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Measurability, spectral densities, and hypertraces in noncommutative geometry

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Abstract. We introduce, in the dual Macaev ideal of compact operators of a Hilbert space, the spectral weight $\rho(L)$ of a positive, self-adjoint operator L having discrete spectrum away from zero. We provide criteria for its measurability and unitarity of its Dixmier traces ($\rho(L)$ is then called spectral density) in terms of the growth of the spectral multiplicities of L or in terms of the asymptotic continuity of the eigenvalue counting function N_L. Existence of meromorphic extensions and residues of the ζ-function ζ_L of a spectral density are provided under summability conditions on spectral multiplicities. The hypertrace property of the states $\Omega_L(\cdot) = \mathrm{Tr}_\omega(\cdot\rho(L))$ on the norm closure of the Lipschitz algebra $\mathcal{A}_L$ follows if the relative multiplicities of L vanish faster than its spectral gaps or if N_L is asymptotically regular.

1. Introduction

Trace theorems for unbounded Fredholm modules $(\mathcal{A}, h, D)$, alias K-cycles or spectral triples, subject to various summability behaviors, date back to the dawning of noncommutative geometry [5–7]. They were proved under finite summability in [6] and hold true also under summability in the dual Macaev ideal in [3]. They were used to deduce hyperfinitness of weak closure of the *-algebra $\mathcal{A}$ in certain representations and to rule out the existence of unbounded Fredholm modules or quasidiagonal approximate units in normed ideals, with specific summability conditions (see [6, 20]). Also, a hypertrace constructed by $(\mathcal{A}, h, D)$ provides a Hilbert bimodule, unitary representation of the universal graded differential algebra $\Omega^*(\mathcal{A})$ [7, Chapter 6.1, Proposition 5].

Here we associate a *spectral weight* $\rho(|D|)$ in the dual Macaev ideal $\mathcal{L}^{(1,\infty)}(h)$, to any unbounded Fredholm module $(\mathcal{A}, h, D)$ and, more in general, to any filtration $\mathcal{F}$ of a Hilbert space h (in the sense of [20]). The spectral weight $\rho(|D|)$ depends, in particular, on the spectral multiplicities of D but not on the location of its eigenvalues.

Under a quite simple assumption on the growth of the filtration, we show measurability of $\rho(|D|)$ and under the *asymptotic continuity* of the eigenvalue counting function $N_{|D|}$, we prove also the unitarity $\mathrm{Tr}_\omega(\rho(|D|)) = 1$ of the Dixmier traces. In this situation, $\rho(|D|)$

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is called the *spectral density* of D and one may deal with the *volume states*

$$\Omega_{|D|}(a) := \mathrm{Tr}_\omega \left(a \cdot \rho(|D|)\right),$$

on the norm closure C^*-algebra A of $\mathcal{A}$, provided by any fixed Dixmier ultrafilter ω.

In commutative terms, i.e., dealing with the standard spectral triple

$$\left(C^\infty(M), L^2(\mathrm{Cl}(M)), D\right)$$

of a compact, closed, Riemannian manifold, taking into account multiplicities only and not the whole spectrum itself, one reconstructs the Riemann probability measure of M loosing information about the volume $V(M)$ and the dimension $d(M)$. On the other hand, this has the advantage to dispense with summability hypotheses and, for example, to recover the unique trace on the reduced C^*-algebra $C^*(\Gamma)$ of a finitely generated, countable discrete group Γ, no matter its growth is. Also, using the density $\rho(|D|)$, one is able to treat, on the same foot, situations like Euclidean domains of infinite volume whose Dirichlet Laplacian has discrete spectrum or certain hypoelliptic ΨDO on compact manifolds, where the asymptotics of the spectrum of D is not *à la Weyl*.

Under summability conditions on the spectral multiplicities of $|D|$, the ζ-function $\zeta_{|D|}$ of the spectral density $\rho(|D|)$ is shown to be meromorphic on a half plane containing $z = 1$ and that its residue is there unitary.

Finally, we show that the volume states $\Omega_{|D|}$ are hypertraces on A provided $N_{|D|} \sim \varphi$ for a nonnegative, increasing $W^{1,1}_{\mathrm{loc}}$-function φ such that the essential limit of φ'/φ vanishes at infinity. This condition is satisfied when the sequence of *relative multiplicities* of $|D|$ vanishes faster than the sequence of its *spectral gaps*.

The work is organized as follows. In Section 2, we introduce the spectral weight $\rho(L)$ of a positive, self-adjoint operator L having discrete spectrum away from zero. Its measurability is proved in terms of the growth of its spectral multiplicities, as a consequence of the asymptotic continuity of the counting function N_L. Sufficient conditions are also given in terms of the nuclearity of the semigroup e^{-tL}. In Section 3, we prove existence of analytic extensions and residues of the ζ-function ζ_L of a density $\rho(L)$, in terms of summability of the multiplicities of L. In Section 4, the volume states $\Omega_L(\cdot) = \mathrm{Tr}_\omega(\cdot\rho(L))$ are introduced and in Section 6 we show that they are hypertraces on the Lipschitz algebra $\mathcal{A}_L$, under asymptotic smoothness of the counting function N_L or when the relative multiplicities vanish faster than the spectral gaps of L. Section 5 is dedicated to various examples concerning (i) K-cycles on compact manifolds given by (hypo)elliptic ΨDO L, (ii) K-cycles on the C^*-algebra $\mathcal{P}(M)$ of scalar, 0-order ΨDO, which are associated to scalar, 1-order ΨDO L, (iii) K-cycles associated to multiplication operators on the group C^*-algebra of countable discrete, (iv) Dirichlet Laplacians of Euclidean domains of infinite volume, (v) Kigami's Laplacians on self-similar fractals, (vi) the Toeplitz C^*-algebra generated by an isometry and the canonical multiplication operator L on natural $\mathbb{N}$ and prime numbers $\mathbb{P}$, and (vii) unbounded Fredholm modules built using Hilbert space filtrations. In the final section (Section 6.5), the structure of the volume states Ω_L on C^*-algebras extensions is briefly outlined.

2. Measurable densities associated to operators with discrete spectrum

In this section, we introduce the spectral weight of a given nonnegative, self-adjoint operator $(L, D(L))$ on a Hilbert space h, having discrete spectrum away from zero, and investigate its measurability in the framework of Connes' noncommutative geometry (NCG). We always keep in mind the situation where $L = |D|$ for a spectral triple $(\mathcal{A}, D, h)$.

2.1. Eigenvalue counting function and multiplicities

In this section, we consider a densely defined, nonnegative, unbounded, self-adjoint operator $(L, D(L))$ on a Hilbert space h with spectrum $\mathrm{sp}(L)$ and spectral measure E^L.

Letting $P_0 := E^L(\{0\})$ be the orthogonal projection onto the kernel of L, we fix the following notations of functional calculus: by convention, for a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the operator $f(L)$ will be 0 on the subspace $P_0(h) = \ker L$ and $f(L(I - P_0))$ on the subspace $(I - P_0)h = (\ker L)^\perp$. For example, with this convention, for $s > 0$, L^{-s} is the nonnegative, densely defined operator defined as 0 on $P_0(h)$ and $(L(I - P_0))^{-s}$ on $(I - P_0)h$.

In this section, we will suppose that L has *discrete spectrum off of its kernel* in the sense that $\mathrm{sp}(L) \setminus \{0\}$ is discrete: this is equivalent to say that L^{-1} is a compact operator in $\mathcal{B}(h)$.

We will adopt two alternative ways for describing the spectrum, out of its kernel:

First way. $\mathrm{sp}(L) \setminus \{0\} = \{0 < \lambda_1(L) \leq \cdots \leq \lambda_n(L) \leq \cdots\}$, where the positive eigenvalues $\lambda_n(L)$ are numbered increasingly with repetition according to their multiplicity.

Second way. $\mathrm{sp}(L) \setminus \{0\} = \{0 < \widetilde{\lambda}_1(L) < \cdots < \widetilde{\lambda}_k(L) < \cdots\}$, where the distinct eigenvalues $\widetilde{\lambda}_k(L)$ are numbered increasingly.

Since L is assumed unbounded, we have

$$\lim_{n\to\infty} \lambda_n(L) = \lim_{k\to\infty} \widetilde{\lambda}_k(L) = +\infty.$$

The *multiplicity* of the eigenvalue $\widetilde{\lambda}_k(L)$ is denoted by $m_k := \mathrm{Tr}(E^L(\{\widetilde{\lambda}_k\}))$ while the *cumulated multiplicity* is defined as $M_k := \mathrm{Tr}(E^L((0, \widetilde{\lambda}_k]))$ so that $M_k = \sum_{j=1}^k m_j$. By convention, $M_0 := 0$. We will refer to the ratio m_k/M_k as the *relative multiplicity* of the eigenvalue $\widetilde{\lambda}_k(L)$. The two labelings correspond through the relation

$$\lambda_n(L) = \widetilde{\lambda}_k(L), \quad M_{k-1} < n \leq M_k.$$

Remark 2.1. We will adopt the simplified notations $\lambda_1, \ldots, \lambda_n, \ldots$ and $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_k, \ldots$ whenever no confusion can arise.

The *eigenvalue counting function* $N_L : \mathbb{R}_+ \to \mathbb{N}$ is defined as

$$N_L(x) := \mathrm{Tr}\left(E^L\big((0, x]\big)\right) = \sharp\big\{n \in \mathbb{N}^* : \lambda_n(L) \leq x\big\},$$

where Tr is the normal, semifinite trace on $B(h)$.

Let us summarize some basic properties of the counting function.

Lemma 2.2. (i) *N_L is a nondecreasing function, right continuous with left limits. For $x \in \mathbb{R}_+$, let us denote $N_L^-(x) = \lim_{\delta\downarrow 0} N_L(x-\delta)$ the left limit function of N_L.*

(ii) *$N_L(x) = M_k$ for $\widetilde{\lambda}_k(L) \le x < \widetilde{\lambda}_{k+1}(L)$.*

(iii) *$\limsup_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} = \limsup_{k\to\infty} \frac{M_k}{M_{k-1}}$.*

Proof. Properties (i) and (ii) are obvious from the definition of N_L. For (iii), it is enough to observe that, for $x \notin \mathrm{sp}(L)$, $N_L^-(x) = N_L(x)$ and that, for $x = \widetilde{\lambda}_k(L)$, $N_L(x) = M_k$ while $N_L^-(x) = M_{k-1}$. ∎

2.2. Spectral weights

Definition 2.3 (Spectral weights). The operator defined as

$$\rho(L) := N_L(L)^{-1}$$

will be called the *spectral weight of* L. As $\mathrm{sp}(L) \setminus \{0\}$ is discrete and unbounded and N_L is nondecreasing, it follows that $\rho(L)$ is nonnegative and compact.

Proposition 2.4. (i) *The eigenvalues of the spectral weight are given by*

$$\mu_n\big(\rho(L)\big) = N_L\big(\lambda_n(L)\big)^{-1} = \frac{1}{M_k} \quad \text{for } M_{k-1} < n \le M_k \text{ and } k \ge 1;$$

(ii) *we also have the bounds*

$$\frac{M_{k-1}}{M_k} \cdot \frac{1}{n} < \mu_n\big(\rho(L)\big) \le \frac{1}{n} \quad \text{for } M_{k-1} < n \le M_k \text{ and } k \ge 1. \tag{2.1}$$

Proof. For (i), apply Lemma 2.2. For (ii), notice that for n as considered, we have

$$\lambda_n(L) = \widetilde{\lambda}_k(L) \quad \text{and} \quad N_L\big(\lambda_n(L)\big) = N_L\big(\widetilde{\lambda}_k(L)\big) = M_k.$$

From one side, $M_k \ge n$ and thus $N_L(\lambda_n(L)) \ge n$. On the other side, $n > M_{k-1}$ and thus

$$N_L\big(\lambda_n(L)\big)^{-1} = M_k^{-1} > \frac{1}{n} \cdot \frac{M_{k-1}}{M_k}.$$ ∎

2.3. Weights by filtrations

For any Borel measurable, strictly increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{x\to\infty} f(x) = +\infty,$$

we have $\rho(f(L)) = \rho(L)$. In fact, $\rho(L)$ depends on L only through the Hilbert space filtration of spectral subspaces of h

$$\{E^L\big((0, \widetilde{\lambda}_k(L)]\big)\}_{k=1}^{+\infty}.$$

In [20, Proposition 5.1], D. V. Voiculescu, motivated by the existence of quasicentral approximate units relative to normed ideals, provided a general construction of spectral triples $(\mathcal{A}, h, D)$ on a C^*-algebra A, represented in a Hilbert space h, associated to given filtrations $h_0 \subset h_1 \subset \cdots \subset h$. He considers the filtration of A given by

$$V_k := \{T \in A : T(h_j) \cup T^*(h_j) \subseteq h_{j+k},\ \forall j \in \mathbb{N}\} \quad \text{for } k \in \mathbb{N},$$

assuming that $\mathcal{A} := \bigcup_{k\in\mathbb{N}} V_k$ is dense in A. Denoting by P_j the projection onto h_j, the Dirac operator is defined as $D := \sum_{j\geq 1}(I - P_j)$ (so that $D = |D|$). The spectrum of D is $\mathbb{N}$ with cumulated multiplicities $M_j := \dim(h_j)$ and the spectral weight is given by

$$\rho(D) = \sum_{k\geq 1}^{+\infty} \frac{1}{M_k} \cdot (P_{k+1} - P_k).$$

2.4. Dixmier traces

In this work, we will consider the nonnormal traces associated with ultrafilters on $\mathbb{N}$, introduced by J. Dixmier [9], making use, in particular, of the approach proposed in [2].

