

NONEXISTENCE OF SOLUTIONS TO QUASILINEAR PARABOLIC EQUATIONS WITH A POTENTIAL IN BOUNDED DOMAINS

GIULIA MEGLIOLI, DARIO D. MONTICELLI, AND FABIO PUNZO

ABSTRACT. We are concerned with nonexistence results for a class of quasilinear parabolic differential problems with a potential in $\Omega \times (0, +\infty)$, where Ω is a bounded domain. In particular, we investigate how the behavior of the potential near the boundary of the domain and the power nonlinearity affect the nonexistence of solutions. Particular attention is devoted to the special case of the semilinear parabolic problem, for which we show that the critical rate of growth of the potential near the boundary ensuring nonexistence is sharp.

1. INTRODUCTION

We investigate nonexistence of nonnegative, nontrivial global weak solutions to quasilinear parabolic inequalities of the following type:

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) \geq V u^q & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u = u_0 & \text{in } \Omega \times \{0\}; \end{cases} \quad (1.1)$$

where Ω is an open bounded connected subset of \mathbb{R}^N , $N \geq 3$, $p > 1$ and $q > \max\{p-1, 1\}$. Furthermore, we assume that $V \in L^1_{loc}(\Omega \times [0, \infty))$, with $V > 0$ a.e. in $\Omega \times (0, +\infty)$, and the initial condition satisfies $u_0 \in L^1_{loc}(\Omega)$, with $u_0 \geq 0$ a.e. in Ω .

Global existence and finite time blow-up of solutions for problem (1.1) has been deeply studied when $\Omega = \mathbb{R}^N$, see e.g. [7–9, 18, 19, 21, 24] and references therein. In particular, in [19], nonexistence of nontrivial weak solutions is proved for problem (1.1) when $\Omega = \mathbb{R}^N$, $V \equiv 1$ and

$$p > \frac{2N}{N+1}, \quad \max\{1, p-1\} < q \leq p-1 + \frac{p}{N}.$$

Moreover, problem (1.1) has been investigated also in the Riemannian setting, see e.g. [1, 15, 23, 28, 30] and references therein. In [15] problem (1.1) is studied when $\Omega = M$ is a complete, N -dimensional, noncompact Riemannian manifold; it is investigated nonexistence of nonnegative nontrivial weak solutions depending on the interplay between the geometry of the underlying manifold, the power nonlinearity and the behavior of the potential at infinity, assuming that $u_0 \in L^1_{loc}(M)$, $u \geq 0$ a.e. in M and $V \in L^1_{loc}(M \times [0, +\infty))$, $V > 0$ a.e. in M .

Furthermore, we mention that nonexistence results of nonnegative nontrivial weak solutions have been also much investigated for solutions to elliptic quasilinear equation of the form

$$\frac{1}{a(x)} \operatorname{div}(a(x)|\nabla u|^{p-2} \nabla u) + V(x)u^q \leq 0 \quad \text{in } M, \quad (1.2)$$

where

$$a > 0, \quad a \in \operatorname{Lip}_{loc}(M), \quad V > 0 \text{ a.e. on } M, \quad V \in L^1_{loc}(M),$$

2020 *Mathematics Subject Classification.* 35A01, 35K92, 35R45.

Key words and phrases. Parabolic inequalities on domains; weighted volume growth; nonexistence of solutions; Green function.

$p > 1$, $q > p - 1$ and M can be the Euclidean space \mathbb{R}^N or a general Riemannian manifold.

We refer to [4, 16–19] for a comprehensive description of results related to problem (1.2), and also to more general problems, on \mathbb{R}^N . Problem (1.2) when M is a complete noncompact Riemannian manifold has been considered e.g. in [10, 11, 14, 26, 27]. In particular, in [14] the authors studied how the geometry of the underlying manifold M and the behavior of the potential V at infinity affect the nonexistence of nonnegative nontrivial weak solutions for inequality (1.2). Finally, we mention that (1.2) posed on an open relatively compact connected domain $\Omega \subset \mathbb{R}^N$ has been studied in [20]. Under the assumptions that

$$a > 0, \quad a \in \text{Lip}_{loc}(\Omega), \quad V > 0 \text{ a.e. on } \Omega, \quad V \in L^1_{loc}(\Omega),$$

$p > 1$, $q > p - 1$, the authors investigate the relation between the behavior of the potential V at the boundary of Ω and nonexistence of nonnegative weak solutions.

In the present paper, we are concerned with nonnegative weak solutions to problem (1.1). Under suitable weighted volume growth assumptions involving V and q , we obtain nonexistence of global weak solutions (see Theorems 2.1, 2.2). The proofs are mainly based on the choice of a family of suitable test functions, depending on two parameters, that enables us to deduce first some appropriate a priori estimates, then that the unique global solution is $u \equiv 0$. Such test functions are defined by adapting to the present situation those used in [15]; however, some important differences occur, since in [15] an unbounded underlying manifold is considered, whereas now we consider a bounded domain. In some sense, the role of *infinity* of [15] is now played by the boundary $\partial\Omega$. Obviously, this implies that such test functions satisfy different properties. To the best of our knowledge, the definition and use of such test functions are new.

As a special case, we consider in particular the semilinear parabolic problem

$$\begin{cases} \partial_t u - \Delta u = Vu^q & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases} \quad (1.3)$$

where $q > 1$, $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω , $V \in L^1_{loc}(\Omega \times [0, +\infty))$, with $V \geq 0$, i.e. problem (1.1) with $p = 2$.

As a consequence of our general results, we infer that nonexistence of global solutions for problems (1.1) and (1.3) prevails, when

$$V(x, t) \geq Cd(x)^{-\sigma_1} \quad \text{for a.e. } x \in \Omega, \quad t \in [0, +\infty)$$

for some $C > 0$ and

$$\sigma_1 > q + 1,$$

where

$$d(x) := \text{dist}(x, \partial\Omega) \quad \text{for any } x \in \overline{\Omega}. \quad (1.4)$$

Furthermore, we show the sharpness of this result for the semilinear problem (1.3) in case $\partial\Omega$ is regular enough and $V = V(x)$ is continuous and independent of t . Indeed, under the assumption that

$$0 \leq V(x) \leq Cd(x)^{-\sigma_1} \quad \text{for all } x \in \Omega$$

for some $C > 0$ and

$$0 \leq \sigma_1 < q + 1,$$

we prove the existence of a global classical solution for problem (1.3) (see Theorem 2.5), if the initial datum u_0 is small enough. This existence result is obtained by means of the sub- and supersolution's method. In particular, we construct a supersolution to problem (1.3), which is actually a supersolution of the associated stationary equation. Such supersolution is obtained as

the fixed point of a suitable contraction map. In order to show that such a fixed point exists, we need to estimate some integrals involving the Green function associated to the Laplace operator $-\Delta$ in Ω (see Lemmas 6.1, 6.2). Finally, we study the *slightly supercritical case*

$$V(x, t) \geq d(x)^{-q-1} f(d(x))^{q-1} \quad \text{for a.e. } x \in \Omega, t \in [0, +\infty),$$

where f is a function satisfying suitable assumptions and such that $\lim_{\varepsilon \rightarrow 0^+} f(\varepsilon) = +\infty$, for which we prove nonexistence of nonnegative nontrivial weak solutions in $\Omega \times (0, +\infty)$. The proof of this result require a different argument with respect to the previous nonexistence results, which makes use of linearity of the operator and of the special form of the potential. Then the critical rate of growth $d(x)^{-q-1}$ as x approaches $\partial\Omega$ is indeed sharp for the nonexistence of solutions to problem (1.3).

The paper is organized as follows. In Section 2 we describe our main results and some consequences for problem (1.1) (see Theorems 2.1, 2.2 and Corollaries 2.3, 2.4); in particular in Subsection 2.1 we give the statements of our results for the semilinear problem (1.3) (see Theorems 2.5, 2.6 and Corollary 2.7). The definition of weak solutions and some preliminary results are stated in Section 3. Finally we prove the results obtained for problem (1.1) in Sections 4 and 5, while the proofs of the results concerning the semilinear problem (1.3) are shown in Sections 6 and 7.

2. STATEMENTS OF THE MAIN RESULTS

We now introduce the following two hypotheses (HP1) and (HP2) under which we will prove nonexistence of weak solutions for problem (1.1). Let $\theta_1 \geq 1$, $\theta_2 \geq 1$, for each $\delta > 0$ we define

$$S := \Omega \times [0, +\infty) \quad \text{and} \quad E_\delta := \left\{ (x, t) \in S : d(x)^{-\theta_2} + t^{\theta_1} \leq \delta^{-\theta_2} \right\}. \quad (2.1)$$

Moreover let

$$\begin{aligned} \bar{s}_1 &:= \frac{q}{q-1} \theta_2, & \bar{s}_2 &:= \frac{1}{q-1}, \\ \bar{s}_3 &:= \frac{pq}{q-p+1} \theta_2, & \bar{s}_4 &:= \frac{p-1}{q-p+1}. \end{aligned} \quad (2.2)$$

(HP1) Assume that there exist constants $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 \geq 0$, $C > 0$, $\delta_0 \in (0, 1)$ and $\varepsilon_0 > 0$ such that

(i) for some $0 < s_2 < \bar{s}_2$

$$\int_{E_{\frac{\delta}{2}} \setminus E_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon} dx dt \leq C \delta^{-\bar{s}_1-C_0\varepsilon} |\log(\delta)|^{s_2} \quad (2.3)$$

for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$;

(ii) for some $0 < s_4 < \bar{s}_4$

$$\int_{E_{\frac{\delta}{2}} \setminus E_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon} dx dt \leq C \delta^{-\bar{s}_3-C_0\varepsilon} |\log(\delta)|^{s_4} \quad (2.4)$$

for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$.

(HP2) Assume that there exist constants $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 \geq 0$, $C > 0$, $\delta_0 \in (0, 1)$ and $\varepsilon_0 > 0$ such that

(i) for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$

$$\int_{E_{\frac{\delta}{2}} \setminus E_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon} dx dt \leq C \delta^{-\bar{s}_1-C_0\varepsilon} |\log(\delta)|^{\bar{s}_2}, \quad (2.5)$$

$$\int_{E_{\frac{\delta}{2}} \setminus E_{\delta}} t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} V^{-\frac{1}{q-1}-\varepsilon} dx dt \leq C \delta^{-\bar{s}_1-C_0\varepsilon} |\log(\delta)|^{\bar{s}_2}; \quad (2.6)$$

(ii) for any $\delta \in (0, \delta_0)$ and for any $\varepsilon \in (0, \varepsilon_0)$

$$\int_{E_{\frac{\delta}{2}} \setminus E_{\delta}} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon} dx dt \leq C \delta^{-\bar{s}_3-C_0\varepsilon} |\log(\delta)|^{\bar{s}_4}, \quad (2.7)$$

$$\int_{E_{\frac{\delta}{2}} \setminus E_{\delta}} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}+\varepsilon\right)} V^{-\frac{p-1}{q-p+1}-\varepsilon} dx dt \leq C \delta^{-\bar{s}_3-C_0\varepsilon} |\log(\delta)|^{\bar{s}_4}. \quad (2.8)$$

We can now state our main results.

Theorem 2.1. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω . Assume that condition (HP1) holds. If u is a nonnegative weak solution of problem (1.1), then $u = 0$ a.e. in S .*

Theorem 2.2. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω . Assume that condition (HP2) holds. If u is a nonnegative weak solution of problem (1.1), then $u = 0$ a.e. in S .*

As a consequence of Theorem 2.1 we introduce Corollary 2.3. Let $d(x)$ be defined as in (1.4) and (2.1) respectively. Moreover we introduce functions $h : \Omega \rightarrow \mathbb{R}$ and $g : (0, +\infty) \rightarrow \mathbb{R}$ such that

$$h(x) \geq C d(x)^{-\sigma_1} (\log(1 + d(x)^{-1}))^{-\delta_1} \quad \text{for a.e. } x \in \Omega, \quad (2.9)$$

$$0 < g(t) \leq C (1+t)^\alpha \quad \text{for a.e. } t \in (0, +\infty), \quad (2.10)$$

where $\sigma_1, \delta_1, \alpha \geq 0$, $C > 0$. We can now state

Corollary 2.3. *Let $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω . Suppose that $V \in L^1_{loc}(\Omega \times [0, +\infty))$ satisfies*

$$V(x, t) \geq h(x)g(t) \quad \text{for a.e. } (x, t) \in S, \quad (2.11)$$

where h and f satisfy (2.9) and (2.10) respectively. Moreover suppose that

$$\begin{aligned} \int_0^T g(t)^{-\frac{1}{q-1}} dt &\leq CT^{\sigma_2} (\log T)^{\delta_2}, \\ \int_0^T g(t)^{-\frac{p-1}{q-p+1}} dt &\leq CT^{\sigma_4}, \end{aligned} \quad (2.12)$$

for $T > 1$, $\sigma_2, \sigma_4, \delta_2 \geq 0$ and $C > 0$. Finally assume that

- (i) $\sigma_1 > q + 1$;
- (ii) $0 \leq \sigma_2 \leq \frac{q}{q-1}$;
- (iii) $\delta_1 < 1$ and $\delta_2 < \frac{1-\delta_1}{q-1}$.

If u is a nonnegative weak solution of problem (1.1), then $u = 0$ a.e. in S .

As an immediate consequence of Corollary 2.3, choosing $g(t) \equiv 1$, $\sigma_2 = \sigma_4 = 1$ and $\delta_1 = \delta_2 = 0$, we obtain the following

Corollary 2.4. *Let $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω . Suppose that $V \in L^1_{loc}(\Omega \times [0, +\infty))$ satisfies*

$$V(x, t) \geq C d(x)^{-\sigma_1} \quad \text{for a.e. } (x, t) \in S, \quad (2.13)$$

with $\sigma_1 > q + 1$. If u is a nonnegative weak solution of problem (1.1), then $u = 0$ a.e. in S .

2.1. Further result for semilinear problems. We prove, for the semilinear problem (1.3), an existence result when $V = V(x)$ is continuous and independent of t and

$$0 \leq V(x) \leq Cd(x)^{-\sigma_1}, \quad x \in \Omega,$$

with

$$0 \leq \sigma_1 < q + 1$$

(see Theorem 2.5). Then we show a nonexistence result that yield that all nonnegative solutions of (1.3) are trivial if V blows up at the boundary $\partial\Omega$ faster than $d(x)^{-q-1}$ (see Theorem 2.6 and Corollary 2.7 for precise statements).

Theorem 2.5. *Suppose that $\partial\Omega$ is of class C^3 and let $u_0 \in C(\Omega)$, $u_0 \geq 0$ in Ω , be such that there exists $\varepsilon > 0$ such that*

$$0 \leq u_0 \leq \varepsilon d(x) \quad \text{for any } x \in \overline{\Omega}. \quad (2.14)$$

Moreover let $V \in C(\Omega)$, $V \geq 0$ in Ω and assume that for some $C > 0$

$$V = V(x) \leq Cd(x)^{-\sigma_1} \quad \text{for any } x \in \overline{\Omega}. \quad (2.15)$$

with

$$0 \leq \sigma_1 < q + 1. \quad (2.16)$$

Then problem (1.3) admits a classical solution u in $\Omega \times (0, +\infty)$ if $\varepsilon > 0$ is small enough.

