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A heuristic algorithm to solve the one-warehouse multiretailer problem with an emission constraint

Matthieu Gruson[a,*], Qihua Zhong[b], Ola Jabali[c] and Raf Jans[b]

[a] *École des Sciences de la gestion, Université du Québec à Montréal, C.P. 8888, succursale Centre-ville, Montréal H3C 3P8, Canada*
[b] *HEC Montréal and GERAD, 3000 Chemin de la Côte-Sainte-Catherine, Montréal H3T 2A7, Canada*
[c] *Politecnico di Milano, Piazza Leonardo da Vinci, 32, Milan 20133, Italy*
E-mail: gruson.matthieu@uqam.ca[Gruson]; qihua.zhong@hec.ca[Zhong]; ola.jabali@polimi.it[Jabali]; raf.jans@hec.ca[Jans]

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Abstract

In this paper, we consider the one-warehouse multiretailer problem with a global carbon emission cap constraint (OWMR-EC). This constraint aims at limiting the carbon emissions related to the production, setup, and inventory-holding operations. We develop a penalized relaxation (PR) method to heuristically solve the considered problem, both with and without the possibility of having initial inventory. This heuristic uses in itself another heuristic that we propose to solve the standard one-warehouse multiretailer problem (OWMR). Our PR method is tested on numerous instances adapted from the literature. Our results indicate that the penalized method is able to find between 87.4% and 89.8% of feasible solutions for this NP-hard problem, with an average optimality gap of 2.1% and 2.2% depending on the algorithms we use to solve the different subproblems involved in the method. The results show that our method is highly effective in terms of run-time and solution quality, when a feasible solution is found. Furthermore, the results indicate that the heuristic for the standard OWMR is also very effective. We further perform a sensitivity analysis on the optimal solutions of the OWMR-EC to better understand the implications of the carbon emission cap constraint. The sensitivity analysis indicates that the marginal cost of reducing carbon emissions increases as the emission cap decreases. The analysis also shows that the correlation between the cost and emission parameters has an important impact on the potential to further lower the emissions, compared to the emission of the minimum cost solution.

Keywords: heuristic; one-warehouse multiretailer; emission constraint; lot sizing

1. Introduction

Over the last decades, there has been a growing interest in incorporating sustainability issues in supply chain management. The main concerns relate to global warming and greenhouse gas (GHG)

*Corresponding author.

emissions which, if left at their current level, will lead to climate changes as stated by Voiland (2009). Therefore, several countries have engaged in reducing these GHG emissions and formalizing it with treaties such as the Kyoto protocol to the United Nations (2009), which sets a GHG reduction goal. More recently, the Paris Agreement of the United Nations (2018) aims at limiting the global temperature increase within 2% by 2100. This agreement has been signed by 195 of the 197 countries that compose the United Nations. At another level, the wish to have a “greener” image has led individual companies to reduce their carbon footprint and engage in more environmental friendly production processes. This shift toward a “greener” image has been observed by Velázquez-Martínez et al. (2014) who indicate that many companies publicly report their level of carbon emissions and indicate their carbon emission target.

Following global concern and actions, operations research (OR) practitioners have put efforts to include these environmental issues in models and methods. Barbosa-Póvoa, et al. (2017) offer a review of OR applications toward the achievement of a greener supply chain. They highlight the numerous applications of OR in green supply chain management at the strategic level and point out that less attention has been paid to the impact OR can have at the tactical or operational level in reducing GHG emission levels. However, as mentioned by Absi et al. (2013), there are many potential benefits at those two levels to reduce GHG emissions by changing operational decisions.

The issues related to green supply chain optimization have drawn the attention of lot sizing researchers (see Suzanne et al., 2020). The basic lot sizing problem (LSP), which works on both tactical and operational levels, determines, given an ordered discrete set of time periods T, the optimal timing and quantities to be produced in order to satisfy a deterministic and dynamic demand for some items. The objective is to minimize the sum of the production setup costs incurred each time there is production, and of the inventory holding costs to carry one item from one period to the next one. This basic LSP has attracted a lot of research since the seminal paper of Wagner and Whitin (1958) who proposed a dynamic programming approach to solve the single-item uncapacitated lot sizing problem (SI-ULSP). The reader is referred to Brahimi et al. (2017) and Pochet and Wolsey (2006) for a review of the work done on the SI-ULSP and its extensions, respectively, to Aloulou et al. (2014) for a review of nondeterministic models, and to Jans and Degraeve (2006) for a review of industrial applications.

Carbon emissions considerations have been introduced into the basic lot sizing models, primarily through constraints that limit the GHG emissions or through penalties in the objective function. The emissions considered relate to production (e.g., by using some machinery), setup (e.g., by using some extra power to set up a machine), and inventory holding (e.g., by using cooling or heating systems). One of the earliest work is the one of Benjaafar et al. (2013) who study the impact of different policies on carbon emissions. The policies include the carbon cap policy (where the total carbon emission is limited by a fixed amount), the carbon tax policy (where there is a tax paid per unit of cabon emitted), the carbon cap-and-trade policy (where companies can emit more than the allowed cap but have to pay for it, or reversely firms that do not emit beyond the cap can sell their unused carbon units), and the carbon offset policy (where it is possible to buy carbon units from independent suppliers and/or invest into projects whose goal is to reduce carbon emissions). They give managerial insights into the impacts of these policies for companies.

Later, Absi et al. (2013) incorporate carbon emission constraints in the basic SI-ULSP model. They impose emission limits globally, per period, and on a rolling horizon basis. They study the complexity of each resulting problem and provide a dynamic programming algorithm for the case with the emission limit per period. In a similar way, Retel Helmrich et al. (2015) address the SI-ULSP with an emission cap constraint imposed on the whole planning horizon. They analyze the complexity of the problem and derive structural properties of the optimal solution. These properties are used in a lagrangian heuristic to provide both lower and upper bounds to the problem. They also develop fully polynomial approximation schemes. Very recently, Dong and Yuan (2023) have addressed the case of lot sizing and supplier selection under a carbon cap-and-trade constraint. The authors argue that supplier selection may be good from an economic point of view but also from a sustainable point of view. They consider demand uncertainty and use a distributionally robust approach to solve their problem.

Carbon emission considerations have also been integrated in LSPs that consider several production modes. In particular, Absi et al. (2016) have extended the work of Absi et al. (2013) to the case where several production modes are available, each mode having its own emission and cost characteristics. The authors prove several structural properties of the optimal solutions and give two dynamic programming algorithms to solve the problem, one where both costs and emission parameters are stationary, and one where only the emission parameters are stationary. In a similar spirit, Hong et al. (2016) consider two production modes: one standard mode and one mode with green technologies. They consider both carbon cap policy and cap-and-trade policy. In the carbon cap policy scheme, they derive structural properties of the optimal solution and use a decomposition approach to develop a dynamic programming algorithm to solve the model to optimality. In the same vein, Phouratsamay and Cheng (2019) study the SI-ULSP with two production modes and a carbon emission cap. One production mode is regular while the other is greener. The authors develop a dynamic programming algorithm to solve the problem.

Other extensions of the basic LSP incorporate carbon emission features. Zouadi et al. (2018) consider carbon emission constraints in an LSP with remanufacturing options. The items remanufactured come from defects during production or from products sent back by the customers. There are carbon emissions considered for both production and the transportation activities. The transportation activities relate to items that are returned by the end customers to the production facility. The authors propose a mathematical model and develop three different heuristics to solve the problem. They compare the results obtained by their heuristics to those obtained by a general purpose solver. Purohit et al. (2016) consider a stochastic LSP and impose carbon emission restrictions under a cap-and-trade system. They further integrate a cycle service-level constraint that imposes a minimum probability that at the end of every period the net inventory will not be negative. They analyze the impact of this service-level constraint and of the coefficient of variation of the demand on the level of emissions.

Carbon emission considerations also appear in multilevel LSPs. In multilevel LSPs, the items are produced in one facility and then sent to the end customer, located in a different place, through other facilities. The transportation activities between each facility and the end customer imply carbon emissions. The work of Memari et al. (2016) is one such example. They address a three-level LSP with emissions coming from the transportation of goods between facilities. They develop a biobjective model that minimizes both the operational costs of the whole system and the carbon emissions. They also consider a just-in-time distribution policy. This policy favors frequent and small shipments to the end customer in order to deliver the items just before they are required. The integration of carbon constraints for multiechelon problems can also be found in the inventory control literature. The interested reader is referred to Sarkar et al. (2016) and Bouchery et al. (2017).

The incorporation of carbon emission constraints in LSPs is also possible from the economic order quantity (EOQ) point of view. The EOQ quantity has been introduced by Harris (1913) and is the quantity that minimizes the setup and inventory holding costs while considering static demand over an infinite time horizon. In particular, Chen et al. (2013) compare the economic lot size obtained with carbon emission considerations to the classical EOQ. They show that it is possible to significantly reduce the carbon emissions with a small increase in cost. They further discuss this emission constrained EOQ under different carbon emission policies. Hua et al. (2011) derive the EOQ under a cap-and-trade policy. They analyze the impact of the emission-related costs of the carbon trade prices and of the carbon cap on ordering decisions. Rather than integrating operational costs and carbon emission costs in a single objective, Bouchery et al. (2012) consider the EOQ model with two different objectives: one that minimizes the costs and the other that minimizes the carbon emissions. They propose a procedure whose goal is to provide a satisfactory solution to a decision maker. They perform a sensitivity analysis to test the robustness of their procedure. They also extend the model they propose to a two-level problem in series. In the two-level problem in series, there is one plant that produces items that are sent to a unique end customer.