We will start with a state ω on $L^\infty(\mathbb{R}_+^*)$ having all the properties of [2, Theorem 1.5]. First, ω is a limit process at $+\infty$ in the sense that

(1) $\operatorname{ess\,lim\,inf}_{t\to+\infty} f(t) \leq \omega(f) \leq \operatorname{ess\,lim\,sup}_{t\to+\infty} f(t)$, $f \in L^\infty(\mathbb{R}_+^*)$,

so that we can write as well $\omega(f) := \omega - \lim_{t\to+\infty} f(t)$.

Then we require that this limit process satisfies the following invariance properties:

(2) $\omega - \lim_{t\to+\infty} f(st) = \omega - \lim_{t\to+\infty} f(t)$, $f \in L^\infty(\mathbb{R}_+^*)$, $s \in \mathbb{R}_+^*$;

(3) $\omega - \lim_{t\to+\infty} f(t^s) = \omega - \lim_{t\to+\infty} f(t)$, $f \in L^\infty(\mathbb{R}_+^*)$, $s \in \mathbb{R}_+^*$;

(4) $\omega - \lim_{t\to\infty} \frac{1}{\operatorname{Log}(t)} \int_1^t f(s)\frac{ds}{s} = \omega - \lim_{t\to+\infty} f(t)$, $f \in L^\infty(\mathbb{R}_+^*)$.

To such ω is associated an ultrafilter on $\mathbb{N}$, still denoted ω, by

$$\omega - \lim_{n\to\infty} f(n) = \omega - \lim_{t\to+\infty} f\big([t]\big), \quad f \in \ell^\infty(\mathbb{N}) \text{ with } [t] = \text{integer part of } t.$$

The associated Dixmier trace is defined on the dual Macaev ideal of compact operators

$$\mathcal{L}^{(1,\infty)}(h) := \left\{T \in \mathcal{K}(h) : \sup_{N\geq 2} \frac{1}{\operatorname{Log} N} \cdot \sum_{n=1}^{N} \mu_n\big(|T|\big) < +\infty\right\}$$

as

$$\operatorname{Tr}_\omega(T) = \omega - \lim_{N\to\infty} \frac{1}{\operatorname{Log} N} \sum_{n=1}^{N} \mu_n(T), \quad T \in \mathcal{L}^{(1,\infty)}(h)_+.$$

The operator T is said to be *measurable* if its Dixmier trace $\operatorname{Tr}_\omega(\rho(L))$ is independent upon the Dixmier ultrafilter ω. Conditions ensuring measurability can be given in terms of Cesaro means (see [6, Proposition 6, Chapter 4.2.β]).

To ω on $L^\infty(\mathbb{R}_+^*)$ as above is associated an alternative limit process $\widetilde{\omega}$ on $\mathbb{R}$ defined as

$$\widetilde{\omega} - \lim_{t\to+\infty} f(t) = \omega - \lim_{t\to+\infty} f\big(\mathrm{Log}(t)\big), \quad f \in L^\infty(\mathbb{R}).$$

The asymptotic behavior of $\frac{1}{\mathrm{Log}\, N}\sum_{n=1}^N \mu_n(T)$ and the limit behavior of $(s-1)\,\mathrm{Tr}(T^s)$ as $s \to 1^+$ are related by the following equality [2, Theorem 3.1]:

$$\mathrm{Tr}_\omega(T) = \widetilde{\omega} - \lim_{r\to+\infty} \frac{1}{r}\,\mathrm{Tr}\big(T^{1+\frac{1}{r}}\big), \quad T \in \mathcal{L}^{(1,\infty)}(h)_+. \tag{2.2}$$

Furthermore, with T as above and any $A \in \mathcal{B}(h)$, we have (see [2, Theorem 3.8])

$$\mathrm{Tr}_\omega(AT) = \widetilde{\omega} - \lim_{r\to+\infty} \frac{1}{r}\,\mathrm{Tr}\big(AT^{1+\frac{1}{r}}\big), \quad T \in \mathcal{L}^{(1,\infty)}(h)_+. \tag{2.3}$$

2.5. Asymptotics for the spectral weights

Proposition 2.5. (i) *The spectral weight belongs to the dual Macaev ideal* $\mathcal{L}^{(1,\infty)}(h)$ *and*

$$\frac{1}{c} \le \mathrm{Tr}_\omega\big(\rho(L)\big) \le 1 \quad \textit{for all Dixmier ultrafilter } \omega,$$

where

$$c := \limsup_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} = \limsup_{k\to\infty} \frac{M_k}{M_{k-1}}.$$

(ii) $\rho(L)^s$ *is trace class for all* $s > 1$ *and* $\limsup_{s\downarrow 1}(s-1)\,\mathrm{Tr}(\rho(L)^s) \le 1$.

(iii) *If*

$$\limsup_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} = \lim_{k\to\infty} \frac{M_k}{M_{k-1}} = 1,$$

then we have

(iii.a) $\mu_n(\rho(L)) \sim 1/n$ *as* $n \to \infty$,

(iii.b) $\mathrm{Tr}_\omega(\rho(L)) = 1$ *for all Dixmier ultrafilter* ω *and the spectral weight* $\rho(L)$ *is measurable,*

(iii.c) $\lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}(\rho(L)^s) = 1$.

Proof. (i) follows from inequality $\mu_n(\rho(L)) \le 1/n$ of Proposition 2.4 and from inequality

$$\liminf_{n\to\infty} \mu_n\big(\rho(L)\big) \ge 1/cn$$

which follows by (2.1). (ii) follows again from inequality $\mu_n(\rho(L)) \le 1/n$ of Proposition 2.4. (iii.a) comes from the double inequality (2.1), while (iii.b) and (iii.c) are straightforward consequences of (iii.a). ∎

Definition 2.6 (Spectral densities). The spectral weight $\rho(L)$ will be called *spectral density* provided it is measurable and $\mathrm{Tr}_\omega(\rho(L)) = 1$ for all Dixmier ultrafilter ω.

We may have measurability of a spectral weight even if it is not a density as follows:

Proposition 2.7. *If* $\lim_{k\to\infty} \frac{M_{k+1}}{M_k} = c > 1$, *then the spectral weight* $\rho(L)$ *is measurable and*

$$\mathrm{Tr}_\omega\big(\rho(L)\big) = \lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}\big(\rho(L)^s\big) = \frac{c-1}{c\,\mathrm{Log}\,c} \quad \textit{for all Dixmier ultrafilter } \omega.$$

Proof. Fix $0<\varepsilon<c-1$ and $K\in\mathbb{N}$ such that $(c-\varepsilon)M_k \le M_{k+1} \le (c+\varepsilon)M_k$ for $k\ge K$. This implies that $(c-\varepsilon)^{k-K}M_K \le M_k \le (c+\varepsilon)^{k-K}M_K$ for $k>K$. For $N>M_K$, let $k(N)$ be the integer such that $M_{k(N)} \le N \le M_{k(N)+1}$. One has

$$\mathrm{Log}\,M_{k(N)} \le \mathrm{Log}\,N \le \mathrm{Log}(M_{k(N)+1}) \le \mathrm{Log}\,M_{k(N)} + \mathrm{Log}(c+\varepsilon)$$

so that $\mathrm{Log}\,N = \mathrm{Log}\,M_{k(N)} + O(1)$ as $N\to+\infty$. Gathering those two results, we get

$$k(N)\,\mathrm{Log}(c-\varepsilon) + O(1) \le \mathrm{Log}\,N \le k(N)\,\mathrm{Log}(c+\varepsilon) + O(1), \quad N\to+\infty. \tag{2.4}$$

Moreover,

$$\begin{aligned}\sum_{n=1}^{N} N_L(\lambda_n)^{-1} &= \sum_{k=1}^{K-1}\sum_{n=M_{k-1}}^{M_k} N_L(\lambda_n)^{-1}\\ &\quad+\sum_{k=K}^{k(N)}\sum_{n=M_{k-1}+1}^{M_k} N_L(\lambda_n)^{-1} + \sum_{n=M_{k(N)+1}}^{N} N_L(\lambda_n)^{-1}\\ &= \sum_{k=1}^{K-1}\sum_{n=M_{k-1}}^{M_k} N_L(\lambda_n)^{-1} + \sum_{k=K}^{k(n)} \frac{M_k - M_{k-1}}{M_k} + \frac{N - M_{k(N)}}{M_{k(N)+1}}.\end{aligned}$$

For $k\ge K$, we have $\frac{c+\varepsilon-1}{c+\varepsilon} \le \frac{M_k - M_{k-1}}{M_k} \le \frac{c-\varepsilon-1}{c-\varepsilon}$, which provides

$$k(N)\frac{c+\varepsilon-1}{c+\varepsilon} + O(1) \le \sum_{n=1}^{N} N_L(\lambda_n)^{-1} \le k(N) \le \frac{c-\varepsilon-1}{c-\varepsilon} + O(1), \quad N\to+\infty.$$

With (2.4), this inequality implies that

$$\begin{aligned}&\frac{1}{k(N)\,\mathrm{Log}(c+\varepsilon)+O(1)}\Big(k(N)\frac{c+\varepsilon-1}{c+\varepsilon} + O(1)\Big)\\ &\quad\le \frac{1}{\mathrm{Log}\,N}\sum_{n=1}^{N} N_L(\lambda_n)^{-1}\\ &\quad\le \frac{1}{k(N)\,\mathrm{Log}(c-\varepsilon)+O(1)}\Big(k(N)\frac{c-\varepsilon-1}{c-\varepsilon} + O(1)\Big)\end{aligned}$$

which provides the first equality $\mathrm{Tr}_\omega(\rho(L)) = \frac{c-1}{c}$.

The second equality $\lim_{s\downarrow 1} \mathrm{Tr}(\rho(L)^s) = \frac{c-1}{c \operatorname{Log} c}$ is obtained through a similar computation. Fix $\varepsilon > 0$ and $K \in \mathbb{N}$ such that $k \geq K \Rightarrow c - \varepsilon \leq M_{k+1}/M_k \leq c + \varepsilon$, hence

$$(c-\varepsilon)^{k-K} M_K \leq M_k \leq (c+\varepsilon)^{k-K} M_K \quad \text{for } k > K.$$

We have also, for $k \geq K+1$, $c - \varepsilon \leq \frac{M_k}{M_{k-1}} = \frac{m_k}{M_{k-1}} + 1 \leq c + \varepsilon$, i.e., $c - 1 - \varepsilon \leq \frac{m_k}{M_{k-1}} \leq c - 1 + \varepsilon$ which implies that

$$\frac{c-1-\varepsilon}{c+\varepsilon} \leq \frac{m_k}{M_k} \leq \frac{c-1+\varepsilon}{c-\varepsilon}.$$

We compute now for $s > 1$

$$\begin{aligned} \mathrm{Tr}\left(\rho(L)^s\right) &= \sum_k m_k M_k^{-s} = \sum_{k=1}^{K} m_k M_k^{-s} + \sum_{k=K+1}^{\infty} m_k M_k^{-s} \\ &= \sum_{k=1}^{K} m_k M_k^{-s} + \sum_{k=K+1}^{\infty} \frac{m_k}{M_k} M_k^{-s+1}. \end{aligned}$$

We have $(s-1)\sum_{k=1}^{K} m_k M_k^{-s} \to 0$ as $s \to 1$, while

$$\begin{aligned} \sum_{k=K+1}^{\infty} \frac{m_k}{M_k} M_k^{-s+1} &\leq \frac{c-1+\varepsilon}{c-\varepsilon}(c+\varepsilon)^{(-s+1)(k-K)} M_K^{-s+1} \\ &= \frac{c-1+\varepsilon}{c-\varepsilon}(c+\varepsilon)^{(-s+1)(-K)} M_K^{-s+1} \frac{1}{1-(c+\varepsilon)^{1-s}} \end{aligned}$$

with $1 - (c+\varepsilon)^{1-s} \sim (s-1)\operatorname{Log}(c+\varepsilon)$ as $s \downarrow 1$.

We have proved $\limsup_{s\downarrow 1}(s-1)\,\mathrm{Tr}(\rho(L)^s) \leq \frac{c-1+\varepsilon}{c-\varepsilon}\frac{1}{\operatorname{Log}(c+\varepsilon)}$. A similar computation provides $\liminf_{s\downarrow 1}(s-1)\,\mathrm{Tr}(\rho(L)^s) \geq \frac{c-1-\varepsilon}{c+\varepsilon}\frac{1}{\operatorname{Log}(c-\varepsilon)}$ and the result. ■

The hypothesis of the above result is verified in discrete free groups (see Example 3.7).

Here is another criterion for $\rho(L)$ having a nonzero Dixmier trace.

Proposition 2.8. *If $M_k \sim f(k)$ $(k \to +\infty)$ with $f \in C^1((0,+\infty))$, then*

$$\limsup_{k\to\infty} \frac{M_{k+1}}{M_k} \leq e^C \quad \textit{with } C := \limsup_{x\to+\infty} \frac{f'(x)}{f(x)}.$$

Hence $\mathrm{Tr}_\omega(\rho(L)) \geq e^{-C}$ *for all Dixmier ultrafilters* ω.

