associated boundary conditions in the interior asymptotic solution. Their main limitation, from a practical implementation viewpoint, is not conceptual but operational: direct use of the formal asymptotic setting can become cumbersome for complex engineering cells and for workflows

built around standard FE models. A second line of work concerns computational homogenization procedures that preserve the spirit of the asymptotic framework while making the characterization directly usable in FE practice. In this respect, the contribution of Cartraud and Messenger [3] is particularly significant, because it provides a coherent formal beam homogenization and representative-cell FE computation. Dizy et al. pushed this engineering perspective further by embedding the homogenization process in a standard Abaqus environment [4]. Their implementation covered the full chain: scripted model generation, enforcement of periodic constraints, application of loading states, and post-processing of the effective properties. Yi et al. likewise emphasized the use of commercial FE software as a black box, showing that periodic-beam homogenization can be carried out on complicated cells without developing ad hoc solvers [5]. The same work also discusses finite-period effects in sandwich structures. Taken together, these studies demonstrate that FE unit-cell homogenization is mature enough for realistic applications, while also revealing recurring implementation issues such as the handling of different discretization technologies, the extraction of the desired constitutive quantities, and the cost of full solid models.

More recent contributions have clarified important aspects of shear-deformable periodic-beam homogenization. Lu et al. [7] showed that conventional periodic boundary conditions applied to a beam RVE may be mechanically inconsistent when nonzero transverse shear is present, because equilibrium requires the corresponding bending moments to vary along the axial direction. They introduced physically consistent boundary conditions and a modified Hill condition that account for this bending-moment redistribution.

Han et al. [6] subsequently proposed a unified equilibrium-based formulation for periodic shear-deformable beams and plates. For beams, the RVE displacement field is represented through rigid-body, constant-strain, linear-strain, and periodic-warping contributions. Kinematic compatibility, interface equilibrium, and energetic equivalence then yield the warping field and a fully coupled 6×6 sectional constitutive matrix. This formulation is particularly close to the present work in its explicit use of equilibrated generalized resultants and the associated affine variation of bending moments under constant transverse shear.

A methodologically distinct route was developed by Liu et al. [8] within the Mechanics of Structure Genome framework. Starting from a three-dimensional periodic structure genome, the method decomposes the original three-dimensional elasticity problem into a microscale structure-genome analysis and a macroscale one-dimensional beam analysis through a variational-asymptotic reduction of the strain energy. Periodicity is imposed on the fluctuating functions at the two end surfaces of the structure genome. The resulting asymptotic strain-energy expression is then transformed into an effective Timoshenko form using the macroscopic beam equilibrium equations, yielding a fully populated 6×6 stiffness matrix. The fluctuating functions are subsequently used to recover local three-dimensional displacement, strain, and stress fields from the macroscopic beam solution. For spanwise heterogeneity, this procedure requires a three-dimensional structure genome rather than the two-dimensional cross-sectional genome appropriate to an axially uniform beam.

The formulations of Han et al. [6] and Liu et al. [8] therefore share important capabilities, including the identification of a fully coupled beam constitutive matrix and the recovery of local three-dimensional fields. Both also employ equilibrium and energetic arguments. Their distinction lies in the primary reduction procedure. Han et al. start from the polynomial RVE deformation modes generated by self-equilibrated generalized resultants, without relying on an asymptotic expansion, and determine the warping and sectional compliance matrices by imposing interface compatibility, local and global equilibrium, and energetic equivalence. Liu et al. instead start from the MSG kinematics and a variational-asymptotic reduction of the three-dimensional strain energy in terms of periodic fluctuating functions; the resulting higher-order energy is subsequently recast in terms of Timoshenko beam strains using the macroscopic equilibrium relations. The distinction is thus methodological rather than associated with the final constitutive matrix or local-field-recovery capabilities.

A related class of implementation-oriented numerical homogenization procedures operates directly on finite-element stiffness matrices and identifies reduced structural properties through strain-energy equivalence. Staszak et al. [16] condensed shell models of thin-walled beams with periodic perforations to obtain equivalent beam stiffness characteristics, and subsequently applied the same general approach to quantify the stiffness reduction caused by local geometric imperfections [17]. Closely related matrix-based procedures have also been used to identify membrane, bending, and transverse-shear stiffnesses of corrugated-core sandwich panels [18], and to replace mixed solid-truss models of prefabricated composite slabs by equivalent shell models [19]. Although the latter applications target plate or shell reductions rather than one-dimensional beams, they illustrate the broader use of condensed finite-element matrices and energetic equivalence for effective-property identification in geometrically complex structural members.

A third, related line includes beam-reduction methodologies that are not focused specifically on spanwise periodicity, but are still relevant because they define alternative ways of deriving one-dimensional models from higher-dimensional descriptions. Variational asymptotic beam formulations and the associated VABS framework perform a two-dimensional analysis on the beam cross section, coupled with a one-dimensional macroscopic beam problem. They have been highly successful for anisotropic and composite members that can be characterized section by section [20], but are conceptually distinct from three-dimensional axial-cell methods required when geometry, material distribution, or reinforcement changes within a period. Extensions of variational-asymptotic reduction toward spanwise-varying configurations have been developed through beam-wise and three-dimensional structure-genome formulations [8,21].

Closely related high-fidelity strategies based on fundamental-state or homogenization-inspired beam descriptions have also been proposed, for example, by Kennedy and Martins [22]. Black-box implementation-oriented strategies have likewise been developed in the broader homogenization setting in order to leverage commercial solvers directly [23]. Other families, such as state-space, transfer-matrix, or symplectic approaches, are particularly informative for finite periodic structures and for the interpretation of Saint-Venant-type solutions [24,25]. Finally, enriched continua have been proposed for lattice-like periodic beams when additional

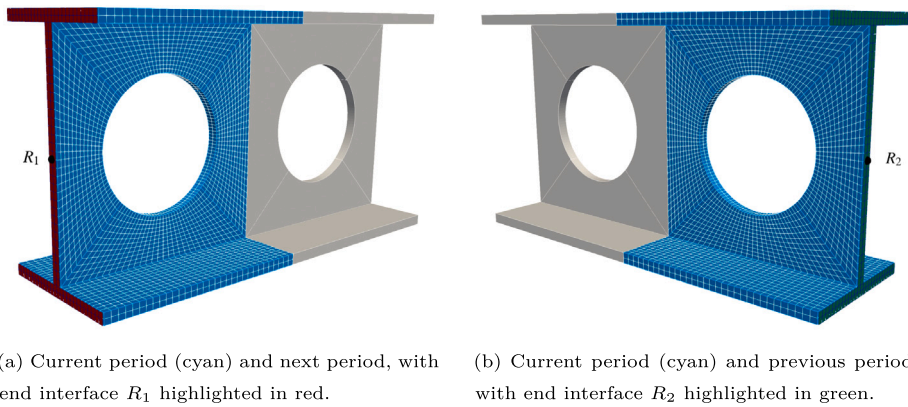


Fig. 1. Definition of the finite-element unit cell and its end interfaces. The current period, selected as the unit cell, is highlighted in cyan. Interface R_1 is highlighted in red in panel (a), which also shows the next period, while interface R_2 is highlighted in green in panel (b), which also shows the previous period.

generalized transmission mechanisms must be represented explicitly [26]. Although the present study does not exactly address the same problem, these contributions are still relevant because they provide important methodological context.

Viewed together, the existing literature shows that both three-dimensional periodic-cell analysis and the identification of fully coupled beam constitutive matrices are established capabilities.

The relationship with the equilibrium-based formulation of Han et al. [6] can be stated more precisely. Both formulations enforce the beam-equilibrium relation between constant transverse shear forces and affine bending moments and eliminate the internal finite-element variables by static condensation. Thereafter, however, the reductions are organized differently. Han et al. determine the periodic nodal warping and sectional compliance matrices through coupled macro-meso compatibility, equilibrium, and energetic-equivalence equations. Their finite-element development assumes solid or solid-shell nodes carrying three translational degrees of freedom. The present implementation instead decomposes each of the two end interfaces into six rigid-handle variables and complementary deformation modes, condenses the latter, and identifies the equivalent compliance from the responses to six beam-equilibrated generalized load states. This handle construction also accommodates conventional shell nodes carrying both translational and rotational degrees of freedom. Moreover, the assembled mass matrix is reduced within the same framework to identify the actual unit-cell inertial properties. The resulting constitutive and inertial quantities are transferred to same-length beam elements and assessed not only in static analyses but also in low-frequency dynamic problems through direct comparisons of natural frequencies and mode shapes. Shell-element handling is thus one component of a broader stiffness-and-inertia characterization and reduced-model validation workflow.

Against this background, the present study is positioned as a unit-cell-based, implementation-oriented reduction procedure. The assembled stiffness and mass matrices of a physical periodic unit cell are reduced to two beam handles, and the unit cell is replaced by an equivalent beam element of the same length. Constitutive and actual inertial properties of the unit cell are obtained within one workflow, using either solid elements or conventional shell elements with translational and rotational interface degrees of freedom. The resulting properties are then assessed directly in finite static and low-frequency modal models. In the recent formulations discussed above, the reported applications focus primarily on stiffness homogenization, static response, and local-field recovery rather than on this combined unit-cell inertial characterization and modal validation.

3. Finite-element characterization of periodic straight beams

3.1. Finite-element unit cell and model partition

The beam characterization is performed on a finite-element unit cell spanning one complete geometric period, whose axial length L_c coincides with the adopted period of the structure. The unit cell is discretized using an FE model and is bounded by two cross-sectional end interfaces orthogonal to the beam axis and consistent with the periodic pattern of the beam, and thus identical in geometry, structural properties, and discretization. This also allows the handle kinematics at the two ends to be defined through the same local construction, once the interface degrees of freedom are ordered consistently. The term *characterization domain* is used below for the more general finite-element domain containing one or more consecutive periods; in the standard case considered here, it coincides with the unit cell. As shown in Fig. 1, the sets of interface degrees of freedom associated with these two end interfaces are denoted by R_1 and R_2 , consistently with the notation adopted in [9]. In the present context, R_1 and R_2 play a double role. On the one hand, they are the two handles through which the macroscopic beam states are imposed on the FE model of the unit cell. On the other hand, their associated reference points define the endpoints of the equivalent one-dimensional beam element corresponding to the characterized period.

After assembly, the FE degrees of freedom of the unit cell are partitioned into interface and internal subsets. Collecting the variables associated with the two handles into a single interface vector, the global displacement vector is written as

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_R \\ \mathbf{u}_L \end{bmatrix}, \quad \mathbf{u}_R = \begin{bmatrix} \mathbf{u}_{R_1} \\ \mathbf{u}_{R_2} \end{bmatrix}, \quad (1)$$

where $\mathbf{u}_{R_1}$ and $\mathbf{u}_{R_2}$ collect the interface degrees of freedom on the two end sections, while $\mathbf{u}_L$ contains all remaining internal degrees of freedom of the unit cell. With the same partition, the linear elastic FE model is described by

$$\begin{bmatrix} \mathbf{K}_{RR} & \mathbf{K}_{RL} \\ \mathbf{K}_{LR} & \mathbf{K}_{LL} \end{bmatrix} \begin{bmatrix} \mathbf{u}_R \\ \mathbf{u}_L \end{bmatrix} = \begin{bmatrix} \mathbf{f}_R \\ \mathbf{0} \end{bmatrix}, \quad (2)$$

where no external loads are applied to the internal set in the characterization problems. Statically eliminating the internal degrees of freedom yields the reduced interface stiffness matrix

$$\mathbf{K}_{BB} = \mathbf{K}_{RR} - \mathbf{K}_{RL} \mathbf{K}_{LL}^{-1} \mathbf{K}_{LR}. \quad (3)$$

In the practical implementation, the same reduced interface operators can also be obtained within a component mode synthesis framework, retaining the interface variables explicitly. This corresponds to the classical fixed-interface component mode synthesis setting introduced by Craig and Bampton [27], which builds on the broader component-mode perspective already established by Hurty [28]. In the following, $\mathbf{K}_{BB}$ denotes the reduced stiffness associated with the retained interface description of the periodic unit cell. From this viewpoint, the present formulation combines the interface-centered characterization developed in the previous work with a standard substructuring reduction framework [9,27].

In practical use, the solver-dependent part of the workflow is limited to the generation of the finite-element input data. The user defines the mesh and material properties of the periodic unit cell, identifies the nodes and active degrees of freedom belonging to the two end sections R_1 and R_2 , and exports the assembled stiffness and mass matrices, together with the mesh information required to associate their rows and columns with the finite-element degrees of freedom. The native matrix format is solver-specific and is handled through an import step. All subsequent operations, including matrix partitioning, condensation, construction of the handle kinematics, and extraction of the effective stiffness and inertial properties, are independent of the finite-element solver. The workflow has been exercised with Abaqus, CalculiX, and Code Aster; other solvers, such as Nastran, can be used provided that the assembled matrices and the corresponding degree-of-freedom and interface mappings are accessible.

After the finite-element data have been imported, the characterization follows the matrix-based sequence developed in [9] and extended to shell discretizations in [10]: the degrees of freedom are partitioned into internal and end-interface sets, the internal and complementary interface variables are condensed, the rigid handle kinematics are introduced, and the equivalent constitutive and inertial properties are extracted. The present work applies this sequence to the two end interfaces of an axial periodic cell; the cell-specific definitions and equations required for this application are given in this section.

3.2. Handle kinematics and generalized beam variables

The interface motion of the periodic unit cell is described by introducing one reference point for each handle, namely one point associated with R_1 and one with R_2 . In the equivalent one-dimensional description, these two points are the start and end nodes of the beam element associated with the characterized unit cell. Let

$$\mathbf{q}_{R_i} \in \mathbb{R}^6, \quad i = 1, 2, \quad (4)$$

denote the vector collecting the generalized rigid variables of handle R_i , namely the three translations and three infinitesimal rotations of its reference point. The displacement field on each handle is decomposed into a beam-consistent rigid contribution and a complementary deformation field accounting for local warping,

$$\mathbf{u}_{R_i} = \mathbf{H}_i \mathbf{q}_{R_i} + \mathbf{C}_i \mathbf{q}_{d_i}, \quad i = 1, 2, \quad (5)$$

where matrix $\mathbf{H}_i$ collects the rigid-body motion shapes associated with the reference point of handle R_i , vector $\mathbf{q}_{d_i}$ contains the amplitudes of the complementary interface deformation modes, and matrix $\mathbf{C}_i$ spans the corresponding deformation space, with $\mathbf{H}_i^T \mathbf{C}_i \equiv \mathbf{0}$ and $[\mathbf{H}_i \ \mathbf{C}_i]$ square and full rank. Since the two end sections of the periodic unit cell are identical in geometry, structural properties, and discretization, and since the reference points $\mathbf{p}_{1_0}$ and $\mathbf{p}_{2_0}$ are chosen at the same relative location of the corresponding sections, the same local construction can be adopted on both handles; after consistent ordering of the interface degrees of freedom, one therefore has $\mathbf{H}_1 \equiv \mathbf{H}_2$ and, accordingly, $\mathbf{C}_1 \equiv \mathbf{C}_2$. In practice, $\mathbf{C}_i$ is obtained as an orthonormal basis of the null space of $\mathbf{H}_i^T$, for instance from a QR factorization or from the singular value decomposition of $\mathbf{H}_i$.

Stacking the two handles gives

$$\mathbf{u}_R = \underbrace{\begin{bmatrix} \mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \mathbf{q}_{R_1} \\ \mathbf{q}_{R_2} \end{bmatrix}}_{\mathbf{q}_b} + \underbrace{\begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} \mathbf{q}_{d_1} \\ \mathbf{q}_{d_2} \end{bmatrix}}_{\mathbf{q}_d}. \quad (6)$$

This decomposition is common to both solid and shell discretizations. For completeness, the shell-specific interface kinematics required by the present formulation are summarized below, including the treatment of nodal rotations and the corresponding rigid interpolation blocks. A detailed derivation and validation of the shell-oriented characterization procedure are provided in [10].

The main differences between the two model classes are reflected in the construction of the rigid interpolation matrices $\mathbf{H}_i$, in the definition of the admissible complementary spaces $\mathbf{C}_i$, and in the resulting warping content of the interface motion, which in the shell-based case also accounts for nodal rotational degrees of freedom of the FE model. The overall identification procedure, however, remains unchanged.

For interface nodes carrying displacements only, as in a standard solid discretization, the node-level block of the rigid interpolation associated with node j on handle R_i is

$$\mathbf{H}_{i_j}^{(\text{solid})} = \begin{bmatrix} \mathbf{I} & -(\mathbf{p}_{i_j} - \mathbf{p}_{i_0}) \times \end{bmatrix}, \quad \mathbf{H}_{i_j}^{(\text{solid})} \in \mathbb{R}^{3 \times 6}, \quad (7)$$

where $\mathbf{p}_{i_0}$ is the reference point of the handle and $\mathbf{p}_{i_j}$ is the position of node j on the same interface section, with $\mathbf{p}_{i_j} - \mathbf{p}_{i_0} = [0 \ y_j \ z_j]^T$ when the beam axis is parallel to axis x_1 .

For shell interface nodes, which carry both translational and rotational variables, the corresponding block also accounts for nodal rotations in the last block row, becoming

$$\mathbf{H}_{i_j}^{(\text{shell})} = \begin{bmatrix} \mathbf{I} & -(\mathbf{p}_{i_j} - \mathbf{p}_{i_0}) \times \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{H}_{i_j}^{(\text{shell})} \in \mathbb{R}^{6 \times 6}. \quad (8)$$

The generalized variables of the equivalent beam element can be described either in nodal form, through the end variables

$$\mathbf{q}_b = \begin{bmatrix} \mathbf{q}_{R_1} \\ \mathbf{q}_{R_2} \end{bmatrix}, \quad (9)$$

or in constitutive form through the generalized strain vector

$$\mathbf{e} = [\epsilon_1 \quad \gamma_2 \quad \gamma_3 \quad \kappa_1 \quad \kappa_2 \quad \kappa_3]^T. \quad (10)$$

For a unit cell of length L_c , the generalized strain state defines the relative beam-consistent motion between the two handles. The rigid contribution in Eq. (6) represents this beam motion, while the complementary term accounts for the local warping required by the detailed FE model to comply with local equilibrium and compatibility.

3.3. Equilibrated generalized load states and compliance identification

The present procedure does not impose conventional nodal periodic boundary conditions between the two end sections. Instead, it formulates a cell-level static equivalence in which the finite-element unit cell is replaced by an equivalent beam segment of the same length. Internal equilibrium and compatibility of the discretized continuum are retained through the assembled finite-element equations and the condensation of the internal degrees of freedom. The two end sections are represented by the handles R_1 and R_2 , which carry the generalized beam motions and their work-conjugate forces and moments. The complementary interface deformation modes are determined from the reduced finite-element equilibrium problem, rather than prescribed through an independently selected rigid or weighted interface constraint. The generalized resultants at the two handles are related by beam equilibrium along the unit cell, as detailed below; in particular, a transverse shear force is accompanied by the corresponding affine variation of the bending moment. The complete derivation of this finite-domain equivalence is provided in [9].