The purpose of this paper is to contribute to the green lot sizing literature by integrating a carbon emission constraint in the one-warehouse multiretailer problem (OWMR). In the OWMR, a central warehouse replenishes several retailers that face a dynamic demand for one or several items over a discrete and finite time horizon. The objective of the problem is to determine the optimal timing and quantities to be produced at the warehouse and to be shipped between the warehouse and the retailers, while minimizing, for the whole system, the sum of production and delivery setup costs, as well as the inventory holding costs. Unlike the SI-ULSP, the OWMR problem works on two levels of a supply chain: the central warehouse and the retailers. The OWMR has been shown to be NP-hard by Arkin et al. (1989). Therefore, the problem we address here, which we call the OWMR-EC, is also an NP-hard problem. The motivation to work on the OWMR-EC is to find an efficient way to solve this NP-hard problem. Specifically, our aim is to develop an easy to reproduce and efficient heuristic.

Our scientific contributions are threefold. First, we fill a gap by integrating a global carbon emission constraint in the OWMR with a finite horizon. To the best of our knowledge, this is the first attempt to impose such constraint in this problem. Note that in an inventory management oriented context, Li and Hai (2019) integrate carbon emission costs in the OWMR with an infinite time horizon. They then propose a power-of-two policy to solve their nonlinear problem. Our second contribution is to develop an efficient heuristic algorithm to solve the OWMR-EC. The heuristic we propose is tested on numerous instances to assess its strengths and weaknesses. The instances we use in the experiments are adapted from existing instances used in the OWMR literature. Furthermore, we perform a sensitivity analysis on these instances to better understand the implications of the emission cap for the OWMR. Our third contribution is a by-product of the general heuristic we develop to solve the OWMR-EC, and is an efficient two-stage heuristic to solve the traditional OWMR.

The remainder of this paper is organized as follows. In Section 2, we give a formal mathematical model for the OWMR-EC. The two-stage heuristic we develop to solve the OWMR is presented in Section 3 while the general heuristic we propose to solve the OWMR-EC is described in Section 4.

Table 1
Parameters used in the mathematical model

Parameter	Definition
ec	Global emission cap over the entire planning horizon
es	Carbon emission for a production setup at the warehouse
ed_t^r	Carbon emissions for a delivery setup, which occurs in period t for retailer r
eh_t^i	Carbon emissions for holding one unit of inventory at facility i in period t
sc_t^i	Setup cost for facility i in period t
hc_t^i	Cost for holding one unit of inventory in facility i at the end of period t

Both heuristics are tested on numerous instances, and the results of these experiments are presented in Section 5, together with the results of the sensitivity analysis. This is followed by the conclusions in Section 6.

2. Mathematical formulation

Let $G = (F, A)$ be a graph with F the set of nodes (facilities in our problem) and A the set of arcs. Let $W = \{w\} \subset F$ be the set containing the unique warehouse and $R \subset F$ be the set of retailers. Following the problem description in Section 1, we have $F = W \cup R$. Note that in the rest of the manuscript, the superscript w will always refer to the single warehouse. Let d_t^r be the demand for retailer r in period $t \in T$ (we consider a single-item setting). Recall that T is the ordered discrete set of time periods. The notion of the demand faced by retailers is extended to the warehouse in the following fashion: $d_t^w = \sum_{r \in R} d_t^r$. We further denote d_{kt}^w as the cumulative demand for the warehouse between periods k and t ($k \leq t$), computed as $d_{kt}^w = \sum_{l=k}^{t} d_l^w$

To account for carbon emissions, we take a similar approach as the one used in Palak et al. (2014) who study the impacts of carbon regulatory policies on mode and supplier selection decisions in the context of a biofuel supply chain. They consider a fixed and a variable component in the transportation related emissions, where the fixed part mainly comes from the loading and unloading processes. They also consider inventory-related emissions due to heating or cooling systems at a facility. In our model, we consider that the emissions from producing one item at the warehouse or transporting it to any retailer are constant through time. Therefore, we do not integrate such related emissions in our model. Indeed, as all demands must be satisfied, the emissions coming from such activities would result in a constant. Thus, our model specifically accounts for emissions from delivery and production setups as well as inventory holding at the warehouse and at the retailers. The former relates to the fixed amount of emissions resulting from the transportation of goods between the warehouse and the retailers, whereas the latter relates to per unit per time emissions resulting from the use of electricity for heating or cooling systems. The emission parameters and cost components used in the mathematical model we propose are presented in Table 1.

The mathematical formulation we give is based on the multicommodity formulation initially proposed by Cunha and Melo (2016) for the OWMR. Other MIP formulations for the OWMR have been proposed in the literature. The interested reader is referred to the works of Solyalı and Süral (2012) and Cunha and Melo (2016) for reformulations of the classical OWMR problem, and to Gruson et al. (2019) for reformulations of its extension to three levels. The choice of the multicommodity formulation is motivated by the empirical results reported by Cunha and Melo (2016), regarding the practical performance of this formulation in terms of computing time and quality of the solution obtained. The idea of the multicommodity formulation is to see the demand d_t^r as a distinct commodity, for each retailer and each time period. For any retailer r, let (i) w_{kt}^{0r} be the amount produced at the warehouse in period k to satisfy d_t^r, (ii) w_{kt}^{1r} be the amount transported to retailer r in period k to satisfy d_t^r, (iii) σ_{kt}^{0r} be the amount stocked at the warehouse at the end of period k to satisfy d_t^r, and (iv) σ_{kt}^{1r} be the amount stocked at retailer r at the end of period k to satisfy d_t^r. Let y_t^w be a boolean setup variable taking value 1 if production occurs at the warehouse in period t, and 0 otherwise. Finally, let y_t^r be a boolean setup variable taking value 1 if there is a delivery to retailer r in period t, and 0 otherwise. The mathematical formulation MC for the OWMR-EC is given as follows:

$$\text{MC} \qquad \text{Min} \sum_{i\in F}\sum_{t\in T} sc_t^i y_t^i + \sum_{i\in R}\sum_{t\in T}\sum_{k=1}^{t-1} hc_k^w \sigma_{kt}^{0i} + \sum_{i\in R}\sum_{t\in T}\sum_{k=1}^{t-1} hc_k^i \sigma_{kt}^{1i} \tag{1}$$

$$\text{s.t. } \sigma_{k-1,t}^{0i} + w_{kt}^{0i} = w_{kt}^{1i} + \sigma_{kt}^{0i} \qquad \forall\, i \in R, k \le t \in T \tag{2}$$

$$\sigma_{k-1,t}^{1i} + w_{kt}^{1i} = \delta_{kt} d_t^i + (1-\delta_{kt})\sigma_{kt}^{1i} \qquad \forall\, i \in R, k \le t \in T \tag{3}$$

$$w_{kt}^{0i} \le d_t^i y_k^w \qquad \forall\, i \in R, k \le t \in T \tag{4}$$

$$w_{kt}^{1i} \le d_t^i y_k^i \qquad \forall\, i \in R, k \le t \in T \tag{5}$$

$$\sum_{t\in T}\left(esy_t^w + \sum_{r\in R} ed_t^r y_t^r\right) + \sum_{t\in T}\left(\sum_{r\in R}\sum_{l=t+1}^{T}\left(eh^w \sigma_{tl}^{0r} + eh^r \sigma_{tl}^{1r}\right)\right) \le ec \tag{6}$$

$$w_{kt}^{0i}, w_{kt}^{1i}, \sigma_{kt}^{0i}, \sigma_{kt}^{1i} \ge 0 \qquad \forall\, i \in R, k \le t \in T \tag{7}$$

$$y_t^i \in \{0; 1\} \qquad \forall\, t \in T, i \in F. \tag{8}$$

The objective function (1) minimizes the sum of the setup costs and of the unit inventory holding costs at all facilities. Constraints (2) are the demand satisfaction constraints at the warehouse level, for each retailer. Constraints (3) ensure that the demand of the retailers is satisfied. Specifically, δ_{kt} is the Kronecker symbol, taking a value of 1 if and only if $k = t$. Constraints (4) and (5) are the setup forcing constraints for the warehouse and the retailers, respectively. Constraint (6) is the global carbon emission cap constraint. Constraints (7) and (8) define the domains of the variables. Note that (1)–(5), (7)–(8) model an OWMR problem. In the sequel, we denote by MC-OWMR the formulation consisting of (1)–(5), (7)–(8).

The MC formulation can be adapted to the possibility of having some initial inventory available at the warehouse. Let I_0 be the amount of initial inventory available at the warehouse. Let σ_t^r be the amount of initial inventory available at the warehouse that is used to satisfy d_t^r. The mathematical formulation MC-II for problem OWMR-EC with initial inventory available at the warehouse is given as follows:

$$\text{Min} \sum_{i \in F} \sum_{t \in T} sc_t^i y_t^i + \sum_{i \in R} \sum_{t \in T} \sum_{k=1}^{t-1} hc_k^w \sigma_{kt}^{0i} + \sum_{i \in R} \sum_{t \in T} \sum_{k=1}^{t-1} hc_k^i \sigma_{kt}^{1i} \tag{9}$$

s.t. (3)–(8)

$$\sigma_t^i + w_{1t}^{0i} = w_{1t}^{1i} + \sigma_{1t}^{0i} \qquad \forall\, i \in R, t \in T \tag{10}$$

$$\sigma_{k-1,t}^{0i} + w_{kt}^{0i} = w_{kt}^{1i} + \sigma_{kt}^{0i} \qquad \forall\, i \in R, 2 \leq k \leq t \in T \tag{11}$$

$$\sum_{r \in R} \sum_{t \in T} \sigma_t^r \leq I_0 \tag{12}$$

$$\sigma_t^i \geq 0 \qquad \forall\, i \in R, t \in T. \tag{13}$$

The objective function (9) minimizes the sum of the setup costs and of the unit inventory holding costs at all facilities. Constraints (10) and (11) are the demand satisfaction constraints at the warehouse level. Constraint (12) defines the limit on the available initial inventory. Constraints (13) define the domains of the decision variables. In the sequel, we denote by MC-OWMR-II the formulation consisting of (3)–(5), (7)–(13).