Proof. Fix $\varepsilon > 0$ and choose $K_\varepsilon \geq 1$ such that for all $k \geq K_\varepsilon$ we have

$$\frac{1-\varepsilon}{1+\varepsilon}\frac{f(k+1)}{f(k)} \leq \frac{M_{k+1}}{M_k} \leq \frac{1+\varepsilon}{1-\varepsilon}\frac{f(k+1)}{f(k)}.$$

It follows that

$$\limsup_{k\to\infty} \frac{M_{k+1}}{M_k} = \limsup_{k\to\infty} \frac{f(k+1)}{f(k)}.$$

Then, setting $C = \limsup_{x\to+\infty} \frac{f'(x)}{f(x)}$, we have $(\mathrm{Log}\, f)'(x) = \frac{f'(x)}{f(x)} \le C + \varepsilon$ for x large enough so that

$$\mathrm{Log}\big(f(k+1)\big) - \mathrm{Log}\big(f(k)\big) \le C + \varepsilon,$$

i.e., $\frac{f(k+1)}{f(k)} \le e^{C+\varepsilon}$ for k large enough too. ∎

2.6. Asymptotic continuity of the eigenvalue counting function and measurability

Here we link the measurability of the spectral weight to the asymptotic continuity of the eigenvalue counting function and to the asymptotic vanishing of the relative multiplicity.

Proposition 2.9. (i) *If the counting function N_L is asymptotically continuous in the sense that there exists a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$N_L(x) \sim \varphi(x), \quad x \to +\infty,$$

then

$$\lim_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} = \lim_{k\to\infty} \frac{M_{k+1}}{M_k} = 1 \quad \textit{or, equivalently,} \quad \lim_{k\to\infty} \frac{m_k}{M_k} = 0.$$

(ii) *Conversely, if* $\lim_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} = 1$ *or* $\lim_{k\to\infty} \frac{M_{k+1}}{M_k} = 1$ *or* $\lim_{k\to\infty} \frac{m_k}{M_k} = 0$, *then N_L is asymptotically continuous.*

In both cases, $\rho(L)$ is a density and the properties (iii) *of Proposition* 2.5 *hold true.*

Proof. (i) For $\varepsilon > 0$ and $x \in \mathbb{R}_+$ large enough, we have

$$(1-\varepsilon)\varphi(x) \le N_L(x) \le (1+\varepsilon)\varphi(x).$$

As φ is continuous, we have as well $(1-\varepsilon)\varphi(x) \le N_L^-(x)$ so that $\frac{N_L(x)}{N_L^-(x)} \le \frac{1+\varepsilon}{1-\varepsilon}$. This implies that

$$\limsup_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} \le \frac{1+\varepsilon}{1-\varepsilon} \quad \text{for all } \varepsilon > 0,$$

and finally $\limsup_{x\to+\infty} \frac{N_L(x)}{N_L^-(x)} = 1$. Lemma 2.2 and Proposition 2.4 provide the result.

(ii) Choose φ continuous, piecewise affine such that $\varphi(\tilde{\lambda}_k) = M_k$. This means that

$$\varphi(x) = M_{k-1} + t(M_k - M_{k-1})$$

for $x \in [\tilde{\lambda}_{k-1}, \tilde{\lambda}_k]$ and $x = \tilde{\lambda}_{k-1} + t(\tilde{\lambda}_k - \tilde{\lambda}_{k-1})$, $0 \le t \le 1$. For such x we have

$$\frac{\varphi(x)}{N_L(x)} = \frac{M_{k-1}}{M_k} + t\left(1 - \frac{M_{k-1}}{M_k}\right) \to 1, \quad x \to +\infty.$$ ∎

Remark 2.10. The condition $1 = \limsup_k \frac{M_{k+1}}{M_k}$ $(= \lim_k \frac{M_{k+1}}{M_k})$ ensuring measurability implies that the weaker condition $\lim_k \sqrt[k]{M_k} = 1$, which in turn is a sufficient condition for the subexponential growth of the spectral multiplicities of L, in this sense that $\lim_k e^{-\beta k} M_k = 0$ for any $\beta > 0$. However, one can find instances of sequences M_k for the cumulated multiplicities having subexponential growth, for which the condition $\lim_k M_{k+1}/M_k = 1$ (and thus asymptotic continuity) is not satisfied.

Combining these results with Karamata's Tauberian theorem (cf. Appendix A), we get a criterion of asymptotic continuity of N_L in terms of regularity of the partition function Z_L.

Proposition 2.11. *Suppose the contraction semigroup* $\{e^{-tL} : t \geq 0\}$ *to be nuclear*

$$Z_L(\beta) := \mathrm{Tr}(e^{-\beta L}) < +\infty \quad \textit{for all } \beta > 0$$

and assume the partition function Z_L *to be regularly varying (cf. Appendix A). Then for some* $c > 0$ *we have*

$$N_L(x) \sim c \cdot Z_L(1/x), \quad x \to +\infty.$$

In particular, under these assumptions, N_L *is asymptotically continuous and* $\rho(L)$ *is a density.*

Proof. Apply Karamata's Tauberian theorem to the measure $\mu := \mathrm{Tr} \circ E^L$ and then apply Proposition 2.9. ■

The result may be applied to *θ-summable spectral triples* $(\mathcal{A}, h, D)$, where $Z_{D^2}(\beta) = \mathrm{Tr}(e^{-\beta D^2}) < +\infty$ for all $\beta > 0$, provided the partition function Z_{D^2} is regularly varying, as a consequence of the identity

$$N_L(x) = N_{D^2}(x^2) \quad \text{for all } x > 0.$$

Remark 2.12 (Physical interpretation of nuclearity and regularity). In applications, L may represent the Hamiltonian of a quantum system. The nuclearity assumption on the semigroup $\{e^{-\beta L} : \beta > 0\}$ is easily seen to be equivalent to the requirement that the mean value of the energy in the Gibbs equilibrium state is finite and non-vanishing at any temperature

$$\langle L \rangle_\beta = -\frac{\dot{Z}_L(\beta)}{Z_L(\beta)} = \frac{\mathrm{Tr}(Le^{-\beta L})}{\mathrm{Tr}(e^{-\beta L})}, \quad \beta > 0.$$

The hypothesis that Z_L is regularly varying requires that for some $\gamma \in \mathbb{R}$

$$\lim_{\beta \to 0^+} \frac{Z_L(s\beta)}{Z_L(\beta)} = s^\gamma, \quad s > 0.$$

If $\dot{Z}_L$ is regularly varying, say $\dot{Z}_L(s\beta) \sim s^{\gamma-1}\dot{Z}_L(\beta)$ for some $\gamma \in \mathbb{R}$ as $\beta \to 0^+$, by de l'Hospital's theorem, then also Z_L is regularly varying

$$\lim_{\beta \to 0^+} \frac{Z_L(s\beta)}{Z_L(\beta)} = s \lim_{\beta \to 0^+} \frac{\dot{Z}_L(s\beta)}{\dot{Z}_L(\beta)} = s^\gamma$$

and the mean energy $\langle L \rangle$ is regularly varying too with

$$\langle L \rangle_{s\beta} = -\frac{\dot{Z}_L(s\beta)}{Z_L(s\beta)} \sim -\frac{s^{\gamma-1}\dot{Z}_L(\beta)}{s^\gamma Z_L(\beta)} = \frac{1}{s} \cdot \langle L \rangle_\beta, \text{ as } \beta \to 0^+, \quad \text{for any fixed } s > 0.$$

3. Meromorphic extensions of zeta functions and residues

The ζ-function of $\rho(L)$ is defined as

$$\zeta_L(s) := \operatorname{Tr}\big(\rho(L)^s\big) = \sum_{n\geq 1} \mu_n\big(\rho(L)\big)^s = \sum_{k\geq 1} m_k \cdot M_k^{-s}$$

for all $s \in \mathbb{C}$ for which the series converges. Its domain and its analytic properties will be found by comparison with the Riemann ζ-function

$$\zeta_0(s) = \sum_{n\geq 1} n^{-s},$$

which, initially defined on the half-plane $\{s \in \mathbb{C} : \Re e(s) > 1\}$, is then extended analytically to $\mathbb{C} \setminus \{1\}$. Recall that $s = 1$ is simple pole for ζ_0 with unital residue.

The following criteria for the asymptotic properties of the ζ-function $\zeta_L(s)$ as $s \to 1$ are based on various growth rates of the spectral multiplicity m_k as $k \to \infty$.

3.1. Criteria for meromorphic extensions of the ζ-function ζ_L

Lemma 3.1. *For $\varepsilon \in [0, 1)$ and $s \in \mathbb{C}$ such that $\Re e(s) \geq 0$, we have*

$$\big|1 - (1-\varepsilon)^s\big| \leq |s| \operatorname{Log}\big((1-\varepsilon)^{-1}\big).$$

Proof. Setting $b := \operatorname{Log}((1-\varepsilon)^{-1})$, $x := \Re e(s) \geq 0$, $f(t) := (1-\varepsilon)^{ts} = e^{-bst}$ for $t \in [0, 1]$, we have

$$\big|1 - (1-\varepsilon)^s\big| = \big|f(1) - f(0)\big| = \left|\int_0^1 f'(t)\, dt\right| = \left|-bs \int_0^1 e^{-bst}\, dt\right|$$
$$\leq b|s| \cdot \int_0^1 |e^{-bst}|\, dt \leq b|s|. \qquad \blacksquare$$

Proposition 3.2. (i) *If N_L is asymptotically continuous, then the ζ-function ζ_L is well defined on the half-plane $\{s \in \mathbb{C} : \Re e(s) > 1\}$ and it admits the limit*

$$\lim_{s\in\mathbb{R},\, s\downarrow 1} (s-1) \operatorname{Tr}\big(\rho(L)^s\big) = 1. \tag{3.1}$$

(ii) *If $\sum_k \frac{m_k^2}{M_k^2} < +\infty$, then ζ_L is analytic on $\{s \in \mathbb{C} : \Re e(s) > 1\}$ and it admits the limit*

$$\lim_{s\in\mathbb{C}, \Re e(s)>1,\, s\to 1} (s-1) \operatorname{Tr}\big(\rho(L)^s\big) = 1. \tag{3.2}$$

(iii) *If $\sum_k \frac{m_k^2}{M_k^{1+\alpha}} < +\infty$ for some $\alpha \in (0, 1)$, then ζ_L extends to a meromorphic function on the half-plane $\{s \in \mathbb{C} : \Re e(s) > \alpha\}$ with a simple pole at $s = 1$ and unital residue.*

Proof. (i) By Proposition 2.5 (iii.a), the asymptotic continuity of N_L implies that

$$\mu_n(\rho(L)) \sim 1/n \quad \text{as } n \to \infty,$$

so that ζ_L is well defined on $\{s \in \mathbb{C} : \Re e(s) > 1\}$. The limit behavior is just the content of Proposition 2.5 (iii.c).

(ii) and (iii) We suppose that the series $\sum_k m_k^2/M_k^{1+\alpha}$ converges for some $\alpha \in (0,1]$. Notice first that the assumption implies that $\lim_{k\to\infty} \frac{m_k}{M_k} = 0$. Hence $\lim_{k\to\infty} \frac{M_{k-1}}{M_k} = 1$, N_L is asymptotically continuous by Proposition 2.9 (ii), and $\mu_n(\rho(L)) \sim 1/n$ as $n \to \infty$ again by Proposition 2.5 (iii.a). Let us write $\delta_n := 1/n - \mu_n(\rho(L))$. According to (2.1), whenever $k \geq 1$ and $M_{k-1} < n \leq M_k$ we have

$$0 \leq \delta_n < \frac{1}{n}\left(1 - \frac{M_{k-1}}{M_k}\right) = \frac{1}{n}\frac{m_k}{M_k} \quad \text{and} \quad 0 \leq n\delta_n < \frac{m_k}{M_k} < 1.$$

Let us estimate the difference

$$\zeta_0(s) - \zeta_L(s) = \sum_{n\geq 1}\left(n^{-s} - (n^{-1} - \delta_n)^s\right) = \sum_{n\geq 1} n^{-s}\left(1 - (1 - n\delta_n)^s\right).$$

Since $1 - n\delta_n \geq M_{k-1}/M_k > 0$ for $k \geq 2$, $M_0 = 0$, $M_1 = m_1 \geq 1$, applying Lemma 3.1, we have, for $s \in \mathbb{C}$ with $\Re e(s) > 1$,

$$\begin{aligned}
\sum_{n>M_1} |n^{-s}|\left|1 - (1 - n\delta_n)^s\right| &\leq |s| \sum_{n\geq M_1} n^{-\Re e(s)} \operatorname{Log}\left((1 - n\delta_n)^{-1}\right) \\
&\leq |s| \sum_{k\geq 2} \sum_{n=M_{k-1}+1}^{M_k} n^{-\Re e(s)} \operatorname{Log}\left((1 - n\delta_n)^{-1}\right) \\
&\leq |s| \sum_{k\geq 2} \sum_{n=M_{k-1}+1}^{M_k} n^{-\Re e(s)} \operatorname{Log}(M_k/M_{k-1}) \\
&\leq |s| \sum_{k\geq 2} \sum_{n=M_{k-1}+1}^{M_k} M_{k-1}^{-\Re e(s)} (m_k/M_{k-1}) \\
&= |s| \sum_{k\geq 2} m_k^2/M_{k-1}^{1+\Re e(s)},
\end{aligned}$$

where, under the current hypothesis, the last series converge as $k \to \infty$ (since $M_k \sim M_{k-1}$) whenever $\mathcal{R}\mathrm{e}(s) \geq \alpha$.