The choice of this equilibrium-based end-resultant description is dictated by mechanical consistency with the target beam model. At the local level, the assembled three-dimensional finite-element equations enforce nodal equilibrium and displacement compatibility, which are retained through condensation. At the global level, the force and moment resultants transmitted across the end interfaces are required to satisfy beam equilibrium. The corresponding beam-level kinematic compatibility is intrinsic to the handle representation: the translations and rotations of the reference points describe the motion of the beam reference line and the orientation of its end sections, while the complementary interface variables account for local warping.

Let the generalized internal resultants at a reference station be defined as

$$\mathbf{g}_{R_{\text{ref}}} = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} \in \mathbb{R}^6, \quad (11)$$

where $\mathbf{f}$ and $\mathbf{m}$ are the force and moment vectors, respectively. In the absence of distributed loads, global equilibrium implies a linear (affine) variation of these resultants along the unit-cell axis,

$$\mathbf{g}_R(\xi) = [\mathbf{I} - (\xi - \xi_{\text{ref}})\mathbf{T}] \mathbf{g}_{R_{\text{ref}}}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{e}_1 \times & \mathbf{0} \end{bmatrix}, \quad (12)$$

where $\mathbf{e}_1$ is the unit vector along the beam axis (axis 1, without loss of generality). The reduced unit-cell problem is then posed for multiple axial placements with respect to the same reference station, enforcing compatibility and the same equilibrated resultant set. Solving the resulting linear system for six independent unit generalized load states yields a linear mapping between generalized resultants and reduced interface displacements,

$$\mathbf{u}_R = \mathbf{X} \mathbf{g}_{R_{\text{ref}}}, \quad (13)$$

where the columns of $\mathbf{X}$ are obtained from the interface responses to the six unit equilibrated load cases. This step follows the same compliance-identification logic previously introduced for uniform beam-section characterization [9,10] and here transferred from an isolated cross section to a periodic unit cell.

The virtual strain work of the reduced unit cell can then be written as

$$\delta \mathcal{W}_s = \delta \mathbf{g}_{R_{\text{ref}}}^T (\mathbf{X}^T \mathbf{K}_{BB} \mathbf{X}) \mathbf{g}_{R_{\text{ref}}}, \quad (14)$$

which defines the equivalent compliance matrix

$$\mathbf{G} = \mathbf{X}^T \mathbf{K}_{BB} \mathbf{X}, \quad \mathbf{D} = \mathbf{G}^{-1}. \quad (15)$$

The matrix $\mathbf{D}$ is the constitutive matrix of the equivalent beam associated with the periodic unit cell. It collects the axial, shear, bending, and torsional stiffnesses, together with any coupling terms induced by the actual geometry and material distribution of the periodic cell.

The same algebraic framework is applied independently of whether the unit cell is discretized by three-dimensional solids or by shell elements. Once the reduced interface operator and the handle kinematics are defined, the identification of the equivalent constitutive matrix follows exactly the same sequence of steps. The inertial characterization is built on the same reduced description and is discussed separately in the following section.

3.4. Identification of inertial properties

The inertial characterization is carried out from the reduced boundary mass matrix $\mathbf{M}_{BB}$ in the same reference frame used for stiffness identification. Since the reference points of the two handles are also the end points of the equivalent beam element, the reduced inertial description of the unit cell can be recast directly into the standard one-dimensional inertial quantities associated with the chosen beam reference line. In particular, the procedure provides the mass per unit length μ , the position of the section center of gravity with respect to the selected reference point, and the inertia matrix per unit length referred to the same point.

Following the same handle-based description adopted for stiffness identification, the rigid-body inertia of the reduced unit cell is first computed as

$$\mathbf{M}_{\text{rb}} = \mathbf{Z}_{\text{rb}}^T \mathbf{M}_{BB} \mathbf{Z}_{\text{rb}}, \quad (16)$$

where $\mathbf{Z}_{\text{rb}}$ maps the six rigid-body linear and angular velocity components of a reference end section to the corresponding rigid-body motion of both end sections of the unit cell. In other words, the whole periodic unit cell is assumed to move as a rigid body, and the reduced mass matrix is projected onto the corresponding interface velocity field. Equating $\mathbf{M}_{\text{rb}}$ to the rigid-body inertia of an equivalent beam element of length L_c yields the inertial quantities per unit length [9]:

$$\mu = \frac{1}{L_c} \mathbf{M}_{\text{rb}}(1, 1), \quad (17a)$$

$$\mathbf{S} = -\mu \frac{L_c}{2} \mathbf{e}_1 - \frac{1}{L_c} \text{ax}(\mathbf{M}_{\text{rb}}(1:3, 4:6)), \quad (17b)$$

$$\mathbf{J} = \frac{1}{L_c} \mathbf{M}_{\text{rb}}(4:6, 4:6) - \mu \frac{L_c^2}{3} \mathbf{e}_1^{\text{xT}} \mathbf{e}_1^{\text{x}} + \frac{L_c}{2} (\mathbf{S}^{\text{xT}} \mathbf{e}_1^{\text{x}} + \mathbf{e}_1^{\text{xT}} \mathbf{S}^{\text{x}}), \quad (17c)$$

where $\mathbf{e}_1$ is the unit vector along the beam axis, $\mathbf{S}$ is the first moment vector identifying the position of the section center of gravity with respect to the chosen reference point, $\mathbf{J}$ is the inertia matrix per unit length referred to the same point, and $\text{ax}(\cdot)$ denotes the axial-vector operator associated with a skew-symmetric matrix.

These expressions are fully consistent with the handle-based kinematic description and therefore yield inertial properties referred to the same beam reference line used for the constitutive matrix $\mathbf{D}$. As in the stiffness identification, the procedure is independent of whether the periodic unit cell is discretized by three-dimensional solid elements or by shell elements, provided that the reduced mass operator and the handle kinematics are consistently defined. The detailed derivation follows the matrix-based formulation introduced in the previous characterization work [9] and later extended to shell models [10].

4. Benchmark configurations and numerical setup

The proposed characterization procedure is assessed on three benchmark families drawn from the literature on periodic beams. The first comprises sandwich beams with periodic cellular cores, for which the literature primarily reports equivalent bending stiffnesses. The second is a thin-walled prismatic box beam with and without transverse ribs, which also permits a direct comparison between solid and shell discretizations. The third is an unsymmetrical corrugated-strip-core sandwich beam for which published reference results include all six diagonal constitutive coefficients and a nonzero torsion-bending coupling term, thereby providing direct validation of transverse-shear and torsional stiffnesses. Together, these benchmarks progressively extend the validation from dominant bending properties to multiple principal stiffnesses and a coupled constitutive response.

It is useful to distinguish the prismatic reference configuration from the genuinely spanwise-periodic cases considered in the numerical assessment. Only the box beam without ribs is uniform along x_1 and can be represented directly by a single conventional cross-sectional model. The perforated sandwich beams, the ribbed box beam, the unsymmetrical corrugated-strip-core sandwich beam, the embossed spar, and the rectangular perforated solid beam instead contain geometric features that occupy only part of each axial period. Their cross-sectional geometry therefore varies with x_1 , and no single fixed two-dimensional section represents the complete periodic cell. A conventional cross-sectional analysis may describe individual axial stations or a separately homogenized

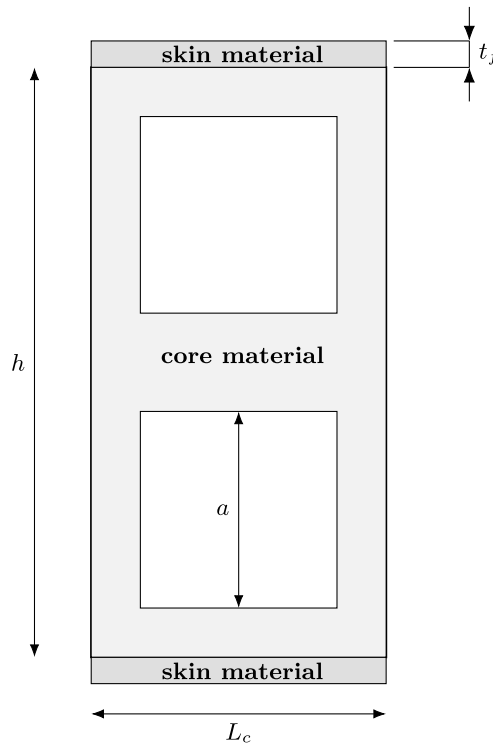


Fig. 2. Longitudinal x_1 - x_3 view of one axial period of the sandwich beam with square perforations. The perforated core geometry varies along x_1 .

section, but it does not directly account for the local axial transitions and load-transfer mechanisms contained within the full periodic cell.

In all cases, the benchmark geometry and input data are summarized by the figures and tables collected in this section, while the comparison of the identified equivalent properties with literature data is deferred to Section 5. Unless otherwise stated, each benchmark is characterized by considering a unit cell whose axial length L_c coincides with the structural period, consistently with the methodology introduced in Section 3. The resulting equivalent beam properties are then compared with the values reported in the corresponding reference studies.

The quantities shown in the following comparison tables are those for which independent reference values are available in the cited studies. The complete constitutive matrices identified for all benchmark cases are reported in Appendix A, together with the three-dimensional unit-cell mass properties of the symmetric sandwich- and box-beam benchmarks. The constitutive matrices of these doubly symmetric cases are diagonal, whereas the complete matrix of the unsymmetrical corrugated-strip-core benchmark also contains symmetry-admissible coupling terms.

4.1. Sandwich beams with periodic perforated cores

The first benchmark family consists of sandwich beams with periodic cores containing through-thickness perforations. These cases are taken from the study by Yi et al. who used them to assess the effective bending stiffness of periodic sandwich beams with different core morphologies and to discuss thickness-related size effects associated with the periodic cell [5]. In the present study, only the square-hole and circular-hole configurations are retained, as they provide two representative and compact benchmark cases for the validation of the proposed periodic-beam characterization.

For both configurations, the beam is periodic along the axial direction, with identical face-sheet material properties and a weaker core material. The adopted geometric and material parameters are reported in Tables 1 and 2, while the corresponding unit-cell geometries are shown in Figs. 2 and 3. In contrast with approaches that first homogenize the core material separately and then apply an ordinary beam model, the present characterization is performed directly on the complete sandwich unit cell, so that the influence of the detailed redistribution of material within the thickness is retained, consistently with the benchmark setting discussed by Yi et al. [5].

In these two cases, the main quantity of interest is the effective bending stiffness associated with the periodic beam. This makes the sandwich benchmarks particularly useful as an initial validation step: although they do not exercise the full constitutive matrix as extensively as the later box-beam case, they test the ability of the proposed procedure to recover the dominant equivalent bending

Table 1

Geometrical and material data for the sandwich beam with square holes (unit cell). Symbols refer to Fig. 2.

Quantity	Symbol	Value
Period length	L_c	1.5 m
Cell width	b	1.5 m
Cell height	h	3.0 m
Face-sheet thickness	t_f	0.1 m
Square-hole side length	a	1.0 m
Young's modulus (faces)	E_f	70 GPa
Poisson ratio (faces)	ν_f	0.34
Young's modulus (core)	E_c	3.5 GPa
Poisson ratio (core)	ν_c	0.34

Geometry and material properties follow the definitions in [5].

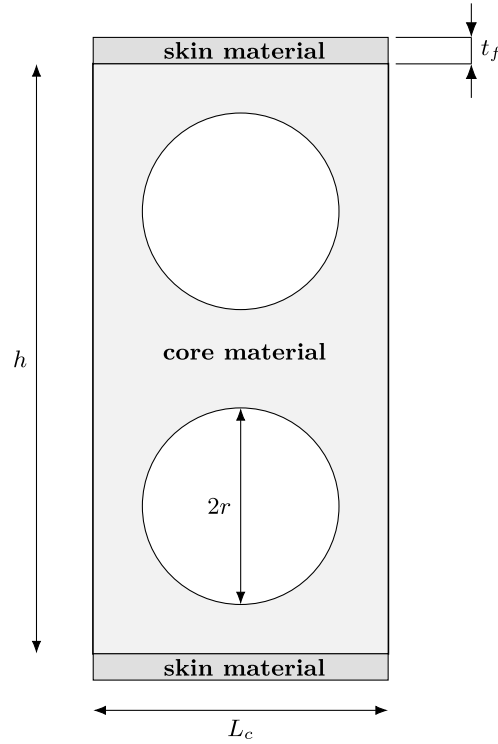


Fig. 3. Longitudinal x_1 - x_3 view of one axial period of the sandwich beam with circular perforations. The perforated core geometry varies along x_1 .

Table 2

Geometrical and material data for the sandwich beam with circular holes (unit cell). Symbols refer to Fig. 3.

Quantity	Symbol	Value
Period length	L_c	1.5 m
Cell width	b	1.5 m
Cell height	h	3.0 m
Face-sheet thickness	t_f	0.1 m
Hole radius	r	0.5614 m
Young's modulus (faces)	E_f	70 GPa
Poisson ratio (faces)	ν_f	0.34
Young's modulus (core)	E_c	3.5 GPa
Poisson ratio (core)	ν_c	0.34

Geometry and material properties follow the definitions in [5].

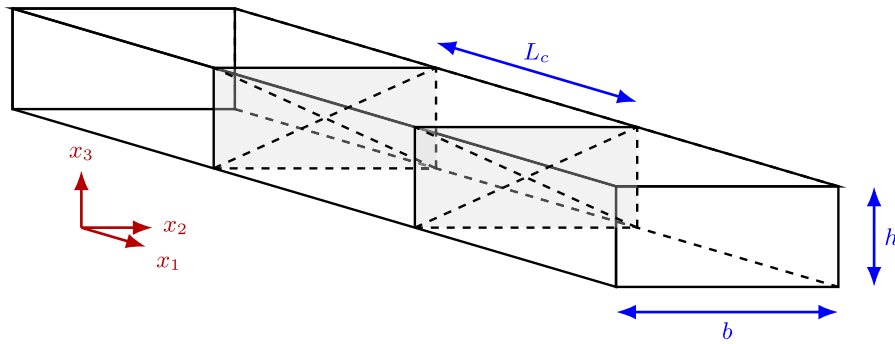


Fig. 4. Schematic of a periodic prismatic beam with ribs. The unit cell (period) has length L_c and the cross-section dimensions are b and h .

Table 3

Geometrical and material data for the prismatic box-beam unit cell.

Quantity	Symbol	Value
Period length	L_c	1.0 m
Box outer width	b	2.0 m
Box outer height	h	1.0 m
Wall/rib thickness	t	0.025 m
Young's modulus	E	70.0 GPa
Poisson ratio	ν	0.3

Geometry follows [4,5].

response of a genuinely periodic microstructured beam. The comparison with literature values is presented in Section 5, where the identified stiffness terms are discussed against the results reported by Yi et al. and by the reference methods considered therein [5].

4.2. Prismatic thin-walled box beam with and without ribs

The second benchmark family comprises two configurations: a prismatic thin-walled box beam without ribs and the corresponding beam with periodic transverse ribs. This case is especially relevant to the present study because it combines a simple and well-documented periodic geometry with a structural layout that is naturally suited to both solid and shell discretizations. Dzy et al. used it as a benchmark for homogenization of slender periodic structures [4]; Yi et al. later studied it in a form that explicitly compares solid and shell FE implementations [5]. Therefore, the benchmark offers a direct opportunity to assess not only the accuracy of the identified equivalent properties but also the computational trade-off between solid and shell models within the proposed framework.

The geometry of the box beam is shown in Fig. 4, while the corresponding geometric and material data are summarized in Table 3. The benchmark is considered in two variants: a box beam without transverse ribs and the corresponding ribbed configuration, in which the rib spacing defines the period of the structure. The material is homogeneous and isotropic, and the walls are sufficiently thin that shell discretizations are expected to provide an efficient alternative to solid models, especially when periodic unit cells are used in a repeated characterization workflow [5]. The same benchmark is therefore well suited to illustrate one of the main practical motivations of the present study, namely the possibility of applying exactly the same handle-based algebraic identification procedure to periodic unit cells modeled either by three-dimensional solids or by shells.

The box beam without ribs is intentionally retained as a prismatic baseline that can also be analyzed using a conventional two-dimensional cross-sectional method. In the ribbed configuration, however, the transverse wall is present only at discrete axial locations, whereas the sections between successive ribs contain only the outer box walls. Consequently, no single fixed cross section represents the complete repeated structure. The axial-cell model retains both the transverse rib and the intervening wall panels, together with their combined contribution to the equivalent beam properties.

For the solid reference models, the periodic unit cell is discretized by three-dimensional FEs. For the shell characterization, two classical shell formulations are considered, namely a discretization based on Kirchhoff–Love elements and a discretization based on Mindlin–Reissner elements. This choice mirrors the shell-based characterization framework and allows us to assess, within the same periodic benchmark, how the identified constitutive properties depend on the underlying thin-walled modeling assumptions. Since the literature already indicates that shell models can reproduce the dominant stiffness terms of this box-beam benchmark with significantly fewer degrees of freedom than solid models [5], this case is also used here to discuss the balance between computational efficiency and constitutive accuracy.

The box-beam benchmark is the only one among the validation cases for which both solid and shell characterizations are carried out systematically in the present study. For this reason, it plays a double role in the paper. First, it serves as a literature benchmark for the validation of the identified equivalent constitutive properties against established reference values [4,5]. Second, it acts as a bridge

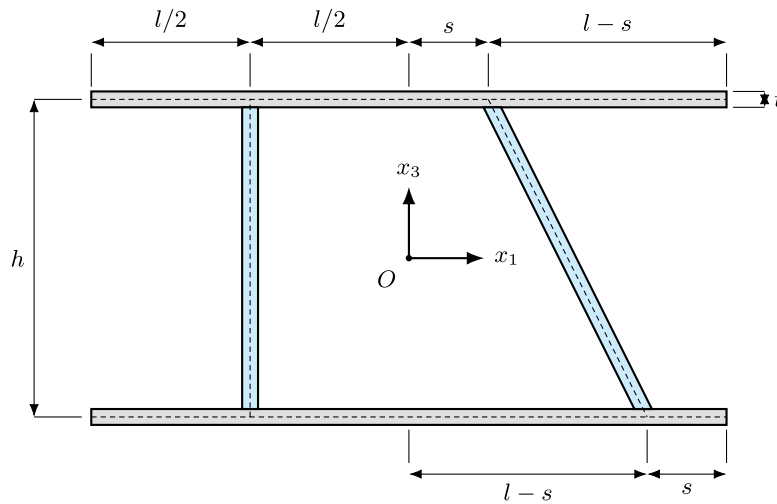


Fig. 5. Longitudinal x_1 - x_3 view of the unsymmetrical corrugated-strip-core sandwich-beam unit cell. The distance h is measured between the face-sheet mid-surfaces. The cell is uniform along x_2 , with width b , and the thickness is slightly exaggerated for clarity.