3. A two-stage heuristic for the OWMR

The following sections detail our proposed two-stage heuristic for the OWMR. We build upon this heuristic to devise a heuristic for the OWMR-EC in Section 4. The objectives of the two stages are to define a production plan for the warehouse and to find a delivery plan for each retailer, respectively. The output of the first stage is used as an input for the second stage.

3.1. Stage one: production plan for the warehouse

The purpose of the first stage is to obtain a production plan for the warehouse. To simplify the complexity of the OWMR, we aggregate all retailers and treat them as one. For this aggregate retailer, we define an aggregate setup cost $sc_t^A = \sum_{r \in R} sc_t^r$, and an aggregate inventory holding cost $hc_t^A = \sum_{r \in R} hc_t^r$. Note that the demand in period t of this aggregated retailer is d_t^w. Treating all retailers as one, we obtain a two-level serial system that can be seen as an OWMR for which $|R| = 1$. We can thus use the MC-OWMR-II and the MC-OWMR formulations if initial inventory is available at the warehouse or not, respectively. We solve these OWMR problems using a general purpose solver and record the production setup decisions at the warehouse. We call this first stage the single retailer aggregation (SRA) stage.

3.2. Stage two: delivery plan for the retailers

In the second stage, we fix the production setup decisions for the warehouse obtained from stage one and proceed to make the delivery plan for each retailer. We first disaggregate the OWMR into $|R|$ independent subproblems. Indeed, when the production setup decisions are fixed and when there is no initial inventory available at the warehouse, constraints (4) do not act as linking constraints anymore and the MC-OWMR formulation reduces to $|R|$ independent subproblems. Let OWMR^r be the subproblem linked to retailer r and let $\widehat{y}^w$ be the optimal values obtained at stage one for the production setup variables at the warehouse. For any retailer r, the problem OWMR^r is modeled as follows:

$$\text{Min} \sum_{t \in T} \left(sc_t^r y_t^r + \sum_{k=1}^{t-1} hc_k^w \sigma_{kt}^{0r} + \sum_{k=1}^{t-1} hc_k^r \sigma_{kt}^{1r} \right) \tag{14}$$

$$\text{s.t. } \sigma_{k-1,t}^{0r} + w_{kt}^{0r} = w_{kt}^{1r} + \sigma_{kt}^{0r} \qquad \forall\, k \le t \in T \tag{15}$$

$$\sigma_{k-1,t}^{1r} + w_{kt}^{1r} = \delta_{kt} d_t^r + (1 - \delta_{kt}) \sigma_{kt}^{1r} \qquad \forall\, k \le t \in T \tag{16}$$

$$w_{kt}^{0r} \le d_t^r \widehat{y}_k^w \qquad \forall\, k \le t \in T \tag{17}$$

$$w_{kt}^{1r} \le d_t^r y_k^r \qquad \forall\, k \le t \in T \tag{18}$$

$$w_{kt}^{0r}, w_{kt}^{1r}, \sigma_{kt}^{0r}, \sigma_{kt}^{1r} \ge 0 \qquad \forall\, k \le t \in T \tag{19}$$

$$y_t^r \in \{0; 1\} \qquad \forall\, t \in T. \tag{20}$$

The objective function (14) minimizes the sum of the setup costs and of the unit inventory holding costs at retailer r. Constraints (15) and (16) are the demand satisfaction constraints at the warehouse and retailer level, respectively. Constraints (17) are the former setup forcing constraints at the warehouse. Constraints (18) are the setup forcing constraints for the retailer. Constraints (19) and (20) are the bound restrictions and binary requirements on the variables, respectively. Note that when there is some initial inventory available at the warehouse we set to 0 the demands of all retailers up to period $t_{\max}$, including $t_{\max}$, where $t_{\max}$ is defined as

$$t_{\max} = \text{argmax}\{t \in T \mid d_{1t}^w \le I_0\}. \tag{21}$$

We further set to 0 the demands $d_{t_{\max}+1}^r$ of all retailers r such that $r \le r_{\max}$ where $r_{\max}$ is defined as

$$r_{\max} = \text{argmax} \left\{ r \in R \mid \sum_{r_1=1}^{r_{\max}} d_{t_{\max}+1}^{r_1} \le I_0 - d_{1t_{\max}}^w \right\}. \tag{22}$$

Algorithm 1. TPRF heuristic

for all $r \in R$ **do**
  $u \leftarrow 1$
  while $u + \rho - 1 < |T|$ **do**
    Optimizing: Optimize OWMR^r over time periods u to $\min\{u + \rho - 1, |T|\}$
    for $u \leq t \leq \min\{u + \rho - 1 - \kappa, |T|\}$ **do**
      Fixing: Fix $y_t^r = \widehat{y}_t^r$
    end for
    $u \leftarrow u + \rho - \kappa$
  end while
end for

We finally set the demand $d_{t_{\max}+1}^{r_{\max}+1}$ to $d_{t_{\max}+1}^{r_{\max}+1} - (I_0 - d_{1t_{\max}}^{w} - \sum_{r_1=1}^{r_{\max}} d_{t_{\max}+1}^{r_1})$. In other words, we divide the available initial inventory between the retailers. We first calculate the maximum number of periods $t_{\max}$ such that the initial inventory at least covers the demand of all retailers between periods 1 and $t_{\max}$. We then sequentially divide the remaining initial inventory among the retailers until no more initial inventory is left. As a consequence, there is no linking quantity between the OWMR^r subproblems. This way, we can always decompose the MC-OWMR into $|R|$ independent subproblems.

To solve each subproblem OWMR^r, we developed a time-partitioning relax-and-fix (TPRF) heuristic and we adapted a dynamic programming recursion for two-level uncapacitated problems. Those two methods are detailed in the following two subsections.

3.2.1. Time-partitioning relax and fix

The first method we developed is a TPRF that contains elements of the time-partitioning (TP) heuristic used by Federgruen and Tzur (1999) for the OWMR and the relax-and-fix (RF) heuristic introduced by Stadtler (2003) for a multilevel LSP with a general product structure and several constrained resources. The TP heuristic decomposes the time horizon into smaller intervals. The original problem is solved on these smaller intervals, and side constraints are added on the boundaries of these intervals to get a feasible solution. The RF heuristic is an iterative approach that works with a limited number of binary setup variables. At each iteration of the RF heuristic, some binary variables are set to a value obtained in previous iterations. The problem obtained is solved to optimality and an additional subset of binary variables is set to their current value for the next iterations. The process stops when there are no more free binary variables.

The TPRF heuristic we propose is an iterative approach that works as follows. The time horizon is decomposed into a series of short time intervals with overlaps, where each short interval consists of several time periods. At each iteration and for each retailer r, we solve the OWMR^r to optimality over the particular short time interval and fix the obtained values of the binary variables that do not overlap with the next short time interval. In the next iteration, we go to the next interval and repeat the procedure until we reach the end of the time horizon. The structure of the TPRF heuristic is outlined in Algorithm 1, where ρ denotes the number of periods in the short time intervals, and κ represents the number of overlapping periods between two consecutive short time intervals. Note that neither ρ nor κ need to be an integer multiple of $|T|$. In Algorithm 1, $\widehat{y}_t^r$ represents the optimal

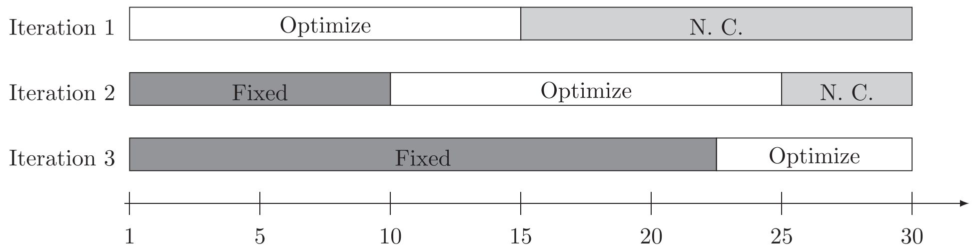

Fig. 1. Illustration of the TPRF heuristic.

value of the setup variables y for retailer r in period t, obtained when solving the OWMR^r on a particular nonoverlapping interval.

Figure 1 illustrates our TPRF heuristic for one subproblem, that is, one problem OWMR^r, with $\rho = 15$, $\kappa = 5$, and a time horizon of 30 periods. In Fig. 1, each row represents an iteration while the time periods are given at the bottom of this figure. In the first iteration, we optimize over the first 15 periods, the other periods being left out of the problem. In the next iteration, we fix the setup decisions for periods 1–10 since they are not part of the overlapping periods. We then optimize over periods 10–25 and the procedure continues until all 30 periods are optimized. Note that in Fig. 1, N. C. stands for "not considered."

To solve the OWMR^r subproblems to optimality over each time interval, we list all the possible delivery plans for each retailer. We then evaluate the cost of each plan and chose the one with the lowest total cost. When $\rho = 10$, there are two possible scenarios (to deliver or not to deliver) at each time period and in total, there are $2^{10} = 1024$ possible delivery plans. In our preliminary experiments, this method has given the best results in terms of computational time compared to the use of a general purpose solver to solve the OWMR^r subproblems to optimality over each interval. We call the heuristic consisting of stages one (SRA heuristic) and two (TPRF heuristic) the SRA-TPRF heuristic.