For any $\varepsilon > 0$ sufficiently small, set

$$\Omega_\varepsilon = \{x \in \Omega \mid d(x) \geq \varepsilon\}. \quad (2.17)$$

Theorem 2.6. *Let $V \in L^1_{loc}(\Omega \times [0, \infty))$, $V > 0$ a.e., and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. Assume that there exists a nonincreasing function $f : (0, \varepsilon_0) \rightarrow [1, \infty)$ such that $\lim_{\varepsilon \rightarrow 0^+} f(\varepsilon) = +\infty$ and such that, for some $C > 0$, for every $\varepsilon > 0$ small enough*

$$\begin{aligned} \int_0^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} V^{-\frac{1}{q-1}} dx dt &\leq C \varepsilon^{\frac{2q}{q-1}}, \\ \int_{\frac{1}{2}f(\varepsilon)}^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}}} V^{-\frac{1}{q-1}} dx dt &\leq C f(\varepsilon)^{\frac{q}{q-1}}. \end{aligned} \quad (2.18)$$

If u is a nonnegative weak solution of problem (1.3), then $u = 0$ a.e. in $\Omega \times (0, +\infty)$.

As a consequence of Theorem 2.6 we have the following

Corollary 2.7. *Suppose that $u_0 \in L^1_{loc}(\Omega)$ with $u_0 \geq 0$ a.e. in Ω . Assume that V satisfies for some $C > 0$*

$$V(x, t) \geq Cd(x)^{-q-1} f(d(x))^{q-1} \quad \text{for a.e. } x \in \Omega, t \in [0, +\infty), \quad (2.19)$$

where $f : (0, \text{diam}(\Omega)] \rightarrow [1, +\infty)$ is nonincreasing in a right-neighborhood of 0 and such that $\lim_{\varepsilon \rightarrow 0^+} f(\varepsilon) = +\infty$. If u is a nonnegative weak solution of problem (1.3), then $u = 0$ a.e. in $\Omega \times (0, +\infty)$.

Remark 2.8. *We note that an example of function f satisfying the assumptions of Corollary 2.7 is*

$$f(r) = \left[\overbrace{\log \circ \log \circ \dots \circ \log}^{m \text{ times}} \left(K + \frac{1}{r} \right) \right]^\beta, \quad r > 0,$$

for any $\beta > 0$, $m \in \mathbb{N}$ and for $K > 0$ sufficiently large.

3. PRELIMINARIES

Let us first give the precise definition of weak solution to problem (1.1) or (1.3).

Definition 3.1. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω . We say that $u \in W^{1,p}_{loc}(\Omega \times [0, +\infty)) \cap L^q_{loc}(\Omega \times [0, +\infty), V dx dt)$ is a weak solution of problem (1.1) if $u \geq 0$ a.e. in $\Omega \times (0, +\infty)$ and for every $\varphi \in \text{Lip}(\Omega \times [0, \infty))$, $\varphi \geq 0$ in $\Omega \times [0, +\infty)$ and with compact support in $\Omega \times [0, \infty)$, one has*

$$\begin{aligned} \int_0^\infty \int_\Omega V u^q \varphi dx dt &\leq \int_0^\infty \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle dx dt \\ &\quad - \int_0^\infty \int_\Omega u \partial_t \varphi dx dt - \int_\Omega u_0 \varphi(x, 0) dx. \end{aligned} \quad (3.1)$$

We now state some preliminary results that will be used in the proofs of Theorems 2.1 and 2.2. We omit here the proofs, that can be found in [15].

Lemma 3.2. *Let $s \geq \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$ be fixed. Then there exists a constant $C > 0$ such that for every $\alpha \in \left(-\min\left\{\frac{1}{2}, \frac{p-1}{2}\right\}, 0\right)$, for every nonnegative weak solution u of problem (1.1) and for every $\varphi \in \text{Lip}(\Omega \times [0, +\infty))$ with compact support, $0 \leq \varphi \leq 1$ one has*

$$\begin{aligned} \frac{1}{2} \int_0^\infty \int_\Omega V u^{q+\alpha} \varphi^s dx dt &+ \frac{3}{4} |\alpha| \int_0^\infty \int_\Omega |\nabla u|^p u^{\alpha-1} \varphi^s dx dt \\ &\leq C \left\{ |\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} dx dt \right. \\ &\quad \left. + \int_0^\infty \int_\Omega |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt \right\}. \end{aligned} \quad (3.2)$$

Lemma 3.3. *Let $s \geq \max\left\{1, \frac{q+1}{q-1}, \frac{2pq}{q-p+1}\right\}$ be fixed. Then there exists a constant $C > 0$ such that for every $\alpha \in \left(-\min\left\{\frac{1}{2}, \frac{p-1}{2}, \frac{q-1}{2}, \frac{q-p+1}{2(p-1)}\right\}, 0\right)$, for every nonnegative weak solution u of problem (1.1) and for every $\varphi \in \text{Lip}(S)$ with compact support and $0 \leq \varphi \leq 1$ one has*

$$\begin{aligned} &\int_0^\infty \int_\Omega V u^q \varphi^s dx dt \\ &\leq C \left[|\alpha|^{-1} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} dx dt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} dx dt \right) \right]^{\frac{p-1}{p}} \\ &\quad \times \left(\int \int_{S \setminus K} V u^q \varphi^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \left(\int \int_{S \setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}} dx dt \right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\ &\quad + C \left(\int \int_{S \setminus K} V u^{q+\alpha} \varphi^s dx dt \right)^{\frac{1}{q+\alpha}} \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q+\alpha-1}} dx dt \right)^{\frac{q+\alpha-1}{q+\alpha}}, \end{aligned} \quad (3.3)$$

where $K := \{(x, t) \in S : \varphi(x, t) = 1\}$ and S has been defined in (2.1).

Corollary 3.4. *Under the hypotheses of Lemma 3.3 one has*

$$\begin{aligned}
& \int_0^\infty \int_\Omega V u^q \varphi^s dx dt \\
& \leq C \left[|\alpha|^{-1} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} dx dt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} dx dt \right) \right]^{\frac{p-1}{p}} \\
& \times \left(\int \int_{S \setminus K} V u^q \varphi^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \left(\int \int_{S \setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}} dx dt \right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
& + C \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} dx dt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} dx dt \right)^{\frac{1}{q+\alpha}} \\
& \times \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q+\alpha-1}} dx dt \right)^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned} \tag{3.4}$$

Lemma 3.5. *Let $s \geq \max \left\{ 1, \frac{q+1}{q-1}, \frac{2pq}{q-p+1} \right\}$ be fixed. Then there exists a constant $C > 0$ such that for every $\alpha \in (-\min \left\{ \frac{1}{2}, \frac{p-1}{2}, \frac{q-1}{2}, \frac{q-p+1}{2(p-1)} \right\}, 0)$, for every nonnegative weak solution u of problem (1.1) and for every $\varphi \in Lip(S)$ with compact support and $0 \leq \varphi \leq 1$ one has*

$$\begin{aligned}
& \int_0^\infty \int_\Omega V u^q \varphi^s dx dt \\
& \leq C \left[|\alpha|^{-1} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} dx dt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} dx dt \right) \right]^{\frac{p-1}{p}} \\
& \times \left(\int \int_{S \setminus K} V u^q \varphi^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{qp}} \left(\int \int_{S \setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}} dx dt \right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
& + C \left(\int \int_{S \setminus K} V u^q \varphi^s dx dt \right)^{\frac{1}{q}} \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t \varphi|^{\frac{q}{q-1}} dx dt \right)^{\frac{q-1}{q}},
\end{aligned} \tag{3.5}$$

where $K := \{(x, t) \in S : \varphi(x, t) = 1\}$ and S has been defined in (2.1).

4. PROOF OF THEOREM 2.1 AND OF COROLLARY 2.3

Proof of Theorem 2.1. For any $\delta > 0$ sufficiently small, let $\alpha := \frac{1}{\log \delta}$. Observe that $\alpha < 0$ and $\alpha \rightarrow 0^-$ for $\delta \rightarrow 0$. We define for any $(x, t) \in S$

$$\varphi(x, t) := \begin{cases} 1 & \text{in } E_\delta \\ \left[\frac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}} \right]^{C_1 \alpha} & \text{in } (E_\delta)^C \end{cases} \tag{4.1}$$

where

$$C_1 > \frac{2(C_0 + \theta_2 + 1)}{\theta_2 q} \tag{4.2}$$

with $C_0 \geq 0$, $\theta_1, \theta_2 \geq 1$ as in (HP1) and E_δ has been defined in (2.1). Moreover, for any $n \in \mathbb{N}$ we define

$$\eta_n(x, t) := \begin{cases} 1 & \text{in } E_{\frac{\delta}{n}} \\ \frac{2^{\theta_2}}{2^{\theta_2}-1} - \frac{1}{2^{\theta_2}-1} \left(\frac{\delta}{n}\right)^{\theta_2} [d(x)^{-\theta_2} + t^{\theta_1}] & \text{in } E_{\frac{\delta}{2n}} \setminus E_{\frac{\delta}{n}} \\ 0 & \text{in } E_{\frac{\delta}{2n}}^C \end{cases}. \quad (4.3)$$

Let

$$\varphi_n(x, t) := \eta_n(x, t) \varphi(x, t). \quad (4.4)$$

Observe that $\varphi_n \in \text{Lip}(S)$ and $0 \leq \varphi \leq 1$. Moreover, for any $a \geq 1$ we have

$$|\partial_t \varphi_n|^a = |\eta_n \partial_t \varphi + \varphi \partial_t \eta_n|^a \leq 2^{a-1} (|\partial_t \varphi|^a + \varphi^a |\partial_t \eta_n|^a). \quad (4.5)$$

$$|\nabla \varphi_n|^a = |\eta_n \nabla \varphi + \varphi \nabla \eta_n|^a \leq 2^{a-1} (|\nabla \varphi|^a + \varphi^a |\nabla \eta_n|^a). \quad (4.6)$$

Let $s \geq \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$, we apply Lemma 3.2 with φ replaced by the family of functions φ_n . Then, for some positive constant C , for every $n \in \mathbb{N}$ and $|\alpha| > 0$ small enough we have

$$\begin{aligned} & \int_0^\infty \int_\Omega V u^{q+\alpha} \varphi_n^s dx dt \\ & \leq C \left\{ |\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} dx dt + \int_0^\infty \int_\Omega |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt \right\} \\ & \leq C |\alpha|^{-\frac{(p-1)q}{q-p+1}} \left[\int_0^\infty \int_\Omega |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} dx dt + \int_0^\infty \int_\Omega \varphi^{\frac{p(q+\alpha)}{q-p+1}} |\nabla \eta_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}} dx dt \right] \\ & + C \left[\int_0^\infty \int_\Omega |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt + \int_0^\infty \int_\Omega \varphi^{\frac{q+\alpha}{q-1}} |\partial_t \eta_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt \right]. \end{aligned}$$

Let us define

$$\tilde{E}_{\delta, n} := E_{\frac{\delta}{2n}} \setminus E_{\frac{\delta}{n}}, \quad (4.7)$$

and

$$I_1 := \int_0^\infty \int_\Omega |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} dx dt, \quad (4.8)$$

$$I_2 := \int \int_{\tilde{E}_{\delta, n}} \varphi^{\frac{p(q+\alpha)}{q-p+1}} |\nabla \eta_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}} dx dt, \quad (4.9)$$

$$I_3 := \int_0^\infty \int_\Omega |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt, \quad (4.10)$$

$$I_4 := \int \int_{\tilde{E}_{\delta, n}} \varphi^{\frac{q+\alpha}{q-1}} |\partial_t \eta_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt. \quad (4.11)$$

Then the latter inequality can be read, for a positive constant C and for every $n \in \mathbb{N}$, as

$$\int_0^\infty \int_\Omega V u^{q+\alpha} \varphi_n^s dx dt \leq C |\alpha|^{-\frac{(p-1)q}{q-p+1}} [I_1 + I_2] + C [I_3 + I_4]. \quad (4.12)$$

In view of (4.1) and (4.3), for $|\alpha| > 0$ small enough noand for every $n \in \mathbb{N}$, we have

$$\begin{aligned} I_2 & \leq \int \int_{\tilde{E}_{\delta, n}} C n^{C_1 \alpha \theta_2} \frac{p(q+\alpha)}{q-p+1} \left(\frac{\delta}{n}\right)^{\theta_2} \frac{p(q+\alpha)}{q-p+1} \left[d(x)^{-\theta_2-1} |\nabla d(x)| \right]^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}} dx dt \\ & \leq C n^{\theta_2} \frac{p(q+\alpha)}{q-p+1} (C_1 \alpha - 1) \delta^{\theta_2} \frac{p(q+\alpha)}{q-p+1} \int \int_{\tilde{E}_{\delta, n}} d(x)^{-(\theta_2+1)} \frac{p(q+\alpha)}{q-p+1} V^{-\frac{p+\alpha+1}{q-p+1}} dx dt. \end{aligned} \quad (4.13)$$

Due to assumption (HP1) – (ii) with $\varepsilon = -\frac{\alpha}{q-p+1} > 0$, (4.13) reduces to

$$I_2 \leq C n^{\theta_2 \frac{p(q+\alpha)}{q-p+1} (C_1 \alpha - 1)} \delta^{\theta_2 \frac{p(q+\alpha)}{q-p+1}} \left(\frac{\delta}{n} \right)^{-\frac{pq\theta_2}{q-p+1} - C_0 \varepsilon} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_4}, \quad (4.14)$$

with s_4 as in (HP1). Now observe that, due (4.2), we have

$$\frac{|\alpha|}{q-p+1} (-\theta_2 p + C_1 p \theta_2 (q+\alpha) - C_0) \geq \frac{|\alpha|}{q-p+1}.$$

Moreover, there exist $\bar{C} > 0$ such that

$$\delta^{\frac{\alpha}{q-p+1} [\theta_2 p + C_0]} = e^{\frac{\alpha}{q-p+1} [\theta_2 p + C_0] \log(\delta)} = e^{\frac{\theta_2 p + C_0}{q-p+1}} \leq \bar{C}.$$

Then from (4.14) we deduce, for some $C > 0$ and $|\alpha| > 0$ small enough

$$I_2 \leq C n^{-\frac{|\alpha|}{q-p+1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_4}. \quad (4.15)$$