This means that the function $\zeta_0 - \zeta_L$ extends as a continuous function on the upper half-plane $\mathcal{R}\mathrm{e}(s) \geq \alpha$ which is analytic on the upper half-plane $\mathcal{R}\mathrm{e}(s) > \alpha$. The thesis follows since ζ_0 is meromorphic with residue one at the simple pole $s = 1$. ■

Proposition 3.3. *If $m_k = O(M_k^\alpha)$, $k \to \infty$, for some $\alpha \in (0,1)$, then ζ_L extends to a meromorphic function on the half-plane $\{s \in \mathbb{C} : \Re e(s) > \alpha\}$ with a simple pole at $s = 1$ and unital residue.*

Proof. The assumption $m_k = O(M_k^\alpha)$ implies that $m_k/M_k = (m_k/M_k^\alpha)M_k^{\alpha-1} \to 0$ as $k \to \infty$, so that, by Proposition 2.9 (ii), N_L is asymptotically continuous. Applying Proposition 2.5 (iii.a), we have $\mu_n(L) \sim 1/n$ as $n \to \infty$ and then, for any fixed $\beta \in (\alpha, 1)$, the following series converge:

$$\sum_{k\geq 1} \frac{m_k^2}{M_k^{1+\beta}} \leq C \sum_{k\geq 1} \frac{m_k}{M_k^{1+\beta-\alpha}} = C \sum_{k\geq 1} \sum_{n=M_{k-1}+1}^{M_k} \mu_n(L)^{-(1+\beta-\alpha)}$$
$$= C \sum_{n\geq 1} \mu_n(L)^{-(1+\beta-\alpha)} < +\infty.$$

Apply Proposition 3.2 to conclude. ■

The hypothesis of the previous proposition can be restated in terms of a bound on the error term in the asymptotically continuous behavior of the counting function.

Proposition 3.4. *For $\alpha \in (0, 1)$, the two statements are equivalent:*

(i) $m_k = O(M_k^\alpha)$, $k \to \infty$;

(ii) *there exists a continuous function φ such that*

$$N_L(x) = \varphi(x) + O\big(\varphi(x)^\alpha\big), \quad x \to +\infty.$$

Proof. Suppose that $m_k = O(M_k^\alpha)$. Notice first that $m_k = o(M_k)$ so that $M_k \sim M_{k+1}$ and N_L is asymptotically continuous. Let φ be the continuous piecewise affine function defined as

$$\varphi(x) = M_k + t(M_{k+1} - M_k) = M_k + t m_{k+1},$$
$$x \in [\widetilde{\lambda}_k, \widetilde{\lambda}_{k+1}],\ x = \widetilde{\lambda}_k + t(\widetilde{\lambda}_{k+1} - \widetilde{\lambda}_k),\ t \in [0, 1].$$

One has $0 \leq \varphi(x) - N_L(x) \leq m_{k+1} = O(M_{k+1}^\alpha) = O(\varphi(x)^\alpha)$, $x \to +\infty$, which provides $N_L(x) = \varphi(x) + O(\varphi(x)^\alpha)$. Conversely, if φ is continuous and such that $N_L(x) = \varphi(x) + O(\varphi(x)^\alpha)$ as $x \to +\infty$, let us define the function $r(x)$ by the formula

$$N_L(x) = \varphi(x)\big(1 + r(x)\big).$$

On one side, one has $r(x) = O(\varphi(x)^{\alpha-1})$ as $x \to +\infty$. On the other side, the function $r(\cdot)$ is right continuous with left limits $r^-(x) := \lim_{\delta\downarrow 0} r(x - \delta)$, in such a way that

$$N_L^-(x) = \varphi(x)\big(1 + r^-(x)\big).$$

Fixing $\delta > 0$ small enough, we have $N_L(\widetilde{\lambda}_{k+1} - \delta) = M_k$ while $N_L(\widetilde{\lambda}_{k+1}) = M_{k+1}$, i.e., $M_k = \varphi(\widetilde{\lambda}_{k+1} - \delta)(1 + r(\widetilde{\lambda}_{k+1} - \delta))$ while $M_{k+1} = \varphi(\widetilde{\lambda}_{k+1})(1 + r(\widetilde{\lambda}_{k+1}))$. Taking the quotient and making δ tend to 0, we get

$$\frac{M_{k+1}}{M_k} = \frac{1 + r(\widetilde{\lambda}_{k+1})}{1 + r^-(\widetilde{\lambda}_{k+1})} = \frac{1 + O(\varphi(x)^{\alpha-1})}{1 + O(\varphi(x)^{\alpha-1})} = 1 + O\big(\varphi(x)^{\alpha-1}\big) = 1 + O(M_k^{\alpha-1})$$

which means that $\frac{m_{k+1}}{M_k} = O(M_k^{\alpha-1})$ and that $m_{k+1} = O(M_k^\alpha) = O(M_{k+1}^\alpha)$ (since $M_{k+1} \sim M_k$). ■

Summarizing, we have the following criterion of meromorphic extension.

Theorem 3.5. *Suppose that there exists* $\alpha \in (0,1)$ *and a continuous function* φ *such that*

$$N_L(x) = \varphi(x) + O\big(\varphi(x)^\alpha\big), \quad x \to +\infty.$$

Then the ζ*-function* $\zeta_L(s) = \mathrm{Tr}(\rho(L)^s)$ *extends as a meromorphic function on the half plane* $\{s \in \mathbb{C} : \Re e(s) > \alpha\}$ *with a simple pole at* $s = 1$ *and unital residue.*

Proof. Apply Propositions 3.4 and 3.3. ∎

3.2. Examples of densities and their ζ-functions

Example 3.6 (Compact smooth manifolds I). Let M be a compact, n-dimensional, orientable, smooth manifold without boundary with cotangent bundle $\pi_* : T^*M \to M$. Let m^* be the symplectic volume measure on T^*M and fix a smooth volume measure m on M. Disintegrating m^* with respect to m by π^*, one gets a family of measures m^*_x on $T^*_x M$ for m-a.e. $x \in M$ such that $m^* = \int_M m^*_x \cdot m(dx)$.

Let L be the Friedrichs extension of an m-symmetric, positive, elliptic, smooth pseudo differential operator of order $k \geq 1$ with classical symbol defined on $C^\infty(M) \subset L^2(M,m)$. Let p be the principal symbol of L, understood as a real, homogeneous polynomial of degree k on T^*M or as a function on the cosphere bundle S^*M. The spectrum of L is discrete and the Weyl asymptotic formula for the eigenvalue counting function of L reads

$$N_L(x) \sim c \cdot x^{n/k}, \quad x \to +\infty,\ c := \frac{1}{(2\pi)^n} \cdot \int_M m^*_x\{p(x,\cdot) < 1\} \cdot m(dx).$$

It follows by Proposition 2.9 that N_L is asymptotically continuous and that $\rho(L)$ is a density. The Hörmander estimates for the remainder term [13, 14] reads

$$N_L(x) = c \cdot x^{n/k} + O(x^{(n-1)/k}), \quad x \to +\infty$$

and, by Theorem 3.5, it follows that the ζ-function ζ_L of the density $\rho(L)$ is meromorphic on $\{s \in \mathbb{C} : \Re e(s) > 1 - 1/n\}$ and it has a simple pole in $s = 1$ with unital residue. The order of the remainder term cannot be improved in general, but in case of a product of 2-spheres $M = S^2 \times S^2$ and for the Laplace–Beltrami operator L, M. Taylor [19] proved the estimate

$$N_L(x) = c \cdot \mathrm{Vol}(S^2 \times S^2) \cdot x^2 + O(x^{4/3}), \quad x \to +\infty,\ c := (4\pi)^{-2}/\Gamma(3)$$

which ensures that ζ_L is meromorphic on $\{s \in \mathbb{C} : \Re e(s) > 2/3\}$. As another example, one can consider the Laplace–Beltrami operator L on the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, where the reminder term is $O(x^{(n-1)/2-\gamma})$ for some $\gamma > 0$. The case $n = 2$ corresponds to the classical Gauss problem: in fact $N_L(x)$ coincides with the number of points in $\mathbb{Z}^2$ falling within the circle of radius $x > 0$. It is known [8, p. 6] that $N_L(x) = \pi x + O(x^\alpha)$ as $x \to +\infty$ with $\alpha \in (1/4, 12/37)$. Still in [19], it is shown that on S^2, the sub-elliptic operator $L = X_1^2 + X_2^2$ given by the sum of squares of two vector fields X_1, X_2, generating rotations around orthogonal axes, has a counting function with a non-Weyl asymptotic behavior

$$N_L(x) = \frac{1}{2} x \cdot \mathrm{Log}(x) + O(x), \quad x \to +\infty.$$

Again by Proposition 2.9, N_L is asymptotically continuous and that $\rho(L)$ is a density. The asymptotic behavior of the counting function N_L of hypoelliptic ΨDO is studied in [17].

One can have a meromorphic extension even if the counting function is not asymptotically continuous:

Example 3.7. Let $\mathbb{F}_p$ be the free group with p generators ($p \geq 2$), ℓ the length function on $\mathbb{F}_p$, and L the multiplication operator by ℓ on the Hilbert space $\ell^2(\mathbb{F}_p)$ (see [11]).

The spectrum of ℓ is $\mathbb{N}$: $\widetilde{\lambda}_k = k$ for $k \geq 1$ with multiplicities $m_k = 2p(2p-1)^{k-1}$ and $M_k = \frac{2p}{2p-2}((2p-1)^k - 1)$. Hence $m_k/M_k \to (2p-2)/(2p-1) > 0$ and, by Proposition 2.9, N_L is not asymptotically continuous. However, since $\lim_{k\to\infty} M_{k+1}/M_k = 2p-1 > 1$, Proposition 2.7 implies that the spectral weight $\rho(L)$ is measurable and that $\mathrm{Tr}_\omega(\rho(L)) = \lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}(\rho(L)^s) = \frac{2p-2}{(2p-1)\,\mathrm{Log}(2p-1)}$. We show that this limit is indeed a residue:

Proposition 3.8. *With the notations of the above example, we have that the zeta function $\zeta_L(s)=\mathrm{Tr}(\rho(L)^s)$ extends as a meromorphic function on the half plane $\{s\in\mathbb{C} : \Re e(s)>0\}$ with a simple pole at $s = 1$ and residue*

$$\mathrm{Res}_{s=1}(\zeta_L) = \lim_{s\in\mathbb{C},\,\mathrm{Re}(s)>0,\,s\to 1} (s-1)\,\mathrm{Tr}\left(\rho(L)^s\right) = \frac{2p-2}{(2p-1)\,\mathrm{Log}(2p-1)}.$$

Proof. Let us compute for $\Re e(s) > 1$

$$\begin{aligned}
\mathrm{Tr}(\rho(L)^s) &= \sum_{k\geq 1} m_k M_k^{-s} \\
&= (2p)^{-s+1}\frac{(2p-2)^s}{2p-1}\sum_{k\geq 1}(2p-1)^k\left((2p-1)^k-1\right)^{-s} \\
&= (2p)^{-s+1}\frac{(2p-2)^s}{2p-1}\sum_{k\geq 1}(2p-1)^{k-ks}\left(1-(2p-1)^{-k}\right)^{-s} \\
&= \varphi(s)\big(Z_1(s) - Z_2(s)\big)
\end{aligned}$$

with

- $\varphi(s) = (2p)^{-s+1}\frac{(2p-2)^s}{2p-1}$: φ extends as an analytic function on the whole complex plane; its value at $s = 1$ is $(2p-2)/(2p-1)$;
- $Z_1(s) = \sum_k (2p-1)^{k-ks} = \frac{(2p-1)^{1-s}}{1-(2p-1)^{1-s}}$: Z_1 extends as a meromorphic function on the whole complex plane with one pole at $s = 1$ which is simple, with residue $\frac{1}{\mathrm{Log}(2p-1)}$;

- $Z_2(s) = \sum_k (2p-1)^{k-ks}(1-(1-(2p-1)^{-k})^s)$: Z_2 appears as a sum of analytic functions, each of them being bounded that way: fix $S > 0$, then there exists a constant C such that

$$\left|1-\left(1-(2p-1)^{-k}\right)^s\right| \le C(2p-1)^{-k}, \quad k \ge 1,\ |s| \le S,$$

hence

$$\left|(2p-1)^{k-ks}\left(1-\left(1-(2p-1)^{-k}\right)^s\right)\right| \le C(2p-1)^{-ks}, \quad k \ge 1,\ |s| \le S.$$

The series $\sum_k (2p-1)^{k-ks}(1-(1-(2p-1)^{-k})^s)$ converges locally uniformly on the upper half plane $\Re e(s) > 0$ and its sum defines an analytic function on the half plane $\Re e(s) > 0$.

Gathering those intermediate results, the proposition is proved. ∎

4. Volume forms associated to spectral weights

4.1. Volume forms

Fix a Dixmier ultrafilter ω on $\mathbb{N}$ as obtained in Section 2.4.

Definition 4.1. The *volume form* on $\mathcal{B}(h)$ associated to L and ω will be the linear form Ω_L:

$$\mathcal{B}(h) \ni T \to \Omega_L(A) = \mathrm{Tr}_\omega\left(T\rho(L)\right).$$

The restriction of Ω_L to a sub-C^*-algebra $A \subset \mathcal{B}(h)$ will be called *volume form* on A.

Volume forms satisfy some obvious properties:

Proposition 4.2. (i) *Ω_L is a positive, hence uniformly continuous linear form on $\mathcal{B}(h)$ with norm less than* $\mathrm{Tr}_\omega(\rho(L)) \le 1$.