3.2.2. Dynamic programming recursion

The second method we use to obtain the retailers replenishment plan is based on the dynamic programming recursion proposed by Melo and Wolsey (2010) to exactly solve a two-level uncapacitated LSP, which is in nature similar to solving a specific OWMR^r. Let $G(t)$ be the minimum cost of the OWMR^r problem restricted from periods 1 to t. Let also $H(j, t)$ be the minimum cost at the retailer level to satisfy all the demands between periods j and t. Finally, let $p_t^w = \sum_{l \geq t} hc_l^w$ and let $p_t^r = \sum_{l \geq t}(hc_l^r - hc_l^w)$. To solve OWMR^r, the recursion is as follows:

$$G(t) = \min_{1 \leq j \leq t}\{G(j-1) + \min_{1 \leq i \leq j}(sc_i^w + p_i^w d_{jt}) + H(j, t)\} \tag{23}$$

$$H(u, t) = \min_{u \leq j \leq t}\{H(u, j-1) + sc_j^r + p_j^r d_{jt}\}. \tag{24}$$

As is, this algorithm does not take into account the initial inventory available at the warehouse, shared among all retailers and which therefore links the different subproblems. Besides, the recursion also decides on the setup values at the warehouse level, whereas we have already made those decisions in stage one. We therefore need to adapt the recursion proposed to fit within our framework. Regarding the initial inventory, we set some demands to 0 as mentioned at the beginning of Section 3.2. Let B be a large number. Regarding the setup decisions at the warehouse level obtained from stage one, we change the setup costs in the recursion as follows:

$$sc_t^w = \begin{cases} 0 & \text{if } \widehat{y}_t^w = 1 \\ B & \text{if } \widehat{y}_t^w = 0. \end{cases} \tag{25}$$

With those two adjustments, we are able to use the recursion provided by Melo and Wolsey (2010). We call the heuristic consisting of stages one (SRA heuristic) and two (DP heuristic) the SRA-DP heuristic.

4. A heuristic for the OWMR-EC

In this section, we develop a penalized relaxation (PR) method to solve the OWMR-EC.

4.1. General idea

Starting from (1) to (8), we relax the emission constraint (6) and penalize it in the objective function (1) with a penalty factor β ($0 \leq \beta \leq 1$). It results in an OWMR, which is solved using the SRA-TPRF or SRA-DP heuristic presented in Section 3, and which has an objective function different from (1). In particular, the new objective function is given as follows:

$$\begin{aligned} \text{Min} \sum_{t \in T} \big((1-\beta) sc_t^w + \beta es\big) y_t^w + \sum_{t \in T} \sum_{r \in R} \big((1-\beta) sc_t^r + \beta ed^r\big) y_t^r \\ + \sum_{t \in T} \sum_{r \in R} \sum_{k=1}^{t-1} \big((1-\beta) hc_k^w + \beta eh_t^w\big) \sigma_{kt}^{0r} + \sum_{t \in T} \sum_{r \in R} \sum_{k=1}^{t-1} \big((1-\beta) hc_k^r + \beta eh_t^r\big) \sigma_{kt}^{1r}. \end{aligned} \tag{26}$$

When β takes the value 0, there is no penalty enforced and we just minimize the costs regardless of the emissions it involves. When β takes the value 1, only emissions are considered and minimizing the objective function will be identical to minimizing the total amount of carbon emissions regardless of the costs.

In the PR heuristic, we iteratively solve a series of OWMR problems with different values of β, the values of parameter β being updated at each iteration. At each iteration, we check if the solution obtained satisfies the emission constraint (6) and if so, we compute its associated cost. After M iterations, the feasible solution with the lowest cost is kept as the final solution of our heuristic. If, for all iterations, we fail to obtain a feasible solution for the original problem, then our heuristic fails to provide a feasible solution to the OWMR-EC.

The solution obtained with the method described above is obtained by a heuristic procedure. Therefore, we may obtain an infeasible solution regarding carbon emissions. To obtain feasibility, we designed an iterated local search (ILS) mechanism at the end of stage two, along with a diversification mechanism during stage one. Indeed, the output of stage one strongly influences the final solution obtained by the PR heuristic, and a change in the output of stage one may be beneficial to lower carbon emissions. The next two sections detail the diversification and the ILS procedures.

4.2. Diversification phase

The first time we use the SRA procedure in stage one, we keep in memory the warehouse setup plan. In case the PR heuristic does not return a feasible solution, we add to the OWMR solved in stage one a diversification constraint indicating that the initial warehouse setup plan must be changed. Let $\widehat{y}_t^{w1}$ be the value of the setup variable at the warehouse in period t the first time SRA is used. In more details, at iteration k of the global PR procedure, we add the following diversification constraint:

$$\sum_{t \in T | \widehat{y}_t^{w1}=1} (1 - y_t^w) + \sum_{t \in T | \widehat{y}_t^{w1}=0} y_t^w = k. \tag{27}$$

Constraint (27) indicates that exactly k setup values must be changed, whether from 0 to 1 or from 1 to 0. We do not orient the search toward any of those two possibilities but instead let the solver optimize based on the objective function (26).

4.3. Intensification phase

Another method we implemented to achieve feasibility is the use of an iterative local search procedure. The procedure is as follows. For each retailer, and for each time period, we analyze the emission gains obtained by changing a setup decision (whether from 0 to 1 or from 1 to 0). If we obtain emission gains, we implement the change and iterate to see if other emission gains can be obtained, that is, the incumbent solution has the change in the retailer setup plan. If, for a certain retailer, there are no emission gains obtained, we perturbate the retailer setup plan and continue with local search iterations until a certain number of iterations have been done. The ILS procedure is illustrated in Algorithm 2. In Algorithm 2, for a specific retailer, we denote by u_t^{next}, u_t^{prev}, and $u_t^{\text{prev},w}$ the closest setup periods after period t at the retailer level, before period t at the retailer level, and before period t at the warehouse level, respectively.

4.4. Pseudo-code

In the iterative process, we apply the bisection method to update the value of the penalty factor, starting with $\beta = 1.0$. We do so in the aim of putting the emphasis on obtaining feasible solutions for OWMR-EC. The PR heuristic is described in Algorithm 3.

Algorithm 2. Iterated local search procedure

for each retailer r **do**
  $it \leftarrow 0$
  while $it < it_{max}$ **do**
    for each period t **do**
      Obtain u_t^{next}, u_t^{prev}, and $u_t^{\text{prev},w}$
      $bestGains \leftarrow 0$
      if $\widehat{y}_t^r = 1$ **then**
      $gains = ed_t^r - \sum_{k=t_{prev}}^{t-1} eh_k^r d^r_{tu_t^{\text{next}}-1} - \sum_{k=t_t^{\text{prev},w}}^{u_t^{\text{prev}}-1} eh_k^w d^r_{tu_t^{\text{next}}-1} + \sum_{k=u_t^{\text{prev},w}}^{t-1} eh_k^w d^r_{tu_t^{\text{next}}}$
      else
      $gains = -ed_t^r + \sum_{k=u^t_{\text{prev}}}^{t-1} eh_k^r d^r_{tu_t^{\text{next}}} + \sum_{k=u_t^{\text{prev},w}}^{u_t^{\text{prev}}} eh_k^w d^r_{tu_t^{\text{next}}} - \sum_{k=u_t^{\text{prev},w}}^{t-1} eh_k^w d^r_{tu_t^{\text{next}}}$
      end if
      if $gains > bestGains$ **then**
      $bestGains \leftarrow gains$
      $bestPeriod \leftarrow t$
    end if
  end for
  if $bestGains > 0$ **then**
    Implement change for $bestPeriod$
  else
    Change setup values for five random periods
  end if
  $it \leftarrow it + 1$
end while
end for

Algorithm 3. PR heuristic

$it \leftarrow 0$
while No feasible solution obtained **and** $it < it_{max}$ **do**
  $\beta \leftarrow 1.0$
  Penalizing: Penalize constraint (6) of OWMR-EC in the objective function (1)
  for $1 \leq l \leq M$ **do**
    Solving: Solve the resulting OWMR using SRA-TPRF or SRA-DP with (26) as its objective function
    if solution is feasible w.r.t the emission constraint (6) **then**
      $\beta \leftarrow \beta - 0.5^l$
      Recording: Record the solution and its cost
    else
      $\beta \leftarrow \beta + 0.5^l$
    end if
  end for
if No feasible solution found **then**
Apply ILS procedure
end if
$it \leftarrow it + 1$
end while
Terminating: Keep the feasible solution with the lowest total cost

5. Computational results

We first report results of the SRA-TPRF and SRA-DP heuristics to solve the OWMR without emission constraint. We then present the results of the PR heuristic to solve the OWMR-EC. We finally perform an analysis of the carbon emissions. To assess the strengths and weaknesses of the methods we designed, we conducted numerical experiments by taking the instances used in Solyalı and Süral (2012). In their experiments, Solyalı and Süral (2012) set the number of retailers $|R|$ equal to 50, 100, or 150, and the length of the time horizon $|T|$ is set equal to 15 or 30. The demand at the retailers is generated in a dynamic way from $U[5, 100]$. For the warehouse, the fixed costs are generated in a static way from $U[1500, 4500]$. For the retailers, the fixed costs are generated in a dynamic way from $U[5, 100]$. All the demands and fixed costs are generated as integer values. The unit inventory holding costs are static and are set to 0.5 for the warehouse. For the retailers, the unit inventory holding costs are also static and are generated from $U[0.5, 1]$. The holding costs take continuous values. The authors also consider the possibility of having some initial inventory available. In this case, the initial inventory is computed as $I_0 = 52 \times |R|$. The authors generated 10 random instances for each combination of settings, resulting in a total of 120 instances.

We further compare the results of our heuristics with the ones obtained by solving the instances using the CPLEX 12.8.0.1 JAVA library. We performed the experiments on a 3.07 GHz Intel Xeon processor with only one thread. We turned off CPLEX's parallel mode and set the MIP optimality tolerance parameter to 10^{-6}. All the other CPLEX parameters are set to their default value. The CPU time limit is set to one hour for the heuristics. No CPU time limit is imposed when instances are solved with CPLEX.