Similarly, in view of (4.1) and (4.3), for $|\alpha| > 0$ small enough and for every $n \in \mathbb{N}$ we have

$$\begin{aligned} I_4 &\leq C \int \int_{\tilde{E}_{\delta,n}} n^{\theta_2 C_1 \alpha \left(\frac{q+\alpha}{q-1} \right)} \left(\frac{\delta}{n} \right)^{\theta_2 \left(\frac{q+\alpha}{q-1} \right)} t^{(\theta_1-1) \frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt \\ &\leq C n^{\theta_2 \left(\frac{q+\alpha}{q-1} \right) (C_1 \alpha - 1)} \delta^{\theta_2 \left(\frac{q+\alpha}{q-1} \right)} \int \int_{\tilde{E}_{\delta,n}} t^{(\theta_1-1) \left(\frac{q+\alpha}{q-1} \right)} V^{-\frac{\alpha+1}{q-1}} dx dt. \end{aligned} \quad (4.16)$$

Due to assumption HP1(i) with $\varepsilon = -\frac{\alpha}{q-1} > 0$, (4.16) reduces to

$$\begin{aligned} I_4 &\leq C n^{\theta_2 \left(\frac{q+\alpha}{q-1} \right) (C_1 \alpha - 1)} \delta^{\theta_2 \left(\frac{q+\alpha}{q-1} \right)} \left(\frac{\delta}{n} \right)^{-\frac{q}{q-1} \theta_2 - C_0 \varepsilon} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_2} \\ &\leq C n^{\frac{1}{q-1} [C_1 \alpha \theta_2 (q+\alpha) - \alpha \theta_2 + C_0 \alpha]} \delta^{\frac{1}{q-1} [\alpha \theta_2 + C_0 \alpha]} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_2}, \end{aligned} \quad (4.17)$$

with s_2 as in (HP1). We now observe that, due to (4.2), we can write

$$n^{-\frac{|\alpha|}{q-1} [C_1 \theta_2 (q+\alpha) - \theta_2 - C_0]} \leq n^{-\frac{|\alpha|}{q-1}}. \quad (4.18)$$

Moreover, observe that there exist $\bar{C} > 0$ such that

$$\delta^{\frac{\alpha}{q-1} (\theta_2 + C_0)} = e^{\frac{\alpha}{q-1} (\theta_2 + C_0) \log(\delta)} = e^{\frac{\theta_2 + C_0}{q-1}} \leq \bar{C}. \quad (4.19)$$

By plugging (4.18) and (4.19) into (4.17) we get for $\delta > 0$ small enough

$$I_4 \leq C n^{-\frac{|\alpha|}{q-1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_2}. \quad (4.20)$$

Let us now consider integral I_1 defined in (4.8). By using the definition of φ in (4.1) we can write

$$\begin{aligned} I_1 &\leq \int \int_{E_\delta^C} \left[C_1 |\alpha| \theta_2 \left(\frac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}} \right)^{C_1 \alpha - 1} \frac{d(x)^{-\theta_2 - 1}}{\delta^{-\theta_2}} \right]^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} dx dt \\ &\leq C \int \int_{E_\delta^C} |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \left[d(x)^{-\theta_2} + t^{\theta_1} \right]^{\frac{(C_1 \alpha - 1)p(q+\alpha)}{q-p+1}} d(x)^{-\frac{(\theta_2+1)p(q+\alpha)}{q-p+1}} \delta^{\frac{\theta_2 C_1 \alpha p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} dx dt. \end{aligned} \quad (4.21)$$

Similarly to (4.19), we can say that there exist $\bar{C} > 0$ such that

$$\delta^{\frac{\theta_2 C_1 \alpha p (q + \alpha)}{q - p + 1}} \leq \bar{C},$$

hence (4.21), for some constant $C > 0$, reduces to

$$I_1 \leq C |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \int \int_{E_\delta^C} V^{-\frac{p+\alpha-1}{q-p+1}} d(x)^{-\frac{(\theta_2+1)p(q+\alpha)}{q-p+1}} \left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right]^{-\frac{\theta_2(C_1\alpha-1)p(q+\alpha)}{q-p+1}} dx dt. \quad (4.22)$$

Claim: *If $f : (0, +\infty) \rightarrow [0, +\infty)$ is a non decreasing function and if (HP1) – (ii) holds then, for any $0 < \varepsilon < \varepsilon_0$ and for any $\delta > 0$ small enough, we can write*

$$\begin{aligned} & \int \int_{E_\delta^C} f \left(\left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right] \right) d(x)^{-(\theta_2+1)p \left(\frac{q}{q-p+1} - \varepsilon \right)} V^{-\frac{p-1}{q-p+1} + \varepsilon} dx dt \\ & \leq C \int_0^{2\delta} f(z) z^{-\frac{pq}{q-p+1} \theta_2 - C_0 \varepsilon - 1} |\log z|^{s_4} dz, \end{aligned} \quad (4.23)$$

for some constant $C > 0$.

To show the claim, we first observe that

$$f \left(\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right) \leq f \left(\frac{\delta}{2^n} \right) \quad \text{in } E_{\frac{\delta}{2^{n+1}}} \setminus E_{\frac{\delta}{2^n}}.$$

Hence, due to HP1(ii), we can write

$$\begin{aligned} & \int \int_{(E_\delta)^C} f \left(\left[d(x)^{-\theta_2} + t^{\theta_1} \right]^{-\frac{1}{\theta_2}} \right) d(x)^{-(\theta_2+1)p \left(\frac{q}{q-p+1} - \varepsilon \right)} V^{-\frac{p-1}{q-p+1} + \varepsilon} dx dt \\ & = \sum_{n=0}^{+\infty} \int \int_{E_{\frac{\delta}{2^{n+1}}} \setminus E_{\frac{\delta}{2^n}}} f \left(\left[d(x)^{-\theta_2} + t^{\theta_1} \right]^{-\frac{1}{\theta_2}} \right) d(x)^{-(\theta_2+1)p \left(\frac{q}{q-p+1} - \varepsilon \right)} V^{-\frac{p-1}{q-p+1} + \varepsilon} dx dt \\ & \leq \sum_{n=0}^{+\infty} f \left(\frac{\delta}{2^n} \right) \int \int_{E_{\frac{\delta}{2^{n+1}}} \setminus E_{\frac{\delta}{2^n}}} d(x)^{-(\theta_2+1)p \left(\frac{q}{q-p+1} - \varepsilon \right)} V^{-\frac{p-1}{q-p+1} + \varepsilon} dx dt \\ & \leq C \sum_{n=0}^{+\infty} f \left(\frac{\delta}{2^n} \right) \left(\frac{\delta}{2^n} \right)^{-\frac{pq}{q-p+1} \theta_2 - C_0 \varepsilon} \left| \log \left(\frac{\delta}{2^n} \right) \right|^{s_4} \\ & \leq C \sum_{n=0}^{+\infty} \int_{\frac{\delta}{2^n}}^{\frac{\delta}{2^{(n-1)}}} f(z) z^{-\frac{pq}{q-p+1} \theta_2 - C_0 \varepsilon - 1} |\log z|^{s_4} dz \\ & = C \int_0^{2\delta} f(z) z^{-\frac{pq}{q-p+1} \theta_2 - C_0 \varepsilon - 1} |\log z|^{s_4} dz. \end{aligned}$$

We now apply (4.23) with $\varepsilon = \frac{|\alpha|}{q-p+1} > 0$ to inequality (4.22). We get

$$I_1 \leq C |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \int_0^{2\delta} z^{-\theta_2 \frac{(C_1\alpha-1)p(q+\alpha)}{q-p+1} - \frac{pq}{q-p+1} \theta_2 + \frac{C_0\alpha}{q-p+1} - 1} |\log z|^{s_4} dz. \quad (4.24)$$

We define

$$b := \frac{1}{q-p+1} \left(-\theta_2 C_1 \alpha p (q + \alpha) + \theta_2 p \alpha + C_0 \alpha \right), \quad (4.25)$$

and due to (4.2), we observe that

$$b \geq \frac{|\alpha|}{q-p+1} > 0.$$

By plugging (4.25) into inequality (4.24) we can write

$$I_1 \leq C |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \int_0^{2\delta} z^{b-1} |\log z|^{s_4} dz. \quad (4.26)$$

Let us now perform a change of variable, we define

$$y := b \log z,$$

hence from (4.26) we deduce

$$\begin{aligned} I_1 &\leq C |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} b^{-s_4-1} \int_{-\infty}^0 e^y |y|^{s_4} dy \\ &\leq C |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \left(\frac{|\alpha|}{q-p+1} \right)^{-s_4-1} \\ &\leq C |\alpha|^{\frac{pq}{q-p+1} - s_4 - 1}. \end{aligned} \quad (4.27)$$

for $|\alpha| > 0$ small enough, with s_4 as in (HP1) – (ii).

Finally, let us consider I_3 defined in (4.10). Due to the definition of φ in (4.1) we get

$$\begin{aligned} I_3 &\leq \int \int_{E_\delta^C} \left[C_1 |\alpha| \theta_1 \left(\frac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}} \right)^{C_1 \alpha - 1} \frac{t^{\theta_1 - 1}}{\delta^{-\theta_2}} \right]^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt \\ &\leq C \int \int_{E_\delta^C} |\alpha|^{\frac{q+\alpha}{q-1}} \left[d(x)^{-\theta_2} + t^{\theta_1} \right]^{\frac{(C_1 \alpha - 1)(q+\alpha)}{q-1}} t^{\frac{(\theta_1 - 1)(q+\alpha)}{q-1}} \delta^{\frac{\theta_2 C_1 \alpha (q+\alpha)}{q-1}} V^{-\frac{\alpha+1}{q-1}} dx dt. \end{aligned} \quad (4.28)$$

Arguing as in (4.19), we can say that there exist $\bar{C} > 0$ such that

$$\delta^{\frac{\theta_2 C_1 \alpha (q+\alpha)}{q-1}} \leq \bar{C}.$$

Hence (4.28), for some constant $C > 0$, reduces to

$$I_3 \leq C |\alpha|^{\frac{q+\alpha}{q-1}} \int \int_{E_\delta^C} V^{-\frac{\alpha+1}{q-1}} t^{\frac{(\theta_1 - 1)(q+\alpha)}{q-1}} \left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right]^{-\theta_2 \frac{(C_1 \alpha - 1)(q+\alpha)}{q-1}} dx dt. \quad (4.29)$$

We have the following

Claim: *If $f : (0, +\infty) \rightarrow [0, +\infty)$ is a non decreasing function and if (HP1) – (i) holds then, for any $0 < \varepsilon < \varepsilon_0$ and for any $\delta > 0$ small enough, we can write*

$$\begin{aligned} &\int \int_{E_\delta^C} f \left(\left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right] \right) t^{(\theta_1 - 1) \left(\frac{q}{q-1} - \varepsilon \right)} V^{-\frac{1}{q-1} + \varepsilon} dx dt \\ &\leq C \int_0^{2\delta} f(z) z^{-\frac{q}{q-1} \theta_2 - C_0 \varepsilon - 1} |\log z|^{s_2} dz, \end{aligned} \quad (4.30)$$

for some constant $C > 0$.

Inequality (4.30) can be proven similarly to (4.23) where one uses (HP1) – (i) instead of (HP1) – (ii). We now apply (4.30) with $\varepsilon = \frac{|\alpha|}{q-1} > 0$ to inequality (4.29). We get

$$I_3 \leq C |\alpha|^{\frac{q+\alpha}{q-1}} \int_0^{2\delta} z^{-\theta_2 (C_1 \alpha - 1) \frac{q+\alpha}{q-1} - \frac{q}{q-1} \theta_2 + \frac{C_0 \alpha}{q-1} - 1} |\log z|^{s_2} dz. \quad (4.31)$$

We define

$$\beta := \frac{1}{q-1} (-\theta_2 C_1 \alpha (q+\alpha) + \theta_2 \alpha + C_0 \alpha), \quad (4.32)$$

and due to (4.2), we have

$$\beta \geq \frac{|\alpha|}{q-1} > 0.$$

By plugging (4.32) into inequality (4.31) and using the change of variables $y = \beta \log z$, we get

$$\begin{aligned} I_3 &\leq C |\alpha|^{\frac{q+\alpha}{q-1}} \int_{-\infty}^0 e^y \left| \frac{y}{\beta} \right|^{s_2} \frac{1}{\beta} dy \\ &\leq C |\alpha|^{\frac{q+\alpha}{q-1}} \beta^{-s_2-1} \\ &\leq C |\alpha|^{\frac{1}{q-1}-s_2}. \end{aligned} \tag{4.33}$$

with s_2 as in (HP1) – (i).

For any $n \in \mathbb{N}$ and $\delta > 0$ small enough, due to inequalities (4.15), (4.20), (4.27) and (4.33), inequality (4.12) reduces to

$$\begin{aligned} \int_0^\infty \int_\Omega V u^{q+\alpha} \varphi_n^s dx dt &\leq C |\alpha|^{-\frac{(p-1)q}{q-p+1}} \left[|\alpha|^{\frac{pq}{q-p+1}-s_4-1} + n^{-\frac{|\alpha|}{q-p+1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_4} \right] \\ &\quad + C \left[|\alpha|^{\frac{1}{q-1}-s_2} + n^{-\frac{|\alpha|}{q-1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{s_2} \right], \end{aligned} \tag{4.34}$$

where $C > 0$ does not depend on δ and n . By taking the limit in (4.34) as $n \rightarrow \infty$ for fixed small enough $\delta > 0$, we get

$$\begin{aligned} 0 &\leq \int \int_{E_\delta} V u^{q+\alpha} dx dt \leq \int_0^\infty \int_\Omega V u^{q+\alpha} \varphi_n^s dx dt \\ &\leq C \left[|\alpha|^{\frac{p-1}{q-p+1}-s_4} + |\alpha|^{\frac{1}{q-1}-s_2} \right]. \end{aligned} \tag{4.35}$$

Observe that, due to the definitions of s_2 in (HP1) – (i) and s_4 in (HP2) – (ii)

$$\frac{1}{q-1} - s_2 > 0, \quad \frac{p-1}{q-p+1} - s_4 > 0.$$

Hence we can take the limit in (4.35) as $\delta \rightarrow 0$, and thus $\alpha \rightarrow 0^-$, obtaining by Fatou's Lemma

$$\int_0^\infty \int_\Omega V u^q dx dt = 0,$$

which concludes the proof. \square

As a consequence of Theorem 2.1 we prove Corollary 2.3.