(ii) *Ω_L vanishes on the C^*-algebra $\mathcal{K}(h)$ of compact operators, and hence defines a positive linear form on the Calkin algebra $\mathcal{B}(h)/\mathcal{K}(h)$.*

(iii) $\Omega_L(T) = \widetilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\,\mathrm{Tr}(T\rho(L)^{1+\frac{1}{r}})$ *(cf. equation (2.2) in Section 2.4).*

Proposition 4.3. *For any measurable function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$N_L(x) \sim \varphi(x), \quad x \to +\infty,$$

(i) *the operator $\varphi(L)^{-1}$ belongs to the ideal $\mathcal{L}^{(1,\infty)}(h)$ and*

$$\mathrm{Tr}_\omega\left(\rho(L)\right) = \mathrm{Tr}_\omega\left(\varphi(L)^{-1}\right) \quad \textit{for all ultrafilter } \omega;$$

(ii) *for any $T \in \mathcal{B}(h)$, the compact operators $T\rho(L)$, $T\varphi(L)^{-1}$ belong to the ideal $\mathcal{L}^{(1,\infty)}(h)$ and the volume form can be represented as*

$$\Omega_L(T) := \mathrm{Tr}_\omega\left(T\rho(L)\right) = \mathrm{Tr}_\omega\left(T\varphi(L)^{-1}\right).$$

Proof. (i) One has $N_L(x)^{-1} \sim \varphi(x)^{-1}$, which we write

$$\varphi(x)^{-1} = N_L(x)^{-1} + g(x)N_L(x)^{-1}$$

for some function g with $\lim_{x\to\infty} g(x) = 0$. We then have $\varphi(L)^{-1} = \rho(L) + g(L)\rho(L)$ with $g(L)$ compact. Hence $g(L)\rho(L)$ belong to the closure $\mathcal{L}_0^{(1,\infty)}(h)$ in $\mathcal{L}^{(1,\infty)}(h)$ of the ideal of finite rank operators, on which the Dixmier trace vanishes and the result follows.

(ii) Notice that $\mathcal{B}(h) \ni T \to \mathrm{Tr}_\omega(T\rho(L))$ is a positive linear form, hence is a norm continuous functional on $\mathcal{B}(h)$. As it is obviously 0 whenever T has finite rank, it vanishes on the C^*-algebra $\mathcal{K}(h)$ of compact operators. Hence, we have $\mathrm{Tr}_\omega(g(L)\rho(L)) = 0$ and, for any $T \in \mathcal{B}(h)$, $\mathrm{Tr}_\omega(Tg(L)\rho(L)) = 0$. ∎

Hereafter are the first examples of such volume states and linear forms.

4.2. Compact smooth manifolds II

(i) In the framework and notations of Example 3.6, consider the action $g \mapsto M_g$ of the commutative C^*-algebra $C(M)$ by pointwise multiplication on $L^2(M,m)$. By the Weyl asymptotic formula $N_L(x) \sim c\cdot x^{n/k} =: \varphi(x)$ as $x \to +\infty$, N_L is asymptotically continuous, $\rho(L)$ is a density, and the volume forms Ω_L are states. Since $\varphi(L)^{-1} = c^{-1}\cdot L^{-n/k}$, by Proposition 4.3 one has $\Omega_L(T) = c^{-1}\cdot \mathrm{Tr}_\omega(TL^{-n/k})$. The restriction of these states to $C(M)$ is represented by probability measures ν_ω: $\Omega_L(M_g) = \int_M g\cdot d\nu_\omega$.

Let $\pi^* : S^*M \to M$ be the cosphere bundle whose fibers are the rays of T^*M and consider a *scalar valued*, elliptic, m-symmetric 1-order ΨDO on M with classical symbol, denoting by D its self-adjoint extension to $L^2(M,m)$.

(ii) Since the operators D and M_g, $g \in C^\infty(M)$, are scalar valued ΨDO, their symbols commute. Also, since they are of order 1 and 0, respectively, by the rules of pseudo differential calculus, the commutators $[D, M_g]$ are 0-order ΨDO, thus bounded. Hence $(C^\infty(M), D, L^2(M,m))$ is a (d,∞)-summable spectral triple on $C(M)$, in the sense of A. Connes [7]. The spectral weight $\rho(|D|)$ is a density and the volume forms $\Omega_{|D|}$ are states on $C(M)$ represented by probability measures ν_ω on M: $\Omega_{|D|}(M_g) = \int_M g\cdot d\nu_\omega$.

(iii) Let $\mathcal{A}$ be the *-algebra of *scalar valued*, 0-order ΨDO on M, acting boundedly on $L^2(M,m)$. Let $\mathcal{P}(M)$ be the C^*-algebra of bounded operators on $L^2(M,m)$ generated by $\mathcal{A}$ (see [12]). Again, since D and operators T in $\mathcal{A}$ are scalar valued, their symbols commute and since they are of order 1 and 0, respectively, the commutators $[T, D]$ have 0-order and are thus bounded. Hence $(\mathcal{A}, D, L^2(M,m))$ is a (d,∞)-summable spectral triple on $\mathcal{P}(\mathcal{M})$. Since

$$0 \to \mathcal{K}\big(L^2(M,m)\big) \to \mathcal{P}(M) \xrightarrow{\sigma} C\big(S^*(M)\big) \to 0$$

is a C^*-algebra extension (see [10, 12]) and the volume states $\Omega_{|D|}$ restricted to $\mathcal{P}(M)$ vanish on the ideal $\mathcal{K}(L^2(M,m))$, it follows that they factorize through suitable probability measures ν^*_ω on the cosphere bundle $S^*(M)$. As the representation $g \mapsto M_g$ is injective, we can identify $C(M)$ with its image in $\mathcal{B}(L^2(M,m))$. Since $C(M) \cap$

$\mathcal{K}(L^2(M,m)) = \{0\}$, $C(M) \subset \mathcal{P}(M)$, and the restriction of the principal symbol map σ is given by $\sigma(M_g) = g \circ \pi^*$ for any $g \in C^\infty(M)$, the measure ν_ω is the image of ν^*_ω under $\pi^* : S^*M \to M$.

4.3. Multiplication operators on discrete groups

Let G be a countable discrete group with unit e and left regular representation λ in the Hilbert space $l^2(G)$. If $\{\delta_g\}_{g\in G}$ is the canonical orthonormal base of $l^2(G)$, we have $\lambda(g)\delta_h = \delta_{gh}$, $g, h \in G$.

Let ℓ be a proper function from G in $\mathbb{R}_+$ and L the operator of multiplication by ℓ on $l^2(G)$: $L\delta_g = \ell(g)\delta_g$. The (discrete) spectrum of L coincides with the image of ℓ: $\mathrm{sp}(L) = \{\ell(g) : g \in G\}$. For any $e \neq g \in G$, $s \in \mathbb{C}$, $\Re e(s) > 1$, we have

$$\mathrm{Tr}\left(\lambda(g)\rho(L)^s\right) = \sum_h \left(\delta_h \mid N_L\left(\ell(g)\right)^{-s}\delta_{gh}\right)_{l^2(G)} = 0.$$

The generic element $a \in C^*_{\mathrm{red}}(G)$ has a Fourier expansion $a = \sum_g a_g\lambda(g)$ and is a uniform limit of elements of which the Fourier expansion has finite support ($a_g = 0$ except for a finite number of g's). For a with finite support, we have $\mathrm{Tr}(a\rho(L)^s) = a_e\,\mathrm{Tr}(\rho(L)^s) = \tau(a)\,\mathrm{Tr}(\rho(L)^s)$, where τ is the canonical trace on $C^*_{\mathrm{red}}(G)$: $\tau(a) = a_e$. By uniform continuity, this formula extends to any $a \in C^*_{\mathrm{red}}(G)$. Finally, applying formula (2.2) of Section 2.4, we get

$$\Omega_L(a) = \mathrm{Tr}_\omega\left(a\rho(L)\right) = \tau(a)\,\mathrm{Tr}_\omega\left(\rho(L)\right), \quad a \in C^*_{\mathrm{red}}(G).$$

Normalizing Ω_L by $\mathrm{Tr}_\omega(\rho(L))$, we get the canonical trace τ for any ultrafilter ω. The multiplicity $m(\lambda)$ of an eigenvalue $\lambda \geq 0$ is the cardinality of the level set $\{g \in G : \ell(g) = \lambda\}$, while its cumulated multiplicity $M(\lambda)$ is the cardinality of the sub-level set $\{g \in G : \ell(g) \leq \lambda\}$. In case $m(\lambda) = o(M(\lambda))$ as $\lambda \to +\infty$, the eigenvalue counting function is asymptotically continuous, $\rho(L)$ is a density, and the volume form Ω_L coincides with the trace state τ for any ultrafilter ω. This is the case of the word length function ℓ of a system of generators for a discrete group G with subexponential growth [8].

Example 4.4. We apply the previous argument to the case where $G = \mathbb{F}_p$ is the free group with p generators and exponential growth and L is the multiplication operator by the length function ℓ. Extending the argument of Section 4.3, we get easily

$$\mathrm{Tr}\left(a\rho(L)^s\right) = a_e\,\mathrm{Tr}\left(\rho(L)^s\right) = \tau(a)\,\mathrm{Tr}\left(\rho(L)^s\right)$$

for $a \in C^*_{\mathrm{red}}(G)$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. Proposition 3.8 allows to reach the volume form as a residue:

$$\begin{aligned}\Omega_L(a) &= \frac{2p-2}{(2p-1)\,\mathrm{Log}(2p-1)}\tau(a) \\ &= \lim_{s\in\mathbb{C},\ \Re e(s)>1,\ s\to 1} (s-1)\,\mathrm{Tr}\left(a\rho(L)^s\right), \quad a \in C^*_{\mathrm{red}}(G).\end{aligned}$$

5. Further examples

5.1. The Toeplitz C*-algebra I

Let $A \subset \mathcal{B}(l^2(\mathbb{N}))$ be the Toeplitz C^*-algebra generated by the shift operator S on $l^2(\mathbb{N})$:

$$Se_n = e_{n+1}, \quad n \in \mathbb{N},$$

and let L be the multiplication operator on $l^2(\mathbb{N})$ given by

$$(Lu)(n) := nu(n), \quad n \in \mathbb{N},\ u \in l^2(\mathbb{N}).$$

Its spectrum $\mathrm{sp}(L) = \mathbb{N}$ is discrete with all multiplicities equal to one and counting function $N_L(x) = [x] + 1$ for $x \geq 0$. Hence $N_L(L) = L + 1$ and $\rho(L) = (L+1)^{-1}$. Since $N_L(x) \sim \varphi(x) := x$ as $x \to +\infty$, N_L is asymptotically continuous and measurability holds true. In this case, ζ_L coincides with the Riemann ζ-function ζ_0 and since the remainder function $N_L(x) - \varphi(x) = 1 - (x - [x])$ is bounded, applying Theorem 3.5 for all $\alpha \in (0,1)$, we obtain the well-known fact that ζ_0 is meromorphic in the open right half plane with a simple pole at $s = 1$ and unital residue. The volume forms

$$\Omega_L(a) = \mathrm{Tr}_\omega\left(a\varphi(L)^{-1}\right) = \mathrm{Tr}_\omega(aL^{-1}), \quad a \in A$$

are states on A vanishing on the ideal $\mathcal{K}(l^2(\mathbb{N}))$ of compact operators. Since A is an extension in the sense of [D]:

$$0 \to \mathcal{K}\left(l^2(\mathbb{N})\right) \to A \xrightarrow{\sigma} C(\mathbb{T}) \to 0,$$

where $\mathbb{T}$ is the unit circle and $\sigma(S)(z) = z$ for $z \in \mathbb{T}$, it follows that the states Ω_L are determined by probability measures m_ω on $\mathbb{T}$, $\Omega_L(a) = \int_{\mathbb{T}} \sigma(a)\,dm_\omega$ for all $a \in A$, and that they are thus traces on A since $C(\mathbb{T})$ is commutative. Since $\mathrm{Tr}(S^k L^{-s}) = \sum_{n\geq 0}(e_n|S^k n^{-s} e_n) = \sum_{n\geq 0} n^{-s}(e_n|e_{n+k}) = \delta_{k,0}\cdot\zeta_0(s)$ for all $s > 1$ and $k \geq 0$, by formula (2.2) we have $\Omega_L(S^k) = \delta_{k,0}$. Hence all measures m_ω coincide with the Haar probability measure m_H for any ultrafilter ω and aL^{-1} is measurable for any $a \in A$.

5.2. The density of Euclidean domains having infinite volume

In Euclidean domains with infinite volume Ω, the Weyl asymptotic formula cannot hold true for the Laplacian L with Dirichlet boundary conditions on $\partial\Omega$, even if the spectrum is discrete. B. Simon determined in [18, Theorem 1.5] the asymptotic behavior of N_L for certain planar domains of infinite volume. For example, when $\Omega := \{(x,y) \in \mathbb{R}^2 : |xy| \leq 1\}$, one has

$$\hat{\mu}_L(t) = Z_L(t) \sim \frac{1}{\pi}\cdot t^{-1}\,\mathrm{Log}(t^{-1}), \quad t \to 0^+$$

from which, by Proposition 2.11, one derives the asymptotic behavior

$$N_L(x) \sim \frac{1}{\pi}\cdot x\cdot \mathrm{Log}(x), \quad x \to +\infty.$$

Hence N_L is asymptotically continuous and, by Proposition 2.9, $\rho(L)$ is a density. The volume states read

$$\Omega_L(T) = \pi \cdot \mathrm{Tr}_\omega \big(TL^{-1}\,\mathrm{Log}^{-1}(L+I)\big), \quad T \in \mathcal{B}\big(L^2(\Omega, dx)\big)$$

and they determine probability measures ν_ω on Ω by $\int_\Omega f \cdot d\nu_\omega := \Omega_L(M_f)$ for $f \in C_0(\Omega)$.