5.1. *Results on the OWMR*

In this section, we report the result of both the SRA-TPRF and SRA-DP heuristic on the OWMR. We analyze the performance of those methods in terms of relative CPU time taken by the heuristic to solve the instances, compared to the CPU time taken by CPLEX to solve the same instances. Let CPU^{DP}, CPU^{TPRF}, and CPU^* be the CPU time taken to solve the instances with the SRA-DP heuristic, the SRA-TPRF heuristic and CPLEX, respectively. The relative time RT, expressed as a percentage, is computed as $\text{RT} = \frac{\text{CPU}^H}{\text{CPU}^*} \times 100$, where H represents one of the two versions of the heuristic. We also report the gap, in terms of cost, between the solution given by CPLEX and the one returned by the two PR heuristics. This gap is computed as $(z^H - z^*)/z^*$, where z^H and z^* are the objective function value of the solution given by the heuristic and CPLEX, respectively. Note that we did not use the ILS procedure nor the diversification mechanism in this section since those two enhancements have been designed to achieve feasibility for OWMR-EC. Besides, we always obtain feasible solutions for the OWMR with both SRA-TPRF and SRA-DP heuristics.

Table 2 displays the results obtained on the OWMR for the two heuristics. In Table 2, the first column gives the number of retailers and the second one gives the length of the time horizon. The third column indicates if there is some initial inventory available. The other four columns display the gap and relative time for SRA-DP (columns four and five), and for SRA-TPRF (columns six and seven). In Table 2, one can see that the two PR versions obtain solutions of high quality,

Table 2
Results obtained for the SRA-DP and SRA-TPRF heuristics on the OWMR

			SRA-DP		SRA-TPRF	
$\lvert R \rvert$	$\lvert T \rvert$	I_0	Gap (%)	RT (%)	Gap (%)	RT (%)
50	15	0	0.19	3.15	0.19	7.24
50	30	0	0.1	1.74	0.22	3.65
100	15	0	0.23	2.3	1.44	5.83
100	30	0	0.3	1.04	0.09	1.43
150	15	0	0.29	1.85	0.29	5.17
150	30	0	1.38	0.62	1.37	1.78
50	15	>0	0.24	7.04	0.24	8.06
50	30	>0	0.12	3.22	0.26	2.11
100	15	>0	0.26	2.42	1.49	3.57
100	30	>0	0.13	1.15	0.13	1.16
150	15	>0	0.31	1.51	0.31	4.67
150	30	>0	1.4	0.83	1.4	2.2
Average			0.42	2.24	0.62	3.91

with an average gap of 0.42% and 0.62% compared to the optimal solution for version SRA-DP and SRA-TPRF, respectively. In terms of CPU time, the SRA-TPRF and SRA-DP heuristics find solutions in much less time compared to CPLEX. On average, the heuristic uses only 2.24% and 3.91% of the CPU time used by CPLEX for version SRA-DP and SRA-TPRF, respectively. This indicates a very good performance of the heuristics, which are quickly able to find solutions of high quality, regardless of the settings of the instance (number of retailers and time periods, presence or absence of initial inventory). We hence conclude that this is an efficient heuristic for the standard OWMR problem.

5.2. Heuristic results

To assess the strengths and weaknesses of our PR method, we conducted numerical experiments by taking the instances used in Solyalı and Süral (2012) and adapted them to have emission parameters. To generate the emission parameters, we consider that the emission intensity of an activity is correlated with the cost of this activity, with a certain level of deviation γ. In the experiments, we set this level of deviation equal to 20%, 50%, and 100%. For each level of deviation, we obtain a range of possible values for the emission intensities. The actual values are then generated using a uniform distribution within this range. For instance, if we have a setup cost of 2000 and a level of deviation of 50%, the associated setup emission parameter will be generated from $U[1000, 3000]$. In most real settings, emissions coming from setups (production and transportation) will be higher than emissions coming from inventory holding. As the setup costs are higher than the inventory holding costs, our choice of values for the emissions parameters allows us to fit within this general framework.

We additionally define a lower bound and an upper bound on the emission cap used in (6). The lower bound is computed as the minimum emission amount needed to provide a feasible solution,

Table 3
Number of feasible solutions obtained by the different methods

Settings			Methods		
$\lvert R\rvert$	$\lvert T\rvert$	I_0	CPLEX	PR-DP	PR-TPRF
50	15	0	630	606	605
50	30	0	630	573	572
100	15	0	630	587	566
100	30	0	630	582	554
150	15	0	630	549	547
150	30	0	630	583	557
50	15	>0	630	576	570
50	30	>0	630	565	508
100	15	>0	630	529	528
100	30	>0	630	556	539
150	15	>0	630	493	491
150	30	>0	630	592	573
Total			7560	6791	6610

while the upper bound is computed as the minimum emission amount needed in a cost optimal solution. The lower bound is obtained by minimizing the carbon emissions while satisfying the demand of the retailers, regardless of the costs. The upper bound is obtained by solving an OWMR instance and computing the emissions. Note that if there are multiple optimal solutions, we take the optimal solution with the lowest emission amount. These two bounds help us define the maximum potential emission reduction (MPER) as MPER = (EUB − ELB)/EUB, where EUB and ELB are the emission upper and lower bound, respectively. In the experiments, we set the emission cap ec as $ec = (1 - \lambda \text{MPER})\text{EUB}$, where λ is varied between 0 and 1 in steps of 0.05. We have a total of 7560 instances to solve.

In the following sections, we display the results obtained for the two versions of the PR heuristic. In the first version, denoted by PR-TPRF, we use the SRA-TPRF heuristic to solve the OWMR problem. In the second version, denoted by PR-DP, we use the SRA-DP method to solve the OWMR problem. In the PR heuristic, we set the number of iterations M equal to 20. We also set the values of the parameters ρ and κ used in the TPRF heuristic to 10 and 5, respectively. In the ILS procedure, the number of iterations it_{max} is set equal to 100. The average results obtained by our heuristic over all instances are shown in Tables 3–5.

We compare the performance with respect to the CPU time (seconds) taken to solve the instances, the number of instances where the heuristic gives a feasible solution and the gap, in terms of cost, between the solution given by CPLEX and the one returned by the PR heuristic. This gap is computed as $(z^H - z^*)/z^*$, where z^H and z^* are the objective function value of the solution given by our heuristic and by CPLEX, respectively. Table 3 gives the average number of feasible solutions obtained by the heuristic over all instances. Table 4 indicates the average gap between the solution given by CPLEX and the one obtained by the different versions of PR, expressed as a percentage of the cost increase between the solution obtained by CPLEX and ours. Note that this gap is computed with respect to the objective function (1) and only over instances where the heuristic actually found a feasible solution. Table 5 reports the average CPU time in seconds taken by our heuristic.

Table 4
Average gap (%) obtained by the different PR versions

Settings			Methods	
$\|R\|$	$\|T\|$	I_0	PR-DP	PR-TPRF
50	15	0	1.45	1.44
50	30	0	1.22	1.2
100	15	0	1.17	1.39
100	30	0	1.63	3.03
150	15	0	3.88	3.88
150	30	0	2.56	2.57
50	15	>0	1.64	1.63
50	30	>0	1.40	1.41
100	15	>0	1.59	1.59
100	30	>0	1.80	1.84
150	15	>0	4.39	4.38
150	30	>0	2.60	2.61
Average			2.09	2.23

Table 5
CPU time (seconds) taken by the different PR versions

Settings			Methods	
$\|R\|$	$\|T\|$	I_0	PR-DP	PR-TPRF
50	15	0	0.43	0.43
50	30	0	1.49	1.49
100	15	0	1.27	1.32
100	30	0	4.61	3.79
150	15	0	3.94	3.99
150	30	0	12.09	11.89
50	15	>0	0.56	0.54
50	30	>0	1.75	1.60
100	15	>0	1.57	1.50
100	30	>0	3.95	3.67
150	15	>0	4.71	4.52
150	30	>0	12.59	12.46
Average			4.08	3.93

In each of these tables, the first column gives the number of retailers and the second one gives the length of the time horizon. The third column indicates if there is some initial inventory available. The rest of the columns display the figures obtained for each possible version of the PR heuristic. In these tables, each cell represents 630 different instances. Note that we report the gaps and CPU time only for the instances for which both PR-TPRF and PR-DP obtained a feasible solution. Detailed results are available in the Appendix of this paper.

One can see in Table 3 that both versions of the PR heuristic cannot always find a feasible solution to the problem. The PR-TPRF version was able to find 6610 feasible solutions out of the 7560 instances, which represents 87.43% of the instances. The PR-DP was able to find a feasible solution

Table 6
Average results obtained over instances where both PR versions find feasible solutions

$\|R\|$	$\|T\|$	I_0	CPU time CPLEX	Solutions found (%)	Gap (%)	PR-DP CPU time (seconds)	Solutions found (%)	Gap (%)	PR-TPRF CPU time (seconds)
50	15	0	4.36	96	1.45	0.43	96	1.44	0.43
50	30	0	53.4	91	1.22	1.49	91	1.20	1.49
100	15	0	14.3	93	1.17	1.27	90	1.39	1.32
100	30	0	285.4	92	1.63	4.61	88	3.03	3.79
150	15	0	32.3	87	3.89	3.94	87	3.88	3.99
150	30	0	414.2	93	2.56	12.09	88	2.57	11.89
50	15	>0	6.8	91	1.64	0.56	90	1.63	0.54
50	30	>0	65.8	90	1.4	1.75	80	1.41	1.6
100	15	>0	24.1	84	1.59	1.57	84	1.59	1.5
100	30	>0	313	88	1.8	3.95	86	1.84	3.67
150	15	>0	59.9	78	4.39	4.71	80	4.38	4.52
150	30	>0	565	94	2.6	12.59	91	2.61	12.46
Average			153.7	89.8	2.09	4.08	87.4	2.22	3.93

for 89.82% of the instances. The better performance of PR-DP can be explained by the fact that it gives an optimal solution for stage two. Detailed results in the Appendix indicate that as the emission amount gets closer to ELB (i.e., when λ gets closer to 1), the number of feasible solutions obtained gets lower. This drawback can be explained by the tightness of the emission constraints (6). Indeed, when ec gets closer to ELB, the problem becomes hard to solve and finding a feasible solution is also a hard task. In the same vein, both versions of our heuristic also find fewer feasible solutions when the number of retailers increases. Surprisingly, the increase in the length of the time horizon leads to an increase in the number of feasible solutions found. This may be explained by the fact that longer planning horizons leads to more iterations in the heuristics. Thus, it allows more flexibility in finding feasible solutions. As far as the initial inventory is concerned, there is no clear impact of this feature on the results of the heuristic.