Proof of Corollary 2.3. We show that under the assumptions of Corollary 2.3, hypothesis (HP1) is satisfied. Let us define

$$\hat{E}_\delta := E_{\frac{\delta}{2}} \setminus E_\delta$$

and observe that

$$\hat{E}_\delta \subset \left\{ d(x) \geq \frac{\delta}{2} \right\} \times \left[0, \left(\frac{\delta}{2} \right)^{-\frac{\theta_2}{\theta_1}} \right] =: \Omega_{\frac{\delta}{2}} \times \left[0, \left(\frac{\delta}{2} \right)^{-\frac{\theta_2}{\theta_1}} \right],$$

where $d(x)$ has been defined in (1.4). Observe that for $\delta > 0$ small enough

$$\begin{aligned}
& \int \int_{\hat{E}_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon} dx dt \\
& \leq \int \int_{\hat{E}_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} [g(t)h(x)]^{-\frac{1}{q-1}+\varepsilon} dx dt \\
& \leq C \int_{\Omega_{\frac{\delta}{2}}} h(x)^{-\frac{1}{q-1}+\varepsilon} dx \int_0^{\left(\frac{\delta}{2}\right)^{-\frac{\theta_2}{\theta_1}}} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} g(t)^{-\frac{1}{q-1}+\varepsilon} dt \\
& \leq C \int_{\Omega_{\frac{\delta}{2}}} \left[d(x)^{-\sigma_1} (\log(1+d(x)^{-1}))^{-\delta_1} \right]^{-\frac{1}{q-1}+\varepsilon} dx \\
& \quad \times \int_0^{\left(\frac{\delta}{2}\right)^{-\frac{\theta_2}{\theta_1}}} g(t)^{-\frac{1}{q-1}} (1+t)^{\alpha\varepsilon} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} dt \\
& \leq C \int_{\Omega_{\frac{\delta}{2}}} d(x)^{\frac{\sigma_1}{q-1}-\varepsilon\sigma_1} (\log(1+d(x)^{-1}))^{\frac{\delta_1}{q-1}-\varepsilon\delta_1} dx \\
& \quad \times \left[\delta^{-\frac{\theta_2}{\theta_1} \left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right) + \alpha\varepsilon \right]} \int_0^{\left(\frac{\delta}{2}\right)^{-\frac{\theta_2}{\theta_1}}} g(t)^{-\frac{1}{q-1}} dt \right] \\
& \leq C |\log(\delta)|^{\frac{\delta_1}{q-1}-\varepsilon\delta_1} \left[\delta^{-\frac{\theta_2}{\theta_1} \left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right) + \alpha\varepsilon \right]} \right] \delta^{-\frac{\theta_2}{\theta_1}\sigma_2} |\log(\delta)|^{\delta_2} \\
& \leq C \delta^{-\frac{\theta_2}{\theta_1} \left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right) + \alpha\varepsilon + \sigma_2 \right]} |\log(\delta)|^{\frac{\delta_1}{q-1}-\varepsilon\delta_1+\delta_2},
\end{aligned} \tag{4.36}$$

for $\theta_1, \theta_2 \geq 1$. For $C_0 > 0$ large and every $\varepsilon > 0$ small enough, condition (2.3) of (HP1) is satisfied because

$$\frac{\theta_2}{\theta_1} \left[\frac{q}{q-1} - \sigma_2 \right] \geq 0 \quad \text{and} \quad \delta_2 + \frac{\delta_1}{q-1} < \bar{s}_2. \tag{4.37}$$

On the other hand, for $\varepsilon, \delta > 0$ sufficiently small

$$\begin{aligned}
& \int \int_{\hat{E}_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon} dx dt \\
& \leq \int \int_{\hat{E}_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} [g(t)h(x)]^{-\frac{p-1}{q-p+1}+\varepsilon} dx dt \\
& \leq \int_{\Omega_{\frac{\delta}{2}}} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} h(x)^{-\frac{p-1}{q-p+1}+\varepsilon} dx \int_0^{\left(\frac{\delta}{2}\right)^{-\frac{\theta_2}{\theta_1}}} g(t)^{-\frac{p-1}{q-p+1}+\varepsilon} dt \\
& \leq C \int_{\Omega_{\frac{\delta}{2}}} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} \left[d(x)^{\sigma_1} (\log(1+d(x)^{-1}))^{\delta_1} \right]^{\frac{p-1}{q-p+1}-\varepsilon} dx \\
& \quad \times \left[\delta^{-\frac{\theta_2}{\theta_1}\alpha\varepsilon} \int_0^{\left(\frac{\delta}{2}\right)^{-\frac{\theta_2}{\theta_1}}} g(t)^{-\frac{p-1}{q-p+1}} dt \right] \\
& \leq C \int_{\Omega_{\frac{\delta}{2}}} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\sigma_1\frac{p-1}{q-p+1}-\varepsilon\sigma_1} (\log(1+d(x)^{-1}))^{\delta_1\frac{p-1}{q-p+1}-\varepsilon\delta_1} dx \\
& \quad \times \left[\delta^{-\frac{\theta_2}{\theta_1}\alpha\varepsilon} \delta^{-\frac{\theta_2}{\theta_1}\sigma_4} \right] \\
& \leq C \delta^{-\frac{\theta_2}{\theta_1}(\alpha\varepsilon+\sigma_4)} |\log(\delta)|^{\delta_1\left(\frac{p-1}{q-p+1}-\varepsilon\right)} \int_{\Omega_{\frac{\delta}{2}}} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\sigma_1\frac{p-1}{q-p+1}-\varepsilon\sigma_1} dx
\end{aligned} \tag{4.38}$$

We define

$$\beta := -(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right) + \sigma_1\frac{p-1}{q-p+1} - \varepsilon\sigma_1$$

and we observe that $\beta < -1$ for θ_2 sufficiently large. Therefore, due to the boundedness of Ω_δ , inequality (4.38) reduces to

$$\int \int_{\bar{E}_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon} dx dt \leq C \delta^{-\frac{\theta_2}{\theta_1}(\alpha\varepsilon+\sigma_4)+\beta+1} |\log(\delta)|^{\delta_1\left(\frac{p-1}{q-p+1}-\varepsilon\right)} \tag{4.39}$$

For $\varepsilon, \delta > 0$ small enough and for $\theta_2/\theta_1 > 0$ small enough, condition (2.4) is satisfied for some large $C_0 > 0$ because the hypotheses of the Corollary 2.3 guarantee that

$$\sigma_1 - \frac{\theta_2}{\theta_1}\sigma_4\frac{q-p+1}{p-1} \geq q+1 \quad \text{and} \quad \delta_1\frac{p-1}{q-p+1} < \bar{s}_4.$$

Thus (HP1) holds and we can apply Theorem 2.1 to obtain the result. \square

5. PROOF OF THEOREM 2.2

Proof of Theorem 2.2. Let us recall the family of functions φ_n defined in (4.4). We claim that $u^q \in L^1(\Omega \times (0, +\infty), V d\mu dt)$. To prove this, we start by showing that for some constants $A > 0$, $B > 0$, $s \geq 1$, for every $\delta > 0$ small enough and every $n \in \mathbb{N}$ we have

$$\int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \leq A \left(\int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \right)^{\frac{p-1}{pq}} + B. \tag{5.1}$$

In order to prove (5.1) we apply Corollary 3.4 with φ replaced by the family of functions φ_n . Let

$$C_1 > \max \left\{ \frac{2(1 + C_0 + \theta_2)}{\theta_2 q}, \frac{2(\theta_2(q-1) + C_0 + 1)}{\theta_2(q-1)q}, \frac{2C_0 + 1}{\theta_2(q-p+1)}, \frac{2C_0 + 1}{\theta_2} \right\}, \quad (5.2)$$

with $C_0 > 0$ and $\theta_2 \geq 1$ as in (HP2). Then for any fixed $s \geq \max \left\{ 1, \frac{q+1}{q-1}, \frac{2pq}{q-p+1} \right\}$, $\delta > 0$ sufficiently small, $\alpha = \frac{1}{\log \delta} < 0$ and for every $n \in \mathbb{N}$, we have

$$\begin{aligned} & \int_0^\infty \int_\Omega V u^q \varphi^s dx dt \\ & \leq C \left[|\alpha|^{-1} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} dx dt \right. \right. \\ & \quad \left. \left. + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} dx dt \right) \right]^{\frac{p-1}{p}} \times \left(\int \int_{E_\delta^C} V u^q \varphi_n^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \\ & \quad \times \left(\int \int_{E_\delta^C} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}} dx dt \right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\ & \quad + C \left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} dx dt \right. \\ & \quad \left. + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} dx dt \right]^{\frac{1}{q+\alpha}} \\ & \quad \times \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q+\alpha-1}} dx dt \right)^{\frac{q+\alpha-1}{q+\alpha}}. \end{aligned} \quad (5.3)$$

where E_δ has been defined in (2.1). We also define

$$J_1 := \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} dx dt; \quad (5.4)$$

$$J_2 := \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} dx dt; \quad (5.5)$$

$$J_3 := \int \int_{E_\delta^C} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}} dx dt; \quad (5.6)$$

$$J_4 := \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q+\alpha-1}} dx dt. \quad (5.7)$$

By using (5.4), (5.5), (5.6) and (5.7), inequality (5.3) reads

$$\begin{aligned}
& \int_0^\infty \int_\Omega V u^q \varphi^s dx dt \\
& \leq C \left[|\alpha|^{-1 - \frac{(p-1)q}{q-p+1}} J_1 \right]^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V u^q \varphi_n^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} J_3^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
& + C \left[|\alpha|^{-1} J_2 \right]^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V u^q \varphi_n^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} J_3^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
& + C \left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 + J_2 \right]^{\frac{1}{q+\alpha}} J_4^{\frac{q+\alpha-1}{q+\alpha}} \\
& \leq C \left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 \right]^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V u^q \varphi_n^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \\
& \times \left[|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3 \right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
& + C J_2^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V u^q \varphi_n^s dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \left[|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3 \right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
& + C \left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 + J_2 \right]^{\frac{1}{q+\alpha}} J_4^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned} \tag{5.8}$$

Let us prove that, for $\delta > 0$ sufficiently small and $|\alpha| = -\frac{1}{\log \delta} > 0$ sufficiently small

$$\limsup_{n \rightarrow \infty} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 \right) \leq C, \tag{5.9}$$

$$\limsup_{n \rightarrow \infty} \left(|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3 \right) \leq C, \tag{5.10}$$

$$\limsup_{n \rightarrow \infty} J_2 \leq C, \tag{5.11}$$

$$\limsup_{n \rightarrow \infty} J_4 \leq C, \tag{5.12}$$

for some $C > 0$ independent of α .

We start by proving (5.9). Observe that

$$J_1 \leq C(I_1 + I_2), \tag{5.13}$$

with I_1 and I_2 defined in (4.8) and (4.9), respectively. Arguing as in the proof of Theorem 2.1, using condition (2.7) in (HP2) – (ii) in place of condition (2.4) in (HP1) – (ii), we obtain, similar to (4.15),

$$I_2 \leq C n^{-\frac{|\alpha|}{q-p+1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_4} \tag{5.14}$$

and, similar to (4.27),

$$I_1 \leq C |\alpha|^{\frac{pq}{q-p+1} - \bar{s}_4 - 1} = C |\alpha|^{\frac{q(p-1)}{q-p+1}}. \tag{5.15}$$

Combining (5.13), (5.14) and (5.15), for some $C > 0$ and for every $n \in \mathbb{N}$, we have

$$|\alpha|^{-\frac{q(p-1)}{q-p+1}} J_1 \leq C \left(1 + |\alpha|^{-\frac{q(p-1)}{q-p+1}} n^{-\frac{|\alpha|}{q-p+1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_4} \right). \quad (5.16)$$

We can compute the limit as $n \rightarrow \infty$ on both sides of (5.16), thus we obtain (5.9).

Now observe that

$$J_2 \leq C(I_3 + I_4), \quad (5.17)$$

with I_3 and I_4 defined in (4.10) and (4.11), respectively. Then arguing as in the proof of Theorem 2.1, due to condition (2.5) in (HP2) – (i) with $\varepsilon = -\frac{\alpha}{q-1} > 0$ we deduce, similar to (4.20), for some positive constant C

$$I_4 \leq C n^{-\frac{|\alpha|}{q-1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2}, \quad (5.18)$$

Moreover, similar to (4.33), we have

$$I_3 \leq C |\alpha|^{\frac{1}{q-1} - \bar{s}_2} = C. \quad (5.19)$$

Combining (5.17), (5.18) and (5.19), for some $C > 0$, every $n \in \mathbb{N}$ and for small enough $|\alpha| > 0$ we have

$$J_2 \leq C \left(1 + n^{-\frac{|\alpha|}{q-1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \right).$$

Letting $n \rightarrow \infty$ we obtain (5.11).

We now proceed to estimate J_4 . Observe that

$$J_4 \leq C(I_5 + I_6), \quad (5.20)$$

where

$$I_5 := \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q+\alpha-1}} dx dt, \quad I_6 := \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} \varphi^{\frac{q+\alpha}{q+\alpha-1}} |\partial_t \eta_m|^{\frac{q+\alpha}{q+\alpha-1}} dx dt.$$

Due to (4.1) we have

$$\begin{aligned} I_5 &\leq C \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \left[\frac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}} \right]^{\frac{(C_1 \alpha - 1)(q+\alpha)}{q+\alpha-1}} \left(\frac{t^{\theta_1-1}}{\delta^{-\theta_2}} \right)^{\frac{q+\alpha}{q+\alpha-1}} dx dt \\ &\leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} \left[d(x)^{-\theta_2} + t^{\theta_1} \right]^{\frac{(C_1 \alpha - 1)(q+\alpha)}{q+\alpha-1}} \delta^{\frac{\theta_2 C_1 \alpha (q+\alpha)}{q+\alpha-1}} t^{(\theta_1-1) \left(\frac{q+\alpha}{q+\alpha-1} \right)} dx dt \\ &\leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} \left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right]^{-\theta_2 (C_1 \alpha - 1) \left(\frac{q}{q-1} - \frac{\alpha}{(q+\alpha-1)(q-1)} \right)} \\ &\quad \times t^{(\theta_1-1) \left(\frac{q}{q-1} - \frac{\alpha}{(q+\alpha-1)(q-1)} \right)} dx dt, \end{aligned} \quad (5.21)$$

where we have used that there exists a positive constant \bar{C} such that

$$\delta^{\theta_2 C_1 \alpha \left(\frac{q+\alpha}{q+\alpha-1} \right)} = e^{\theta_2 C_1 \alpha \left(\frac{q+\alpha}{q+\alpha-1} \right) \log \delta} = e^{\theta_2 C_1 \left(\frac{q+\alpha}{q+\alpha-1} \right)} \leq \bar{C}.$$

Claim: *If $f : (0, +\infty) \rightarrow [0, +\infty)$ is a non decreasing function and if (HP2)–(i) holds then, for any $0 < \varepsilon < \varepsilon_0$ and for any $\delta > 0$ small enough, we can write*

$$\begin{aligned} \int \int_{E_\delta^C} f \left(\left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right] \right) t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} V^{-\frac{1}{q-1}-\varepsilon} dx dt \\ \leq C \int_0^{2\delta} f(z) z^{-\bar{s}_1 - C_0\varepsilon - 1} |\log z|^{\bar{s}_2} dz, \end{aligned} \quad (5.22)$$

for some constant $C > 0$ with \bar{s}_1 and \bar{s}_2 as in (2.2).