Table 4

Geometrical and material data for the unsymmetrical corrugated-strip-core sandwich-beam unit cell. Symbols refer to Fig. 5.

Quantity	Symbol	Value
Period length	$L_c = 2l$	4.0 m
Half-period parameter	l	2.0 m
Corrugation offset	s	0.5 m
Width along x_2	b	2.0 m
Face-sheet mid-surface spacing	h	2.0 m
Common face-sheet and web thickness	t	0.1 m
Young's modulus	E	206 GPa
Poisson ratio	ν	0.3

Geometry and material properties follow [8,29]. The dimensions of the corrugated core are defined with respect to the structural mid-surfaces.

between the previously developed solid-based and shell-based characterization procedures, showing that the same unit-cell-based framework can be consistently applied to both model classes.

4.3. Unsymmetrical corrugated-strip-core sandwich beam

The third benchmark family consists of the unsymmetrical corrugated-strip-core sandwich beam originally considered by Xu et al. [29] and subsequently analyzed by Liu et al. using the Mechanics of Structure Genome formulation [8]. This case complements the preceding benchmarks because the published reference data comprise all six diagonal constitutive coefficients and the nonzero coupling coefficient C_{46}^b , thus providing direct comparisons for transverse shear, torsion, extension, and bending.

The longitudinal geometry of the unit cell is shown in Fig. 5, and the corresponding dimensions and material properties are summarized in Table 4. The cell extends from $x_1 = -l$ to $x_1 = +l$, with the beam reference line passing through the central point O . The geometry is uniform along x_2 .

The baseline finite-element model is discretized using 12008 quadratic brick elements. In this model, the two end interfaces R_1 and R_2 contain 1980 interface degrees of freedom in total. A refined model containing 21280 quadratic brick elements is also considered to provide a limited two-level mesh-refinement check. Its size is close to that of the MSG model reported by Liu et al. [8], which comprised 21120 quadratic hexahedral elements. The baseline results are retained in the benchmark comparison, while the effect of mesh refinement is discussed in Section 5.4.

4.4. Numerical models and reported quantities

All benchmark cases are characterized through the methodology of Section 3, using one periodic unit cell per analysis. For each unit cell, the FE model is partitioned into the two interface handles and the internal degrees of freedom, reduced by means of the Craig-Bampton procedure, and post-processed to identify the equivalent beam constitutive matrix. The resulting quantities

Table 5

Benchmark of the effective bending stiffness coefficient for the sandwich beam with square holes. Reference values (Analytical, VABS, NIAH) are reported in [5]. Present results are obtained from one-period characterization using 3D solids with linear and quadratic brick elements. Values are reported in 10^{10} Pa, consistently with [5].

Source	$K_{\text{bend}} / 10^{10}$ Pa
Analytical	5.883
VABS	5.576
NIAH	5.576
Present (linear bricks)	5.577
Present (quadratic bricks)	5.575

Table 6

Benchmark of the effective bending stiffness coefficient for the sandwich beam with circular holes. Reference values (Analytical, VABS, NIAH) are reported in [5]. Present results are obtained from one-period characterization using 3D solids with linear and quadratic brick elements. Values are reported in 10^{10} Pa, consistently with [5].

Source	$K_{\text{bend}} / 10^{10}$ Pa
Analytical	5.671
VABS	5.537
NIAH	5.554
Present (linear bricks)	5.560
Present (quadratic bricks)	5.553

are reported in the same notation used by the corresponding reference studies, so that direct comparisons can be made without ambiguity.

For the sandwich beams, the comparison focuses on the dominant bending stiffness reported in the literature benchmark. For the thin-walled box beam, the comparison includes the main axial, torsional, and bending stiffness terms, both in the configurations with and without ribs. For the unsymmetrical corrugated-strip-core beam, the comparison covers all six diagonal coefficients and the nonzero torsion–bending coupling coefficient D_{46} reported in the reference studies, thereby directly assessing both transverse-shear stiffnesses and the torsional stiffness.

Representative cell and interface dimensions are reported for the periodic models. The reduction in algebraic model size achieved in the finite structural analyses, together with the distinction between the one-time characterization and the recurring one-dimensional analyses, is discussed in Section 7.5.

The corresponding numerical comparisons are presented in Section 5.

Whenever literature values are taken from references that use slightly different notational conventions, the comparison is performed after consistent identification of the corresponding generalized beam quantities. In this way, the benchmark section remains focused on the actual structural configurations and numerical models, while the next section can concentrate on the validation of the extracted equivalent properties.

5. Validation against literature benchmarks

This section compares the equivalent properties identified by the proposed procedure with reference results available in the literature for the benchmark configurations introduced in Section 4. The discussion is organized according to the three benchmark families. First, the sandwich beams with periodic perforated cores are used to validate the recovery of the dominant bending stiffness. Second, the thin-walled prismatic box beam is used to assess the identification of the main axial, bending, and torsional terms, with particular attention to the comparison between solid and shell discretizations. Finally, the unsymmetrical corrugated-strip-core sandwich beam extends the validation to both transverse-shear stiffnesses, torsion, and a nonzero torsion–bending coupling coefficient. The corresponding benchmark values are taken from [4,5]; the corrugated-beam reference values are reported in [8,29].

In addition to the benchmark terms directly comparable with the literature, the complete constitutive matrices identified for the validation cases are reported in Appendix A, so as to document the full output of the proposed procedure and facilitate future comparisons.

5.1. Sandwich beams with square and circular perforated cores

The results obtained for the sandwich beams with square-hole and circular-hole cores are reported in Tables 5 and 6. In both cases, the comparison is focused on the effective bending stiffness, consistently with the reference data reported by Yi et al. [5]. These benchmarks are especially useful because they test the capability of the method to represent periodic heterogeneity distributed

Table 7

Benchmark of effective stiffness coefficients for the prismatic box beam without ribs. Values are taken from [5] and [4] (as reported therein).

	Analytical	FE	VABS	Dizy	NIAH (Yi)	NIAH shell	CB-ANBA shell	CB-ANBA 3D
EA/GN	10.3	10.3	10.3	10.3	10.3	10.3	10.3	10.3
$GJ/(GN\text{ m}^2)$	1.79	1.71	1.71	1.71	1.71	1.72	1.69	1.71
$EI_{33}/(GN\text{ m}^2)$	1.91	1.91	1.91	1.91	1.91	1.91	1.91	1.91
$EI_{22}/(GN\text{ m}^2)$	5.58	5.58	5.58	5.58	5.58	5.57	5.58	5.58

Note: The literature values are reported with the same precision as in the cited sources. For the axial stiffness, the underlying analytical value is $EA = 10.325\text{ GN}$; both CB-ANBA shell and CB-ANBA 3D recover this value, while the reference columns report the rounded value 10.3.

Table 8

Benchmark of effective stiffness coefficients for the prismatic box beam with ribs. Values are taken from [5] and [4] (as reported therein).

	FE	Dizy	NIAH (Yi)	NIAH shell	CB-ANBA shell	CB-ANBA 3D
EA/GN	10.5	10.5	10.5	10.4	10.5	10.5
$GJ/(GN\text{ m}^2)$	1.72	1.72	1.72	1.72	1.69	1.72
$EI_{33}/(GN\text{ m}^2)$	1.94	1.94	1.94	1.92	1.93	1.94
$EI_{22}/(GN\text{ m}^2)$	5.62	5.62	5.65	5.62	5.64	5.65

not only along the beam axis but also across the thickness, where the redistribution of the weak core material influences the global flexural response.

For the square-hole core, the identified bending stiffness is in very good agreement with the literature values obtained by the black-box FE implementation of asymptotic homogenization and with the VABS-based reference reported in [5]. The same trend is observed for the circular-hole core. In both configurations, the discrepancy with the benchmark values remains small and confirms that the present unit-cell-based formulation correctly captures the dominant flexural contribution of the periodic sandwich beam. This agreement is fully consistent with the observations of Yi et al. who reported a high level of correspondence between their implementation and the VABS formulation homogenization method described in [5]. The more approximate analytical route, which first homogenizes the core material separately, produced stiffer predictions in their study.

From the present perspective, these two cases are important not because they exercise the entire constitutive matrix, but because they verify that the proposed handle-based identification remains accurate when the periodic cell contains substantial material redistribution through the thickness. This point is relevant for engineering sandwich layouts, where the free upper and lower surfaces of the beam make the direct use of standard three-dimensional material homogenization less appropriate, as also discussed in [5]. The present results show that a direct characterization of the full periodic unit cell provides equivalent bending properties consistent with the dedicated benchmark literature.

5.2. Thin-walled box beam without and with ribs

The validation of the thin-walled box beam is reported in Tables 7 and 8. The identified constitutive terms are compared against three sets of reference values: analytical thin-walled beam theory, the full-beam and unit-cell results of Dizy et al. [4], and the solid- and shell-based calculations of Yi et al. [5]. This benchmark is more demanding than the sandwich cases because it involves several dominant stiffness terms and, in the ribbed configuration, periodic reinforcement that slightly modifies the beam's global constitutive response.

For the box beam without ribs, the agreement of the present results with the literature is excellent for the dominant axial and bending terms. The torsional stiffness also follows the expected trend and remains close to the values reported by the reference studies. This is consistent with the benchmark data of Dizy et al. where the homogenized stiffness constants were found to very closely match both analytical thin-walled theory and UM/VABS for the configuration [4] without ribs. Therefore, the present characterization recovers the constitutive behavior of the simple thin-walled periodic cell with the expected level of accuracy.

The ribbed box beam provides a more informative validation case because the transverse reinforcement perturbs the constitutive response only moderately, making the benchmark sensitive to the identification method's ability to capture relatively small stiffness increments without introducing spurious effects. In the literature, both Dizy et al. and Yi et al. reported that the addition of the transverse wall leads to a modest but clearly detectable increase in the dominant stiffness terms [4,5]. The present results correctly reproduce this trend: the ribbed configuration is identified as slightly stiffer than the one without ribs, with variations consistent with the benchmark references. This point is important because it shows that the method is not merely reproducing the constitutive behavior of the base box section, but is also sensitive to the incremental effect of the periodic reinforcement.

5.3. Comparison between solid and shell discretizations

A central aspect of the present study is that the same algebraic identification framework can be applied to periodic unit cells modeled either by three-dimensional solid elements or by shell elements. The box-beam benchmark is particularly well suited to illustrate this point. Yi et al. explicitly showed that shell models can reproduce the dominant stiffness terms of the thin-walled periodic box with very small differences relative to solid models, while drastically reducing the model size [5].

Table 9

Comparison of the effective stiffness coefficients identified for the unsymmetrical corrugated-strip-core sandwich beam. The present values are obtained with the baseline 12008-element discretization. The MSG and Cheng values are reported by Liu et al. [8]; the Cheng results originate from [29]. The signed differences are evaluated as $(D_{\text{present}} - D_{\text{ref}})/D_{\text{ref}}$.

Term	Unit	Present	MSG	Δ_{MSG}	Cheng	Δ_{Cheng}
D_{11}	N	$8.3284 \cdot 10^{10}$	$8.325 \cdot 10^{10}$	+0.04%	$8.320 \cdot 10^{10}$	+0.10%
D_{22}	N	$2.6097 \cdot 10^{10}$	$2.614 \cdot 10^{10}$	-0.17%	$2.660 \cdot 10^{10}$	-1.89%
D_{33}	N	$1.1671 \cdot 10^8$	$1.157 \cdot 10^8$	+0.87%	$1.149 \cdot 10^8$	+1.57%
D_{44}	N m ²	$3.7181 \cdot 10^8$	$3.696 \cdot 10^8$	+0.60%	$3.679 \cdot 10^8$	+1.06%
D_{55}	N m ²	$8.3239 \cdot 10^{10}$	$8.320 \cdot 10^{10}$	+0.05%	$8.318 \cdot 10^{10}$	+0.07%
D_{66}	N m ²	$2.7676 \cdot 10^{10}$	$2.766 \cdot 10^{10}$	+0.06%	$2.765 \cdot 10^{10}$	+0.10%
D_{46}	N m ²	$5.3303 \cdot 10^7$	$5.375 \cdot 10^7$	-0.83%	$5.328 \cdot 10^7$	+0.04%

Note: The reference values are retained with the numerical precision reported in [8]. In the notation of that study, $D_{ij} \equiv C_{ij}^b$ for the coefficients listed above.

The present results are fully consistent with this picture. For the box beams with and without ribs, the shell-based characterizations reproduce the dominant constitutive terms obtained from the solid models with only small differences. In particular, the bending and axial terms remain extremely close, while the torsional term is somewhat more sensitive, as expected for thin-walled closed sections and for ribbed layouts where the local representation of the reinforcement differs slightly between shell and solid idealizations. This behavior mirrors the interpretation already given by Yi et al. [5], who noted that the shell model tends to be slightly more compliant in the ribbed case. The reason is that solid elements can represent the stiffness contribution across the rib thickness more completely than shell idealizations.

Within the shell family, both the Kirchhoff–Love and Mindlin–Reissner formulations remain close to the solid reference for the dominant constitutive terms of the thin-walled cells considered here. The choice between the two formulations should nevertheless reflect the expected importance of transverse-shear deformation. Kirchhoff–Love shells are appropriate for sufficiently thin, membrane- and bending-dominated walls, whereas Mindlin–Reissner shells should be preferred when the wall thickness or the quantities of interest make shear-related stiffness terms, torsion, or associated coupling terms sensitive to transverse shear. A detailed thickness-dependent assessment of these effects within the same characterization framework is provided in [10]. The close agreement obtained in the present benchmark therefore reflects its thin-walled geometry and should not be interpreted as general interchangeability of the two shell theories.

For the thin configurations examined here, the main advantage of the shell models is computational: they preserve the quality of the dominant constitutive identification while substantially reducing the size of the unit-cell problem.

5.4. Unsymmetrical corrugated-strip-core sandwich beam

The stiffness coefficients identified for the unsymmetrical corrugated-strip-core sandwich beam are compared with the MSG and Cheng results reported by Liu et al. [8], the latter originating from the formulation of Xu et al. [29]. The generalized strain ordering adopted in those studies coincides with the ordering used here. Consequently, the coefficients denoted by C_{ij}^b in the reference study correspond directly to the entries D_{ij} of the present constitutive matrix. The comparison is restricted to the seven coefficients for which published reference values are available.

The agreement is close for all seven published coefficients. The maximum absolute difference with respect to the MSG results is 0.87%, for D_{33} , while the maximum difference with respect to Cheng’s method is 1.89%, for D_{22} . In particular, both transverse-shear stiffnesses D_{22} and D_{33} , the torsional stiffness D_{44} , and the nonzero torsion–bending coupling coefficient D_{46} are reproduced within 2% of both reference solutions. This case therefore extends the benchmark validation beyond the axial and bending-dominated coefficients considered in the preceding examples and provides a direct assessment of the shear- and torsion-related terms. Although the present procedure identifies the complete symmetric 6×6 constitutive matrix, the remaining coupling coefficients are not included in the quantitative comparison because independent numerical reference values are not reported in the cited studies.

A limited two-level mesh-refinement check was also performed by increasing the discretization from 12008 to 21280 quadratic brick elements. Relative to the baseline values, the seven coefficients included in Table 9 change by at most 0.665%, with the largest difference occurring for D_{33} . Among the additional coupling coefficients highlighted in Appendix A, D_{13} , D_{24} , and D_{26} change by 0.75%, 1.87%, and 3.40%, respectively. The principal stiffness terms are therefore stable with respect to this refinement, whereas the smaller coupling coefficients show the expected greater sensitivity to mesh resolution.

5.5. Discussion of benchmark validation

Taken together, the benchmark results support four main observations. First, for the sandwich beams, the present formulation recovers the dominant effective bending stiffness with good accuracy relative to the published benchmark values. Second, for the thin-walled box beam, the method correctly captures the main axial, torsional, and bending terms, as well as the moderate stiffness changes induced by the transverse ribs. Third, the unsymmetrical corrugated-strip-core benchmark directly validates both transverse-shear stiffnesses, torsion, and a nonzero torsion–bending coupling coefficient against two published methods, with differences below 2% for all seven compared coefficients. The limited two-level mesh-refinement check further confirms the stability of these principal

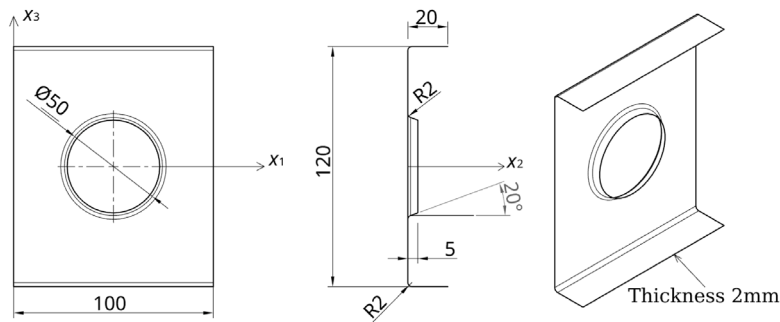


Fig. 6. Spar-like periodic C-section with an embossed web opening: dimensions and axes orientation.

stiffness terms. Fourth, the same identification workflow remains applicable to both solid and shell discretizations, with shell models providing a favorable compromise between accuracy and computational cost for thin-walled periodic structures.

These conclusions are significant for the positioning of the present study with respect to the literature. The comparison with [5] confirms that the proposed approach attains a level of accuracy comparable to dedicated FE implementations of asymptotic homogenization on the selected benchmarks. At the same time, the comparison with [4] shows that the equivalent constitutive properties extracted from the periodic unit cell are consistent with established homogenization workflows already published. The corrugated-strip-core comparison extends this validation to the more challenging shear- and torsion-related coefficients and to a nonzero coupling term. Therefore, the present method stands on solid validation grounds before being applied, in the next section, to more complex periodic structures that constitute the main new demonstrator of the paper.

It is also worth stressing that the benchmark section is intentionally focused on constitutive validation. The inertial characterization introduced in Section 3 is not assessed here because the selected literature benchmarks report stiffness quantities only. Its role becomes more relevant in the subsequent one-dimensional simulations, where the identified equivalent beam properties are used in a predictive setting beyond the mere comparison of constitutive constants.

6. Characterization of a spar-like periodic beam

After the validation against literature benchmarks, the proposed framework is applied to a more complex periodic structure representative of lightweight aerostructural layouts. The selected case is a spar-like beam with a thin-walled open section and repeated local embossed features along the span. Compared with the literature benchmarks, this structure is introduced not primarily to compare selected coefficients with published values, but to assess the method on a more application-oriented periodic geometry and to study the sensitivity to the definition of the periodic unit cell.