The quality of the solutions found by the different versions of the PR heuristic can be evaluated by the gaps given in Table 4. Note that detailed results are available in the Appendix. In Table 4, one can see that the gaps obtained are low for the PR-DP and PR-TPRF, with an overall average of 2.09% and 2.23%, respectively. This indicates that the solutions found by both versions are of high quality. This result is impressive when we contrast it with the CPU time taken by both versions as illustrated in Table 5. Indeed, one can see in Table 5 that the CPU time taken by PR-DP and PR-TPRF is 4.08 and 3.93 seconds on average, respectively. The maximum average CPU time taken is also quite low (12.59 seconds), showing that the PR is very fast, regardless of the size of the instance to solve. This speed is the major strength of our heuristic. Detailed results for the CPU time are available in the Appendix. The average CPU time taken by the PR-DP and PR-TPRF version is on average 37.7 and 39 times lower compared to the CPU time taken by CPLEX, respectively. As a manager may want to analyze the effect of those λ on operations, we do think that this difference in CPU time represents a main strength of our heuristic.

To illustrate the results of the two PR versions from another perspective, we report in Table 6 the average performance of both the PR versions and CPLEX. In Table 6, the average number of

Table 7
Average values obtained for MPER and CMER over all instances

Level of deviation			20%		50%		100%	
$\|R\|$	$\|T\|$	I_0	MPER	CMER	MPER	CMER	MPER	CMER
50	15	0	0.43	0.56	3.09	3.68	18.93	31.47
100	15	0	0.44	0.43	2.69	3.51	12.14	20.39
150	15	0	0.56	0.51	4.57	5.2	16.76	24.68
50	30	0	0.55	0.54	3.04	3.18	15.81	20.5
100	30	0	0.55	0.64	3	3.55	15.28	23.97
150	30	0	0.54	0.54	4.85	4.48	13.8	24.23
50	15	>0	0.53	0.46	3.08	3.82	19.34	32.59
100	15	>0	0.48	0.46	2.74	3.65	12.2	21.06
150	15	>0	0.54	0.53	4.55	5.03	16.5	24.2
50	30	>0	0.54	0.52	2.98	3.19	15.83	20.74
100	30	>0	0.58	0.63	3.07	3.47	15.38	24.45
150	30	>0	0.54	0.53	4.85	4.28	13.85	24.54
Average			0.52	0.53	3.54	3.9	15.49	24.4

feasible solutions found (in percentage), the average gap (in percentage), the CPU time taken by CPLEX (in seconds) and the CPU time taken by the two PR versions (in seconds) are reported in columns 5–10. The CPU time of CPLEX is calculated over the instances for which both PR versions could find a feasible solution. In Table 6, one can see that the CPU time taken by our heuristic is drastically lower than the one of CPLEX. Whereas CPLEX takes on average 153.7 seconds, the PR-DP heuristic takes only 4.08 seconds, and the PR-TPRF heuristic only 3.93 seconds. The average gap is also relatively low, between 1.2% and 4.39%. This comparison between the results obtained with CPLEX and the ones obtained by our PR heuristics emphasizes the strengths of the PR heuristic, which are its speed and its ability to provide solutions of high quality, when found.

5.3. *Analysis of carbon emissions*

In this section, we provide an analysis of the impact of the emission cap constraint on the optimal solutions obtained with CPLEX for the OWMR-EC.

5.3.1. *Cost of the maximum potential reduction*

We discuss here the cost of achieving the MPER introduced previously in Section 5.2. To do so, we define the cost of maximum emission reduction (CMER) as the gap between the cost of the minimal emission solution and the minimal cost solution. Let CLES be the cost of the solution with the lowest possible emissions, that is, the cost of the solution for which ec = ELB. Let z^* be the cost of the solution that minimizes the operational costs and has the lowest possible emissions among those cost optimal solutions. The CMER is computed as CMER = $(\text{CLES} - z^*)/z^*$. Table 7 illustrates the average values obtained over all our instances. In Table 7, each row represents a particular setting while the columns represents the average values obtained for MPER and CMER, expressed as a percentage, for each level of emission deviation considered in our experiments, respectively.

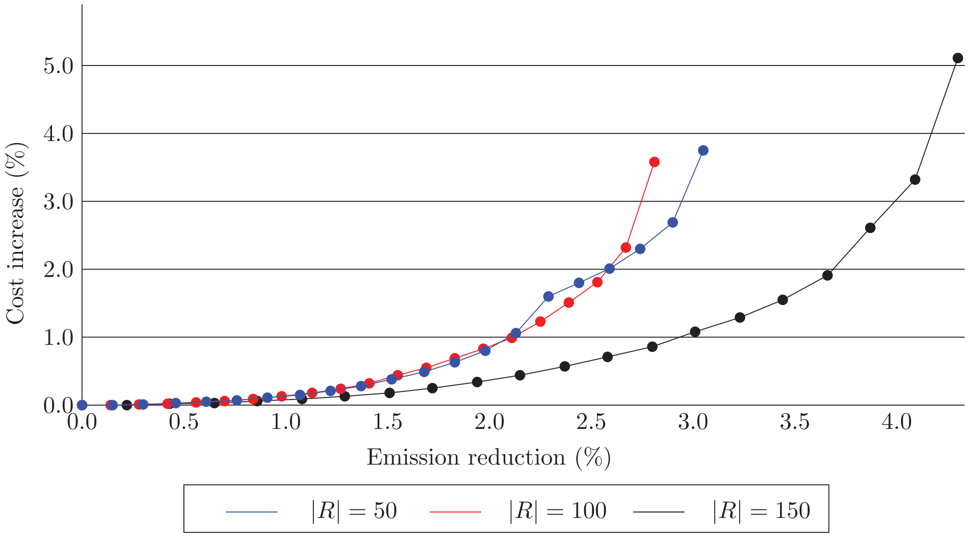

Fig. 2. Average trade-off curve over 20 instances for varying number of retailers with 15 time periods and an emission deviation of 50%.

One can see in Table 7 that, as the emission parameters deviate farther from the corresponding costs (i.e., when the emission parameters and the corresponding costs are less correlated), the MPER increases, meaning that we can achieve much higher emission reductions. The values obtained for CMER increase at an even faster rate compared to MPER, indicating that the marginal cost of emission reduction is increasing. On the contrary, when the emission parameters and the corresponding costs are more correlated, there is little possibility for emission reduction, and the cost increase is rather low. These results indicate that when the emission factors are closely related to the cost factors, a minimization of the costs will also lead to a good result with respect to the emissions, with little opportunity to further lower the emissions. However, when the emissions factors can deviate largely from the cost factors (i.e., they are not much correlated), then a minimization of the cost will not necessarily lead to a good solution with respect to the emissions, and there is substantial room to further improve the emissions, but at a high additional cost.

5.3.2. Curve of the cost-emission trade-off

When imposing emission restrictions, one wants to know what will be the cost of imposing such restrictions. Figure 2 illustrates, on the y-axis, the cost increase depending on the desired emission reduction, shown on the x-axis. The cost increase is expressed as a percentage of extra cost compared to the cost of the minimum cost solution, that is, the optimal solution of OWMR-EC for which $ec = \text{EUB}$. In Fig. 2, there are three curves representing each possible number of retailers. The blue curve corresponds to 50 retailers, the red curve to 100 retailers, and the black curve to 150 retailers. Each curve represents the aggregated trade-off curve over 20 instances (10 with initial inventory and 10 without initial inventory available at the warehouse), with an emission deviation level of 50% and with a planning horizon of 15 periods. In Fig. 2, one can see that the marginal cost of emission reduction tends to increase as higher percentage of emission

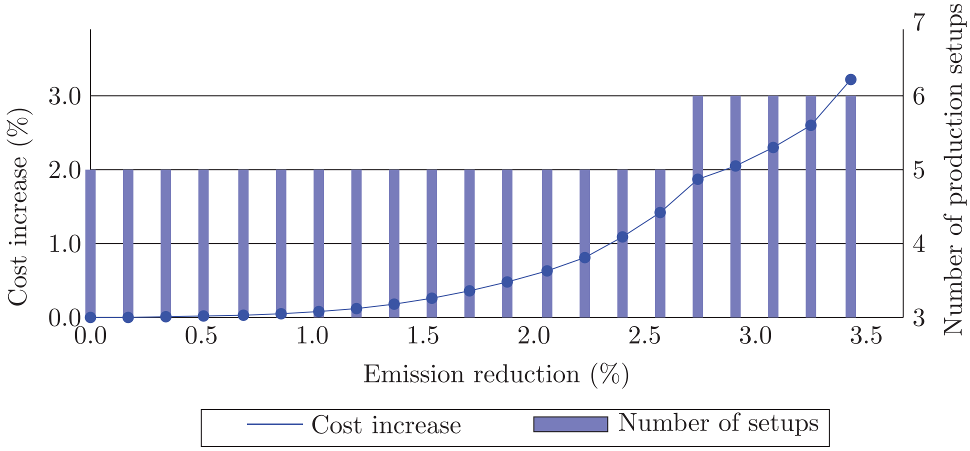

Fig. 3. Trade-off curve for one particular instance with 50 retailers, 15 time periods, and an emission reduction of 50%.

reduction must be achieved. Indeed, as the emission reduction gets higher, the cost increases at an increasing speed. For instance, on the black curve, the first 2.0% emission reduction only incurs about 0.4% extra cost while for 4.0% emission reduction it incurs an increase in cost of about 3%.