Inequality (5.22) can be proven similarly to (4.23), where one uses condition (2.6) in (HP2)–(i) instead of (HP1)–(i). By using the latter claim with $\varepsilon = \frac{|\alpha|}{(q+\alpha-1)(q-1)} > 0$ we obtain

$$I_5 \leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int_0^{2\delta} z^{-\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right) - \bar{s}_1 - C_0\varepsilon - 1} |\log z|^{\bar{s}_2} dz.$$

Then observe that, due to (5.2), for $|\alpha| > 0$ small

$$-\theta_2(C_1\alpha-1) \left(\frac{q+\alpha}{q+\alpha-1} \right) - \bar{s}_1 - C_0\varepsilon \geq \frac{|\alpha|}{(q-1)^2} =: b$$

Now we define

$$y := b \log z,$$

then there exists $\bar{C} > 0$ such that for $|\alpha| > 0$ small

$$\begin{aligned} I_5 &\leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int_{-\infty}^0 e^y \left| \frac{y}{b} \right|^{\bar{s}_2} \frac{1}{b} dy \\ &\leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} b^{-\bar{s}_2-1} \int_{-\infty}^0 e^y |y|^{\bar{s}_2} dy \\ &\leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \left(\frac{|\alpha|}{(q-1)^2} \right)^{-\bar{s}_2-1} \\ &\leq C |\alpha|^{\frac{q+\alpha}{q+\alpha-1} - \frac{1}{q-1} - 1} \leq \bar{C}. \end{aligned} \quad (5.23)$$

On the other hand, due to (4.1) and condition (2.6) in (HP2)–(i) with $\varepsilon = \frac{|\alpha|}{(q+\alpha-1)(q-1)}$, by using the definition of $\tilde{E}_{\delta,n}$ in (4.7), for every $n \in \mathbb{N}$ we have

$$\begin{aligned} I_6 &\leq C \int \int_{\tilde{E}_{\delta,n}} V^{-\frac{1}{q+\alpha-1}} \left[\left(\frac{\delta}{n} \right)^{\theta_2} t^{\theta_1-1} \right]^{\frac{q+\alpha}{q+\alpha-1}} n^{\theta_2\alpha C_1 \left(\frac{q+\alpha}{q+\alpha-1} \right)} dx dt \\ &\leq C n^{\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right)} \delta^{\theta_2\left(\frac{q+\alpha}{q+\alpha-1}\right)} \int \int_{\tilde{E}_{\delta,n}} V^{-\frac{1}{q-1}-\varepsilon} t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} dx dt \\ &\leq C n^{\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right)} \delta^{\theta_2\left(\frac{q+\alpha}{q+\alpha-1}\right)} \left(\frac{\delta}{n} \right)^{-\bar{s}_1 - C_0\varepsilon} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \\ &\leq C n^{-\frac{|\alpha|}{q+\alpha-1} \left[\theta_2 C_1 (q+\alpha) + \frac{\theta_2}{q-1} - \frac{C_0}{q-1} \right]} \delta^{\frac{|\alpha|}{(q+\alpha-1)(q-1)} [\theta_2 - C_0]} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \end{aligned} \quad (5.24)$$

Now observe that there exists a positive constant \bar{C} such that

$$\delta^{\frac{|\alpha|}{(q+\alpha-1)(q-1)} [\theta_2 - C_0]} = e^{\frac{|\alpha|}{(q+\alpha-1)(q-1)} [\theta_2 - C_0] \log \delta} = e^{\frac{C_0 - \theta_2}{(q+\alpha-1)(q-1)}} \leq \bar{C}, \quad (5.25)$$

and due to (5.2)

$$-\frac{|\alpha|}{q+\alpha-1} \left[\theta_2 C_1 (q+\alpha) + \frac{\theta_2}{q-1} - \frac{C_0}{q-1} \right] \leq -\frac{|\alpha|}{(q-1)^2}. \quad (5.26)$$

Combining (5.25) and (5.26) with (5.24) we obtain

$$I_6 \leq C n^{-\frac{|\alpha|}{(q-1)^2}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2}. \quad (5.27)$$

We now substitute (5.23) and (5.27) into inequality (5.20) thus we have, for some $C > 0$ and for every $n \in \mathbb{N}$

$$J_4 \leq C \left[1 + n^{-\frac{|\alpha|}{(q-1)^2}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \right].$$

Letting $n \rightarrow \infty$ we get (5.12).

In order to estimate integral J_3 defined in (5.6), we define, for sufficiently small $|\alpha| > 0$, the positive constant λ

$$\lambda := \frac{|\alpha|q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]}. \quad (5.28)$$

Observe that, for sufficiently small $|\alpha| > 0$

$$\frac{|\alpha|q(p-1)}{(q-p+1)^2} < \lambda < \frac{2|\alpha|q(p-1)}{(q-p+1)^2}, \quad (5.29)$$

and

$$\frac{pq}{q-(1-\alpha)(p-1)} = \frac{\bar{s}_3}{\theta_2} + \lambda p, \quad (5.30)$$

where \bar{s}_3 has been defined in (2.2) and $\theta_2 \geq 1$ as in (HP2). Thus by the definition of φ_n in (4.4) and by (5.28), for sufficiently small $|\alpha| > 0$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned} J_3 &\leq C \int \int_{E_\delta^C} V^{-\lambda-\bar{s}_4} |\nabla \varphi|^{\frac{\bar{s}_3}{\theta_2} + \lambda p} dx dt + C \int \int_{\tilde{E}_{\delta,n}} V^{-\lambda-\bar{s}_4} (\varphi |\nabla \eta_n|)^{\frac{\bar{s}_3}{\theta_2} + \lambda p} dx dt \\ &=: C(I_7 + I_8), \end{aligned} \quad (5.31)$$

where $\tilde{E}_{\delta,n}$ has been defined in (4.7). Due to the very definition of φ and η_n in (4.1) and (4.3) respectively, and by (5.30) we get

$$\begin{aligned} I_8 &\leq C \int \int_{\tilde{E}_{\delta,n}} V^{-\lambda-\bar{s}_4} n^{C_1 \alpha \theta_2 \left(\frac{\bar{s}_3}{\theta_2} + \lambda p \right)} \left(\frac{\delta}{n} \right)^{\theta_2 \left(\frac{\bar{s}_3}{\theta_2} + \lambda p \right)} d(x)^{-(\theta_2+1) \left(\frac{\bar{s}_3}{\theta_2} + \lambda p \right)} dx dt \\ &\leq C n^{(C_1 \alpha - 1)(\bar{s}_3 + \lambda p \theta_2)} \delta^{\bar{s}_3 + \lambda p \theta_2} \int \int_{\tilde{E}_{\delta,n}} V^{-\lambda-\bar{s}_4} d(x)^{-(\theta_2+1)p \left(\frac{q}{q-p+1} + \lambda \right)} dx dt \end{aligned}$$

Now we use condition (2.8) in (HP2) – (ii) with $\varepsilon = \lambda$ and we obtain, for every $n \in \mathbb{N}$ and for sufficiently small $\delta > 0$

$$\begin{aligned} I_8 &\leq C n^{(C_1 \alpha - 1)p \theta_2 \left(\frac{q}{q-p+1} + \lambda \right)} \delta^p \theta_2 \left(\frac{q}{q-p+1} + \lambda \right) \left(\frac{\delta}{n} \right)^{-\frac{pq}{q-p+1} \theta_2 - C_0 \lambda} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_4} \\ &\leq C n^{C_1 \alpha p \theta_2 \left(\frac{q}{q-p+1} + \lambda \right) - \lambda p \theta_2 + C_0 \lambda} \delta^p \theta_2 \lambda^{-C_0 \lambda} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_4}. \end{aligned}$$

Due to the definition of λ in (5.28), inequality (5.29) and the definition of C_1 in (5.2), for sufficiently small $|\alpha| > 0$ we write

$$\begin{aligned}
& C_1 \alpha p \theta_2 \left(\frac{q}{q-p+1} + \lambda \right) - \lambda p \theta_2 + C_0 \lambda \\
&= (C_1 \alpha - 1) p \theta_2 \frac{|\alpha| q (p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} + C_1 \alpha \frac{p q \theta_2}{q-p+1} + \frac{C_0 |\alpha| q (p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} \\
&\leq (C_1 \alpha - 1) p \theta_2 \frac{|\alpha| q (p-1)}{(q-p+1)^2} + C_1 \alpha \frac{p q \theta_2}{q-p+1} + \frac{2C_0 |\alpha| q (p-1)}{(q-p+1)^2} \\
&= C_1 \alpha \theta_2 \left[\frac{|\alpha| q p^2}{(q-p+1)^2} - \frac{|\alpha| q p}{(q-p+1)^2} + \frac{q p}{q-p+1} \right] - \frac{|\alpha| q (p-1)}{(q-p+1)^2} [p \theta_2 - 2C_0] \\
&= -\frac{|\alpha|}{(q-p+1)^2} [C_1 \theta_2 p q (q + (p-1)(|\alpha| - 1)) + (p \theta_2 - 2C_0) q (p-1)] \\
&\leq -\frac{|\alpha| q}{(q-p+1)^2} [C_1 \theta_2 p (q-p+1) - 2C_0 p] \\
&\leq -\frac{|\alpha| q p}{(q-p+1)^2}.
\end{aligned}$$

Moreover, since $\alpha = \frac{1}{\log \delta} < 0$, there exists \bar{C} such that

$$\delta^{\lambda(p\theta_2 - C_0)} = e^{\lambda(p\theta_2 - C_0) \log \delta} < e^{\frac{-|\alpha| q (p-1)}{(q-p+1)^2} (p\theta_2 - C_0) |\log \delta|} \leq \bar{C}$$

Therefore we obtain the following bound on I_8

$$I_8 \leq C n^{-\frac{|\alpha| p q}{(q-p+1)^2}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_4}. \quad (5.32)$$

On the other hand, by using the definition of φ in (4.1) we can write

$$I_7 \leq C |\alpha|^{\frac{p q}{q-(1-\alpha)(p-1)}} \int \int_{E_\delta^C} V^{-\lambda - \bar{s}_4} \left[\left(\frac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}} \right)^{C_1 \alpha - 1} \delta^{\theta_2} d(x)^{-(\theta_2+1)} \right]^{\frac{\bar{s}_3}{\theta_2} + \lambda p} dx dt,$$

and we observe that there exists $\bar{C} > 0$ such that for $|\alpha| > 0$ small

$$\delta^{C_1 \alpha \theta_2 \left(\frac{\bar{s}_3}{\theta_2} + \lambda p \right)} = \delta^{C_1 \alpha \theta_2 \left(\frac{p q}{q-(1-\alpha)(p-1)} \right)} < \delta^{C_1 \alpha \theta_2 \left(\frac{2 p q}{q-p+1} \right)} = e^{C_1 \alpha \theta_2 \left(\frac{2 p q}{q-p+1} \right) \log \delta} \leq \bar{C}.$$

Therefore we get

$$I_7 \leq C |\alpha|^{\frac{p q}{q-(1-\alpha)(p-1)}} \int \int_{E_\delta^C} V^{-\lambda - \bar{s}_4} \left[d(x)^{-\theta_2} + t^{\theta_1} \right]^{(C_1 \alpha - 1) \left(\frac{\bar{s}_3}{\theta_2} + \lambda p \right)} d(x)^{-(\theta_2+1) \left(\frac{\bar{s}_3}{\theta_2} + \lambda p \right)} dx dt,$$

We now state the following

Claim: *Let $f : (0, +\infty) \rightarrow [0, +\infty)$ be a non decreasing function and suppose that (HP2)–(ii) holds. Then, for any $0 < \varepsilon < \varepsilon_0$ and for any $\delta > 0$ small enough, we can write*

$$\begin{aligned}
& \int \int_{E_\delta^C} f \left(\left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right] \right) d(x)^{-(\theta_2+1)p \left(\frac{q}{q-p+1} + \varepsilon \right)} V^{-\frac{p-1}{q-p+1} - \varepsilon} dx dt \\
& \leq C \int_0^{2\delta} f(z) z^{-\bar{s}_3 - C_0 \varepsilon - 1} |\log z|^{\bar{s}_4} dz,
\end{aligned} \quad (5.33)$$

for some constant $C > 0$ with \bar{s}_3 and \bar{s}_4 as in (2.2).

Inequality (5.33) can be proven similarly to (4.23), where one uses condition (2.8) in (HP2) – (ii) instead of (HP1) – (ii). By using the latter claim with $\varepsilon = \lambda$ we get

$$I_7 \leq C |\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int_0^{2\delta} z^{-\theta_2(C_1\alpha-1)\left(\frac{\bar{s}_3}{\theta_2}+\lambda p\right)-\bar{s}_3-C_0\lambda-1} |\log z|^{\bar{s}_4} dz \quad (5.34)$$

Observe that, since $\alpha < 0$ and due to (5.2)

$$\begin{aligned} & -\theta_2(C_1\alpha-1)\left(\frac{\bar{s}_3}{\theta_2}+\lambda p\right)-\bar{s}_3-C_0\lambda \\ &= -\theta_2 C_1 \alpha \frac{pq}{q-(1-\alpha)(p-1)} + p\theta_2 \frac{|\alpha|q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} - C_0 \frac{|\alpha|q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} \\ &\geq |\alpha|\theta_2 C_1 \frac{pq}{(q-p+1)^2} + p\theta_2 \frac{|\alpha|q(p-1)}{(q-p+1)^2} - C_0 \frac{2|\alpha|q(p-1)}{(q-p+1)^2} \\ &\geq \frac{|\alpha|q(p-1)}{(q-p+1)^2} \{\theta_2 C_1 - 2C_0\} \\ &\geq \frac{|\alpha|q(p-1)}{(q-p+1)^2} =: a. \end{aligned}$$

We now set $y := a \log z$ then, by using the definition of \bar{s}_4 in (2.2), from (5.34) we deduce

$$I_7 \leq C |\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} a^{-\bar{s}_4-1} \int_{-\infty}^0 e^y |y|^{\bar{s}_4} dy \leq C |\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)} - \frac{q}{q-p+1}}. \quad (5.35)$$

Combining together (5.31), (5.32) and (5.35), for any $\delta > 0$ small enough and for every $n \in \mathbb{N}$ we have

$$|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} J_3 \leq C |\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} \left[|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)} - \frac{q}{q-p+1}} + n^{-\frac{|\alpha|pq}{(q-p+1)^2}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_4} \right].$$

Then letting $n \rightarrow \infty$, for every $\delta > 0$ small enough we obtain (5.10). Now using (5.9), (5.10), (5.11) and (5.12) in (5.8), for any $\delta > 0$ sufficiently small and for every $n \in \mathbb{N}$ we get

$$\begin{aligned} \int_0^\infty \int_\Omega \varphi_n^s u^q V d\mu dt &\leq C' \left(\int \int_{E_\delta^c} \varphi_n^s u^q V dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} + C'' \\ &\leq C' \left(\int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} + C'' \\ &\leq C' \left(1 + \int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \right)^{\frac{p-1}{p}} + C'' \\ &\leq A \left(\int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \right)^{\frac{p-1}{p}} + B, \end{aligned}$$

where A, B are positive constants independent of n, δ and $\frac{p-1}{p} \in (0, 1)$. This easily implies that there exists $C > 0$ such that, for sufficiently small $\delta > 0$ and for every $n \in \mathbb{N}$

$$\int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \leq C. \quad (5.36)$$