The structure is periodic along the beam axis, and each characterized unit cell is chosen so that its ends coincide with two interface sections consistent with the repeated pattern. The corresponding geometry is shown in Fig. 6. As in the previous sections, the characterization is performed directly on the FE model of the unit cell by introducing the two handles associated with the end sections, reducing the model by Craig–Bampton decomposition, and identifying the equivalent constitutive matrix through equilibrated generalized load states.

6.1. Geometry and periodic cuts

A distinctive feature of the spar-like beam is that the origin of the periodic unit cell can be translated along the beam axis while preserving the same assembled periodic structure. To assess the sensitivity of the characterized properties to this choice, six alternative unit cells of equal length are considered. Cut A corresponds to the reflection-symmetric configuration in which the embossed feature is centered within the unit cell. Four intermediate configurations are obtained by translating the feature by $\delta = 5.000$ mm, $1.000 \cdot 10^1$ mm, $1.500 \cdot 10^1$ mm and $2.000 \cdot 10^1$ mm from this centered position, without intersecting the end sections. Cut B represents the second reflection-symmetric limiting configuration, $\delta = 5.000 \cdot 10^1$ mm, in which the feature is divided between the two ends of the periodic unit cell. Fig. 7 illustrates Cut A, one representative intermediate translation, and Cut B. All six choices reproduce the same infinite periodic structure, but they allocate the local geometric feature differently within the selected finite interval.

All six configurations are extracted from shell models with the same global discretization, comprising 26 864 elements. For Cut A and the four intermediate translations, each retained end interface contains 161 nodes arranged in one connected loop, giving 322 retained nodes over $R_1 \cup R_2$. At Cut B, each interface contains 150 nodes distributed over two disconnected loops, giving 300 retained nodes in total. The five configurations with $0 \leq \delta \leq 20$ mm therefore form a controlled translation study with unchanged mesh size and interface topology. Cut B is retained as an extreme admissible configuration in which the local feature also changes the connectivity of the retained end sections.

This sequence of cuts separates the gradual effect of translating the local feature within the unit cell from the additional interface-topology change occurring at Cut B. The comparison is relevant in practical models of periodic thin-walled structures, where the

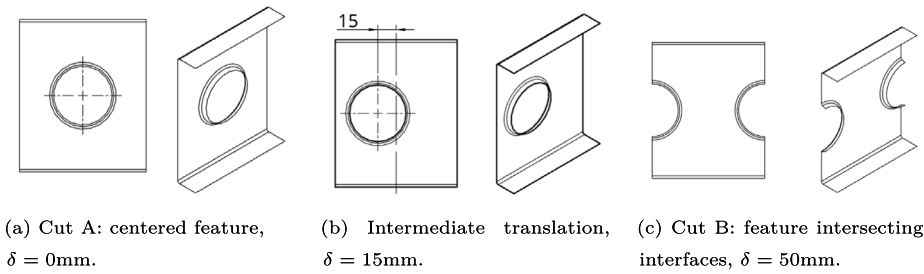


Fig. 7. Spar-like periodic C-section: representative positions of the periodic cut. The four intermediate configurations translate the local feature from Cut A toward Cut B without intersecting the retained end sections.

Table 10

Kirchhoff–Love: diagonal stiffness terms and their sensitivity to the unit-cell cut definition.

Term	$D_{ii}^{(A)}$	$D_{ii}^{(B)}$	$\Delta_{\text{cut}} (\%)$
D_{11}/N	$1.703 \cdot 10^7$	$1.703 \cdot 10^7$	0.00
D_{22}/N	$9.819 \cdot 10^5$	$8.390 \cdot 10^5$	−14.55
D_{33}/N	$2.989 \cdot 10^6$	$3.141 \cdot 10^6$	5.10
$D_{44}/(\text{N m}^2)$	$8.811 \cdot 10^1$	$9.180 \cdot 10^1$	4.18
$D_{55}/(\text{N m}^2)$	$3.875 \cdot 10^4$	$3.877 \cdot 10^4$	0.05
$D_{66}/(\text{N m}^2)$	$7.524 \cdot 10^2$	$7.521 \cdot 10^2$	−0.03

Table 11

Mindlin–Reissner: diagonal stiffness terms and their sensitivity to the unit-cell cut definition.

Term	$D_{ii}^{(A)}$	$D_{ii}^{(B)}$	$\Delta_{\text{cut}} (\%)$
D_{11}/N	$1.699 \cdot 10^7$	$1.699 \cdot 10^7$	0.00
D_{22}/N	$9.821 \cdot 10^5$	$8.400 \cdot 10^5$	−14.47
D_{33}/N	$2.955 \cdot 10^6$	$3.103 \cdot 10^6$	5.01
$D_{44}/(\text{N m}^2)$	$8.734 \cdot 10^1$	$9.109 \cdot 10^1$	4.30
$D_{55}/(\text{N m}^2)$	$3.875 \cdot 10^4$	$3.875 \cdot 10^4$	0.02
$D_{66}/(\text{N m}^2)$	$7.519 \cdot 10^2$	$7.517 \cdot 10^2$	−0.03

location of the cut planes may be influenced by meshing convenience or local geometry rather than by a unique analytical definition of the unit cell. The configurations are used below to determine which equivalent beam properties are robust with respect to the cut location and which retain a measurable dependence on the distribution of the local feature within the selected finite interval.

6.2. Constitutive matrices of the limiting cut configurations

The constitutive matrices obtained for Cut A and Cut B are first examined in detail. These configurations form the two reflection-symmetric limits of the translated cuts introduced in Section 6.1. The diagonal coefficients are reported in Tables 10 and 11, while the most relevant off-diagonal terms are collected in Tables 12 and 13. Results are given for both the Kirchhoff–Love and Mindlin–Reissner descriptions, providing a detailed comparison of the endpoint configurations before the systematic translation study presented in Section 6.3.

To interpret the off-diagonal terms, it is useful to introduce a normalized coupling indicator. For each off-diagonal coefficient D_{ij} , $i \neq j$, we define

$$c_{ij} = \frac{|D_{ij}|}{\sqrt{D_{ii} D_{jj}}}, \quad i \neq j. \quad (18)$$

The quantity c_{ij} provides a dimensionless measure of the relative importance of the coupling term with respect to the corresponding diagonal terms. In the discussion below, couplings associated with small values of c_{ij} are referred to as weak couplings, since even visible relative differences between two characterizations may correspond to terms that remain small compared with the dominant constitutive contributions. Conversely, larger values of c_{ij} indicate stronger couplings and therefore coefficients that are more structurally relevant.

This normalization is especially useful when comparing alternative cuts, because it helps distinguish between differences affecting structurally dominant couplings and differences confined to coefficients of minor practical influence.

The endpoint comparison shows that the axial and bending coefficients and the extension–bending coupling D_{16} remain very close for Cut A and Cut B. Measurable differences are instead found in the transverse-shear coefficients D_{22} and D_{33} , the shear–torsion coupling D_{34} , and the reference-axis torsional coefficient D_{44} . The agreement between the Kirchhoff–Love and Mindlin–Reissner comparisons shows that these Cut A/Cut B trends are not specific to one shell kinematic description.

Table 12

Kirchhoff-Love: off-diagonal stiffness terms retained by (18). The coupling indicator c_{ij} (computed for Cut A) quantifies the coupling strength relative to the diagonal scale.

Term	$D_{ij}^{(A)}$	$D_{ij}^{(B)}$	Δ_{cut} (%)	$c_{ij}^{(A)}$
Strong couplings				
$D_{16}/(\text{N m})$	$5.663 \cdot 10^4$	$5.664 \cdot 10^4$	0.003	$5.000 \cdot 10^{-1}$
$D_{34}/(\text{N m})$	$1.513 \cdot 10^4$	$1.588 \cdot 10^4$	4.972	$9.320 \cdot 10^{-1}$
Minor couplings (small c_{ij})				
$D_{26}/(\text{N m})$	$4.570 \cdot 10^1$	$5.997 \cdot 10^1$	31.24	$1.680 \cdot 10^{-3}$
$D_{35}/(\text{N m})$	$-8.777 \cdot 10^1$	$-2.039 \cdot 10^1$	-76.78	$2.580 \cdot 10^{-4}$

Table 13

Mindlin-Reissner: off-diagonal stiffness terms retained by (18). The coupling indicator c_{ij} (computed for Cut A) quantifies the coupling strength relative to the diagonal scale.

Term	$D_{ij}^{(A)}$	$D_{ij}^{(B)}$	Δ_{cut} (%)	$c_{ij}^{(A)}$
Strong couplings				
$D_{16}/(\text{N m})$	$5.647 \cdot 10^4$	$5.647 \cdot 10^4$	0.002	$5.000 \cdot 10^{-1}$
$D_{34}/(\text{N m})$	$1.516 \cdot 10^4$	$1.591 \cdot 10^4$	4.921	$9.440 \cdot 10^{-1}$
Minor couplings (small c_{ij})				
$D_{26}/(\text{N m})$	$2.402 \cdot 10^1$	$3.336 \cdot 10^1$	38.90	$8.840 \cdot 10^{-4}$
$D_{35}/(\text{N m})$	$-3.931 \cdot 10^1$	$-1.059 \cdot 10^1$	-73.06	$1.160 \cdot 10^{-4}$

Table 14

Mindlin-Reissner translated-cut study. The entries denoted by Δ are signed percentage differences from Cut A. The last column reports the normalized extension-shear coupling indicator rather than a percentage difference because $D_{12}(0)$ is nearly zero.

Configuration	δ (mm)	ΔD_{11} (%)	ΔD_{22} (%)	ΔD_{33} (%)	ΔD_{34} (%)	ΔD_{44} (%)	ΔGJ (%)	c_{12}
Cut A	0	0.000	0.000	0.000	0.000	0.000	0.000	$6.757 \cdot 10^{-6}$
Offset 5	5	0.011	-0.269	0.077	0.083	0.081	0.022	$1.063 \cdot 10^{-2}$
Offset 10	10	0.041	-1.130	0.314	0.325	0.300	0.008	$2.014 \cdot 10^{-2}$
Offset 15	15	0.077	-2.546	0.735	0.751	0.680	-0.030	$2.752 \cdot 10^{-2}$
Offset 20	20	0.105	-4.436	1.365	1.386	1.249	-0.034	$3.198 \cdot 10^{-2}$
Cut B	50	0.000	-14.47	5.010	4.921	4.296	-0.065	$1.718 \cdot 10^{-4}$

The endpoint comparison alone does not isolate the effect of translating the unit-cell origin. In Cut B, the local feature intersects both retained interfaces and changes their connectivity, whereas Cut A has simply connected interfaces located between consecutive features. Cut A and Cut B are therefore interpreted as the two limiting configurations of the admissible cut range. The systematic translation study presented in Section 6.3 introduces four intermediate cuts with unchanged interface topology, allowing the gradual effect of moving the feature toward the interfaces to be distinguished from the additional topology change occurring at Cut B. The complete endpoint constitutive matrices are collected in Appendix B.

6.3. Sensitivity to translation of the periodic cut

The translated cuts introduced in Section 6.1 are used here to quantify the sensitivity of the identified constitutive and inertial properties. Cut A is taken as the reference, while Cut B is interpreted separately from the four controlled intermediate translations because it also changes the connectivity of the retained interfaces.

For each constitutive coefficient, the signed relative difference from Cut A is defined as

$$\Delta D_{ij}(\delta) = 100 \frac{D_{ij}(\delta) - D_{ij}(0)}{D_{ij}(0)}. \quad (19)$$

The same convention is used for the stiffness and positive inertia components. Since D_{12} is nearly zero for Cut A, its variation is instead described by the normalized coupling indicator c_{12} of Eq. (18). Table 14 reports the coefficients that exhibit the most relevant cut dependence.

The axial stiffness remains highly stable, with a maximum difference of 0.105%, while the differences in D_{16} , D_{55} , and D_{66} remain below 0.03% over the full range. The transverse-shear terms are more sensitive to the location of the feature. Within the controlled range, D_{22} decreases by 4.436% and D_{33} increases by 1.365% at $\delta = 20\text{mm}$. At Cut B, where interface connectivity also changes, the corresponding differences reach -14.470% and 5.010%.

The intermediate translations break the axial reflection symmetry of the centered cell and introduce an extension-shear coupling D_{12} . Its normalized indicator increases smoothly to $c_{12} = 0.032$ at $\delta = 2.000 \cdot 10^1 \text{ mm}$. Cut B restores reflection symmetry because the feature is divided between the two ends, and c_{12} consequently returns to 1.7×10^{-4} . The remaining non-reported couplings have $c_{ij} < 0.002$ and do not show a comparable systematic trend.

Table 15

Translated-cut comparison of the unit-cell mass properties. Inertia components are barycentric and expressed in the same fixed beam axes. Percentage differences are evaluated relative to Cut A.

Configuration	δ (mm)	Δx_G (mm)	$\Delta I_{11}^{(3D)}$ (%)	$\Delta I_{22}^{(3D)}$ (%)	$\Delta I_{33}^{(3D)}$ (%)	$10^7 I_{12}^{(3D)}/(\text{kg m}^2)$
Cut A	0	0.000	0.0000	0.000	0.000	0.000
Offset 5	5	0.415	-0.0002	-0.080	-0.244	-1.480
Offset 10	10	0.831	-0.0007	-0.319	-0.976	-2.972
Offset 15	15	1.246	-0.0016	-0.719	-2.195	-4.463
Offset 20	20	1.662	-0.0029	-1.278	-3.901	-5.953
Cut B	50	0.000	-0.0078	-5.101	-15.54	0.000

The constitutive matrix is expressed about the selected beam reference axis. For an open section, the shear–torsion coupling can be interpreted through the shear center, defined as the point in the transverse plane through which a shear resultant can be applied without producing a torsional resultant in the equivalent beam model. Its coordinates and the corresponding torsional rigidity are obtained from the constitutive matrix using the standard change-of-reference relations of [30].

For the present configurations, D_{23} and D_{24} are numerically negligible and the principal shear axes coincide with the adopted transverse axes. The general transformation therefore reduces to

$$GJ = D_{44} - \frac{D_{34}^2}{D_{33}}. \quad (20)$$

Under the same alignment conditions, the relevant shear-center coordinate is $|x_{2,sc}| = |D_{34}|/D_{33}$. It remains between 5.126 mm and 5.132 mm over the six cut configurations. The shear-center-referred torsional rigidity GJ changes by only 0.065% between Cut A and Cut B. Thus, the 4.296% variation of the reference-axis coefficient D_{44} mainly reflects the cut-dependent contribution of the transverse-shear coupling rather than a comparable variation of the intrinsic torsional rigidity.

Among the dominant constitutive terms, the transverse-shear coefficient D_{22} exhibits the largest residual dependence on the cut position. Unlike the apparent variation of D_{44} , this dependence cannot be removed through a change of reference axis.

For the present isolated open thin-walled C-section, this coefficient is associated with the weak transverse-shear direction. The corresponding response involves out-of-plane deformation of substantial portions of the shell walls and local cross-sectional distortion rather than membrane action alone. At the local wall level, Kirchhoff–Love theory neglects shear deformation transverse to the shell midsurface, whereas Mindlin–Reissner theory represents it through a first-order approximation; both formulations nevertheless retain in-plane membrane shear and wall bending. Because the reduction to a single equivalent beam coefficient cannot retain the complete local deformation field, the identified shear stiffness may remain sensitive to these local mechanisms and to the location of the unit-cell boundaries.

The nearly identical Cut A–Cut B variations obtained with the Kirchhoff–Love and Mindlin–Reissner models, -14.55% and -14.47%, respectively, show that this residual dependence is not an artifact of one particular shell formulation. The sensitivity is also localized: between Cut A and Cut B, the axial and bending coefficients D_{11} , D_{55} , and D_{66} vary by no more than approximately 0.05%, while the shear-center-referred torsional rigidity varies by only 0.065%. The remaining sensitivity is concentrated in the transverse-shear block, with changes of approximately 5% in D_{33} and 14.5% in D_{22} . This residual dependence distinguishes the same-length unit-cell equivalence adopted here from a strictly homogenized constitutive result, for which invariance under translation of the unit-cell origin would be expected.

The resulting D_{22} should therefore be interpreted as an effective shear stiffness identified from the selected unit cell rather than as a strictly cut-invariant sectional constant. A similar limitation for shear properties associated with deformation normal to the principal shell surfaces was identified in the preceding shell-characterization study [10]. For the slender members considered here, transverse-shear deformation provides only a secondary contribution to the global static and low-frequency response. The observed sensitivity may become more relevant for short or deep members, shear-dominated loading, or configurations in which local sectional deformation is significant.

The same six unit cells were compared in terms of their actual mass properties. Table 15 reports the axial center-of-gravity shift and the fixed-axis components of the barycentric inertia tensor. For every configuration, the inertia tensor is evaluated about the center of gravity of that unit cell and expressed in the same beam axes.

The unit-cell mass remains constant to within $4.5 \times 10^{-5}\%$. For the four interior translations, the axial center-of-gravity shift is linear in δ . At Cut B the feature is divided symmetrically between the two ends of the finite interval, so the axial center of gravity returns to the cell midpoint. The transverse center-of-gravity coordinate changes by less than 0.0014%, while the other transverse coordinate remains zero by symmetry.

The barycentric inertia about the beam axis, $I_{11}^{(3D)}$, changes by less than 0.008%. The transverse components are more sensitive to the material allocated within the selected finite interval: at $\delta = 20\text{mm}$, their differences are -1.278% and -3.901%, while Cut B gives -5.101% and -15.540%. The loss of axial reflection symmetry in the intermediate cells also produces a nonzero barycentric product $I_{12}^{(3D)}$, which returns to zero for the two reflection-symmetric endpoint configurations.

These inertia variations are not generated by the interface condensation: they are obtained directly from the finite-element mass distribution. They reflect the finite-domain nature of the reduction. Translating the unit-cell boundary preserves the total mass and the assembled infinite periodic distribution, but it changes the first and second moments of the material assigned to one finite

Table 16

Direct comparison between shell models: relative difference of diagonal terms between RM and KL, computed as $A_{\text{model}} = 100(D^{\text{RM}} - D^{\text{KL}})/D^{\text{KL}}$ for each cut.

Term	A_{model} (Cut A) (%)	A_{model} (Cut B) (%)
D_{11}/N	-0.247	-0.247
D_{22}/N	0.015	0.112
D_{33}/N	-1.127	-1.212
$D_{44}/(N \text{ m}^2)$	-0.879	-0.771
$D_{55}/(N \text{ m}^2)$	-0.015	-0.042
$D_{66}/(N \text{ m}^2)$	-0.058	-0.056

interval. The translated-cut study therefore shows that the axial, bending, elastic-axis, and shear-center-referred torsional properties are highly robust with respect to the selected cut. The symmetry-dependent coupling terms and the actual unit-cell inertia tensor vary consistently with the allocation of the local feature within the selected interval. The transverse-shear block is less robust, with D_{22} showing the largest residual sensitivity. The results constitute a case-specific numerical assessment rather than a general cut-invariance theorem. In practical applications, the interfaces should preferably be placed away from pronounced local discontinuities and selected so that their topology remains unchanged.