One can see that the curves illustrated in Fig. 2 are piecewise convex, that is, they are convex on certain intervals. This phenomenon is explained in Fig. 3. The curve plotted in Fig. 3 represents, on the left y-axis, the increase in cost depending on the emission reduction wanted. This cost increase is expressed as a percentage compared to the cost of the optimal solution of OWMR-EC for which $ec = \text{EUB}$. On the right y-axis, we have represented the number of periods with a production setup at the warehouse, depending on the emission reduction wanted, shown on the x-axis. Figure 3 has been obtained with one particular instance where there is no initial inventory available at the warehouse, 50 retailers, 15 time periods, and an emission deviation level of 50%. This is done to avoid the aggregation effect.

In Fig. 3, the curve is also piecewise convex in two sections. This indicates that, within each section, the marginal cost of reducing emissions increases. Moreover, one can see that each section corresponds to one specific setup plan at the warehouse. In the first convex section, from 0% emission reduction to 2.7% emission reduction, there are 5 periods with a production setup out of 15. From 2.7% emission reduction to 3.5% emission reduction, there are 6 periods with a production setup out of 15. This change in the setup production plan explains the origin of the piecewise convexity of the curves illustrated in Fig. 2.

6. Conclusions

We have developed the PR heuristic to solve the OWMR problem with a carbon emission constraint (OWMR-EC). The PR heuristic is an iterative procedure that solves a series of OWMRs obtained from the original OWMR-EC where the carbon emission cap constraint has been relaxed and penalized in the objective function. At the end of each iteration of the PR heuristic, the penalties are updated. The OWMR we obtain by relaxing the emission constraint is solved using a two-stage heuristic we have also developed. In the first stage, all the retailers are aggregated into a single one and we obtain a production plan for the warehouse (SRA heuristic). In the second stage, we iteratively solve a series of small OWMRs for each retailer and we obtain the transportation plans for each retailer. We either solve the resulting $OWMR^r$ by means of a heuristic procedure called TPRF heuristic, or by means of an exact dynamic programming approach. We have tested the two heuristics on the classical instances from the literature. The results of the heuristics are impressive in terms of quality of the solution obtained in a short amount of time. We have also tested our PR heuristic on numerous instances adapted from the literature on the OWMR. The PR heuristic appears to be very fast, which makes it useful in practice for companies that want to reduce their carbon footprint. The PR-DP version was also able to provide feasible solutions for 89.8% of the instances tested, while the PR-TPRF version was able to provide feasible solutions for 87.4% of the instances. Besides, when a feasible solution was obtained, it was of high quality. Indeed, compared to the optimal solution given by a general purpose solver, an average gap of 2.1% and 2.2% was achieved by the PR-DP and PR-TPRF heuristic, respectively. It often fails however to provide feasible solutions when the emission cap is very tight.

We have also performed a sensitivity analysis to better understand the implications of the emission restrictions. We have seen that the cost of emission reduction depends on the level of correlation between the emissions and the operational costs. When those two parameters have a low correlation, one can achieve much higher emission reductions, but the cost of this reduction is also large. On the contrary, when those two parameters are highly correlated, little emission reduction can be achieved. This sensitivity analysis can be used by practitioners to have an idea of the cost increase for emission reductions. We have also noted that the cost to achieve emission reduction rises at an increased pace, meaning that the last percentage of emission reduction is more expensive than the first one. Finally, we have noted that the cost-emission trade-off curve is piecewise convex. The origin of this phenomenon lies in the difference of setup plans in each section where the curve is convex.

In future research, we may explore two avenues. First, we can impose other emission constraints proposed in the literature in the context of the OWMR. Our heuristic may be adapted to handle these different policies. Second, the PR heuristic we developed shares similarities with the Lagrangian relaxation method. Future research could focus on developing a Lagrangian relaxation to our problem setting, in order to obtain a lower bound on the cost of the optimal solution.

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Appendix

Table A1–A6

Table A1
Number of feasible solutions obtained by the PR-DP method

Settings			λ																				
$\|R\|$	$\|T\|$	I_0	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	Total
50	15	0	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	29	7	606
50	30	0	30	30	30	29	29	29	29	29	29	29	29	29	29	28	27	27	27	27	26	26	5	573
100	15	0	30	30	30	30	30	30	30	30	29	29	29	29	29	29	29	28	28	26	26	25	11	587
100	30	0	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	26	5	582
150	15	0	30	30	30	30	30	28	28	28	28	28	28	28	27	26	26	26	26	25	24	22	1	549
150	30	0	30	30	30	30	30	30	30	30	29	29	29	29	28	28	28	28	28	28	27	25	7	583
50	15	>0	29	29	29	29	29	29	29	29	29	29	29	29	29	29	28	28	28	27	26	24	9	576
50	30	>0	30	30	30	29	29	29	29	29	29	29	28	28	28	28	27	27	27	26	24	23	6	565
100	15	>0	29	29	29	29	29	29	28	28	27	26	26	26	26	26	26	25	25	23	22	16	5	529
100	30	>0	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	27	27	27	25	24	6	556
150	15	>0	28	27	27	26	28	26	26	26	25	25	25	25	24	24	24	23	23	22	20	18	1	493
150	30	>0	30	30	30	30	30	30	30	30	30	30	30	30	29	29	29	28	28	28	26	24	11	592
Total			353	352	352	349	351	347	346	346	342	341	340	340	336	334	331	326	326	318	305	282	74	6791

Table A2
Number of feasible solutions obtained by the PR-TPRF method

Settings			λ																				
$\|R\|$	$\|T\|$	I_0	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	Total
50	15	0	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	28	7	605
50	30	0	30	30	30	29	29	29	29	29	29	29	29	29	29	28	28	27	27	27	26	25	4	572
100	15	0	30	30	30	30	30	30	30	30	29	29	29	28	28	28	28	27	27	25	23	21	4	566
100	30	0	29	29	29	29	29	29	28	28	28	28	28	28	28	28	27	26	26	26	25	22	4	554
150	15	0	30	30	30	30	29	28	28	28	28	28	28	28	27	26	26	26	26	25	24	21	1	547
150	30	0	30	30	30	30	29	29	29	29	28	28	28	28	27	27	27	26	26	26	25	22	3	557
50	15	>0	29	29	29	29	29	29	29	29	29	29	29	29	29	29	28	28	28	27	25	22	6	570
50	30	>0	30	28	27	27	29	26	26	26	27	27	27	24	24	25	22	23	23	24	23	17	3	508
100	15	>0	29	29	29	29	29	29	28	28	27	26	26	26	26	26	26	25	25	23	22	17	3	528
100	30	>0	29	29	29	29	29	28	27	27	27	27	27	27	27	26	26	26	26	25	23	22	3	539
150	15	>0	28	27	26	26	26	27	26	26	25	25	25	25	24	24	24	23	23	22	20	18	1	491
150	30	>0	30	30	30	30	30	30	30	30	29	29	29	29	28	28	28	27	27	26	25	24	4	573
Average			354	351	349	348	348	344	340	340	336	335	335	331	327	325	320	314	314	306	291	259	43	6610

Table A3
Average gap (%) obtained by the PR-DP method

Settings			λ																					Average
$\|R\|$	$\|T\|$	I_0	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	
50	15	0	1.5	1.5	1.5	1.5	1.4	1.4	1.3	1.2	1	1	0.8	0.8	1.1	2	1.7	1.8	2.4	2.2	2.1	1.2	0	1.4
50	30	0	1.3	1.3	1.3	1.3	1.3	1.2	1.2	1.1	1	0.9	0.8	0.8	0.6	0.5	0.7	0.9	1.3	2.1	2.7	2.8	0	1.2
100	15	0	1.4	1.4	1.4	1.4	1.3	1.3	1.2	1.2	1.2	1.1	1	0.8	0.7	0.7	0.8	0.7	0.8	1.6	2.2	2.1	0	1.2
100	30	0	1.6	1.6	1.6	1.6	1.5	1.5	1.4	1.3	1.2	1.2	1.8	1.6	1.3	1	1.3	1.7	1.8	2.1	2.3	2.6	0	1.5
150	15	0	3.3	3.3	3.2	3.2	3.2	3.3	3.6	3.5	4.5	4.7	4.5	3.8	4.3	4.9	4.2	4.2	4.3	4.9	4.1	3.2	0	3.7
150	30	0	2.3	2.2	2.2	2.2	2.2	2.2	2.2	2.5	2.6	2.8	2.6	2.5	2.5	2.3	2.3	2.3	2.9	2.9	3.9	3.2	0	2.4
50	15	>0	1.5	1.5	1.5	1.4	1.4	1.4	1.3	1.3	1.3	1.2	1.2	1.3	2.3	2.1	2	2.2	2.6	2.4	1.8	1.3	0	1.6
50	30	>0	1.4	1.4	1.4	1.4	1.4	1.4	1.3	1.2	1.2	1.1	0.9	0.8	0.7	0.8	1.7	1.8	2.1	2.9	3.8	3.9	2	1.7
100	15	>0	1.5	1.5	1.5	1.5	1.4	1.4	1.3	1.3	1.2	1.2	1	0.8	1	1.3	1.5	1.6	2	3.2	4.4	2.5	0	1.6
100	30	>0	1.7	1.7	1.7	1.6	1.6	1.5	1.4	1.4	1.2	1.3	1.6	1.8	1.4	1.3	1.6	2.1	2.3	2.9	3.8	3.1	2.4	1.9
150	15	>0	3.4	3.4	3.6	3.7	3.5	3.8	3.8	3.8	5.1	5.4	5.3	4.6	4.8	5.7	4.8	4.9	5	5.3	4.8	3.6	0	4.2
150	30	>0	2.3	2.2	2.2	2.2	2.2	2.3	2.4	2.6	2.5	2.8	2.7	2.5	2.6	2.4	2.3	2.4	2.9	2.8	4.3	3.3	0.1	2.5
Average			1.9	1.9	1.9	1.9	1.9	1.9	1.9	1.9	2	2.1	2	1.9	1.9	2.1	2.1	2.2	2.5	3	3.3	2.7	2.8	2.2