We then have

$$\int \int_{E_\delta} u^q V dx dt \leq \int_0^\infty \int_\Omega \varphi_n^s u^q V dx dt \leq C.$$

Thus letting $\delta \rightarrow 0$ we obtain that

$$u^q \in L^1(\Omega \times (0, \infty); V \, dxdt) \quad (5.37)$$

Now, we want to show that

$$\int_0^\infty \int_\Omega u^q V \, dxdt = 0.$$

We use Lemma 3.5 where φ is replaced by φ_n

$$\begin{aligned} \int \int_{E_\delta} u^q V \, dxdt &\leq \int_0^\infty \int_\Omega \varphi_n^s u^q V \, dxdt \\ &\leq C \left[|\alpha|^{-1 - \frac{q(p-1)}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} \, dxdt \right. \\ &\quad + |\alpha|^{-1} \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} \, dxdt \left. \right]^{\frac{p-1}{p}} \left(\int \int_{E_\delta^C} \varphi_n^s u^q V \, dxdt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \\ &\quad \times \left[\int \int_{E_\delta^C} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}} \, dxdt \right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\ &\quad + C \left[\int \int_{E_\delta^C} \varphi_n^s u^q V \, dxdt \right]^{\frac{1}{q}} \left[\int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t \varphi_n|^{\frac{q}{q-1}} \, dxdt \right]^{\frac{q-1}{q}} \\ &\leq C \left[|\alpha|^{-\frac{q(p-1)}{q-p+1}} J_1 \right]^{\frac{p-1}{p}} \left(\int \int_{E_\delta^C} \varphi_n^s u^q V \, dxdt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \left[|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} J_3 \right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\ &\quad + C J_2^{\frac{p-1}{p}} \left(\int \int_{E_\delta^C} \varphi_n^s u^q V \, dxdt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \left[|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} J_3 \right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\ &\quad + C \left(\int \int_{E_\delta^C} \varphi_n^s u^q V \, dxdt \right)^{\frac{1}{q}} J_5^{\frac{q-1}{q}}, \end{aligned} \quad (5.38)$$

where J_1, J_2, J_3 have been defined in (5.3), (5.4), (5.5) and

$$J_5 := \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t \varphi_n|^{\frac{q}{q-1}} \, dxdt.$$

Due to the definition of φ_n in (4.4) we have

$$\begin{aligned} J_5 &\leq C \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t \varphi|^{\frac{q}{q-1}} \, dxdt + \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \varphi^{\frac{q}{q-1}} |\partial_t \eta_m|^{\frac{q}{q-1}} \, dxdt \\ &:= C(I_9 + I_{10}). \end{aligned} \quad (5.39)$$

By (4.1) we have

$$I_9 \leq C |\alpha|^{\frac{q}{q-1}} \int \int_{E_\delta^C} V^{-\frac{1}{q-1}} \left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right]^{-\theta_2(C_1\alpha-1)\frac{q}{q-1}} t^{(\theta_1-1)\frac{q}{q-1}} \, dxdt \quad (5.40)$$

We now state the following

Claim: *Let $f : (0, +\infty) \rightarrow [0, +\infty)$ be a non decreasing function and suppose that (HP2)-(i) holds. Then, for any $\delta > 0$ small enough, we can write*

$$\begin{aligned} \int \int_{E_\delta^C} f \left(\left[\left(d(x)^{-\theta_2} + t^{\theta_1} \right)^{-\frac{1}{\theta_2}} \right] \right) t^{(\theta_1-1)\left(\frac{q}{q-1}\right)} V^{-\frac{1}{q-1}} dx dt \\ \leq C \int_0^{2\delta} f(z) z^{-\bar{s}_1-1} |\log z|^{\bar{s}_2} dz, \end{aligned} \quad (5.41)$$

for some constant $C > 0$ with \bar{s}_1 and \bar{s}_2 as in (2.2).

Inequality (5.41) can be proven similarly to (4.23) where one uses the condition (HP2) – (i) with $\varepsilon = 0$ instead of (HP1) – (ii). We now use the latter claim in (5.40), thus we have

$$\begin{aligned} I_9 &\leq C |\alpha|^{\frac{q}{q-1}} \int_0^{2\delta} z^{-\theta_2(C_1\alpha-1)\frac{q}{q-1}-\frac{q}{q-1}\theta_2-1} |\log z|^{\bar{s}_2} dz \\ &\leq C |\alpha|^{\frac{q}{q-1}} \int_0^{2\delta} z^{-\theta_2 C_1 \alpha \frac{q}{q-1} - 1} |\log z|^{\bar{s}_2} dz \\ &\leq C |\alpha|^{\frac{q}{q-1}} \int_{-\infty}^0 e^y \left| \frac{y}{\gamma} \right|^{\bar{s}_2} \frac{1}{\gamma} dy \\ &\leq C |\alpha|^{\frac{q}{q-1} - \bar{s}_2 - 1} \\ &\leq C \end{aligned} \quad (5.42)$$

where

$$\gamma := |\alpha| \theta_2 C_1 \frac{q}{q-1} \quad \text{and} \quad y := \gamma \log z.$$

On the other hand, by (4.3) we have

$$\begin{aligned} I_{10} &\leq C \int \int_{\tilde{E}_{\delta,n}} V^{-\frac{1}{q-1}} \left[n^{\theta_2} C_1 \alpha \left(\frac{\delta}{n} \right)^{\theta_2} t^{\theta_1-1} \right]^{\frac{q}{q-1}} dx dt \\ &\leq C n^{\theta_2(C_1\alpha-1)\frac{q}{q-1}} \delta^{\theta_2\frac{q}{q-1}} \int \int_{\tilde{E}_{\delta,n}} V^{-\frac{1}{q-1}} t^{(\theta_1-1)\frac{q}{q-1}} dx dt \end{aligned}$$

Then, due to (HP2) – (ii) with $\varepsilon = 0$ we have

$$\begin{aligned} I_{10} &\leq C n^{\theta_2(C_1\alpha-1)\frac{q}{q-1} + \frac{q}{q-1}\theta_2} \delta^{\theta_2\frac{q}{q-1} - \frac{q}{q-1}\theta_2} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \\ &\leq n^{-|\alpha|\theta_2 C_1 \frac{q}{q-1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \end{aligned} \quad (5.43)$$

Now, combining (5.39), (5.42) and (5.43) we get

$$J_5 \leq C \left[1 + n^{-|\alpha|\theta_2 C_1 \frac{q}{q-1}} \left| \log \left(\frac{\delta}{n} \right) \right|^{\bar{s}_2} \right]$$

By letting $n \rightarrow \infty$ we obtain

$$\limsup_{n \rightarrow +\infty} J_5 \leq C \quad (5.44)$$

Finally we use inequalities (5.9), (5.10), (5.11) and (5.44) into (5.38) and, passing to the lim sup as $n \rightarrow \infty$, we obtain for some constant $C > 0$

$$\int \int_{E_\delta} u^q V dx dt \leq C \left[\left(\int \int_{E_\delta^C} u^q V dx dt \right)^{\frac{(1-\alpha)(p-1)}{p}} + \left(\int \int_{E_\delta^C} u^q V dx dt \right)^{\frac{1}{q}} \right]. \quad (5.45)$$

Now we can pass to the limit in (5.45) as $\delta \rightarrow 0$, and thus as $\alpha \rightarrow 0^-$, and conclude by using Fatou's Lemma and (5.37) that

$$\int_0^\infty \int_\Omega u^q V \, dx dt = 0.$$

Thus $u = 0$ a.e. in $\Omega \times [0, \infty)$. □

6. PROOF OF THEOREM 2.5

Throughout this section we always assume that $\partial\Omega$ is of class C^3 . We now introduce two Lemmas that will be used in the proof of Theorem 2.5. Let us first observe that, under the assumptions of Theorem 2.5, the Green function $G(x, y)$ associated to the laplacian operator $-\Delta$ satisfies the following bound

$$G(x, y) \leq C \min \left\{ 1, \frac{d(x)d(y)}{|x-y|^2} \right\} |x-y|^{2-N}, \quad (6.1)$$

for some $C > 0$ and $d(x)$ as in (1.4). See [12], [31]; see also [3], [5].

Lemma 6.1. *Suppose that (6.1) holds and define*

$$\psi(x) := \int_\Omega G(x, y) d(y)^\beta \, dy, \quad (6.2)$$

for $\beta > -1$. Then there exist $c = c(\beta) > 0$ such that

$$0 \leq \psi(x) \leq c d(x) \quad \text{for every } x \in \Omega, \quad (6.3)$$

Proof. Let us fix $x \in \Omega$ such that $d(x) > 0$. Then, for any $y \in \Omega$

$$d(y) \geq 2|x-y|, \quad (6.4)$$

or

$$d(y) \leq 2|x-y|. \quad (6.5)$$

Therefore we write

$$\psi(x) = \int_{\{d(y) \geq 2|x-y|\}} G(x, y) d(y)^\beta \, dy + \int_{\{d(y) \leq 2|x-y|\}} G(x, y) d(y)^\beta \, dy$$

Moreover observe that, for any $z \in \partial\Omega$,

$$|y-z| \leq |x-z| + |y-x|.$$

If we fix $z \in \partial\Omega$ such that $d(x) = |x-z|$ then the latter can be rewritten as

$$|y-z| \leq d(x) + |y-x|. \quad (6.6)$$

Combining (6.4) and (6.6), it follows that

$$2|x-y| \leq d(y) \leq |y-z| \leq d(x) + |y-x| \implies |x-y| \leq d(x). \quad (6.7)$$

If $\beta > 0$ we have

$$\int_\Omega G(x, y) d(y)^\beta \, dy \leq (\text{diam } \Omega)^\beta \int_\Omega G(x, y) \, dy,$$

thus w.l.o.g. we can consider only the case when $-1 < \beta \leq 0$. Then, due to (6.1), (6.4) and (6.7)

$$\begin{aligned}
0 &\leq \int_{\{d(y) \geq 2|x-y|\}} G(x, y) d(y)^\beta dy \\
&\leq c \int_{\{d(y) \geq 2|x-y|\}} \frac{d(y)^\beta}{|x-y|^{N-2}} dy \\
&\leq c \int_{\{d(y) \geq 2|x-y|\}} \frac{d(x)d(y)^\beta}{|x-y|^{N-1}} dy \\
&\leq c \int_{\{d(y) \geq 2|x-y|\}} \frac{d(x)}{|x-y|^{N-1-\beta}} dy.
\end{aligned}$$

Now, since $-1 < \beta \leq 0$

$$c \int_{\{d(y) \geq 2|x-y|\}} \frac{d(x)}{|x-y|^{N-1-\beta}} dy \leq c d(x) \int_{B_R(x)} \frac{1}{|x-y|^{N-1-\beta}} dy \leq c d(x), \quad (6.8)$$

where $R := \text{diam}(\Omega)$. Similarly, due to (6.1) and (6.5)

$$\begin{aligned}
0 &\leq \int_{\{d(y) \leq 2|x-y|\}} G(x, y) d(y)^\beta dy \\
&\leq c \int_{\{d(y) \leq 2|x-y|\}} \frac{d(x)d(y)^{1+\beta}}{|x-y|^N} dy \\
&\leq c \int_{\{d(y) \leq 2|x-y|\}} \frac{d(x)}{|x-y|^{N-(1+\beta)}} dy \\
&\leq c d(x) \int_{B_R(x)} \frac{1}{|x-y|^{N-(1+\beta)}} dy \\
&\leq c d(x)
\end{aligned} \quad (6.9)$$

Finally, due to (6.8) and (6.9), for any $x \in \Omega$, there exists $c = c(\beta)$ such that

$$0 \leq \psi(x) \leq c d(x).$$

□

Lemma 6.2. *Suppose that (6.1) holds. Let us recall the definition of ψ in (6.2) and suppose that*

$$\beta > -2. \quad (6.10)$$

Then there exist $M > 0$ such that

$$0 \leq \psi(x) \leq M \quad \text{for any } x \in \Omega, \quad (6.11)$$

Proof. By Lemma 6.1 we only need to consider the case $-2 < \beta \leq -1$. For every $\varepsilon > 0$ small enough, let Ω_ε be defined as in (2.17). Moreover let $G_\varepsilon(x, y)$ be the Green function associated to the operator $-\Delta$ for $x, y \in \Omega_\varepsilon$. For every $\varepsilon > 0$, let

$$u_\varepsilon(x) := \int_{\Omega_\varepsilon} G_\varepsilon(x, y) d(y)^\beta dy. \quad (6.12)$$

Observe that, for every $\varepsilon > 0$, $u_\varepsilon \in C^\infty(\text{Int}(\Omega_\varepsilon)) \cap C^0(\Omega_\varepsilon)$, $u_\varepsilon > 0$ in $\text{Int}(\Omega_\varepsilon)$ and it solves the following problem

$$\begin{cases} -\Delta u_\varepsilon(x) = d(x)^\beta & \text{in } \text{Int}(\Omega_\varepsilon) \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Moreover, due to assumption (6.10), see [22], there exists $v : \bar{\Omega} \rightarrow \mathbb{R}$, $v \in C^0(\bar{\Omega})$, $v > 0$ in Ω such that v is a solution to problem

$$\begin{cases} -\Delta v(x) = d(x)^\beta & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that, due to the maximum principle, it follows that

$$0 < u_\varepsilon < v \quad \text{in } \text{Int}(\Omega_\varepsilon) \quad \text{for any } \varepsilon > 0. \quad (6.13)$$

Moreover, for $0 < \varepsilon_1 < \varepsilon_2$ one has

$$u_{\varepsilon_2}(x) \leq u_{\varepsilon_1}(x) \quad \text{for any } x \in \Omega_{\varepsilon_2}. \quad (6.14)$$

Hence, the family of functions $\{u_\varepsilon\}_{\varepsilon>0}$, due to (6.13) and (6.14), admits a finite limit for $\varepsilon \rightarrow 0$, in particular we write

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x) = w(x) \quad \text{for any } x \in \Omega, \quad (6.15)$$

and $0 < w(x) \leq v(x)$ for any $x \in \Omega$. Now observe that

$$G_\varepsilon(x, y) \nearrow G(x, y) \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for any } x, y \in \Omega.$$

It follows by the Monotone Convergence Theorem that for any $\varepsilon > 0$ one has

$$u_\varepsilon(x) = \int_{\Omega} G_\varepsilon(x, y) d(y)^\beta dy \longrightarrow \int_{\Omega} G(x, y) d(y)^\beta dy \quad \text{as } \varepsilon \rightarrow 0. \quad (6.16)$$

Hence, due to (6.13), (6.15) and (6.16), for any $x \in \Omega$ we can write

$$w(x) = \int_{\Omega} G(x, y) d(y)^\beta dy, \quad \text{and} \quad 0 \leq \int_{\Omega} G(x, y) d(y)^\beta dy \leq v(x).$$