6.4. Influence of the thin-walled formulation

Since the spar-like structure is intrinsically thin-walled, shell discretizations provide the most natural route for its characterization. In the present study, both a Kirchhoff–Love and a Mindlin–Reissner formulation are considered, consistently with the shell-based extension of the methodology. Their comparison for the dominant constitutive coefficients is summarized in Table 16.

The comparison assesses the influence of the adopted shell kinematics on the identified equivalent properties and verifies that the principal trends observed between the two limiting configurations are not specific to one shell formulation.

The translated-cut study of Section 6.3 makes use of the Mindlin–Reissner model. Here, in order to compare it with the Kirchhoff–Love one, the comparison is restricted to the limiting configurations Cut A and Cut B and is used to assess whether the differences observed between them depend on the selected shell kinematics.

The results confirm that both shell formulations provide a coherent description of the dominant equivalent behavior of the spar-like periodic beam.

The largest difference is found for D_{33} , for which the Mindlin–Reissner result is approximately 1.2% lower than the Kirchhoff–Love value. The difference in D_{44} remains below 0.9%, while those in the axial, bending, and remaining shear coefficients are below approximately 0.25%.

These results are consistent with the observations made for the benchmark box beam and confirm that, for the thin-walled configurations considered here, the principal equivalent properties are only weakly affected by the choice between Kirchhoff–Love and Mindlin–Reissner shell kinematics. This agreement should nevertheless be distinguished from the sensitivity to the location of the periodic cut discussed in Section 6.3. In particular, the residual cut dependence of D_{22} concerns the representation of the corresponding global shear response and is not affected by the shell formulation.

6.5. Discussion

The spar-like case highlights three practical aspects of the proposed framework. First, it shows that the characterization procedure can be applied to a realistic periodic thin-walled geometry. Second, the systematic translated-cut study shows that the axial, bending, elastic-axis, and shear-center-referred torsional properties are highly robust with respect to the location of the periodic cut. Measurable sensitivity remains in the transverse-shear block, in couplings that arise when the selected unit cell loses reflection symmetry, and in the transverse barycentric inertias, which depend on the allocation of the local feature within the selected finite interval. Third, the comparison between Kirchhoff–Love and Mindlin–Reissner models confirms that the same handle-based algebraic procedure remains applicable under different shell kinematic assumptions and that the principal endpoint trends are not an artifact of a single shell formulation.

Among the dominant coefficients, D_{22} is the least robust with respect to the cut position. Unlike the apparent variation of the reference-axis torsional coefficient D_{44} , its sensitivity cannot be removed by a change of reference point. For the present open thin-walled section, the corresponding equivalent beam shear stiffness may retain some dependence on local deformation mechanisms and on the location of the unit-cell boundaries, even though the Mindlin–Reissner shell formulation includes transverse shear deformation. This limitation is consistent with the observations reported in the preceding shell-characterization study and is expected to have limited influence on the bending-dominated response of the slender members considered here, while becoming more relevant for short or shear-dominated configurations.

For the present spar-like case, the analysis is intentionally restricted to single-period unit cells and to the sensitivity associated with translating the periodic cut. The results do not imply a universal cut-invariance property. They instead provide practical guidance for defining the finite-element unit cell: the retained interfaces should preferably be placed away from pronounced local stiffness or mass discontinuities and should preserve their connectivity as the cut is translated.

7. Use of the characterized properties in one-dimensional analyses

The previous sections focused on the identification of equivalent properties from a periodic unit cell and on the interpretation of the resulting constitutive matrices. The final step is to assess how these properties can be used in actual one-dimensional structural analyses of finite periodic beams. This section therefore considers both static and dynamic comparisons between equivalent beam models and detailed finite-element reference models.

The spar-like periodic beam introduced in Section 6 is first retained as the main application case. This structure is particularly useful because it combines several features that are relevant in practice: an open thin-walled section, local periodic details, sensitivity to the choice of the periodic cut, and a nontrivial shear-center location. At the same time, it should be stressed that such a spar-like component is not generally intended to operate as an isolated structural member. In realistic aerostructural layouts, it would typically be integrated into a more complete assembly, for example through ribs, skins, or a closed box configuration. As a consequence, the isolated spar is a demanding test case for the equivalent beam model, especially in dynamics, because the detailed shell reference model may contain local sectional deformation patterns that are not expected to be represented by a one-dimensional description.

For this reason, the dynamic assessment is complemented by a second, more controlled periodic beam: a rectangular solid beam with a periodic through-hole. This case is less representative of a complete aerostructural layout, but it provides a cleaner verification of the dynamic use of the identified constitutive and inertial properties. In particular, its vibration modes are more clearly beam-like in the frequency range of interest, so that the comparison between the detailed finite-element model and the equivalent one-dimensional model is less affected by local shell deformation modes.

7.1. Static cantilever comparison for the spar-like beam

The static cantilever comparison provides the first predictive use of the characterized properties. The equivalent one-dimensional model is built by assigning to the beam elements the constitutive matrix identified from the spar-like periodic unit cell, while the reference solution is obtained from a detailed finite-element model of the corresponding finite periodic structure. This comparison is used to verify that the identified constitutive matrix reproduces the dominant global response of the structure when the loading condition is defined consistently with the equivalent beam description.

7.1.1. Use of the identified shear center in the static loading

The shear-center location was introduced in Section 6.3, where its sensitivity to the periodic-cut position was also examined. The static comparison uses the Mindlin–Reissner properties of Cut A. For this configuration, the identified shear-center coordinate is approximately $x_{2,sc} = 5.131\text{mm}$ with respect to the selected beam reference line.

For the static cantilever validation discussed in Section 7.1.2, the generalized force and moment resultants of the one-dimensional model are applied at the reference point at which the constitutive matrix is expressed. Because the full matrix retains the shear–torsion coupling associated with that reference point, a transverse force applied there may produce both bending and torsional rotation.

The detailed finite-element model is loaded consistently with the same generalized resultant. The central solution obtained during the characterization is used to reconstruct a distribution of nodal forces on the tip section whose resultant force and moment coincide with those prescribed in the one-dimensional model. Applying the transverse force without a compensating torsional moment would therefore provide a valid bending–torsion comparison, since both models would be expected to develop the twist associated with the identified coupling.

In the present validation, the shear-center position derived from **D** is instead used to apply the compensating torsional moment

$$M_1 = F_3 x_{2,sc}. \quad (21)$$

The resulting force–moment pair, expressed at the reference point, is equivalent to applying the transverse force through the shear center. The tip rotation about the beam axis is retained as a diagnostic quantity in both models. This compensated loading simultaneously assesses the shear–torsion coupling of the extracted constitutive matrix, the derived shear-center position, and the reconstruction of an equivalent nodal load on the detailed finite-element section. A force applied at any other geometric or physical attachment point could be treated analogously by expressing the corresponding force and moment resultants at the selected reference point.

The corresponding derived beam quantities, including the shear-center coordinates, are reported in [Appendix B](#).

7.1.2. Static cantilever comparison between 1D and 3D models

A static cantilever benchmark is considered in order to assess the predictive capability of the equivalent one-dimensional model obtained from the characterized periodic unit cell.

Two cantilever lengths are considered, corresponding to beams composed of 4 and 8 repeated periods. This choice is intended to assess the predictive capability of the equivalent one-dimensional model under different relative contributions of bending and shear deformation: the shorter configuration is more sensitive to shear flexibility, whereas the longer one is increasingly dominated by bending.

The test configuration is shown in [Fig. 8](#). The equivalent beam is built by assigning to each one-dimensional element the constitutive matrix identified from the spar-like periodic unit cell, while the fully resolved reference model is obtained from the corresponding detailed three-dimensional discretization of the entire periodic structure.

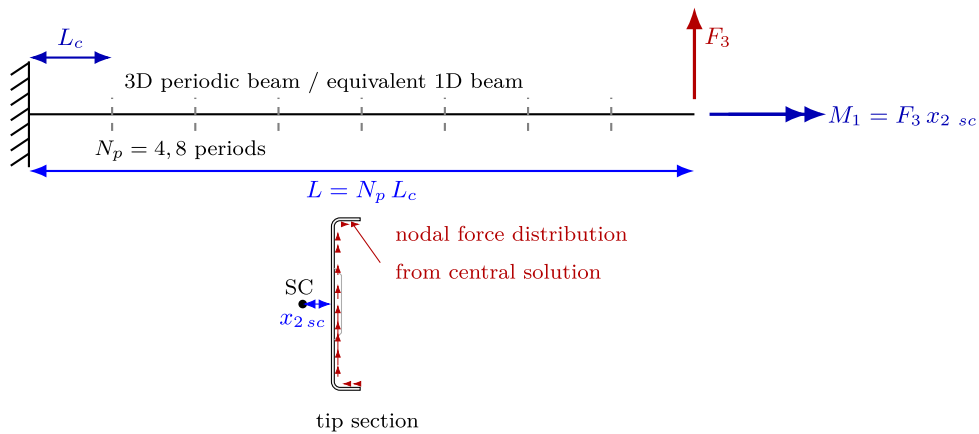


Fig. 8. Static cantilever benchmark used for the comparison between the full 3D periodic beam and the equivalent 1D MBDyn model. The transverse load is applied through the shear center by adding the equivalent torsional moment $M_1 = F_3 x_{2,sc}$. In the 3D model, the tip load is distributed according to the force pattern associated with the central solution.

Table 17

Static cantilever benchmark: response quantities for the full 3D model, the equivalent 1D MBDyn model, and the theoretical beam estimate.

N_p	Quantity	Unit	3D	1D (MBDyn)	Theory
4	$u_{x_3}(L)$	m	$6.853 \cdot 10^{-4}$	$6.857 \cdot 10^{-4}$	$6.860 \cdot 10^{-4}$
4	$r_{x_2}(L)$	rad	$-2.033 \cdot 10^{-3}$	$-2.0647 \cdot 10^{-3}$	$-2.060 \cdot 10^{-3}$
4	$r_{x_1}(L)$	rad	$1.020 \cdot 10^{-3}$	$4.5379 \cdot 10^{-5}$	0.0
8	$u_{x_3}(L)$	m	$4.667 \cdot 10^{-3}$	$4.675 \cdot 10^{-3}$	$4.676 \cdot 10^{-3}$
8	$r_{x_2}(L)$	rad	$-8.212 \cdot 10^{-3}$	$-8.2589 \cdot 10^{-3}$	$-8.260 \cdot 10^{-3}$
8	$r_{x_1}(L)$	rad	$-1.200 \cdot 10^{-3}$	$1.2943 \cdot 10^{-5}$	0.0

Table 18

Static cantilever benchmark: signed percentage differences with respect to the theoretical beam estimate, computed as $\delta = 100(q - q_{th})/q_{th}$. Relative differences are not reported for $r_{x_1}(L)$, since the theoretical reference is zero and the quantity is used mainly as a qualitative indicator of residual torsion.

N_p	Quantity	Unit	3D (%)	1D (MBDyn) (%)
4	$u_{x_3}(L)$	m	-0.1021	-0.0437
4	$r_{x_2}(L)$	rad	-1.3107	0.2282
4	$r_{x_1}(L)$	rad	see discussion	
8	$u_{x_3}(L)$	m	-0.1925	-0.0214
8	$r_{x_2}(L)$	rad	-0.5811	-0.0133
8	$r_{x_1}(L)$	rad	see discussion	

Table 19

Theoretical decomposition of the tip displacement into bending and shear contributions, according to $u_z(L) = \frac{F_3 L^3}{3D_{33}} + \frac{F_3 L}{D_{33}}$.

N_p	Bending	Shear	Total	Bending (%)
4	$5.506 \cdot 10^{-4}$	$1.354 \cdot 10^{-4}$	$6.860 \cdot 10^{-4}$	80.3
8	$4.405 \cdot 10^{-3}$	$2.710 \cdot 10^{-4}$	$4.676 \cdot 10^{-3}$	94.2

The quantities of interest are the tip displacements and rotations of the cantilever, together with their deviation from the corresponding three-dimensional reference values. The numerical results are reported in [Tables 17](#) and [18](#). To support the interpretation of these comparisons, [Table 19](#) also reports the theoretical decomposition of the static response into bending- and shear-related contributions. This is useful because the 4-period and 8-period configurations represent two different structural regimes: in the shorter beam, shear flexibility contributes more significantly to the overall tip response, whereas in the longer beam the response approaches the slender-beam limit and is dominated by bending.

The agreement between the equivalent one-dimensional model and the full three-dimensional simulation is very good for the main static response quantities in both the 4-period and 8-period configurations. This is relevant because the two cases have different relative contributions of shear and bending, and because the spar-like beam is not a simplified benchmark geometry from the literature, but an application-oriented periodic structure with local details and a nontrivial definition of the periodic cell. The

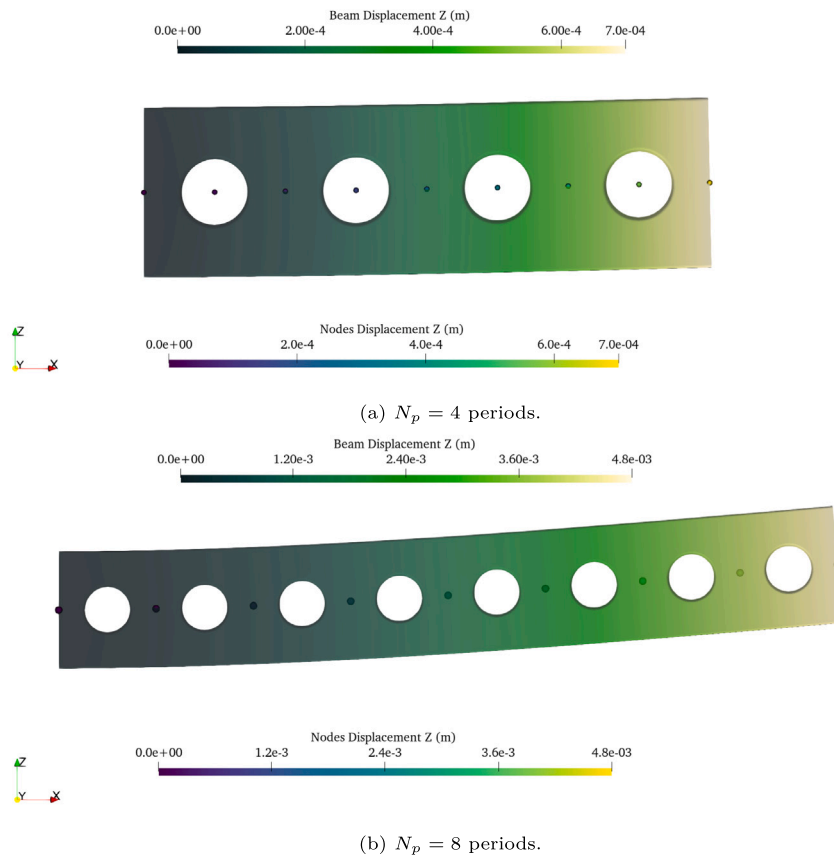


Fig. 9. Comparison of deformed configurations for the full 3D model and the equivalent 1D MBDyn model (10× amplification).

cantilever comparison therefore shows that the identified equivalent properties are not only formally meaningful, but also predictive in a structural setting.

A qualitative comparison of the deformed configurations is provided in Fig. 9. As expected, the largest differences are concentrated near the beam ends, where end effects and local three-dimensional features are most pronounced and cannot be exactly represented by a uniform equivalent beam model. However, away from the boundaries the overall deformation pattern is reproduced satisfactorily, which is entirely consistent with the homogenization rationale underlying the unit-cell-based characterization procedure.

7.2. Modal comparison procedure

All dynamic comparisons in this section are performed in a free–free configuration, to avoid issues associated with prescribed boundary conditions. The modal analyses therefore include six rigid-body modes, which are discarded before comparing the elastic vibration modes. The comparison focuses on the first deformable modes, consistently with the intended use of the characterized properties for low-frequency dynamics governed by global beam-like deformation patterns.

The objective of the dynamic comparison is to verify whether the constitutive and inertial quantities identified from a single periodic unit cell can reproduce the first global elastic modes of finite periodic beams. In this sense, the low-frequency modal verifications complement the static validation discussed above by exercising the inertial part of the characterization, which is not involved in the benchmark comparisons of Section 5.

The one-dimensional models are built in MBDyn¹ [31] using three-node finite-volume beam elements [32,33]. The eigensolution is computed following the procedure described in [34]. The constitutive and inertial properties assigned to the beam elements are those identified from the corresponding periodic unit cell.

Mode shapes are compared by reducing the detailed finite-element solution to a set of beam-compatible section variables, so that the three-dimensional and one-dimensional modes are expressed on comparable degrees of freedom.

¹ A free, general-purpose multibody solver, <https://www.mbdyn.org> (last accessed May 2026).

For each elastic mode of the detailed finite-element model, the three-dimensional displacement field is sampled at a set of sections corresponding to the stations used in the one-dimensional model. At each selected section, the nodal displacement field is approximated by the rigid motion of an equivalent beam section,

$$\mathbf{u}(\mathbf{r}) \approx \mathbf{u}_0 + \boldsymbol{\theta} \times (\mathbf{r} - \mathbf{r}_0), \quad (22)$$

where $\mathbf{u}_0$ and $\boldsymbol{\theta}$ are, respectively, the translation and rotation of the equivalent section, and $\mathbf{r}_0$ is the reference point adopted for the fit. In matrix form, the displacement field on section s is written as

$$\mathbf{u}_s \approx \mathbf{H}_s \mathbf{q}_s^{(6)}, \quad (23)$$

where $\mathbf{H}_s$ is the rigid interpolation matrix of the section, built consistently with the handle kinematics introduced in Section 3. Since the fit is performed on nodal displacements, $\mathbf{H}_s \in \mathbb{R}^{3n_s \times 6}$, where n_s is the number of nodes on the selected section. For shell models, this corresponds to using the displacement block of the shell handle interpolation. The least-squares estimate of the equivalent section variables is therefore

$$\mathbf{q}_s^{(6)} = \mathbf{H}_s^+ \mathbf{u}_s, \quad \mathbf{H}_s^+ = (\mathbf{H}_s^T \mathbf{H}_s)^{-1} \mathbf{H}_s^T, \quad (24)$$

assuming that $\mathbf{H}_s$ has full column rank.

This provides, at each comparison section, the six-component vector

$$\mathbf{q}^{(6)} = [u_{x_1} \quad u_{x_2} \quad u_{x_3} \quad \theta_{x_1} \quad \theta_{x_2} \quad \theta_{x_3}]^T. \quad (25)$$

The same six-component vector is obtained directly from the corresponding one-dimensional beam mode at the same station.