Table A4
Average gap (%) obtained by the PR-TPRF method

Settings			λ																					
$\|R\|$	$\|T\|$	I_0	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	Average
50	15	0	1.5	1.5	1.5	1.5	1.4	1.4	1.3	1.2	1	1	0.8	0.8	1.1	2	1.7	1.8	2.2	2.2	2	1.3	0	1.4
50	30	0	1.3	1.3	1.3	1.3	1.2	1.2	1.1	1	1	0.9	0.8	0.7	0.6	0.5	0.6	0.8	0.9	1.8	2.3	2.2	0	1.1
100	15	0	1.4	1.4	1.4	1.4	1.3	1.3	1.2	1.2	1.1	1.1	1	0.8	0.8	1	1.1	1	1.9	2.5	2	1.4	0	1.3
100	30	0	3.3	3.2	3.3	3.3	3.2	3.1	3	3	2.8	2.8	3.1	3	2.6	2.4	2.3	2.5	2.3	2.6	2.1	2.1	0	2.7
150	15	0	3.3	3.3	3.3	3.2	3.2	3.1	3.4	3.3	4.2	4.5	4.2	3.5	3.8	4.3	3.6	3.7	3.7	4	3.3	2.1	0	3.4
150	30	0	2.3	2.2	2.2	2.2	2.2	2.2	2.2	2.5	2.5	2.7	2.5	2.4	2.4	2.2	2.1	2.1	2.7	2.6	3.4	2.3	0	2.3
50	15	>0	1.5	1.5	1.4	1.4	1.3	1.4	1.3	1.2	1.3	1.1	1.2	1.3	2.2	2	1.8	2.1	2.3	2.2	1.6	1	0	1.5
50	30	>0	1.4	1.4	1.4	1.4	1.4	1.3	1.2	1.1	1.1	1	0.9	0.7	0.6	0.7	0.6	0.8	1	1.8	2.2	1.8	2	1.7
100	15	>0	1.5	1.5	1.4	1.4	1.4	1.3	1.2	1.2	1.1	1	0.9	0.8	0.8	1.2	1.3	1.3	2.2	2.5	2.9	1.2	0	1.3
100	30	>0	1.7	1.7	1.7	1.6	1.6	1.5	1.4	1.3	1.2	1.3	1.5	1.7	1.6	1.3	1.6	1.9	2.1	2.2	2.8	1.8	0	1.6
150	15	>0	3.2	3.1	3.1	3.2	3.1	3.4	3.3	3.2	4.3	4.6	4.4	3.8	3.9	4.5	3.8	3.8	3.9	3.9	3.2	2.2	0	3.4
150	30	>0	2.3	2.2	2.2	2.2	2.2	2.3	2.4	2.6	2.5	2.8	2.6	2.5	2.4	2.2	2.3	2.2	2.7	2.7	3.6	2.8	0	2.4
Average			2.1	2	2	2	2	2	1.9	1.9	2	2.1	2	1.8	1.9	2	1.9	2	2.3	2.6	2.6	1.8	2.7	2.1

Table A5
CPU time (seconds) taken by the PR-DP method

Settings			λ																				
$\lvert R\rvert$	$\lvert T\rvert$	I_0	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	Average
50	15	0	0.4	0.4	0.4	0.4	0.3	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.5	0.7	0.7	0.9	4.1	0.6
50	30	0	1.3	1.2	1.2	1.6	1.5	1.5	1.5	1.5	1.5	1.5	2.1	2.2	2.2	2.4	2.9	3.2	3.2	3.2	4	5.4	13.7	2.8
100	15	0	1.2	1	1	1.1	1.1	1	1	1	1.3	1.3	1.3	1.7	1.7	1.9	1.9	2.2	2.2	2.6	2.9	3.6	5.7	1.8
100	30	0	3.2	3	3.7	3.3	3.3	3.4	4.3	4.8	5	5.7	10	8.5	12.2	9.6	9.8	10.5	10.9	11.1	11	12.7	24.2	8.1
150	15	0	3.5	3.4	3.4	3.4	3.5	3.7	3.7	3.7	3.7	3.7	3.8	4	4.7	5.5	5.6	5.6	6	6.4	6.8	7.7	10.7	4.9
150	30	0	9.3	9.5	9.5	9.2	11.2	11.4	11.3	12.1	12.5	12.6	13.8	15	16.2	16.8	17.4	17.8	18.1	18.2	19.4	20.4	32.3	14.9
50	15	>0	0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.8	0.7	0.7	0.7	0.7	0.9	1	1.2	1.5	1.7	2.2	4.7	1.1
50	30	>0	1.6	1.4	1.5	1.9	2.3	1.7	1.7	1.7	2.4	2.4	2.9	3.6	3.6	4.2	4.7	5.6	9.9	9.1	11.1	13.5	25.2	5.3
100	15	>0	2.4	1.6	1.7	1.7	1.6	1.6	2	2.1	2.2	3.6	2.9	2.8	2.9	3.1	3.6	3.7	3.2	3.8	4.7	5.8	8.6	3.1
100	30	>0	4.4	3.8	4	5.5	5.1	6.9	9.3	12.4	10.6	11.4	10.7	11.4	11.7	11.3	10.9	11.5	11.7	12.1	15.7	17.3	28.7	10.8
150	15	>0	5.1	4.9	5.9	7	4.2	5.1	5.1	5	5.6	5.6	5.4	5.5	6.2	6.9	6.8	8	8.4	9.1	9.7	10.3	12.9	6.8
150	30	>0	10.1	9.9	9.9	11.3	11.2	10.6	11.7	11.5	12.1	12.2	13.5	13.7	15.4	21.2	18.4	18	18.1	18.3	19.9	21.5	28.8	15.1
Average			3.6	3.4	3.6	3.9	3.8	4	4.4	4.7	4.8	5.1	5.6	5.8	6.5	7	6.9	7.3	7.8	8	9	10.1	16.6	6.3

Table A6
CPU time (s) taken by the PR-TPRF method

Settings			λ																					
$\|R\|$	$\|T\|$	I_0	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	Average
50	15	0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.5	0.7	0.7	0.9	4.3	0.6
50	30	0	1.2	1.1	1.1	1.7	1.6	1.7	1.6	1.6	1.6	1.6	2.3	2.4	2.3	2.7	2.6	3.4	3.4	3.4	4.3	5.7	15	3
100	15	0	1.2	1.1	1	1.1	1.1	1.1	1.1	1.1	1.3	1.3	1.3	1.8	1.8	2	2	2.3	2.4	2.8	3.5	3.9	7.7	2
100	30	0	3	3	3.6	3.5	3.4	4	4.5	4.4	4.5	4.7	5.3	5.6	5.8	6.3	6.8	8.1	8.2	8.2	9.3	12.5	24.4	6.6
150	15	0	3.5	3.5	3.5	3.5	3.5	3.7	3.8	3.8	3.7	3.8	3.8	4.1	4.8	5.6	5.7	5.7	6.1	6.4	6.8	7.7	10.7	4.9
150	30	0	9.3	9.4	9.2	9.2	11.3	11.5	11.5	12.4	12.7	12.9	14.1	15.4	16.5	16.9	17.4	17.9	17.8	18.1	19.3	20.3	32.3	15
50	15	>0	0.8	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	1	1.1	1.2	1.5	1.9	2.2	4.7	1.1
50	30	>0	1.3	1.2	1.2	1.8	1.8	1.7	1.8	1.8	2.5	2.5	3	3.6	3	3.1	4.5	4.9	4.9	5.5	6.5	8.5	16.4	3.9
100	15	>0	1.6	1.5	1.5	1.5	1.5	1.5	1.9	1.9	2.1	2.4	2.5	3	3.1	3.2	3.3	3.5	3.6	4.1	5.1	6	9.1	3
100	30	>0	4.2	3.9	4.3	3.8	4.1	4.5	9.1	5.7	6.1	6.6	6.4	6.3	6.4	7.3	7.3	7.4	7.7	9	10.6	13.3	23.7	7.5
150	15	>0	4.6	4.6	5.1	5	5	4.7	5.1	5	5.4	5.4	5.5	5.6	6.4	7	7	7.6	7.8	8.2	9.2	9.4	12.1	6.5
150	30	>0	10.1	10	9.8	9.7	10.5	10.5	11.7	11.5	11.9	12.1	14.1	14.2	15.5	16	16.6	17.6	17.9	18.5	19.7	20.5	28.2	14.6
Average			3.4	3.4	3.5	3.5	3.8	3.8	4.4	4.2	4.4	4.5	4.9	5.2	5.6	5.9	6.2	6.7	6.8	7.2	8.1	9.3	15.7	5.7