5.3. Kigami's Laplacians on post critically finite (P.C.F.) fractals

Let K be a P.C.F., self-similar fractal set and $(\mathcal{E}, \mathcal{F})$ the Dirichlet form associated to a fixed regular harmonic structure (with energy weights $0 < r_1, \dots, r_m < 1$) in the J. Kigami's sense [16]. This quadratic form is closable with respect to any Bernoulli measure m on K (with weights $0 < \mu_1, \dots, \mu_m < 1$ such that $\sum_{i=1}^m \mu_i = 1$) and we denote by L the densely defined, nonnegative, self-adjoint operator on $L^2(K, m)$ associated to its closure. Set $\gamma_i := \sqrt{r_i \mu_i}$ and define the *spectral dimension* d_S as the unique positive number such that

$$\sum_{i=1}^m \gamma_i^{d_S} = 1.$$

In the *non-arithmetic* case, where $\sum_{i=1}^m \mathbb{Z}\,\mathrm{Log}\,\gamma_i$ is a dense additive subgroup of $\mathbb{R}$, the asymptotics of the counting function N_L follows a power law similar to the Weyl one for compact Riemannian manifolds (see [16, Theorem 2.4]):

$$N_L(x) \sim c \cdot x^{d_S/2}, \quad x \to +\infty,$$

where

$$c := \Big[-\Big(\sum_{i=1}^m \gamma_i^{d_S}\,\mathrm{Log}\,\gamma_i \Big)^{-1} \cdot \int_{\mathbb{R}} e^{-d_S t} R(e^{2t})dt \Big]$$

and

$$R(x) := N_L(x) - \sum_{i=1}^m N_L(r_i \mu_i x).$$

Consequently, the spectral weight $\rho(L)$ is measurable and it is in fact a density. Evaluating the volume forms Ω_L on the multiplication operators $M_g \in \mathcal{B}(L^2(K, m))$ by continuous functions $g \in C(K)$, one gets positive states on $C(K)$ represented by probability measures ν_ω on K

$$\Omega_L(M_g) = c^{-1} \cdot \mathrm{Tr}_\omega(M_g \cdot L^{-d_S/2}) = \int_K g \cdot d\nu_\omega.$$

6. Volume traces from densities

We denote by $\mathcal{A}_L$ the so-called *Lipschitz algebra* of L, i.e.,

$$\mathcal{A}_L := \big\{a \in \mathcal{B}(h),\ [a, L] \text{ is bounded}\big\}.$$

6.1. Statement of the results

The purpose of the whole section is to prove the following theorem, together with two main corollaries, providing conditions which ensure the existence of *hypertraces or amenable traces* (see [1, 6]). In case of a finitely-summable spectral triple [5, p. 68], the result, known as Connes' trace theorem, is proved in [6, Theorem 8 and Remark 10 (b)]. In case of a (d,∞)-summable spectral triple, a proof of the result, stated in [7, Chapter IV.2, Proposition 15], is provided in [3].

Notice that the condition $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ in the theorem below means that φ has a derivative φ', in the sense of distributions, which is a locally L^1-function, and implies that φ is continuous and a primitive of φ': $\varphi(x) = \varphi(0) + \int_0^x \varphi'(t)dt$, $x \in \mathbb{R}_+$.

Theorem 6.1 (Trace theorem). *Suppose that the counting function satisfies*

$$N_L(x) \sim \varphi(x), \quad x \to +\infty$$

for some nonnegative, increasing function $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ *such that*

$$\operatorname*{ess\text{-}lim\,sup}_{x\to+\infty} \varphi'(x)/\varphi(x) = 0. \tag{6.1}$$

Then the following limit properties hold true.

(1.a) *For* $s > 1$, $\varphi(L)^{-s}$ *is trace class and* $\lim_{s\downarrow 1}(s-1)\operatorname{Tr}(\varphi(L)^{-s}) = 1$.

(1.b) *For* $a \in \mathcal{A}_L$, *one has*

$$\lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big(\big|[a,\varphi(L)^{-s}]\big|\big) = 0.$$

(1.c) *For* $a \in \mathcal{A}_L$ *and* $b \in \mathcal{B}(h)$, *one has*

$$\lim_{s\downarrow 1}(s-1)\operatorname{Tr}\big((ab-ba)\varphi(L)^{-s}\big) = 0.$$

(2) *Hypertrace properties.*

For $s > 1$ *define the linear functionals*

$$\omega_s(b) := (s-1)\operatorname{Tr}\big(b\varphi(L)^{-s}\big), \quad b \in \mathcal{B}(h).$$

As $s \downarrow 1$, *the limit point set of* $\{\omega_s \in \mathcal{B}(h)^* : s > 1\}$ *is not empty and any such limit linear form* τ *is a state on* $\mathcal{B}(h)$ *with the following properties:*

(2.a) τ *vanishes on the algebra* $\mathcal{K}(h)$ *of compact operators;*

(2.b) τ *is a hypertrace on the uniform closure* A *of the Lipschitz algebra* $\mathcal{A}_L$:

$$\tau(ba) = \tau(ab), \quad a \in A,\ b \in \mathcal{B}(h);$$

(2.c) *the restriction of* τ *to* A *is a tracial state.*

(3) *Any volume form* $\Omega_L(a) = \operatorname{Tr}_\omega(a\rho(L))$ *on* $\mathcal{B}(h)$ *is a hypertrace and a tracial state on* A *(with* ω *as in Section 2.4).*

The first corollary is just a variation on the conclusions, with the same assumptions:

Corollary 6.2. *Suppose that the counting function satisfies*

$$N_L(x) \sim \varphi(x), \quad x \to +\infty$$

for some nonnegative, increasing function $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$\varphi' \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+), \quad \operatorname*{ess\text{-}lim\,sup}_{x\to+\infty} \varphi'(x)/\varphi(x) = 0.$$

Then the following limit properties hold true.

(1.a) *For $s > 1$, $\rho(L)^s$ is trace class and* $\lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}(\rho(L)^s) = 1$.

(1.b) *For $a \in \mathcal{A}_L$, one has*

$$\lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}\left(\left|\left[a, \rho(L)^s\right]\right|\right) = 0.$$

(1.c) *For $a \in \mathcal{A}_L$ and $b \in \mathcal{B}(h)$, one has*

$$\lim_{s\downarrow 1}(s-1)\,\mathrm{Tr}\left((ab - ba)\rho(L)^s\right) = 0.$$

For $s > 1$ define the linear functionals

$$\omega_s(b) := (s-1)\,\mathrm{Tr}\left(b\rho(L)^s\right), \quad b \in \mathcal{B}(h).$$

As $s \downarrow 1$, the limit point set of $\{\omega_s \in \mathcal{B}(h)^* : s > 1\}$ is not empty and any such limit linear form τ is a state on $\mathcal{B}(h)$ with the following properties:

(2.a) *τ vanishes on the algebra $\mathcal{K}(h)$ of compact operators;*

(2.b) *τ is a hypertrace on the uniform closure A of the Lipschitz algebra $\mathcal{A}_L$:*

$$\tau(ba) = \tau(ab), \quad a \in A,\ b \in \mathcal{B}(h); \tag{6.2}$$

(2.c) *the restriction of τ to A is a tracial state.*

(3) *Any volume form $\Omega_L(a) = \mathrm{Tr}_\omega(a\rho(L))$ on $\mathcal{B}(h)$ is a hypertrace and a tracial state on A (with ω as in Section 2.4).*

The second corollary provides a sufficient condition for the assumptions above to hold true. It requires that the relative multiplicities vanish faster than the spectral gaps of L:

Corollary 6.3. *Suppose that the following conditions on the spectrum of L are satisfied:*

$$\lim_{k\to\infty} m_k/M_k = 0 \quad \text{and} \quad \frac{m_k}{M_k} = o\big(\widetilde{\lambda}_{k+1}(L) - \widetilde{\lambda}_k(L)\big) \text{ as } x \to +\infty.$$

Then the assumptions of Corollary 6.2 are satisfied, so that all of its conclusions hold true. In particular, they hold true if N_L is asymptotically continuous and the spectral gaps are uniformly bounded away from zero

$$\liminf_{k\to\infty}\big(\widetilde{\lambda}_{k+1}(L) - \widetilde{\lambda}_k(L)\big) > 0.$$

Passing from the operator L to a monotone functional calculus $f(L)$ of it, multiplicities remain unchanged but gaps $f(\widetilde{\lambda}_{k+1}(L)) - f(\widetilde{\lambda}_k(L))$ may vary. However, conditions involving gaps in the corollary above are still satisfied, for example, if $f \in C^1(\mathbb{R}_+)$ and $\inf f' > 0$.

The theorem, together with its corollaries, will be proved in Sections 6.3 and 6.4, after some examples and comments.

Example 6.4 (The Toeplitz C^*-algebra II). Let $\mathbb{P} = \{2, 3, 5, \dots\}$ be the set of prime numbers and let $S' : l^2(\mathbb{P}) \to l^2(\mathbb{P})$ be the shift operator defined as $(S'u)(2) = 0$ and $(S'u)(p) = u(p')$, where $p' \in \mathbb{P}$ denotes the greatest prime strictly less than $p \in \mathbb{P}$ for $p \geq 3$. S' is an isometry $S'^* S' = I$ which generates the Toeplitz C^*-algebra $A' \subset \mathcal{B}(l^2(\mathbb{P}))$.

Let J be the operator on $l^2(\mathbb{P})$ defined by

$$(Ju)(p) := pu(p), \quad p \in \mathbb{P},\ u \in l^2(\mathbb{P}).$$

We have $\mathrm{Sp}(J) = \mathbb{P}$, all multiplicities equal to one, and, by the prime number theorem, $N_J(x) \sim \varphi(x) := x/\mathrm{Log}(x)$ for $x \to +\infty$ so that N_J is asymptotically continuous and $\rho(J)$ is a density. Since $\varphi'(x) = (\mathrm{Log}(x))^{-1} - (\mathrm{Log}(x))^{-2} > 0$ for $x > 3$, φ is strictly increasing and since $\varphi'(x)/\varphi(x) = x^{-1} - (x\,\mathrm{Log}(x))^{-1} \to 0$ as $x \to +\infty$, by Theorem 6.1 we have that the volume state on $\mathcal{B}(l^2(\mathbb{P}))$:

$$\Omega_J(a) = \mathrm{Tr}_\omega\left(a\varphi(J)^{-1}\right) = \mathrm{Tr}_\omega(a\,\mathrm{Log}\,J/J), \quad a \in A',$$

is a hypertrace on the Toeplitz C^*-algebra, vanishing on the ideal $\mathcal{K}(l^2(\mathbb{P}))$. Since for $s > 1$

$$\begin{aligned}\mathrm{Tr}\left(S'^k\varphi(J)^{-s}\right) &= \sum_{p\in\mathbb{P}} \left(\delta_p | S'^k\varphi(J)^{-s}\delta_p\right) = \sum_{p\in\mathbb{P}} \left(p/\mathrm{Log}(p)\right)^{-s} (\delta_p | S'^k\delta_p) \\ &= \delta_{k,0}\cdot \sum_{p\in\mathbb{P}} \left(p/\mathrm{Log}(p)\right)^{-s} \sim \delta_{k,0}\cdot(s-1)^{-1} \quad \text{as } s \to 1^+,\end{aligned}$$

by formula (2.3) we have $\Omega_J(S'^k) = \delta_{k,0}$ for any Dixmier ultrafilter ω. Analogously to the situation of Section 5.1, one has $A'/\mathcal{K}(\mathbb{P}) \simeq C(\mathbb{T})$ and the induced measure on the circle $\mathbb{T}$ is again the Haar probability measure.