In addition, a reduced vector is considered,

$$\mathbf{q}^{(3)} = [u_{x_2} \quad u_{x_3} \quad \theta_{x_1}]^T, \quad (26)$$

because these components are the most directly associated with the dominant bending and torsional deformation patterns considered in the present comparisons.

For mode j of the three-dimensional model and mode k of the one-dimensional model, the section quantities are stacked over the n_s comparison sections as

$$\hat{\boldsymbol{\phi}}_j^{3D,(6)} = \begin{bmatrix} \mathbf{q}_{j,1}^{3D,(6)} \\ \vdots \\ \mathbf{q}_{j,n_s}^{3D,(6)} \end{bmatrix}, \quad \hat{\boldsymbol{\phi}}_k^{1D,(6)} = \begin{bmatrix} \mathbf{q}_{k,1}^{1D,(6)} \\ \vdots \\ \mathbf{q}_{k,n_s}^{1D,(6)} \end{bmatrix}, \quad (27)$$

and analogously for the three-component vectors,

$$\hat{\boldsymbol{\phi}}_j^{3D,(3)} = \begin{bmatrix} \mathbf{q}_{j,1}^{3D,(3)} \\ \vdots \\ \mathbf{q}_{j,n_s}^{3D,(3)} \end{bmatrix}, \quad \hat{\boldsymbol{\phi}}_k^{1D,(3)} = \begin{bmatrix} \mathbf{q}_{k,1}^{1D,(3)} \\ \vdots \\ \mathbf{q}_{k,n_s}^{1D,(3)} \end{bmatrix}. \quad (28)$$

The modal correlation is quantified by means of the Modal Assurance Criterion (MAC), which is widely used to assess the consistency between modal vectors obtained from different models or test configurations [35,36]. Given two modal vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, the MAC is defined as

$$\text{MAC}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \frac{|\boldsymbol{\phi}^H \boldsymbol{\psi}|^2}{(\boldsymbol{\phi}^H \boldsymbol{\phi})(\boldsymbol{\psi}^H \boldsymbol{\psi})}, \quad (29)$$

where $(\cdot)^H$ denotes the Hermitian transpose. The two reduced indicators used in the following are then defined as

$$\begin{aligned} \text{MAC}_3(j, k) &= \text{MAC}(\hat{\boldsymbol{\phi}}_j^{3D,(3)}, \hat{\boldsymbol{\phi}}_k^{1D,(3)}), \\ \text{MAC}_6(j, k) &= \text{MAC}(\hat{\boldsymbol{\phi}}_j^{3D,(6)}, \hat{\boldsymbol{\phi}}_k^{1D,(6)}). \end{aligned} \quad (30)$$

The MAC_3 value focuses on the transverse displacement and twist components most directly involved in the dominant bending-torsion response, whereas MAC_6 also includes axial displacement and the two bending rotations, and is therefore a more restrictive reduced comparison.

The quality of the rigid-section approximation is quantified through the normalized rigid-section residual

$$\eta_{\text{RS}}^{(j,s)} = \frac{\|\mathbf{u}_{j,s}^{3D} - \mathbf{u}_{j,s}^{\text{rig}}\|_2}{\|\mathbf{u}_{j,s}^{3D}\|_2}, \quad (31)$$

where j denotes the mode index and s the comparison section. Here, $\mathbf{u}_{j,s}^{3D}$ is the actual nodal displacement field of the detailed model on section s , while $\mathbf{u}_{j,s}^{\text{rig}}$ is the displacement field reconstructed from the fitted rigid-section motion in Eq. (22). Small values of $\bar{\eta}_{\text{RS}}$ indicate that the section behaves in a beam-like manner, whereas larger values reveal a more pronounced internal deformation of

the section and therefore a less accurate reduction to one-dimensional generalized variables. In the modal comparison tables, the reported quantity is the section-averaged value

$$\bar{\eta}_{RS}^{(j)} = \frac{1}{n_s} \sum_{s=1}^{n_s} \eta_{RS}^{(j,s)}, \quad (32)$$

where n_s is the number of comparison sections.

A complementary expanded comparison is also performed. In this case, the one-dimensional mode is mapped back to the nodes of each comparison section using the rigid-section kinematics,

$$\tilde{\mathbf{u}}_{k,s}^{\text{ID}}(\mathbf{r}) = \mathbf{u}_{0,k,s}^{\text{ID}} + \boldsymbol{\theta}_{k,s}^{\text{ID}} \times (\mathbf{r} - \mathbf{r}_0), \quad (33)$$

where $\mathbf{u}_{0,k,s}^{\text{ID}}$ and $\boldsymbol{\theta}_{k,s}^{\text{ID}}$ are taken from the one-dimensional mode at section s . Stacking the reconstructed nodal displacement fields over all comparison sections gives an expanded one-dimensional modal vector $\tilde{\boldsymbol{\phi}}_k^{\text{ID}}$, which can be compared directly with the corresponding stacked three-dimensional nodal displacement vector $\tilde{\boldsymbol{\phi}}_j^{\text{3D}}$. The expanded modal assurance criterion is defined as

$$\text{MAC}_{\text{exp}}(j, k) = \text{MAC}(\tilde{\boldsymbol{\phi}}_j^{\text{3D}}, \tilde{\boldsymbol{\phi}}_k^{\text{ID}}). \quad (34)$$

The expanded comparison is more demanding than the reduced one because it evaluates the agreement in the nodal displacement space after imposing rigid-section kinematics on the one-dimensional mode. The reduced and expanded indicators are therefore related, but they are not algebraically equivalent. To see this, consider one comparison section and omit the mode and section indices for clarity. The three-dimensional displacement field can be decomposed into the rigid-section part identified by the projection and a residual non-rigid contribution orthogonal to it,

$$\mathbf{u}^{\text{3D}} = \mathbf{H}\mathbf{q}_R + \mathbf{u}_d, \quad \mathbf{q}_R = \mathbf{H}^+ \mathbf{u}^{\text{3D}}, \quad \mathbf{H}^T \mathbf{u}_d = \mathbf{0}. \quad (35)$$

where $\mathbf{q}_R = \mathbf{H}^+ \mathbf{u}^{\text{3D}}$ and $\mathbf{u}_d = \mathbf{C}\mathbf{q}_d$ contains the part of the section displacement that cannot be represented by a rigid-section motion. The reduced comparison is performed after projection onto the beam variables. Therefore, for the six-component reduced vector, the MAC has the form

$$\text{MAC}_{\text{red}} = \frac{|\mathbf{q}_R^H \mathbf{q}|^2}{(\mathbf{q}_R^H \mathbf{q}_R) (\mathbf{q}^H \mathbf{q})}, \quad (36)$$

where $\mathbf{q}$ denotes the corresponding one-dimensional beam variables. In this reduced space, the non-rigid residual $\mathbf{u}_d$ does not enter explicitly; its magnitude is instead monitored separately through the residual measure η_{RS} .

In the expanded comparison, the one-dimensional variables are mapped to the section nodes before computing the MAC. Introducing

$$\mathbf{M}_H = \mathbf{H}^T \mathbf{H}, \quad (37)$$

the expanded indicator can be written as

$$\text{MAC}_{\text{exp}} = \frac{|\mathbf{q}_R^H \mathbf{M}_H \mathbf{q}|^2}{(\mathbf{q}_R^H \mathbf{M}_H \mathbf{q}_R + \mathbf{u}_d^H \mathbf{u}_d) (\mathbf{q}^H \mathbf{M}_H \mathbf{q})}. \quad (38)$$

Thus, the expanded MAC measures the correlation in nodal displacement space. The rigid-section part is weighted by the geometric matrix $\mathbf{H}^T \mathbf{H}$, while the denominator also contains the energy-like norm of the non-rigid section residual. Consequently, when the detailed mode contains significant local cross-sectional deformation, MAC_{exp} is expected to decrease even if the beam-compatible part of the mode is well correlated in the reduced variables. This explains why the expanded comparison is generally more demanding than MAC_3 and MAC_6 .

The procedure is applied below to two finite periodic beams: first to the spar-like shell structure introduced above, which represents the main application case, and then to a rectangular solid beam with a periodic through-hole, which provides a more controlled dynamic verification.

7.3. Free-free dynamics of the spar-like beam

The first dynamic comparison is performed on the eight-period spar-like beam already considered in the static validation. The three-dimensional reference model is the shell model discussed in Section 6, whereas the equivalent one-dimensional model is obtained by assigning to each beam element the constitutive and inertial properties identified from a single periodic unit cell. The corresponding characterized properties are reported in Appendix B.

This case is deliberately demanding. The spar-like structure is an open thin-walled component and, when analyzed in isolation, may exhibit local shell deformation, plate-like patterns, and cross-sectional distortions that are not represented by the kinematics of an equivalent beam model.

Table 20

Comparison between selected free-free elastic modes of the shell model and of the equivalent beam model for the eight-period spar-like beam. The relative frequency difference is computed as $(f_{1D} - f_{3D})/f_{3D}$. The last row reports a representative shell mode that is not retained as a meaningful beam-like correspondence.

3D mode	Mode family	f_{3D} (Hz)	f_{1D} (Hz)	Δf (%)	$\bar{\eta}_{RS}$	MAC ₃ red.	MAC ₆ red.	MAC ₃ exp.	MAC ₆ exp.
1	first torsion	54.17	49.04	-9.47	0.105	0.982	0.982	0.978	0.978
2	first bending about x_3	143.17	146.54	2.35	0.052	1.000	0.999	0.995	0.995
4	second bending about x_3	328.68	390.77	18.89	0.327	0.999	0.966	0.902	0.901
7	local shell mode, not beam-like	549.45	–	–	0.990	0.113	0.098	0.002	0.002

Table 20 reports the most meaningful correspondences between the shell and beam models after removal of the six rigid-body modes. The retained pairs correspond to the first torsional mode, the first bending mode about x_3 , and the second bending mode about x_3 . A representative shell mode that is not retained as a beam-like correspondence is also included in the last row, in order to illustrate the presence of local three-dimensional deformation patterns that are outside the kinematic scope of the equivalent beam model. The first torsional mode is predicted with a frequency difference of about 9%, while both the reduced and expanded MAC values remain close to 0.98, indicating a clear correspondence in the beam-compatible modal content. The first bending mode about x_3 shows the best agreement, with a frequency difference below 3%, nearly unitary reduced MAC values, expanded MAC values close to 0.995, and a low average fitting error. The second bending mode about x_3 is also strongly correlated in the reduced space, although the larger frequency difference and the lower expanded MAC indicate that the shell displacement field contains non-negligible contributions that are not fully reproduced by the one-dimensional expansion. Conversely, the non-beam-like shell mode reported in the last row exhibits a large rigid-section fitting error and very low reduced and expanded MAC values, confirming that it should not be interpreted as a meaningful counterpart of any beam mode.

The results confirm that the equivalent beam model captures the first global beam-like modes of the spar-like structure with good accuracy. The second bending mode about x_3 shows that a high reduced MAC can still be associated with a larger frequency discrepancy and with a lower expanded MAC, indicating that part of the detailed shell displacement field is not fully represented by the one-dimensional expansion. This trend is consistent with the expected progressive loss of validity of a rigid-section beam description for higher modes or for modes involving local deformation of the thin-walled section.

Fig. 10 shows representative mode-shape comparisons for the four retained pairs. The visual agreement is consistent with the values reported in Table 20: the first torsional and first bending modes show the clearest correspondence, whereas the second bending mode illustrates the situation in which the reduced modal shape remains strongly correlated even though the frequency error is larger.

Overall, the spar-like dynamic comparison confirms that the identified properties can be used to reproduce the first global free-free elastic modes of a realistic periodic thin-walled structure. At the same time, it also highlights the limits of applying an isolated one-dimensional beam model to an open shell component whose detailed spectrum contains local deformation mechanisms. This motivates the additional dynamic comparison of Section 7.4, where a solid periodic beam is used as a more controlled verification case.

7.4. Free-free dynamics of a rectangular perforated periodic beam

To complement the spar-like shell example, a second dynamic comparison is performed on a rectangular solid periodic beam with a through-hole. The purpose of this case is not to introduce a new characterization procedure, but to provide a more controlled verification of the dynamic use of the identified periodic-cell properties. Unlike the open thin-walled spar, this structure is discretized with three-dimensional solid elements and its low-frequency spectrum is dominated by clearly beam-like global modes. It therefore helps separate the dynamic performance of the equivalent one-dimensional model from the local shell effects discussed in Section 7.3.

The same modal-comparison procedure described in Section 7.2 is used here. The three-dimensional modes are reduced to equivalent rigid-section variables at the comparison stations and are also compared with the one-dimensional modes after expansion of the latter onto the three-dimensional section nodes.

7.4.1. Geometry and material

The periodic cell is shown in Fig. 11. It consists of a rectangular solid section of dimensions 24 mm by 40 mm, with a through-hole of diameter 24 mm located at mid-period. The period length is 60 mm. The finite structure considered in the modal comparison is obtained by repeating the cell for twelve periods, resulting in a total length of 720 mm.

Because the through-hole intersects only part of the period, the cross-sectional geometry varies along x_1 , from the solid end sections to the perforated region around mid-period. A single conventional two-dimensional cross-sectional model cannot therefore represent the complete cell without introducing an additional smearing or homogenization assumption.

The material is aluminum, modeled as linear elastic and isotropic with

$$E = 70 \text{ GPa}, \quad \nu = 0.30, \quad \rho = 2700 \text{ kg/m}^3. \quad (39)$$

The equivalent elastic and inertial properties identified from the periodic cell are reported in Appendix C.

Owing to the two orthogonal symmetry planes of the geometry and the corresponding choice of reference axes, the constitutive matrix is diagonal. The six nonzero constitutive coefficients and the three-dimensional unit-cell mass properties are reported in the appendix.

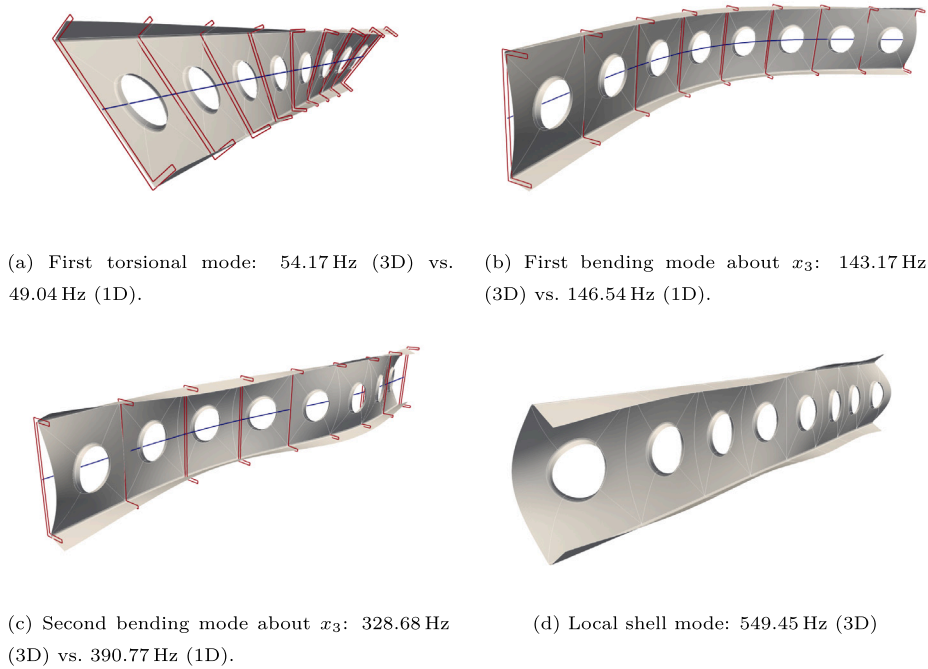


Fig. 10. Comparison between representative free-free elastic modes of the three-dimensional shell model and of the equivalent one-dimensional beam model for the eight-period spar-like beam.

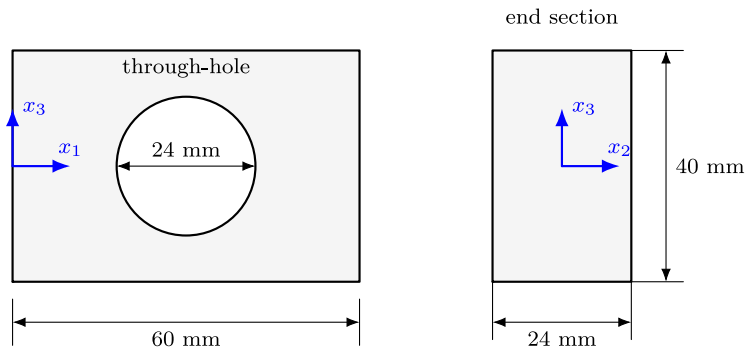


Fig. 11. Geometry of the rectangular perforated periodic cell used for the solid-beam dynamic comparison.

Table 21

Mesh convergence of the one-period rectangular perforated solid-beam unit cell. The quantity ϵ_D is the maximum relative difference among the six diagonal constitutive coefficients, evaluated with respect to the finest discretization.

Mesh	Total DOFs	$R_1 \cup R_2$ DOFs	ϵ_D (%)
Coarse	22 660	1038	$2.03 \cdot 10^{-1}$
Intermediate	71 570	2202	$2.56 \cdot 10^{-2}$
Fine	164 000	3798	$5.22 \cdot 10^{-3}$
Finest	477 500	7398	reference

7.4.2. Mesh convergence and multiperiod consistency

Mesh sensitivity was assessed using four discretizations of the one-period unit cell with 20-node quadratic hexahedral elements. The models range from 22 656 to 477 504 total degrees of freedom and from 1038 to 7398 degrees of freedom retained on the two end interfaces $R_1 \cup R_2$. Table 21 reports the maximum relative difference among the six diagonal constitutive coefficients, using the finest discretization as reference. The complete constitutive and inertial results for the four meshes are reported in Appendix C.

Table 22

Multiperiod consistency of the rectangular perforated solid-beam characterization. The mass and axial-inertia ratios refer to the actual three-dimensional properties of each finite characterization domain.

n_p	L (m)	$\varepsilon_D^{(n_p)}$ (%)	$M^{(n_p)}/M^{(1)}$	$I_{11}^{(n_p)}/I_{11}^{(1)}$
1	0.06	reference	1	1
2	0.12	$2.3 \cdot 10^{-9}$	2	2
4	0.24	$2.2 \cdot 10^{-7}$	4	4

The maximum difference is 0.203% for the coarsest mesh, 0.0256% for the intermediate mesh, and 0.00522% for the fine mesh. For the coarse and intermediate discretizations, the largest variation occurs in the transverse-shear coefficient D_{33} . Over the complete refinement range, the unit-cell mass and barycentric inertia vary by less than $8 \times 10^{-5}\%$. The finest one-period discretization provides the properties used in the modal comparison reported below.