Finally, since $v \in C^0(\bar{\Omega})$, there exists $M > 0$ such that

$$0 \leq \int_{\Omega} G(x, y) d(y)^\beta dy \leq M.$$

□

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. We want to construct a subsolution and a supersolution to problem (1.3) which will be denoted by \underline{u} and \bar{u} respectively. We set

$$\underline{u} \equiv 0.$$

On the other hand, in order to construct \bar{u} , let us define, for any $\lambda > 0$

$$S_\lambda = \{v \in C^0(\bar{\Omega}) : 0 \leq v(x) \leq \lambda d(x), \forall x \in \Omega\}. \quad (6.17)$$

with $d(x)$ as in (1.4). Moreover we define the map $T : S_\lambda \rightarrow S_\lambda$

$$Tv(x) = \lambda^q \int_{\Omega} G(x, y) dy + \int_{\Omega} G(x, y) V(y) v(y)^q dy. \quad (6.18)$$

We prove that T is well defined and that it is a contraction map for $\lambda > 0$ small enough. Observe that, by Lemma 6.1 with $\beta = 0$, one has for some $c_1 > 0$

$$0 \leq \lambda^q \int_{\Omega} G(x, y) dy \leq c_1 \lambda^q d(x), \quad \text{for every } x \in \Omega. \quad (6.19)$$

Similarly, due to (2.15), Lemma 6.1 with $\beta = -\sigma_1 + q$ and (2.16), for some $c_2 > 0$

$$0 \leq \int_{\Omega} G(x, y) V(y) v(y)^q dy \leq c \lambda^q \int_{\Omega} G(x, y) d(y)^{-\sigma_1+q} dy \leq c_2 \lambda^q d(x). \quad (6.20)$$

By using (6.19) and (6.20), (6.18) yields for some $C > 0$ and $\lambda > 0$ small enough

$$0 \leq Tv(x) \leq C\lambda^q d(x) \leq \lambda d(x) \quad \text{for any } x \in \Omega.$$

Hence, for a sufficiently small $\lambda > 0$, the function $Tv : \bar{\Omega} \rightarrow \mathbb{R}$ is continuous and thus the map $T : S_\lambda \rightarrow S_\lambda$ is well defined. Let us now show that T is a contraction map, for $\lambda > 0$ small enough. Fix $w, v \in S_\lambda$, then for any $x \in \Omega$

$$\begin{aligned} |Tw(x) - Tv(x)| &\leq \int_{\Omega} G(x, y)V(y)|w^q(y) - v^q(y)| dy \\ &\leq \int_{\Omega} G(x, y)V(y)q\xi(y)^{q-1}|w(y) - v(y)| dy, \end{aligned}$$

for some $\xi(y)$ between $w(y)$ and $v(y)$. Then $0 \leq \xi(y) \leq \lambda d(y)$ and hence, due to Lemma 6.2 with $\beta = -\sigma_1 + q - 1$ and (2.16),

$$\begin{aligned} |Tw(x) - Tv(x)| &\leq C \left(\int_{\Omega} G(x, y)d(y)^{-\sigma_1+q-1} dy \right) \lambda^{q-1} \|w - v\|_{L^\infty(\Omega)} \\ &\leq CM \lambda^{q-1} \|w - v\|_{L^\infty(\Omega)}. \end{aligned}$$

Thus we have, for $\lambda > 0$ small enough,

$$\|Tw - Tv\|_{L^\infty(\Omega)} \leq \frac{1}{2} \|w - v\|_{L^\infty(\Omega)},$$

hence T is a contraction map. Therefore, there exists $\varphi \in S_\lambda$ such that $\varphi = T\varphi$. In particular, we have

- (i) $0 \leq \varphi(x) \leq \lambda d(x)$ for any $x \in \bar{\Omega}$;
- (ii) φ is a solution of

$$\begin{cases} -\Delta\varphi = \lambda^q + V\varphi^q & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

- (iii) $\varphi > 0$ in Ω .

We now set $\bar{u}(x, t) = \varphi(x)$ and show that \bar{u} is a supersolution to problem (1.3). Observe that

- (i) $\partial_t \bar{u} - \Delta \bar{u} = -\Delta \varphi = \lambda^q + V\varphi^q \geq V\bar{u}^q$ in $\Omega \times (0, +\infty)$;
- (ii) $\bar{u}(x, t) = \varphi(x) = 0$ for any $x \in \partial\Omega, t \in (0, +\infty)$;
- (iii) $\bar{u} > 0$ in $\Omega \times (0, +\infty)$;
- (iv) $0 \leq u_0(x) \leq \bar{u}(x, 0)$ for any $x \in \Omega$, if ε is small enough; indeed we can apply the Hopf's Lemma and if n denotes the inward normal unit vector to $\partial\Omega$ deduce that

$$\frac{\partial \varphi}{\partial n}(x) > 0, \quad \text{for any } x \in \partial\Omega.$$

Then, due to the compactness of $\bar{\Omega}$ and the continuity of φ in Ω , we observe that there exists $\alpha > 0$ such that

$$\varphi(x) \geq \alpha d(x) \quad \text{for any } x \in \bar{\Omega}.$$

Now, if $\varepsilon > 0$ in (2.14) is sufficiently small, we have that

$$0 \leq u_0(x) \leq \varepsilon d(x) \leq \alpha d(x) \leq \varphi(x) = \bar{u}(x, 0) \quad \text{for any } x \in \bar{\Omega}.$$

Thus $\bar{u} : \bar{\Omega} \times [0, +\infty) \rightarrow \mathbb{R}$ is a supersolution to problem (1.3), such that $\bar{u} \geq \underline{u}$ in $\bar{\Omega} \times [0, +\infty)$. Finally, we conclude that there exists a solution $u : \Omega \times [0, +\infty) \rightarrow \mathbb{R}$ of problem (1.3) such that

$$0 \leq u(x) \leq \bar{u}(x) \quad \text{for any } x \in \bar{\Omega}.$$

□

7. PROOF OF THEOREM 2.6 AND OF COROLLARY 2.7

We introduce some auxiliary Lemmas that are needed in the proof of Theorem 2.6.

Lemma 7.1. *Let $V \in L^1_{loc}(\Omega)$, with $V(x) > 0$ a.e., and assume that the initial condition satisfies $u_0 \in L^1_{loc}(\Omega)$, with $u_0 \geq 0$ a.e. Let $u \geq 0$ be a weak solution of problem (1.3). If $\alpha > \frac{2q}{q-1}$ and $\psi \in C^{2,1}_{x,t}(\Omega \times [0, T])$, $\psi \geq 0$ a.e. in $\Omega \times [0, T]$ with compact support in $\Omega \times [0, T]$, then*

$$\begin{aligned} \int_0^T \int_{\Omega} u^q V \psi^\alpha \, dxdt &\leq 2^{\frac{1}{q-1}} \left\{ \int_0^T \int_{\Omega} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha(\alpha-1)| |\nabla \psi|^2 + \alpha \psi \Delta \psi \Big|_{\frac{q-1}{q}} \, dxdt \right. \\ &\quad \left. + \int_0^T \int_{\Omega} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha \psi \psi_t|_{\frac{q-1}{q}} \, dxdt \right\}. \end{aligned} \quad (7.1)$$

Proof. Using the definition of weak solution to problem (1.3) and Young inequality with coefficients q and $\frac{q}{q-1}$ we have

$$\begin{aligned} \int_0^T \int_{\Omega} u^q V \psi^\alpha \, dxdt &\leq \int_0^T \int_{\Omega} u |(\psi^\alpha)_t + \Delta(\psi^\alpha)| \, dxdt - \int_{\Omega} u_0(x) \psi^\alpha(x, 0) \, dx \\ &\leq \frac{1}{q} \int_0^T \int_{\Omega} u^q V \psi^\alpha \, dxdt + \frac{q-1}{q} \int_0^T \int_{\Omega} (V \psi^\alpha)^{-\frac{1}{q-1}} |(\psi^\alpha)_t + \Delta(\psi^\alpha)|_{\frac{q-1}{q}} \, dxdt \end{aligned}$$

Reordering terms we get

$$\begin{aligned} \int_0^T \int_{\Omega} u^q V \psi^\alpha \, dxdt &\leq \int_0^T \int_{\Omega} (V \psi^\alpha)^{-\frac{1}{q-1}} |\alpha \psi^{\alpha-1} \psi_t + \alpha(\alpha-1) \psi^{\alpha-2} |\nabla \psi|^2 + \alpha \psi^{\alpha-1} \Delta \psi|_{\frac{q-1}{q}} \, dxdt \\ &\leq \int_0^T \int_{\Omega} V^{-\frac{1}{q-1}} \psi^{-\frac{\alpha}{q-1} + \frac{q(\alpha-2)}{q-1}} |\alpha \psi \psi_t + \alpha(\alpha-1) |\nabla \psi|^2 + \alpha \psi \Delta \psi|_{\frac{q-1}{q}} \, dxdt \\ &\leq 2^{\frac{1}{q-1}} \left\{ \int_0^T \int_{\Omega} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha(\alpha-1)| |\nabla \psi|^2 + \alpha \psi \Delta \psi \Big|_{\frac{q-1}{q}} \, dxdt \right. \\ &\quad \left. + \int_0^T \int_{\Omega} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha \psi \psi_t|_{\frac{q-1}{q}} \, dxdt \right\} \end{aligned}$$

This proves the thesis. \square

Lemma 7.2. *Let the assumptions of Lemma 7.1 hold. Moreover let $K \subset \Omega \times [0, T]$ be a compact set and let ψ be such that $\psi \equiv 1$ in K . Let $S_k := (\Omega \times [0, T]) \setminus K$ then*

$$\begin{aligned} \int_0^T \int_{\Omega} u^q V \psi^\alpha \, dxdt &\leq 2^{\frac{1}{q}} \left(\int \int_{S_k} u^q V \psi^\alpha \, dxdt \right)^{\frac{1}{q}} \\ &\quad \times \left\{ \left[\int \int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha(\alpha-1)| |\nabla \psi|^2 + \alpha \psi \Delta \psi \Big|_{\frac{q-1}{q}} \, dxdt \right]^{\frac{q-1}{q}} \right. \\ &\quad \left. + \left[\int \int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha \psi \psi_t|_{\frac{q-1}{q}} \, dxdt \right]^{\frac{q-1}{q}} \right\}. \end{aligned} \quad (7.2)$$

Proof. Similarly to the proof of Lemma 7.1, using the definition of weak solution of problem (1.3) and Hölder inequality with coefficients q and $\frac{q}{q-1}$ we get

$$\begin{aligned}
\int_0^T \int_{\Omega} u^q V \psi^\alpha dx dt &\leq \left(\int \int_{S_K} u^q V \psi^\alpha dx dt \right)^{\frac{1}{q}} \left(\int \int_{S_K} V^{-\frac{1}{q-1}} \psi^{-\frac{\alpha}{q-1}} |(\psi^\alpha)_t + \Delta(\psi^\alpha)|^{\frac{q}{q-1}} dx dt \right)^{\frac{q-1}{q}} \\
&= \left(\int \int_{S_k} u^q V \psi^\alpha dx dt \right)^{\frac{1}{q}} \\
&\times \left(\int \int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha \psi \psi_t + \alpha(\alpha-1)|\nabla \psi|^2 + \alpha \psi \Delta \psi|^{\frac{q}{q-1}} dx dt \right)^{\frac{q-1}{q}} \\
&\leq 2^{\frac{1}{q}} \left(\int \int_{S_k} u^q V \psi^\alpha dx dt \right)^{\frac{1}{q}} \\
&\times \left\{ \left[\int \int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha(\alpha-1)|\nabla \psi|^2 + \alpha \psi \Delta \psi|^{\frac{q}{q-1}} dx dt \right]^{\frac{q-1}{q}} \right. \\
&\left. + \left[\int \int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha \psi \psi_t|^{\frac{q}{q-1}} dx dt \right]^{\frac{q-1}{q}} \right\}
\end{aligned}$$

This proves the thesis. \square

We now need to introduce the so called *Whitney distance* $\delta : \Omega \rightarrow \mathbb{R}^+$, which is a function in $C^\infty(\Omega)$, regardless of the regularity of $\partial\Omega$, such that for all $x \in \Omega$

$$\begin{aligned}
c_0^{-1} d(x) &\leq \delta(x) \leq c_0 d(x), \\
|\nabla \delta(x)| &\leq c_0, \\
|\Delta \delta(x)| &\leq c_0 \delta^{-1}(x),
\end{aligned} \tag{7.3}$$

where $d(x)$ has been defined in (1.4) and $c_0 > 0$ is a constant. These properties of the Whitney distance can be found, e.g., in [2, 25].