To compare the situation described in Section 5.1 to the present one, let us notice first that, since

$$\begin{aligned}[L, S]e_n &= LSe_n - SLe_n = Le_{n+1} - nSe_n \\ &= (n+1)e_{n+1} - ne_{n+1} = e_{n+1} = Se_n, \quad n \in \mathbb{N},\end{aligned}$$

we have $[L, S] = S$ so that the *-algebra $\mathcal{A}_L$ generated by S contains all commutators $[L, a]$ for any $a \in \mathcal{A}_L$. On the other hand, the commutator $[J, S']$ is unbounded since

$$\begin{aligned}\left([J, S']u\right)(p) &= p(S'u)(p) - S'(Ju)(p) \\ &= pu(p') - (Ju)(p') = (p - p')u(p'), \quad 3 \leq p \in \mathbb{P}\end{aligned}$$

and it is known that the *prime gap* $g(p') := p - p'$ can be arbitrarily large. Moreover, $[\mathrm{Log}\, J, S']$ is compact. In fact, for $3 \le p \in \mathbb{P}$

$$\begin{aligned}\big([\mathrm{Log}\, J, S']u\big)(p) &= (\mathrm{Log}\, p)(S'u)(p) - S\big((\mathrm{Log}\, J)u\big)(p) = pu(p') - \big((\mathrm{Log}\, J)u\big)(p') \\ &= \big(\mathrm{Log}(p/p')\big)u(p') = \big(\mathrm{Log}(p/p')\big)(S'u)(p)\end{aligned}$$

and it is known that $\lim_{p\to+\infty} p/p' = \lim_{p'\to+\infty}(1 - g(p')/p') = 1$. However, A. E. Ingham [15] showed that there exists $\alpha \in (0, 3/8)$ such that $p - p' \le p'^{1-\alpha}$ for sufficiently large p. Hence, for this fixed value of α and for sufficiently large p, we have

$$0 \le p^\alpha - p'^\alpha = p'^\alpha\Big[\Big(1 + \frac{p-p'}{p'}\Big)^\alpha - 1\Big] \le p'^\alpha \alpha \frac{p-p'}{p'} = \alpha \frac{p-p'}{p'^{1-\alpha}} \le \alpha$$

so that, for some constant $C \ge \alpha$, we have $0 \le p^\alpha - p'^\alpha \le C$ for all $p \in \mathbb{P}$. Then $([J^\alpha, S']u)(2) = J^\alpha(S'u)(2) - S'(J^\alpha u)(2) = 0$ and for $3 \le p \in \mathbb{P}$

$$\begin{aligned}\big([J^\alpha, S']u\big)(p) &= p^\alpha(S'u)(p) - S'(Ju)(p) = p^\alpha u(p') - (Ju)(p') \\ &= (p^\alpha - p'^\alpha)a(p') = (p^\alpha - p'^\alpha)\cdot(S'u)(p).\end{aligned}$$

It follows that $\|[J^\alpha, S']\| \le C$ and that all commutators $[J^\alpha, a]$ are bounded for any a in the *-subalgebra $\mathcal{A}'_{J^\alpha} \subset A'$ generated by S'. Notice that $\Omega_J = \Omega_{J^\alpha}$ since J^α is an increasing, unbounded function of J. In conclusion, even if the C^*-algebras A and A' are isomorphic and their hypertraces correspond to $\Omega_L \simeq \Omega_J$, these structures differ from a metric point of view since their *Lipschitz* algebras are not isomorphic $\mathcal{A}_L \nsim \mathcal{A}'_{J^\alpha}$.

Example 6.5. The Dirac operator D of a spectral triple $(\mathcal{A}, h, D)$ defined on a C^*-algebra A, represented in a Hilbert space h, and associated to a filtration of h as in Section 2.3, has spectrum $\mathbb{N}$. All spectral gaps are equal to 1 so that the second condition in Corollary 6.3 is satisfied. As soon as the growth of the filtration satisfies

$$\lim_{k\to+\infty} M_{k+1}/M_k = 1,$$

the spectral weight $\rho(D)$ is then a density and the volume states Ω_D are hypertraces for any Dixmier ultrafilter ω.

6.2. Relationship with subexponential growth

Here, we assume that the assumptions of Theorem 6.1 hold true. As a first consequence, we show that L has subexponential spectral growth rate. Let us recall (cf. [4]) that the subexponential growth of L means that the semigroup $\{e^{-tL}\}_{t>0}$ is nuclear (i.e., trace class), and that it is proved there [4, Lemma 3.13] that, for an operator with discrete spectrum, it is equivalent to the limit property $\lim_{n\to\infty} \sqrt[n]{N_L(n)} = 1$. This equivalence remains true for an operator having discrete spectrum off of its kernel, provided one adopts the notation of Section 2.1 for the functional calculus of L.

Lemma 6.6. *For any $\beta > 0$, the partition function is finite $Z_L(\beta) := \mathrm{Tr}(e^{-\beta L}) < +\infty$.*

Proof. Condition (6.1) on φ implies that, for any fixed $\beta > 0$, the function $x \to e^{-\beta x}\varphi(x)$ has a derivative $(\varphi'(x) - \beta\varphi(x))e^{-\beta x}$ which is eventually almost everywhere negative. Hence the function $x \to e^{-\beta x}\varphi(x)$ is nonnegative and eventually decreasing, so that it admits a finite limit at $+\infty$. Consequently, for all fixed $\beta > 0$, $\lim_{x\to+\infty} e^{-2\beta x}\varphi(x) = 0$. In particular, $\lim_{n\to\infty} e^{-2\beta n}\varphi(n) = 0$, so that $\limsup_{n\to\infty} \sqrt[n]{\varphi(n)} \le e^{2\beta}$. Since this holds for any $\beta > 0$, we get $\limsup_{n\to\infty} \sqrt[n]{\varphi(n)} \le 1$ (and indeed $\lim_{n\to\infty} \sqrt[n]{\varphi(n)} = 1$). By equivalence of functions, we have as well $\lim_{n\to\infty} {}^n\sqrt{N_L(n)} = 1$. Applying [4, Lemma 3.13], we get the result. ∎

6.3. Preparatory results

In this section, we assume the hypotheses of Theorem 6.1. We fix some $a \in \mathcal{A}_L$.

Lemma 6.7. *For $s > 1$, $\varphi(L)^{-s}$ is trace class, with* $\lim_{s\to1^+}(s-1)\operatorname{Tr}(\varphi(L)^{-s}) = 1$.

Proof. The assumptions made imply that φ is a continuous function, which in turn implies that $\lim_k M_k/M_{k-1} = 1$ (by Proposition 2.9). Then, by Proposition 2.5, we get

$$N_L\big(\lambda_n(L)\big) \sim n$$

as $n \to +\infty$ (eigenvalues numbered with repetition according to the multiplicity) and thus $\lambda_n(\varphi(L)^{-1}) \sim 1/n$. The result follows easily. ∎

The assumptions on φ lead to the following technical result.

Lemma 6.8. *For $s > 1$ and $N \in \mathbb{N}^*$, one has*

$$\sup_{k>\ell\ge N} \frac{\varphi\big(\widetilde{\lambda}_k(L)\big)^s - \varphi\big(\widetilde{\lambda}_\ell(L)\big)^s}{\varphi\big(\widetilde{\lambda}_k(L)\big)^s\big(\widetilde{\lambda}_k(L) - \widetilde{\lambda}_\ell(L)\big)} \le \operatorname{ess-}\sup_{x\ge\widetilde{\lambda}_N(L)} s\frac{\varphi'(x)}{\varphi(x)}.$$

Proof. To short notation, set $\lambda_k := \lambda_k(L)$, etc. Keeping in mind the fact that φ is increasing, we write for $k > \ell \ge N$

$$\begin{aligned}
\frac{\varphi(\widetilde{\lambda}_k)^s - \varphi(\widetilde{\lambda}_\ell)^s}{\varphi(\widetilde{\lambda}_k)^s} &= \int_{\widetilde{\lambda}_\ell}^{\widetilde{\lambda}_k} \frac{s\varphi(x)^{s-1}\varphi'(x)}{\varphi(\widetilde{\lambda}_k)^s}dx \\
&\le \int_{\widetilde{\lambda}_\ell}^{\widetilde{\lambda}_k} \frac{s\varphi(x)^{s-1}\varphi'(x)}{\varphi(x)^s}dx \\
&\le s(\widetilde{\lambda}_k - \widetilde{\lambda}_\ell)\operatorname{ess-}\sup_{\widetilde{\lambda}_\ell\le x\le\widetilde{\lambda}_k} \varphi'(x)/\varphi(x) \\
&\le s(\widetilde{\lambda}_k - \widetilde{\lambda}_\ell)\operatorname{ess-}\sup_{\widetilde{\lambda}_N\le x} \varphi'(x)/\varphi(x).
\end{aligned}$$ ∎

Let $L^2(h)$ be the space of Hilbert–Schmidt operators on h with norm

$$\|\Phi\|_2 = \operatorname{Tr}(\Phi^*\Phi)^{1/2}$$

and corresponding scalar product $\langle\Psi, \Phi\rangle_2 = \operatorname{Tr}(\Psi^*\Phi)$. For $\ell \ge 1$, let us denote by π_ℓ the orthogonal projection in h onto the eigenspace of L corresponding to the eigenvalue $\lambda_\ell(L)$.

Lemma 6.9. *Let* $\{\alpha_{k,\ell}\}_{k,\ell\geq 1}\subset\mathbb{C}$ *be a bounded set and* T *a bounded operator on* h. *Then*

$$\sum_{k,\ell}\alpha_{k,l}\pi_k T\pi_\ell\varphi(L)^{-s/2},\qquad s>1,$$

is a Hilbert–Schmidt operator and the following estimate holds true:

$$\Big\|\sum_{k,\ell}\alpha_{k,l}\pi_k T\pi_\ell\varphi(L)^{-s/2}\Big\|_2\leq\big(\sup_{k,\ell}|\alpha_{k,\ell}|\big)\cdot\|T\|\cdot\mathrm{Tr}\,\big(\varphi(L)^{-s}\big)^{1/2}.$$

Proof. As the right and left actions of $\mathcal{B}(h)$ on $L^2(h)$ commute, for each $k,\ell\geq 1$ we define an orthogonal projection $p_{k,\ell}$ in $\mathcal{B}(L^2(h))$ by

$$p_{k,\ell}\Phi=\pi_k\Phi\pi_\ell,\qquad\Phi\in L^2(h).$$

We have obviously $\sum_{k,\ell}p_{k,\ell}=I$, so that the operator norm of $\sum_{k,\ell}\alpha_{k,\ell}p_{k,\ell}$ acting on $L^2(h)$ is $\sup_{k,\ell}|\alpha_{k,\ell}|$. We get the result writing

$$\sum_{k,\ell}\alpha_{k,l}\pi_k T\pi_\ell\varphi(L)^{-s/2}=\sum_{k,\ell}\alpha_{k,l}\pi_k T\varphi(L)^{-s/2}\pi_\ell=\Big(\sum_{k,\ell}\alpha_{k,\ell}p_{k,\ell}\Big)T\varphi(L)^{-s/2}.\ \blacksquare$$

Proposition 6.10. *For any* $s>1$ *and* $N\geq 1$, *set*

$$Y_N(s)=\sum_{k>\ell\geq N}\frac{\varphi\big(\widetilde{\lambda}_k(L)\big)^s-\varphi\big(\widetilde{\lambda}_\ell(L)\big)^s}{\varphi\big(\widetilde{\lambda}_k(L)\big)^s\big(\widetilde{\lambda}_k(L)-\widetilde{\lambda}_\ell(L)\big)}\pi_k[L,a]\pi_\ell\varphi(L)^{-s/2}.$$

(i) *One has* $Y_N(s)\in L^2(h)$ *with*

$$\big\|Y_N(s)\big\|_2\leq s\big(\sup_{x\geq\widetilde{\lambda}_N}\varphi'(x)/\varphi(x)\big)\big\|[L,a]\big\|\,\mathrm{Tr}\,\big(\varphi^{-s}(L)\big)^{1/2}.$$

(ii) $\lim_{s\downarrow 1}(s-1)^{1/2}\|Y_1(s)\|_2=0.$

Proof. To short notation, set $\lambda_k:=\lambda_k(L)$, etc. For (i), apply Lemmas 6.8 and 6.9. (ii) Fix $\varepsilon>0$ and $N\geq 1$ such that

$$\text{ess-}\sup_{x\geq\widetilde{\lambda}_N}\varphi'(x)/\varphi(x)\leq\varepsilon.$$

On one hand,

$$Y_1(s)-Y_N(s)=\sum_{1\leq\ell\leq N,\,k\geq\ell}\frac{\varphi(\widetilde{\lambda}_k)^s-\varphi(\widetilde{\lambda}_\ell)^s}{\varphi(\widetilde{\lambda}_k)^s(\widetilde{\lambda}_k-\widetilde{\lambda}_\ell)}\pi_k[L,a]\pi_\ell\varphi(L)^{-s/2}$$

has a Hilbert–Schmidt norm less than $C\|(\sum_{\ell=1}^N\pi_\ell)\varphi(L)^{-s/2}\|_2$ for some constant C depending only on φ and $\|[L,a]\|$, by Lemma 6.9. Hence $\|Y_1(s)-Y_N(s)\|_2$ is bounded independently of s. For s close enough to 1, we then have $(s-1)^{1/2}\|Y_1(s)-Y_N(s)\|_2\leq\varepsilon$.

On the other hand, applying (i), we get $\|Y_N(s)\|_2 \le \varepsilon s \|[L,a]\| \operatorname{Tr}(\varphi^{-s})^{1/2}$. Applying Lemma 6.7, we get that, for s close enough to 1, $(s-1)^{1/2} Y_N(s)$ has a Hilbert–Schmidt norm less than $\varepsilon \cdot s \cdot \|[L,a]\| \cdot (1+\varepsilon)$.

Summing up, we get that, for s close enough to 1, $(s-1)^{1/2} Y_1(s)$ has Hilbert–Schmidt norm less that $\varepsilon \cdot (s\|[L,a]\|(1+\varepsilon)+1)$. ∎

Proposition 6.11. *Setting*

$$Z_1(s) = \sum_{1 \le k \le \ell} \varphi(L)^{-s/2} \frac{\varphi\big(\tilde{\lambda}_k(L)\big)^s - \varphi\big(\tilde{\lambda}_\ell(L)\big)^s}{\varphi\big(\tilde{\lambda}_\ell(L)\big)^s \big(\tilde{\lambda}_k(L) - \tilde{\lambda}_\ell(L)\big)} \pi_k [L,a] \pi_\ell,$$

we have

(i) $Z_1(s) \in L^2(h)$ *whenever* $s > 1$,

(ii) $\lim_{s \downarrow 1} (s-1)^{1/2} \|Z_1(s)\|_2 = 0$.