The effect of grouping consecutive periods into a single characterization domain was assessed separately using models containing $n_p = 1, 2$, and 4 complete periods. The intermediate mesh resolution was retained within each period, and the two ends of every characterization domain were located at equivalent periodic sections. Table 22 reports the maximum difference among the six constitutive coefficients relative to the one-period result, together with the ratios of the actual characterization-domain mass and axial barycentric inertia to their one-period values. Here, $I_{ii}^{(n_p)}$ denotes the three-dimensional barycentric inertia of a finite characterization domain containing n_p complete periods, as obtained from the original finite-element mass matrix; it is not normalized by the domain length. The complete constitutive and inertial results are reported in Appendix C.

The six constitutive coefficients are numerically indistinguishable for the three characterization-domain lengths at the reported accuracy. The mass of the two- and four-period characterization domains is, respectively, two and four times that of the one-period unit cell, and the barycentric inertia about the beam axis, $I_{11}^{(n_p)}$, follows the same linear scaling. The transverse barycentric inertias include the expected parallel-axis contribution caused by the increasing axial separation of the individual periods. Specifically, their values satisfy

$$I_{\alpha\alpha}^{(n_p)} = n_p I_{\alpha\alpha}^{(1)} + M^{(1)} L_c^2 \frac{n_p (n_p^2 - 1)}{12}, \quad \alpha = 2, 3. \quad (40)$$

The computed mass and inertia values agree with these direct aggregation relations at the reported accuracy. Thus, grouping consecutive complete periods does not alter the identified constitutive properties, while each characterization retains the actual mass properties of the finite characterization domain being replaced.

7.4.3. Modal comparison results

The comparison is carried out on the twelve-period free-free structure. The modal quantities are evaluated at the two end sections and at the intermediate period interfaces, which correspond to the nodes of the equivalent one-dimensional model. Table 23 reports the mode correspondences identified below 2.5 kHz, including natural frequencies, relative frequency differences, rigid-section fitting errors, reduced MAC values, and expanded MAC values.

The modal correspondence is very strong for all reported modes. The reduced-space MAC values are close to unity both in the three-component and six-component comparisons. The same conclusion is obtained from the expanded comparison, where the one-dimensional modes reconstructed on the three-dimensional section nodes retain nearly identical MAC values. This confirms that the agreement is not an artifact of the reduced section variables, but is also visible when the beam modes are interpreted as three-dimensional rigid-section displacement fields.

The frequency agreement is also good. The relative differences remain below about 1.2% for the first four matched modes, apart from the closely spaced modes around 1.09 kHz, where the ordering of the fourth and fifth one-dimensional modes is interchanged with respect to the three-dimensional ordering. The largest discrepancy in the reported set is observed for the seventh mode, with a frequency difference of approximately 3.25%. The first torsional mode at 1431.17 Hz exhibits a larger section fitting error than the bending modes, although the MAC values remain essentially unitary. This indicates that the mode contains a stronger non-rigid body motion contribution at the section level (the out-of-plane warping displacement that is characteristic of torsion), while preserving a global beam-like deformation pattern that is accurately captured by the one-dimensional model.

Fig. 12 shows representative mode-shape comparisons for the matched modes listed in Table 23. The agreement is visually consistent with the quantitative indicators: the bending families are reproduced with very high correlation, and the torsional mode retains the expected global deformation despite the larger local fitting error.

Overall, the rectangular perforated beam provides a clean dynamic verification case. The fitting errors remain small for most modes, the reduced and expanded MAC values are mutually consistent, and the frequency agreement remains close over the reported frequency range. These results support the use of the identified periodic-cell properties in equivalent one-dimensional beam models for global free-free modal analyses of solid periodic structures.

The first observation is that the modal correspondence is very strong for all the reported modes. The reduced-space MAC values are close to unity in both the MAC3 and MAC6 spaces. The same conclusion is obtained from the expanded comparison, where the 1D modes reconstructed on the three-dimensional section nodes retain nearly identical MAC values. This agreement suggests that the matching is not an artifact of the reduction procedure, but is also visible when the beam modes are interpreted as three-dimensional rigid-section displacement fields.

Table 23

Modal correspondence between the 3D model and the 1D equivalent beam model for the rectangular perforated periodic beam. The fit error is the rigid section error associated with the three-dimensional to one-dimensional reduction.

3D mode	Mode family	f_{3D} (Hz)	f_{1D} (Hz)	Δf (%)	$\bar{\eta}_{RS}$	MAC ₃ red.	MAC ₆ red.	MAC ₃ exp.	MAC ₆ exp.
1	first bending about x_3	206.14	205.57	0.28	$1.189 \cdot 10^{-3}$	1.0000	1.0000	1.0000	1.0000
2	first bending about x_2	414.33	412.86	0.35	$1.757 \cdot 10^{-3}$	1.0000	1.0000	1.0000	1.0000
3	second bending about x_3	563.91	558.44	0.97	$2.388 \cdot 10^{-3}$	0.9999	0.9999	0.9999	0.9999
4	second bending about x_2	1090.76	1078.71	1.11	$3.807 \cdot 10^{-3}$	0.9999	0.9999	0.9998	0.9999
5	third bending about x_3	1094.27	1072.58	1.98	$4.996 \cdot 10^{-3}$	0.9997	0.9997	0.9997	0.9997
6	first torsion	1431.17	1427.89	0.23	$9.395 \cdot 10^{-2}$	1.0000	1.0000	1.0000	0.9998
7	third bending about x_2	1785.99	1727.88	3.25	$8.147 \cdot 10^{-3}$	0.9994	0.9994	0.9994	0.9994

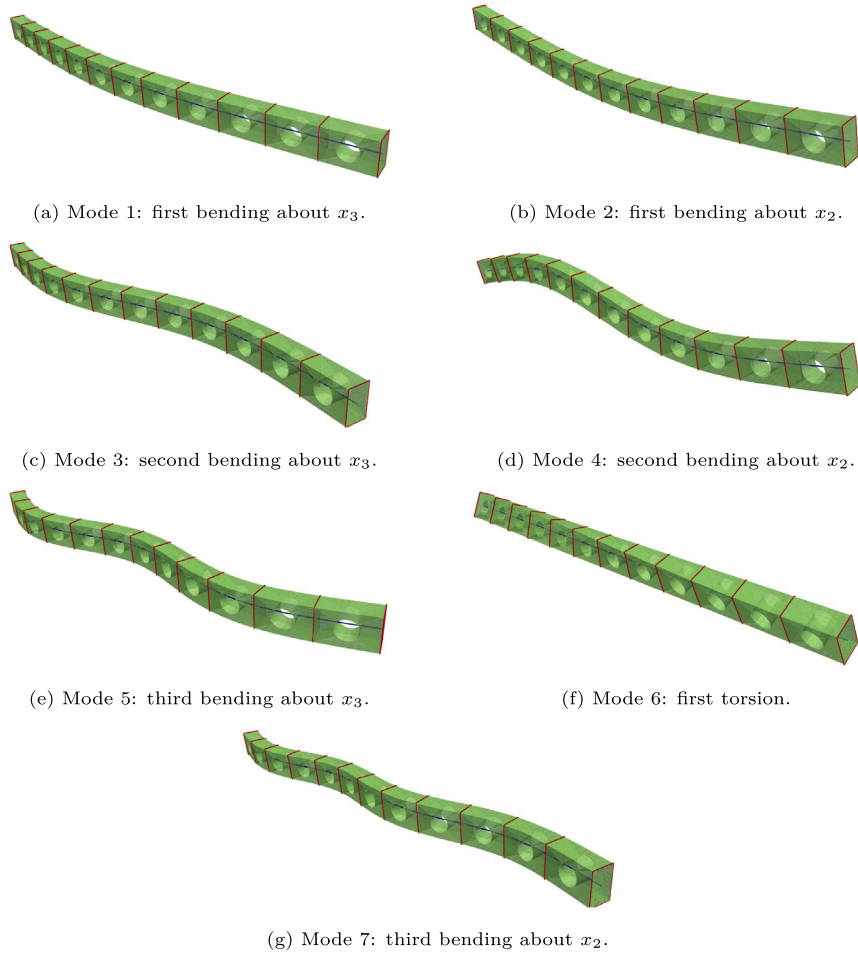


Fig. 12. Representative mode-shape comparisons between the 3D model (green) and the 1D equivalent beam model (axis in blue and sections used for MAC in red).

7.5. Computational considerations and model-size reduction

The computational assessment should distinguish the one-time characterization of a detailed periodic cell from the recurring analyses of the resulting one-dimensional structural model. Denoting by C_{off} the combined cost of finite-element matrix generation, Craig–Bampton condensation, and beam-property identification, and by N_a the number of subsequent structural analyses, the two workflows may be represented schematically as

$$C_{\text{reduced}} = C_{\text{off}} + N_a C_{1D}, \quad C_{\text{detailed}} = N_a C_{\text{FE}}. \quad (41)$$

Table 24

Representative algebraic dimensions for the rectangular perforated solid beam. The detailed model is assembled by merging the coincident interfaces of consecutive periods.

Quantity	Value
One-period finite-element DOFs	71 568
Internal DOFs eliminated by Craig–Bampton reduction	69 366
Retained $R_1 \cup R_2$ DOFs	2202
Order of the condensed interface matrices	2202×2202
Number of periods in the detailed validation model	12
Detailed twelve-period model DOFs	846 705
Equivalent beam nodes	25
Equivalent beam DOFs	150

This expression is intended only to distinguish offline and recurring operations; it does not assume a particular complexity exponent for the linear or eigenvalue solvers. The relative costs depend on matrix sparsity, solver type, factorization reuse, implementation, and computing platform.

In the present implementation, the assembled stiffness and mass matrices are generated by a conventional finite-element solver and imported into the authors' characterization tool, which currently performs both the Craig–Bampton reduction and the subsequent identification. The initial reduction eliminates the internal degrees of freedom and retains the two end interfaces $R_1 \cup R_2$. The computational dimension of the subsequent characterization is therefore governed primarily by the interface size, rather than by the complete three-dimensional or shell discretization. The two interfaces are finally represented by 12 rigid-handle variables, from which the 6×6 beam constitutive matrix and the corresponding inertial properties are obtained.

A representative quantitative assessment is provided by the intermediate mesh of the rectangular perforated solid beam. One periodic cell contains $n_c = 71\,568$ degrees of freedom, of which 2202 belong to $R_1 \cup R_2$; the Craig–Bampton step therefore eliminates 69 366 internal degrees of freedom and produces interface matrices of order 2202. Since the two end interfaces have equal size, one interface contains $n_f = 1101$ degrees of freedom. When N cells are assembled, the right interface of one cell is merged with the left interface of the following cell, and the detailed model size is

$$n_{\text{FE}}(N) = N(n_c - n_f) + n_f. \quad (42)$$

The one-dimensional models use one three-node beam element per period. Adjacent elements share their end nodes, so that an N -period model has $2N + 1$ beam nodes and

$$n_{\text{1D}}(N) = 6(2N + 1) \quad (43)$$

algebraic degrees of freedom. Table 24 summarizes the resulting dimensions for the twelve-period rectangular perforated beam used in the detailed-to-reduced comparison.

The reduction in algebraic model size is therefore more than three orders of magnitude for this example. This ratio is independent of processor speed and software optimization and is consequently a more portable indicator than wall-clock time. A similar downstream reduction applies to the eight-period spar model: its cells contain 26 864 shell elements and each regular end interface contains 161 nodes, whereas the corresponding one-dimensional model has 17 nodes and 102 degrees of freedom.

New wall-clock and peak-memory values are not used here as intrinsic performance measures. The present characterization code is a research implementation, and the measured values depend strongly on hardware, sparse-matrix libraries, ordering strategies, matrix input/output, and the degree of software optimization. A direct implementation-level comparison was previously reported for a prismatic solid section [9].

For an axially uniform beam, a dedicated two-dimensional cross-sectional formulation such as VABS [20] or ANBA [37] is therefore generally the more economical choice. The three-dimensional axial cell becomes useful when geometry, thickness, material distribution, holes, ribs, or local reinforcements vary within the physical period, so that no single fixed cross section represents the structure. In such cases, the one-time cell characterization retains the axial transitions and load-transfer mechanisms and can subsequently be reused in long structures, multiple load cases, modal analyses, parameter studies, or optimization loops. For laminated thin-walled structures, shell models provide an additional reduction by replacing the explicit through-thickness ply discretization with laminate constitutive properties while retaining the actual spanwise-periodic geometry; agreement with corresponding detailed three-dimensional models was demonstrated in the shell-characterization study [10].

7.6. Discussion

Taken together, the static and dynamic one-dimensional analyses clarify the intended use of the proposed characterization framework. The periodic unit cell is first reduced to equivalent constitutive and inertial properties through the handle-based identification procedure. These quantities are then assigned to an equivalent beam model, which can be used to predict the global response of finite periodic structures within the regimes for which a one-dimensional description is meaningful.

The static cantilever comparison shows that the characterized constitutive matrix captures the dominant response of the spar-like periodic structure with good accuracy. This case is particularly relevant because it involves an open thin-walled section, a

nontrivial shear-center location, and a realistic periodic geometry. The analysis also shows the practical importance of applying loads consistently with the identified beam properties, since the shear-center position can be used to avoid introducing spurious torsional effects in the comparison.

The dynamic comparisons play a complementary role by exercising the inertial part of the characterization. For the spar-like beam, the first global bending and torsional modes are reproduced with reasonable accuracy, but the comparison also reveals the expected limitations of an isolated beam representation for an open shell structure. Some modes of the detailed model contain local shell deformation or cross-sectional distortion and therefore do not admit a meaningful one-dimensional counterpart. This behavior is not a failure of the characterization procedure, but rather a consequence of the richer kinematics available in the shell model.

The rectangular perforated solid beam provides a more controlled dynamic verification. In this case, the low-frequency modes are more clearly beam-like, the section fitting errors remain small for most modes, and both the reduced and expanded MAC values confirm a strong correspondence between the three-dimensional and one-dimensional models. The improved agreement obtained for this case supports the use of the identified constitutive and inertial properties for low-frequency free-free modal analyses when the relevant modes are governed by global beam deformation.

Overall, the results indicate that the proposed framework provides predictive equivalent one-dimensional models for the global static and low-frequency dynamic behavior of periodic beams. At the same time, the comparisons clarify the limits of the approach: the equivalent beam model should not be interpreted as a surrogate for the full three-dimensional or shell spectrum, but as a reduced representation of the beam-like part of the structural response.

8. Conclusions

This paper presented a finite-element framework for the characterization of periodic straight beams from representative axial cells. Building on a previously developed matrix-based characterization strategy for beam-like structures, the specific contribution of the present work is its formulation and assessment for cells with spanwise periodic heterogeneity. The framework provides the full equivalent beam constitutive matrix and the associated inertial properties, and applies the same algebraic procedure to both three-dimensional solid and shell finite-element models.

The verified findings can first be summarized from the benchmark studies. For sandwich beams with perforated cores and thin-walled box beams with and without transverse ribs, the identified dominant stiffness coefficients were consistent with the available literature results. For the thin-walled box beams, the solid- and shell-based characterizations also produced closely matching dominant constitutive terms, showing that shell discretizations can be used without altering the overall identification procedure for the thin configurations considered here. For the unsymmetrical corrugated-strip-core beam, the validation was extended to all six diagonal coefficients and the nonzero torsion-bending coupling coefficient D_{46} , including direct comparisons of both transverse-shear stiffnesses and torsion. All seven coefficients agreed within 2% with both reference methods, while the limited two-level mesh-refinement check changed them by at most 0.665%.

The spar-like periodic beam provided a more demanding application case. A systematic translation of the periodic cut showed that the axial stiffness, bending stiffnesses, extension-bending coupling, and shear-center-referred torsional rigidity remain highly stable. The transverse-shear block and the extension-shear coupling were more sensitive when the local feature approached the retained interfaces, and the largest differences occurred for the interface-intersecting Cut B configuration. The total unit-cell mass remained invariant, whereas the transverse barycentric inertias changed with the allocation of the local feature within the selected finite cell. These results quantify both the practical robustness and the limitations of the unit-cell-based characterization and indicate that interfaces should preferably be placed away from strong local discontinuities. The identified constitutive matrix also provided the shear-center location used to define mechanically consistent loading conditions for the static comparison.

The predictive comparisons further verified the use of the characterized properties in equivalent one-dimensional models of finite periodic beams. The static cantilever analysis reproduced the dominant global response of the spar-like structure. In the free-free modal comparisons, the open thin-walled spar highlighted both the predictive capability and the limits of an isolated one-dimensional representation, whereas the rectangular perforated solid beam provided a controlled verification case with strong agreement in frequencies and mode shapes for the first global beam-like modes.

From a practical standpoint, the framework provides a route from standard finite-element stiffness and mass matrices to equivalent beam properties that can be transferred directly to reduced one-dimensional structural models. Its unified use with solid and shell discretizations allows the finite-element representation of the periodic cell to be selected according to its geometry and modeling requirements, while retaining the same characterization workflow.

The characterization represents a one-time offline computation. Once the effective constitutive and inertial properties have been identified, they can be reused without repeating the detailed-cell reduction in multiple static load cases, low-frequency dynamic analyses, parametric studies, or different structures assembled from the same periodic cell. This reuse is particularly advantageous for long structures and for applications requiring repeated dynamic or parametric analyses, since the initial characterization cost is amortized over the subsequent one-dimensional computations.

This computational advantage must nevertheless be interpreted within the intended range of validity of the equivalent model, which is restricted to global static response and low-frequency dynamics governed by beam-like deformation. The model is not intended to reproduce local sectional or shell deformation modes, higher-frequency wave-propagation effects, strongly dispersive regimes, or frequency-dependent dynamic homogenization. These limitations become particularly relevant when local deformation is comparable to the global beam motion, as observed for some modes of the isolated open thin-walled spar. Future work will address more complete structural assemblies, including ribbed and closed-box configurations, and will further investigate the role of periodic inertial characterization in dynamic reduced-order models.

Table A.25

Diagonal constitutive coefficients for the sandwich-beam benchmark cases.

	Square-hole core	Circular-hole core
D_{11}/N	$2.658 \cdot 10^{10}$	$2.651 \cdot 10^{10}$
D_{22}/N	$8.329 \cdot 10^9$	$8.442 \cdot 10^9$
D_{33}/N	$5.418 \cdot 10^8$	$9.994 \cdot 10^8$
$D_{44}/(N\text{ m}^2)$	$7.205 \cdot 10^8$	$9.704 \cdot 10^8$
$D_{55}/(N\text{ m}^2)$	$5.575 \cdot 10^{10}$	$5.553 \cdot 10^{10}$
$D_{66}/(N\text{ m}^2)$	$4.978 \cdot 10^9$	$4.979 \cdot 10^9$

Table A.26

Three-dimensional unit-cell mass properties for the sandwich-beam benchmark cases. All values are expressed in SI units.