Lemma 7.3. *Let $V \in L^1_{loc}(\Omega \times [0, \infty))$, $V > 0$ a.e., and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. Assume that there exists a nonincreasing function $f : (0, \varepsilon_0) \rightarrow [1, \infty)$ such that $\lim_{\varepsilon \rightarrow 0^+} f(\varepsilon) = +\infty$ and such that for every $\varepsilon > 0$ small enough conditions (2.18) hold. Let $u \geq 0$ be a weak solution of problem (1.3), then*

$$\int_0^{+\infty} \int_{\Omega} u^q V dx dt < +\infty \tag{7.4}$$

Proof. For every $\varepsilon > 0$ small enough, we consider a smooth function $g_\varepsilon : [0, \infty) \rightarrow \mathbb{R}$ such that $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon \equiv 1$ in $[\varepsilon, +\infty)$, $\text{supp } g_\varepsilon \subset [\frac{\varepsilon}{2}, +\infty)$, $0 \leq g'_\varepsilon \leq \frac{C}{\varepsilon}$ and $|g''_\varepsilon| \leq \frac{C}{\varepsilon^2}$ for some constant $C > 0$. We also introduce η a smooth function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $[0, \frac{1}{2}f(\varepsilon)]$, $\text{supp } \eta \subset [0, f(\varepsilon))$ and $-\frac{C}{f(\varepsilon)} \leq \eta' \leq 0$. Now let

$$\psi_\varepsilon(x, t) := \phi_\varepsilon(x) \eta(t), \tag{7.5}$$

where

$$\phi_\varepsilon(x) := g_\varepsilon(\delta(x)) \tag{7.6}$$

where δ is the Whitney distance introduced in (7.3). Observe that, due to (7.5), (7.6) and (7.3) for every $x \in \Omega$, $t \in [0, T)$ we have

$$\begin{aligned} |\nabla \psi_\varepsilon(x, t)| &= |g'_\varepsilon(\delta(x))\eta(t)\nabla\delta(x)| \leq \frac{C}{\varepsilon}, \\ |\Delta \psi_\varepsilon(x, t)| &= |g''_\varepsilon(\delta(x))\eta(t)|\nabla\delta(x)|^2 + g'_\varepsilon(\delta(x))\eta(t)\Delta\delta(x)| \leq \frac{C}{\varepsilon^2}, \end{aligned} \quad (7.7)$$

for some constant $C > 0$. Hence for every $x \in \Omega$, $t \in [0, T)$ we have

$$|(\psi_\varepsilon)_t| \leq \frac{C}{f(\varepsilon)}, \quad |\alpha(\alpha - 1)|\nabla\psi_\varepsilon|^2 + \alpha\psi_\varepsilon\Delta\psi_\varepsilon|^{\frac{q}{q-1}} \leq \frac{C}{\varepsilon^{\frac{2q}{q-1}}}. \quad (7.8)$$

Let $\tilde{\Omega}_\varepsilon = \{x \in \Omega \mid \delta(x) \geq \varepsilon\}$ and note that by (7.3) for every $r > 0$ we have

$$\tilde{\Omega}_r \subset \Omega_{\frac{r}{c_0}}, \quad \Omega_r \subset \tilde{\Omega}_{\frac{r}{c_0}}.$$

We now observe, applying Lemma 7.1 with the test function ψ_ε defined in (7.5), that

$$\begin{aligned} \int_0^{\frac{1}{2}f(\varepsilon)} \int_{\tilde{\Omega}_\varepsilon} u^q V \, dxdt &\leq \int_0^{+\infty} \int_{\Omega} u^q \psi_\varepsilon^\alpha V \, dxdt \\ &\leq C \left\{ \int_0^{+\infty} \int_{\Omega} V^{-\frac{1}{q-1}} \psi_\varepsilon^{\alpha - \frac{2q}{q-1}} |\alpha(\alpha - 1)|\nabla\psi_\varepsilon|^2 + \alpha\psi_\varepsilon\Delta\psi_\varepsilon|^{\frac{q}{q-1}} \, dxdt \right. \\ &\quad \left. + \int_0^{+\infty} \int_{\Omega} V^{-\frac{1}{q-1}} \psi_\varepsilon^{\alpha - \frac{2q}{q-1}} |\alpha\psi_\varepsilon(\psi_\varepsilon)_t|^{\frac{q}{q-1}} \, dxdt \right\} \\ &=: C(I_1 + I_2). \end{aligned} \quad (7.9)$$

Now, due to the definition of ψ_ε in (7.5) and by (2.18) and (7.8), for every small enough $\varepsilon > 0$ we have

$$\begin{aligned} I_1 &\leq \int_0^{f(\varepsilon)} \int_{\tilde{\Omega}_{\frac{\varepsilon}{2}} \setminus \tilde{\Omega}_\varepsilon} V^{-\frac{1}{q-1}} [|\alpha(\alpha - 1)|\nabla\psi_\varepsilon|^2 + \alpha\psi_\varepsilon\Delta\psi_\varepsilon]^{\frac{q}{q-1}} \, dxdt \\ &\leq \frac{C}{\varepsilon^{\frac{2q}{q-1}}} \int_0^{f(\varepsilon)} \int_{\Omega_{\frac{2\varepsilon}{2c_0}} \setminus \Omega_{c_0\varepsilon}} V^{-\frac{1}{q-1}} \, dxdt \\ &\leq \frac{C}{\varepsilon^{\frac{2q}{q-1}}} \sum_{k=0}^N \int_0^{f(\varepsilon)} \int_{\Omega_{\frac{2^{k-1}\varepsilon}{c_0}} \setminus \Omega_{\frac{2^k\varepsilon}{c_0}}} V^{-\frac{1}{q-1}} \, dxdt \\ &\leq \frac{C}{\varepsilon^{\frac{2q}{q-1}}} \sum_{k=0}^N \left(\frac{2^k\varepsilon}{c_0} \right)^{\frac{2q}{q-1}} \leq C, \end{aligned} \quad (7.10)$$

where we set $N = [2 \log_2 c_0] + 1$. Similarly, due to (7.5) and by (2.18) and (7.8), we have

$$I_2 \leq \frac{C}{(f(\varepsilon))^{\frac{q}{q-1}}} \int_{\frac{1}{2}f(\varepsilon)}^f \int_{\Omega_{\frac{\varepsilon}{2}}} V^{-\frac{1}{q-1}} \, dxdt \leq C. \quad (7.11)$$

By substituting (7.10) and (7.11) into (7.9) and letting $\varepsilon \rightarrow 0$ we obtain the thesis. \square

We are now ready to prove Theorem 2.6.

Proof of Theorem 2.6. For small enough $\varepsilon > 0$ consider the test function ψ_ε defined in (7.5). Define

$$K_\varepsilon := \tilde{\Omega}_\varepsilon \times \left[0, \frac{1}{2}f(\varepsilon)\right]; \quad (7.12)$$

and

$$S_{K_\varepsilon} := (\Omega \times [0, +\infty)) \setminus K_\varepsilon. \quad (7.13)$$

Observe that $\psi_\varepsilon \equiv 1$ on K_ε , hence we can apply Lemma 7.2 with the test function ψ_ε and we have

$$\begin{aligned} \int_0^{\frac{1}{2}f(\varepsilon)} \int_{\tilde{\Omega}_\varepsilon} u^q V \, dxdt &\leq \int_0^{+\infty} \int_{\Omega} u^q \psi_\varepsilon^\alpha V \, dxdt \\ &\leq C \left(\int \int_{S_{K_\varepsilon}} u^q V \psi_\varepsilon^\alpha \, dxdt \right)^{\frac{1}{q}} \\ &\times \left\{ \left[\int \int_{S_{K_\varepsilon}} V^{-\frac{1}{q-1}} \psi_\varepsilon^{\alpha - \frac{2q}{q-1}} |\alpha(\alpha-1)| |\nabla \psi_\varepsilon|^2 + \alpha \psi_\varepsilon \Delta \psi_\varepsilon \Big| \frac{q}{q-1} \, dxdt \right]^{\frac{q-1}{q}} \right. \\ &\left. + \left[\int \int_{S_{K_\varepsilon}} V^{-\frac{1}{q-1}} \psi_\varepsilon^{\alpha - \frac{2q}{q-1}} |\alpha \psi_\varepsilon(\psi_\varepsilon)_t| \frac{q}{q-1} \, dxdt \right]^{\frac{q-1}{q}} \right\} \\ &=: C(I_1 + I_2) \left(\int \int_{S_{K_\varepsilon}} u^q V \psi_\varepsilon^\alpha \, dxdt \right)^{\frac{1}{q}}. \end{aligned} \quad (7.14)$$

Now we can argue as in Lemma 7.3 and prove that there exists $C > 0$ such that

$$I_1 \leq C, \quad I_2 \leq C.$$

Thus we have

$$\int_0^{\frac{1}{2}f(\varepsilon)} \int_{\tilde{\Omega}_\varepsilon} u^q V \, dxdt \leq C \left(\int \int_{S_{K_\varepsilon}} u^q V \, dxdt \right)^{\frac{1}{q}}.$$

Letting $\varepsilon \rightarrow 0$ by Lemma 7.3 we obtain

$$\int_0^{+\infty} \int_{\Omega} u^q V \, dxdt = 0, \quad (7.15)$$

which proves the thesis. \square

Proof of Corollary 2.7. By (2.19) and the assumptions on f , for $\varepsilon > 0$ small enough we have

$$\begin{aligned} \int_0^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} V^{-\frac{1}{q-1}} \, dxdt &\leq C f(\varepsilon) \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} d(x)^{\frac{q+1}{q-1}} f(d(x))^{-1} \, dx \\ &\leq C \varepsilon^{\frac{q+1}{q-1}} \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} dx \leq C \varepsilon^{\frac{2q}{q-1}} \end{aligned}$$

and

$$\begin{aligned} \int_{\frac{1}{2}f(\varepsilon)}^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}}} V^{-\frac{1}{q-1}} \, dxdt &\leq C f(\varepsilon) \int_{\Omega_{\frac{\varepsilon}{2}}} d(x)^{\frac{q+1}{q-1}} f(d(x))^{-1} \, dx \\ &\leq C f(\varepsilon) \leq C f(\varepsilon)^{\frac{q}{q-1}}. \end{aligned}$$

Thus conditions (2.18) are satisfied and by Theorem 2.6 $u \equiv 0$ a.e. in $\Omega \times [0, \infty)$. \square

Acknowledgements. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). D.D.M. and F.P. are partially supported by 2020 GNAMPA project "Equazioni Ellittiche e Paraboliche ed Analisi Geometrica". F.P. is supported by the PRIN-201758MTR2 project "Direct and inverse problems for partial differential equations: theoretical aspects and applications."

REFERENCES

- [1] C. Bandle, M.A. Pozio, A. Tesi, "The Fujita exponent for the Cauchy problem in the hyperbolic space". *J. Differ. Eq.*, **251** (2011) 2143–2163.
- [2] C. Bandle, V. Moroz, W. Reichel, "Large solutions to semilinear elliptic equations with Hardy potential and exponential nonlinearity". *Around the Research of Vladimir Maz'ya II. International Mathematical Series, vol. 12* Springer New York (2010).
- [3] E. B. Davies, "The equivalence of certain heat kernel and Green function bounds". *Journal of Functional Analysis*, **71** (1987), 88–103.
- [4] L. D'Ambrosio, V. Mitidieri, "A priori estimates, positivity results and nonexistence theorems for quasilinear degenerate elliptic inequalities". *Adv. Math.*, **224** (2010), 967–1020.
- [5] S. Filippas, L. Moschini, A. Tertikas, "Sharp two-sided kernel estimates for critical Schrödinger operators on bounded domains". *Communications in Mathematical Physics*, **273** (2007), 237–281.
- [6] R. L. Foote, "Regularity of the distance function". *Proceedings of the America Mathematical Society*, **92** (1984), 153–155.
- [7] V.A. Galaktionov, "Conditions for the absence of global solutions for a class of quasilinear parabolic equations". *Zh. Vychisl. Mat. i Mat. Fiz.*, **22** (1982), 322–338.
- [8] V.A. Galaktionov, "Blow-up for quasilinear heat equations with critical Fujita's exponents". *Proc. R. Soc. Edinb. Sect. A*, **124** (1994), 517–525.
- [9] V.A. Galaktionov, H-A- Levine "A general approach to critical Fujita exponents in nonlinear parabolic problems". *Nonlinear Anal.*, **34** (1998), 1005–1027.
- [10] Grigor'yan, A., Kondratiev, V.A, "On the existence of positive solutions of semilinear elliptic inequalities on Riemannian manifolds". In: *Around the Research of Vladimir Maz'ya. II. Int. Math. Ser. (N.Y.)*, **12** (2010) 203–218. Springer, New York.
- [11] Grigor'yan, A., Sun, Y., "On non-negative solutions of the inequality $\Delta u + u\sigma \leq 0$ on Riemannian manifolds". *Commun. Pure Appl. Math.* **67**, (2014) 1336–1352.
- [12] H. Hueber, M. Sieveking, "Uniform bounds for quotients of Green functions in $C^{1,1}$ domains". *Ann. Inst. Fourier (Grenoble)*, **32** (1982), 105–117.
- [13] S. G. Krantz, H. R. Parks, "Distance to C^k Hypersurfaces". *Journal of Differential Equations*, **40** (1981), 116–120.
- [14] P. Mastrolia, D.D. Monticelli, F. Punzo, "Non existence results for elliptic differential inequalities with a potential on Riemannian manifolds". *Calc. Var. PDE*, **54** (2015) 1345–1372.
- [15] P. Mastrolia, D.D. Monticelli, F. Punzo, "Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds". *Mathematische Annalen*, **367** (2017), 929–963.
- [16] E. Mitidieri, S.I. Pohozaev, "Absence of global positive solutions of quasilinear elliptic inequalities". *Dokl. Akad. Nauk.*, **359** (1998), 456–460.
- [17] E. Mitidieri, S.I. Pohozaev, "Nonexistence of positive solutions for quasilinear elliptic problems in \mathbb{R}^N ". *Tr. Mat. Inst. Steklova*, **227** (1999), 192–222.
- [18] E. Mitidieri, S.I. Pohozaev, "A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities". *Tr. Mat. Inst. Steklova*, **234** (2001), 1–384.
- [19] E. Mitidieri, S.I. Pohozaev, "Towards a unified approach to nonexistence of solutions for a class of differential inequalities". *Milan J. Math.*, **72** (2004), 129–162.
- [20] D.D. Monticelli, F. Punzo, "Nonexistence results to elliptic differential inequalities with a potential in bounded domains". *Discrete and Continuous Dynamical System, Vol. 38* No. 2 (2018) 675–695.

- [21] S.I. Pohozaev, A. Tesi “Nonexistence of local solutions to semilinear partial differential inequalities”. Ann. Inst. H. Poincaré Anal. Non Linéaire, **21** (2004), 487–502.
- [22] M.A. Pozio, F. Punzo, A. Tesi, “Criteria for well-posedness of degenerate elliptic and parabolic problems”. Journal of Mathématiques Pures et Appliquées, 2008.
- [23] F. Punzo, “Blow-up of solutions to semilinear parabolic equations on Riemannian manifolds with negative sectional curvature”. J. Math. Anal. Appl, **387** (2012), 815–827.
- [24] F. Punzo, A. Tesi, “On a semilinear parabolic equation with inverse-square potential”. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **21**, (2010), 359–396.
- [25] E.M. Stein, “Singular integrals and differentiability properties of functions”. Princeton University Press, (1970).
- [26] Y. Sun, “Uniqueness results for nonnegative solutions to semilinear inequalities on Riemannian manifolds”. J. Math. Anal. Appl., **419** (2014) 646–661.
- [27] Y. Sun, “On nonexistence of positive solutions of quasilinear inequality on Riemannian manifolds”. Proc. Amer. Math. Soc., **143** (2015) 2969–2984.
- [28] Q. Gu, Y. Sun, J. Xiao, F. Xu, “Global positive solution to a semi-linear parabolic equation with potential on Riemannian manifold”. Calc. Var., **59** (2020) 170.
- [29] Q. S. Zhang, “A new critical phenomenon for semilinear parabolic problems”. Journal of Mathematical Analysis and Applications, **219** (1998), 125–139.
- [30] Q. S. Zhang, “Blow-up results for nonlinear parabolic equations on manifolds”. Duke Mathematical Journal, Vol. 97, No. 3 (1999), 515–539.
- [31] Z. Zhao, “Green function for Schrodinger operator and conditioned Feymann-Kac Guage”. Journal of Mathematical Analysis and Applications, **116** (1986), 309–334.

(G. Meglioli) DIPARTIMENTO DI MATEMATICA
POLITECNICO DI MILANO, MILANO, ITALY
Email address: `giulia.meglioli@polimi.it`

(D.D. Monticelli) DIPARTIMENTO DI MATEMATICA
POLITECNICO DI MILANO, MILANO, ITALY
Email address: `dario.monticelli@polimi.it`

(F. Punzo) DIPARTIMENTO DI MATEMATICA
POLITECNICO DI MILANO, MILANO, ITALY
Email address: `fabio.punzo@polimi.it`