Proof. Up to a sign, $Z_1(s)^*$ is given by the same formula as $Y_1(s)$, with $\bar{s}$ substituted to s and a^* substituted to a. Apply Proposition 6.10 (i) and (ii). ∎

Lemma 6.12 (Chain rule). *For any $f \in C(\mathbb{R})$ and $a \in \mathcal{A}_L$ we have, for $k, \ell \ge 1$, $k \ne \ell$,*

(i) $\pi_k [a, f(L)] \pi_k = 0$ *and* $\pi_k [a, f(L)] \pi_\ell = (f(\tilde{\lambda}_k) - f(\tilde{\lambda}_\ell)) \pi_k a \pi_\ell$,

(ii) $\pi_k [L,a] \pi_\ell = (\tilde{\lambda}_k - \tilde{\lambda}_\ell) \pi_k a \pi_\ell$,

(iii) $\pi_k [a, f(L)] \pi_\ell = \frac{(f(\tilde{\lambda}_k(L)) - f(\tilde{\lambda}_\ell(L)))}{\tilde{\lambda}_k(L) - \tilde{\lambda}_\ell(L)} \pi_k [L,a] \pi_\ell$.

In other words, by easily understood abuse of notation, we can write

$$\big[a, f(L)\big] = \sum_{k \ne \ell} \frac{\big(f\big(\tilde{\lambda}_k(L)\big) - f\big(\tilde{\lambda}_\ell(L)\big)\big)}{\tilde{\lambda}_k(L) - \tilde{\lambda}_\ell(L)} \pi_k [L,a] \pi_\ell.$$

Proof. (i) and (ii) are straightforward. (iii) is an obvious combination of (i) and (ii). ∎

Proposition 6.13. *For any $a \in \mathcal{A}_L$ (i.e., $[a, L]$ is bounded), one has*

$$\lim_{s \downarrow 1} \operatorname{Tr}\big(\big|\big[a, \varphi(L)^{-s}\big]\big|\big) = 0. \tag{6.3}$$

Proof. To short notation, set $\lambda_k := \lambda_k(L)$, etc. Lemma 6.12 allows us to write

$$\begin{aligned}
\big[a, \varphi(L)^{-s}\big] &= \sum_{k \ne \ell} \frac{\big(\varphi(\tilde{\lambda}_k)^{-s} - \varphi(\tilde{\lambda}_\ell)^{-s}\big)}{\tilde{\lambda}_k - \tilde{\lambda}_\ell} \pi_k [L,a] \pi_\ell \\
&= \sum_{k \ne \ell} \frac{\big(\varphi(\tilde{\lambda}_k)^{s} - \varphi(\tilde{\lambda}_\ell)^{s}\big)}{\varphi(\tilde{\lambda}_k)^{-s} (\tilde{\lambda}_k - \tilde{\lambda}_\ell) \varphi(\tilde{\lambda}_\ell)^{-s}} \pi_k [L,a] \pi_\ell \\
&= X^+(s) + X^-(s)
\end{aligned}$$

with

$$
\begin{aligned}
X^+(s) &= \sum_{k>\ell\geq 1} \frac{\big(\varphi(\widetilde{\lambda}_k)^s - \varphi(\widetilde{\lambda}_\ell)^s\big)}{\varphi(\widetilde{\lambda}_k)^{-s}(\widetilde{\lambda}_k - \widetilde{\lambda}_\ell)\varphi(\widetilde{\lambda}_\ell)^{-s}} \pi_k[L,a]\pi_\ell \\
&= \sum_{k>\ell\geq 1} \frac{\big(\varphi(\widetilde{\lambda}_k)^s - \varphi(\widetilde{\lambda}_\ell)^s\big)}{\varphi(\widetilde{\lambda}_k)^{-s}(\widetilde{\lambda}_k - \widetilde{\lambda}_\ell)} \pi_k[L,a]\pi_\ell\varphi(L)^{-s} \\
&= Y_1(s)\varphi(L)^{-s/2}
\end{aligned}
$$

while

$$
\begin{aligned}
X^-(s) &= \sum_{1\leq k<\ell} \frac{\big(\varphi(\widetilde{\lambda}_k)^s - \varphi(\widetilde{\lambda}_\ell)^s\big)}{\varphi(\widetilde{\lambda}_k)^{-s}(\widetilde{\lambda}_k - \widetilde{\lambda}_\ell)\varphi(\widetilde{\lambda}_\ell)^{-s}} \pi_k[L,a]\pi_\ell \\
&= \sum_{1\leq k<\ell} \varphi(L)^{-s} \frac{\big(\varphi(\widetilde{\lambda}_k)^s - \varphi(\widetilde{\lambda}_\ell)^s\big)}{\varphi(\widetilde{\lambda}_\ell)^{-s}(\widetilde{\lambda}_k - \widetilde{\lambda}_\ell)} \pi_k[L,a]\pi_\ell \\
&= \varphi(L)^{-s/2} Z_1(s).
\end{aligned}
$$

Let $X^+(s) = u^+(s)|X^+(s)|$ be the polar decomposition of $X_+(s)$. Applying Lemma 6.7 and Proposition 6.10 (2), we get

$$
\begin{aligned}
\operatorname{Tr}\big(\big|X^+(s)\big|\big) &= \operatorname{Tr}\big(u^+(s)^* X^+(s)\big) \\
&= \operatorname{Tr}\big(u^+(s)^* Y_1(s)\varphi(L)^{-s/2}\big) \\
&= \operatorname{Tr}\big(\varphi(L)^{-s/2} u^+(s)^* Y_1(s)\big) \\
&\leq \big\|u^+(s)\varphi(L)^{-\bar{s}/2}\big\|_2 \big\|Y_1(s)\big\|_2 \\
&= O\big(\big(\operatorname{Re}(s) - 1\big)^{-1/2}\big) o\big(\big(\operatorname{Re}(s) - 1\big)^{-1/2}\big) = o\big(\operatorname{Re}(s) - 1\big)^{-1},
\end{aligned}
$$

which proves $(s-1)\operatorname{Tr}(|X^+(s)|) \to 0$ as $s \downarrow 1$.

A similar argument, *mutatis mutandis*, provides $(s-1)\operatorname{Tr}(|X^-(s)|) \to 0$ as $s \downarrow 1$. ■

Lemma 6.14. *If T is a compact operator, then*

$$
\lim_{s\downarrow 1} (s-1)\operatorname{Tr}(T\varphi(L)^{-s}) = 0.
$$

Proof. Fix $\varepsilon > 0$ and T_0 a finite rank operator such that $\|T - T_0\| \leq \varepsilon$. On one hand, $\lim_{s\downarrow 1} \operatorname{Tr}(T_0\varphi(L)^{-s}) = \operatorname{Tr}(T_0\varphi(L)^{-1})$ exists, so that $\lim_{s\downarrow 1}(s-1)\operatorname{Tr}(T_0\varphi(L)^{-s}) = 0$, which means that $|\operatorname{Tr}(T_0\varphi(L)^{-s})| \leq \varepsilon$ for s close to 1. On the other hand, one has for $s > 1$,

$$
(s-1)\big|\operatorname{Tr}\big((T - T_0)\varphi(L)^{-s}\big)\big| \leq \varepsilon(s-1)\operatorname{Tr}\big(\varphi(L)^{-s}\big)
$$

with, by Lemma 6.7, $(s-1)\operatorname{Tr}(\varphi(L)^{-s}) \leq 1 + \varepsilon$ for $s > 1$ close to 1. Summing up, we have $(s-1)|\operatorname{Tr}(T\varphi(L)^{-s})| \leq \varepsilon(2+\varepsilon)$ for $s > 1$ close to 1. ■

6.4. Proofs of the theorem and its corollaries

Proof of Theorem 6.1. (1.a) is Lemma 6.7. (1.b) is Proposition 6.13 and (1.c) is an obvious consequence of (1.b). In (2), the fact that the Ω_s are bounded as $s \to 1+$ and that a limit linear form is a state is a consequence of Lemma 6.7. (2.a) comes from Lemma 6.14. (2.b) and (2.c) come from (1.b) and (1.c). ■

Proof of Corollary 6.2. We have $\varphi^{-1} = N_L^{-1}(1+g)$ with $g : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing at infinity. This implies that $\varphi(L)^{-1} = N_L(L)^{-1}(I+T)$ with T a compact operator commuting with L, $N_L(L)$, and $\varphi(L)$. Apply Lemma 6.14 repeatedly for substituting $N_L(L)^{-1}$ by $\varphi(L)^{-1}$ in every successive item of Theorem 6.1. ■

Proof of Corollary 6.3. Let φ be the continuous piecewise affine function on $\mathbb{R}_+$ interpolating affinely between the points $\widetilde{\lambda}_k(L)$ and $\widetilde{\lambda}_{k+1}(L)$, i.e.,

$$\varphi(x) = M_k + \big(x - \widetilde{\lambda}_k(L)\big)\frac{M_{k+1} - M_k}{\widetilde{\lambda}_{k+1}(L) - \widetilde{\lambda}_k(L)} \quad \text{whenever } x \in \big[\widetilde{\lambda}_k(L), \widetilde{\lambda}_{k+1}(L)\big].$$

This is the function constructed in Proposition 2.9, where it is shown to be asymptotically equivalent to N_L, provided that M_{k+1}/M_k tends to 1 as $k \to \infty$.

φ is differentiable on each interval $(\widetilde{\lambda}_k, \widetilde{\lambda}_{k+1})$ with derivative

$$\varphi'(x) = \frac{M_{k+1} - M_k}{\widetilde{\lambda}_{k+1} - \widetilde{\lambda}_k}.$$

Moreover, for $x \in (\widetilde{\lambda}_k, \widetilde{\lambda}_{k+1})$ we have $\varphi(x) \geq M_k$ and $\frac{\varphi'(x)}{\varphi(x)} \leq (\frac{M_{k+1}}{M_k} - 1)\frac{1}{\widetilde{\lambda}_{k+1} - \widetilde{\lambda}_k}$, and by hypothesis we have $\lim_{x\to+\infty} \frac{\varphi'(x)}{\varphi(x)} = 0$. ■

6.5. Densities on C*-algebras extensions

We conclude with a remark concerning densities and their volume forms on C^*-algebras extensions $A \subset \mathcal{B}(h)$ in the sense of [10, 12]

$$0 \to \mathcal{K} \to A \xrightarrow{\sigma} C(X) \to 0,$$

where $\mathcal{K}$ is the elementary C^*-algebra represented in h with finite multiplicity and X is a compact metrizable space. This framework includes the Toeplitz extension and the extension generated by scalar, 0-order ΨDO on compact manifolds.

Proposition 6.15 (Volume forms on extension). *Assume the counting function N_L to be asymptotically continuous. Then, for any fixed Dixmier ultrafilter ω,*

(i) *the volume form*

$$\Omega_L : \mathcal{B}(h) \to \mathbb{C}, \quad \Omega_L(T) := \operatorname{Tr}_\omega\big(T\rho(L)\big)$$

is a state vanishing on the ideal $\mathcal{K}(h)$ of compact operators and thus it factorizes through a state on the Calkin algebra $\mathcal{Q}(h) = \mathcal{B}(h)/\mathcal{K}(h)$;

(ii) *the restriction of* Ω_L *to* A *is a trace that factorizes through a probability measure* μ_ω *on* X

$$\Omega_L(a) = \int_X \big(\sigma(a)\big)(x)\mu_\omega(dx), \quad a \in A.$$

Under the assumptions of Theorem 6.1 or Corollary 6.3, we also have

(iii) Ω_L *is a hypertrace (or amenable trace state) vanishing on the ideal* $\mathcal{K}$*;*

(iv) *there exists a conditional expectation* $E^L_\omega : \mathcal{B}(h) \to L^\infty(X, \mu_\omega)$ *such that*

$$\Omega_L(T) = \int_X E^L_\omega(T) \cdot d\mu_\omega, \quad T \in \mathcal{B}(h).$$

Proof. Straightforward. ∎

A. Appendix

A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *regularly varying* if there exist the limits

$$k(s) := \lim_{t\to+\infty} \frac{f(st)}{f(t)} \in (0, +\infty) \quad \forall s > 0.$$

If $k(s) = 1$ for all $s > 0$, then f is said to be *slowly varying*. Necessarily, k must have the form $k(s) = s^\gamma$ for some $\gamma \in \mathbb{R}$, called the *index of regular variation* ($f \in R_\gamma$) and $f(t) = t^\gamma \ell(t)$ for some slowly varying function $\ell \in R_0$.

Theorem A.1 (Karamata's characterization). *The following characterization holds true:* $f \in R_\gamma$ *if and only if for some* $\sigma > -(\gamma + 1)$ *one has*

$$\lim_{t\to+\infty} \frac{t^{\gamma+1} f(t)}{\int_0^t x^\sigma f(x)dx} = \sigma + \gamma + 1.$$

Theorem A.2 (Karamata's Tauberian theorem). *Let* μ *be a positive Borel measure on* $[0, +\infty)$ *such that*

$$\int_0^{+\infty} e^{-tx}\mu(dx) < +\infty \quad \text{for all } t > 0$$

and suppose that it has a regularly varying Laplace transform (with index $\gamma \in \mathbb{R}$*)*

$$\widehat{\mu}(t) := \int_0^{+\infty} e^{-tx}\mu(dx), \quad t > 0.$$

Then the function $N_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *defined as* $N_\mu(a) := \mu([0, a))$ *has the following asymptotics:*

$$N_\mu(a) = \mu\big([0, a)\big) \sim \frac{\widehat{\mu}(1/a)}{\Gamma(\gamma + 1)}, \quad a \to +\infty.$$

Notice that the function $a \mapsto \widehat{\mu}(1/a)$ is continuously differentiable as it is $\widehat{\mu}$:

$$\frac{d\widehat{\mu}}{dt}(t) = -\int_0^{+\infty} xe^{-tx}\mu(dx), \quad t > 0.$$

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