	Square-hole core	Circular-hole core
M/kg	$4.965 \cdot 10^3$	$4.995 \cdot 10^3$
$I_{11}^{(3D)}/(\text{kg m}^2)$	$6.976 \cdot 10^3$	$7.014 \cdot 10^3$
$I_{22}^{(3D)}/(\text{kg m}^2)$	$7.288 \cdot 10^3$	$7.337 \cdot 10^3$
$I_{33}^{(3D)}/(\text{kg m}^2)$	$2.174 \cdot 10^3$	$2.196 \cdot 10^3$

Note: The inertia tensor components are reported as three-dimensional unit-cell quantities referred to the unit-cell center of gravity.

CRedit authorship contribution statement

Claudio Caccia: Writing – original draft, Validation, Software, Data curation. **Pierangelo Masarati:** Writing – review & editing, Writing – original draft, Supervision, Formal analysis, Conceptualization. **Marco Morandini:** Writing – review & editing, Methodology, Investigation.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author(s) used OpenAI's ChatGPT for language editing, text refinement, and assistance in improving the clarity and organization of selected parts of the manuscript. The authors reviewed and edited the output as needed and take full responsibility for the content of the published article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Complete constitutive matrices for the benchmark cases

The validation discussed in Section 5 focuses on the constitutive terms that are directly comparable with the quantities reported in the benchmark literature. Since the proposed procedure identifies the full constitutive matrix of each periodic unit cell, the complete matrices obtained for the benchmark cases are collected in this appendix for completeness. Their inclusion serves two purposes. First, it documents the full output of the characterization beyond the dominant terms usually reported in benchmark studies. Second, it provides a reference dataset that may be useful for future comparisons with alternative homogenization procedures, discretization strategies, or beam formulations.

A.1. Perforated-core sandwich and prismatic box beams

The perforated-core sandwich-beam and prismatic-box benchmark families possess two orthogonal planes of symmetry, and the beam reference line and transverse axes are selected consistently with these symmetries. Their constitutive matrices are therefore diagonal: all extension–shear, extension–bending, shear–torsion, shear–bending, torsion–bending, and bending–bending coupling coefficients vanish identically. Consequently, the six diagonal coefficients constitute the complete 6×6 constitutive matrices for these symmetric cases. The same symmetries place the center of mass on the reference line and cause the products of inertia to vanish. In contrast, the unsymmetrical corrugated-strip-core benchmark exhibits additional coupling terms, and its complete constitutive matrix is therefore reported separately in Table A.29.

The diagonal constitutive coefficients of the perforated-core sandwich-beam and prismatic-box benchmarks are reported in Tables A.25 and A.27, respectively. The complete constitutive matrix of the unsymmetrical corrugated-strip-core benchmark is given in Table A.29. The corresponding three-dimensional unit-cell mass properties of the symmetric benchmark cases are reported in Tables A.26 and A.28.

Table A.27

Diagonal constitutive coefficients identified for the prismatic box-beam benchmark cases. All values are expressed in SI units.

	Without ribs	With ribs
D_{11}/N	$1.032 \cdot 10^{10}$	$1.053 \cdot 10^{10}$
D_{22}/N	$2.449 \cdot 10^9$	$2.459 \cdot 10^9$
D_{33}/N	$8.872 \cdot 10^8$	$8.925 \cdot 10^8$
$D_{44}/(\text{N m}^2)$	$1.709 \cdot 10^9$	$1.719 \cdot 10^9$
$D_{55}/(\text{N m}^2)$	$1.914 \cdot 10^9$	$1.936 \cdot 10^9$
$D_{66}/(\text{N m}^2)$	$5.576 \cdot 10^9$	$5.649 \cdot 10^9$

Table A.28

Three-dimensional unit-cell mass properties for the prismatic box-beam benchmark cases. All values are expressed in SI units.

	Without ribs	With ribs
M/kg	$3.983 \cdot 10^2$	$5.233 \cdot 10^2$
$I_{11}^{(3D)}/(\text{kg m}^2)$	$2.889 \cdot 10^2$	$3.379 \cdot 10^2$
$I_{22}^{(3D)}/(\text{kg m}^2)$	$1.070 \cdot 10^2$	$1.164 \cdot 10^2$
$I_{33}^{(3D)}/(\text{kg m}^2)$	$2.483 \cdot 10^2$	$2.879 \cdot 10^2$

Note: The inertia tensor components are reported as three-dimensional unit-cell quantities referred to the unit-cell center of gravity.

Table A.29

Complete constitutive matrix identified for the unsymmetrical corrugated-strip-core sandwich-beam unit cell using the baseline 12 008-element discretization.

$8.328 \cdot 10^{10}$		$-4.363 \cdot 10^7$		$1.015 \cdot 10^4$	
	$2.610 \cdot 10^{10}$		$-3.599 \cdot 10^7$		$-2.637 \cdot 10^7$
		$1.167 \cdot 10^8$		$-5.599 \cdot 10^4$	
			$3.718 \cdot 10^8$		$5.330 \cdot 10^7$
				$8.324 \cdot 10^{10}$	
					$2.768 \cdot 10^{10}$

Note: Only upper-triangular coefficients with absolute value not smaller than 1 are reported. Blank entries denote lower-triangular terms or values below the reporting threshold. Coefficients in the translational 3×3 block have units of N, translation–rotation coupling coefficients have units of Nm, and coefficients in the rotational 3×3 block have units of Nm^2 .

A.2. Unsymmetrical corrugated-strip-core sandwich beam

The complete constitutive matrix identified for the unsymmetrical corrugated-strip-core benchmark using the baseline 12 008-element discretization is reported in Table A.29. Only the upper-triangular terms are shown because the matrix is symmetric to numerical precision. The seven coefficients for which published reference values are available are compared in Table 9; the additional terms reported here document the complete output of the present characterization and provide reference data for future comparisons.

Among the coefficients not tabulated in the reference comparison, the most significant are D_{13} , D_{24} , and D_{26} . Their occurrence is qualitatively consistent with the geometric asymmetry introduced by the inclined core web. In the adopted ordering, D_{13} couples axial extension with transverse shear along x_3 ; D_{24} couples transverse shear along x_2 with torsion, consistent with a small offset between the selected reference line and the shear center; and D_{26} couples transverse shear along x_2 with bending about x_3 . The pattern of zero and nonzero coefficients respects the remaining reflection symmetry about the x_1 - x_3 plane. These terms are therefore symmetry-admissible and physically interpretable. Their changes under mesh refinement are 0.75%, 1.87%, and 3.40%, respectively, indicating that their precise magnitudes are more mesh-sensitive than the principal stiffness terms.

Comparison with the complete MSG matrix² confirms the same symmetry-admissible coupling pattern. The coefficient D_{13} agrees closely, with values of -4.363×10^7 N from the present method and -4.400×10^7 N from MSG, corresponding to a difference of 0.85%. Larger differences are found for D_{24} and D_{26} : the present values are -3.599×10^7 N m and -2.637×10^7 N m, respectively, whereas MSG gives 7.536×10^5 N m and -1.071×10^5 N m. Using $c_{ij} = |D_{ij}|/\sqrt{D_{ii}D_{jj}}$, their normalized magnitudes are 1.155% and 0.098% for the present matrix, compared with 0.024% and 0.0004% for MSG. These differences do not affect the close agreement obtained for the seven published coefficients, but they remain relevant to the weaker shear-related couplings and, in the case of D_{24} , to the inferred shear-center location. The limited mesh-refinement check does not remove these differences. A conclusive assessment would require a dedicated finite-beam analysis with independently defined load transfer and checks of beam length, boundary effects, and mesh resolution; this verification is left for future work.

² The complete set of off-diagonal coefficients associated with the MSG solution of Liu et al. [8] was kindly provided by the authors through personal communication and is used here with their permission. Their cooperation is gratefully acknowledged.

Table B.30

Complete constitutive matrix for the spar-like beam, Cut A, Kirchhoff–Love formulation.

$1.703 \cdot 10^7$	$2.275 \cdot 10^1$ $9.819 \cdot 10^5$				$5.663 \cdot 10^4$ $4.570 \cdot 10^1$
		$2.989 \cdot 10^6$	$1.513 \cdot 10^4$ $8.811 \cdot 10^1$	$-8.777 \cdot 10^1$ $3.875 \cdot 10^4$	
					$7.524 \cdot 10^2$

Note: Only upper-triangular terms with absolute value not smaller than 1 are reported. Blank entries indicate lower-triangular terms or values below the reporting threshold. Values are expressed in SI units.

Table B.31

Complete constitutive matrix for the spar-like beam, Cut B, Kirchhoff–Love formulation.

$1.703 \cdot 10^7$	$1.105 \cdot 10^2$ $8.390 \cdot 10^5$				$5.664 \cdot 10^4$ $5.997 \cdot 10^1$
		$3.141 \cdot 10^6$	$1.588 \cdot 10^4$ $9.180 \cdot 10^1$	$-2.039 \cdot 10^1$ $3.877 \cdot 10^4$	
					$7.521 \cdot 10^2$

Note: Only upper-triangular terms with absolute value not smaller than 1 are reported. Blank entries indicate lower-triangular terms or values below the reporting threshold. Values are expressed in SI units.

Table B.32

Complete constitutive matrix for the spar-like beam, Cut A, Mindlin–Reissner formulation.

$1.699 \cdot 10^7$	$2.760 \cdot 10^1$ $9.821 \cdot 10^5$				$5.647 \cdot 10^4$ $2.402 \cdot 10^1$
		$2.955 \cdot 10^6$	$1.516 \cdot 10^4$ $8.734 \cdot 10^1$	$-3.931 \cdot 10^1$ $3.875 \cdot 10^4$	
					$7.519 \cdot 10^2$

Note: Only upper-triangular terms with absolute value not smaller than 1 are reported. Blank entries indicate lower-triangular terms or values below the reporting threshold. Values are expressed in SI units.

Table B.33

Complete constitutive matrix for the spar-like beam, Cut B, Mindlin–Reissner formulation.

$1.699 \cdot 10^7$	$-6.489 \cdot 10^2$ $8.400 \cdot 10^5$				$5.647 \cdot 10^4$ $3.336 \cdot 10^1$
		$3.103 \cdot 10^6$	$1.591 \cdot 10^4$ $9.109 \cdot 10^1$	$-1.059 \cdot 10^1$ $3.875 \cdot 10^4$	
					$7.517 \cdot 10^2$

Note: Only upper-triangular terms with absolute value not smaller than 1 are reported. Blank entries indicate lower-triangular terms or values below the reporting threshold. Values are expressed in SI units.

Appendix B. Elastic and inertial properties of the spar-like periodic beam

This appendix collects the equivalent one-dimensional properties identified for the spar-like periodic beam. The reported data complement the discussion presented in Sections 6 and 7 by providing the complete constitutive and inertial quantities associated with the characterized periodic unit cell. Their inclusion serves two purposes. First, it documents the full output of the characterization procedure beyond the subset of coefficients emphasized in the main text. Second, it provides a compact reference dataset for future comparisons with alternative unit-cell definitions, discretization strategies, and one-dimensional beam models.

B.1. Equivalent constitutive matrices

Tables B.30, B.31, B.32, and B.33 report the detailed equivalent constitutive matrices identified for the spar-like periodic beam using the two alternative cut definitions and the two shell kinematic descriptions considered in the paper.

B.2. Derived beam quantities

The derived beam quantities obtained from the constitutive matrices are summarized in Table B.34. These include, in particular, the shear-center coordinates with respect to the chosen beam reference line. As discussed in the main text, these quantities are useful not only for physical interpretation of the identified coupling terms, but also for the consistent definition of loading conditions in the one-dimensional and three-dimensional validation tests.

Table B.34

Derived beam quantities for the spar-like periodic beam.

Cut	Formulation	$x_{2,sc}$ (m)	$x_{3,sc}$ (m)
Cut A	Kirchhoff–Love	$5.061 \cdot 10^{-3}$	$-7.479 \cdot 10^{-1} 1$
Cut A	Mindlin–Reissner	$5.131 \cdot 10^{-3}$	$-3.323 \cdot 10^{-1} 1$
Cut B	Kirchhoff–Love	$5.055 \cdot 10^{-3}$	$-1.111 \cdot 10^{-1} 0$
Cut B	Mindlin–Reissner	$5.126 \cdot 10^{-3}$	$-4.121 \cdot 10^{-1} 2$

Note: The shear-center coordinates are computed from the stiffness matrix using the formulation in [30].

Table B.35

Mass, center-of-gravity coordinates, and barycentric three-dimensional inertia components of the $L_c = 100$ mm spar unit cells. All values are extracted from the Mindlin–Reissner mass matrices.

	Cut A	Cut B
M /kg	$7.88920 \cdot 10^{-2}$	$7.88920 \cdot 10^{-2}$
$x_{1,G}$ /m	$5.00000 \cdot 10^{-2}$	$5.000000 \cdot 10^{-2}$
$x_{2,G}$ /m	$-2.88106 \cdot 10^{-3}$	$-2.88107 \cdot 10^{-3}$
$I_{11}^{(3D)}$ /(kg m ²)	$1.53835 \cdot 10^{-4}$	$1.53823 \cdot 10^{-4}$
$I_{22}^{(3D)}$ /(kg m ²)	$2.22295 \cdot 10^{-4}$	$2.10956 \cdot 10^{-4}$
$I_{33}^{(3D)}$ /(kg m ²)	$7.29863 \cdot 10^{-5}$	$6.16443 \cdot 10^{-5}$

Note: The reported quantities are properties of the complete finite unit cell. The inertia tensor is evaluated about the center of gravity of each configuration and expressed in the same fixed beam axes. The products $I_{ij}^{(3D)}$, for $i \neq j$, vanish by symmetry and are omitted. The Cut A/Cut B differences in the transverse barycentric inertias reflect the different axial allocation of the local feature.

Table C.36

Complete nonzero constitutive coefficients of the one-period rectangular perforated solid-beam unit cell for the four mesh discretizations. Powers of ten are factored into the column headings.

Mesh	$D_{11}/(10^7 \text{ N})$	$D_{22}/(10^7 \text{ N})$	$D_{33}/(10^6 \text{ N})$	$D_{44}/(10^3 \text{ N m}^2)$	$D_{55}/(10^3 \text{ N m}^2)$	$D_{66}/(10^3 \text{ N m}^2)$
Coarse	3.5929	4.3289	6.2733	1.8209	8.0245	1.9164
Intermediate	3.5914	4.3324	6.2622	1.8200	8.0232	1.9159
Fine	3.5911	4.3331	6.2603	1.8198	8.0229	1.9158
Finest	3.5911	4.3330	6.2606	1.8198	8.0229	1.9159

B.3. Equivalent inertial properties

The mass properties associated with the spar-like periodic beam are reported in Table B.35. All quantities are extracted from the assembled mass matrices of the Mindlin–Reissner model used for the constitutive characterization and the translated-cut study. The table reports the actual mass M of the finite unit cell, its nonzero center-of-gravity coordinates, and the diagonal barycentric three-dimensional inertia components expressed in the fixed beam axes.

Appendix C. Rectangular perforated periodic beam

This appendix reports the detailed constitutive and inertial results associated with the mesh-convergence and multiperiod studies of Section 7.4.2. The finest one-period discretization provides the properties used in the modal comparison of Section 7.4.3. Owing to the two orthogonal symmetry planes of the geometry and the corresponding choice of reference axes, all constitutive coupling coefficients and products of inertia vanish. For every one-period discretization, the center of gravity is located at $(x_G, y_G, z_G) = (0.03, 0, 0)$ m.

C.1. Mesh-convergence results

Tables C.36 and C.37 report, respectively, the six nonzero constitutive coefficients and the three-dimensional mass properties obtained with the four one-period discretizations.

The complete values confirm that the small residual mesh dependence is concentrated in the constitutive coefficients, particularly in transverse shear, whereas the mass and barycentric inertia are already stable on the coarsest discretization.

Table C.37

Three-dimensional mass and principal barycentric inertia components of the one-period rectangular perforated solid-beam unit cell for the four mesh discretizations.

Mesh	M/kg	$I_{11}^{(3D)}/\text{kg m}^2$	$I_{22}^{(3D)}/\text{kg m}^2$	$I_{33}^{(3D)}/\text{kg m}^2$
Coarse	0.1262053	$2.573853 \cdot 10^{-5}$	$6.528135 \cdot 10^{-5}$	$5.165853 \cdot 10^{-5}$
Intermediate	0.1262052	$2.573852 \cdot 10^{-5}$	$6.528133 \cdot 10^{-5}$	$5.165852 \cdot 10^{-5}$
Fine	0.1262052	$2.573852 \cdot 10^{-5}$	$6.528133 \cdot 10^{-5}$	$5.165852 \cdot 10^{-5}$
Finest	0.1262052	$2.573852 \cdot 10^{-5}$	$6.528133 \cdot 10^{-5}$	$5.165852 \cdot 10^{-5}$

Table C.38

Complete nonzero constitutive coefficients obtained from characterization domains containing one, two, and four complete periods. Powers of ten are factored into the column headings.

n_p	$D_{11}/(10^7 \text{ N})$	$D_{22}/(10^7 \text{ N})$	$D_{33}/(10^6 \text{ N})$	$D_{44}/(10^3 \text{ N m}^2)$	$D_{55}/(10^3 \text{ N m}^2)$	$D_{66}/(10^3 \text{ N m}^2)$
1	3.591366	4.332369	6.262216	1.819965	8.023150	1.915921
2	3.591366	4.332369	6.262216	1.819965	8.023150	1.915921
4	3.591366	4.332369	6.262216	1.819965	8.023150	1.915921

Table C.39

Actual mass and principal barycentric inertia components of characterization domains containing one, two, and four complete periods. The quantities are not normalized by the domain length.

n_p	L/m	M/kg	$I_{11}^{(3D)}/\text{kg m}^2$	$I_{22}^{(3D)}/\text{kg m}^2$	$I_{33}^{(3D)}/\text{kg m}^2$
1	0.06	0.12620519	$2.573852 \cdot 10^{-5}$	$6.528 \cdot 10^{-5}$	$5.166 \cdot 10^{-5}$
2	0.12	0.25241038	$5.147703 \cdot 10^{-5}$	$3.577 \cdot 10^{-4}$	$3.305 \cdot 10^{-4}$
4	0.24	0.50482075	$1.029541 \cdot 10^{-4}$	$2.533 \cdot 10^{-3}$	$2.478 \cdot 10^{-3}$

C.2. Multiperiod results

Tables C.38 and C.39 report the constitutive and three-dimensional mass properties obtained from characterization domains containing one, two, and four complete periods. The intermediate mesh resolution of Table 21 was retained within each period.

The two- and four-period characterization domains have, respectively, twice and four times the one-period mass and axial barycentric inertia. The transverse barycentric inertia components additionally contain the parallel-axis contribution associated with the increased axial separation of the individual periods, as discussed in Section 7.4.2.

These data provide the complete constitutive and inertial results used in the convergence and multiperiod assessments of the rectangular perforated periodic beam.

Data availability

Data will be made available on request.

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