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ORIGINAL RESEARCH

An upper bound for the inter-exit time of two jobs in an *m*-machine flow shop

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Abstract
This paper addresses the class of permutation flow shop scheduling where jobs, after their completion, must be grouped in batches. This is a common scheme in industrial environments, where products undergo multiple process steps in different shops and, when completed, must be transported to customers or the next production step. A new optimisation criterion is used, the *inter-exit time*, i.e., the difference between the completion times of two jobs. An upper bound is proposed and demonstrated for a general permutation flow shop with m machines.

Keywords Flow shop · Scheduling · Inter-exit time

1 Introduction

Flow shops are probably the most common architecture for manufacturing systems and, consequently, one of the most addressed scheduling problems (Emmons & Vairaktarakis, 2013). In a flow shop system, the processing of a product is usually decomposed into a set of steps, providing a way to structure the manufacturing process through a limited number of indivisible operations to be planned and scheduled. Each operation has specific requirements for production equipment, skills of the operators, raw materials or work-in-progress, and consumables, and also entails a processing time (deterministic or stochastic) to be accomplished. In flow shops, operations are executed according to a given sequence that is the same for all the products (jobs) to be processed. Thus, the processing of a job goes through a sequence of stages operated by specific machines (or machine groups), and all the jobs visit machines in the same given order.

Different jobs in the same stage can have different processing times when multiple part types are processed. Consequently, the sequence of jobs being processed could affect the value of time-based performance indicators, e.g., makespan or job completion time. In this context, the scheduling of both the jobs and the single operations has been widely studied by covering different classes of flow shops scheduling problems (Pinedo, 2008). A specific

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subclass of the general flow shop scheduling problem is the *permutation flow shop* (Emmons & Vairaktarakis, 2013), where the sequencing of the jobs is the same for all the machines.

In this work, we address a scheduling problem within the class of permutation flow shops where the jobs, after their completion, must be grouped in batches. This is a very common scheme in manufacturing, where the products to be processed undergo various process steps in different shops. In these cases, a number of parts are grouped at the end of the flow shop before proceeding to the following production step. While the batch is being completed, i.e., from the time the first part is ready until the arrival of the last part, some resources could be kept idle. An example is a case where multiple parts must be loaded on a transporter (e.g., a truck or a ship) as soon as they are processed in the flow shop. In this scenario, the transporter has to wait until the loading is completed before operating the transport. This is particularly relevant for large parts that cannot be laid at the end of the shop due to the lack of space or dedicated equipment, e.g., a fixture.

To address this class of problems, a different measure of performance is investigated, i.e., the inter-exit time, measuring the difference between the completion time of two jobs in a schedule. Thus, given a schedule in a permutation flow shop, and a pair of jobs in positions h and k with $h < k$, the inter-exit time between these jobs, $C_{[k]} - C_{[h]}$, has to be estimated.

This manuscript proposes an upper bound for the inter-exit time of any two jobs in a m-machine flow shop. The upper bound can be estimated by considering the characteristics of the jobs sequenced in the positions from h to k, without the need for any further information and hypothesis on the remaining jobs in the schedule. The proposed upper bound can support the development of optimisation methods for scheduling approaches requiring jobs to be grouped in batches. E.g., considering the same m-machine flow shop scheduling problem but involving multiple batches to be processed, with the aim to minimize a function of the batch completion time (i.e., the total/average batch completion time, or a measure of their variability, e.g., the variance). In these cases, two nested problems arise, first forming the batches (e.g., deciding the jobs that will stay in the same batch) and then sequencing the processing of these batches in the flow shop.

The availability of an upper bound for the inter-exit time can be easily coupled with a lower bound for the same measure of performance. In fact, a lower bound for the inter-exit time can be easily obtained by just considering the sum of the processing times of the jobs (excluding the first one) on the last machine of the flow shop. The availability of both an upper and lower bound can support more advanced scheduling approaches, e.g., branch-and-bound algorithms.

The structure of the paper is the following: an analysis of the scheduling literature for this class of flow shops is addressed in Sects. 2, 3 provides a formal model for the considered scheduling problem, Sect. 4 formulates the theorem for the proposed upper bound and provide its demonstration. Sections 5 ans 6 reports a preliminary analysis on the tighthness of the proposed bound and analyse its relevance to support decision approaches. Finally, Sect. 7 presents conclusions and future directions.

2 State of the art

The flow shop scheduling problem has been extensively studied in the literature, leading to a wide range of contributions related to specific system features (e.g., hybrid, flexible, etc.) and optimization criteria (makespan, total completion time, maximum lateness, etc.). Most of these criteria are regular performance measures, i.e., functions that are nondecreasing in the completion times $C_1, \ldots, C_n$ of the jobs to schedule (Pinedo, 2008).

In this manuscript, the optimisation of a different criterion is addressed, i.e., the inter-exit time, defined as the time between the completion time of the two jobs in a sequence (Angius et al., 2016). This criterion, being the difference between two completion times, is not regular (Emmons & Vairaktarakis, 2013) and, according to our knowledge, has not been addressed in the literature. Its relevance derives from manufacturing applications, where parts to be processed are grouped in batches before proceeding to subsequent manufacturing processes. Results in this field are scheduling approaches addressing the *interstage batch delivery* problem (Agnetis et al., 2014). In this problem, after being processed, jobs must be grouped in a batch and transported to the next production stage or delivered to the customer. Therefore, while scheduling the processing of jobs, production (sequencing) and distribution decisions (batching) have to be jointly considered. This class of problems was first defined as the *IPODS - integrated production and outbound distribution scheduling* (Potts, 1980), and the related literature analysed (Chen, 2010). In this class of problems, optimization criteria usually consider the point of view of the customers receiving the delivery of product batches after the production phase. Examples are the minimization of the sum of arrival times of the batches after the delivery (Wan & Zhang, 2014), or the maximum delivery completion time (Ng & Lu, 2012). The criterion addressed in this paper takes a different point of view that has not been considered in the literature related to the *IPODS* problem.

A related scheduling problem is the one addressing lot streaming, where a lot of jobs have to be processed in a flow shop. In such cases, a lot can proceed to the downstream machine in the flow shop only when all the jobs in the lot have been processed. Lot streaming approaches (Emmons & Vairaktarakis, 2013) aim to split a lot into sub-lots to reduce the time a given machine has to wait before the upstream machine in the flow shop has completed the lot processing. The main objective is to allow overlapping for the processing of sub-lots to accelerate progress and reduce the makespan. Lot streaming approaches typically refer to scheduling problems where the jobs in a lot are identical and different theoretical results and approaches exist with respect to the dimension of the sub-lots, no-wait constraint, idle times, etc. (Sarin & Jaiprakash, 2007).

Contributions in this area linked to the problem investigated have been addressing the sequencing of parts of different types in lots (Agnetis, 1997), in a flow shop in which parts are not allowed to wait (*no-wait lot streaming*), to minimize the total completion time. Lot streaming with different types of jobs has also been considered (Hall et al., 2003; Kumar et al., 2000) where the main objective is coping with setups, e.g., modelling this issue as a *Travel Salesman Problem* to minimize the makespan (Kumar et al., 2000).

Referring to objective functions considering a weighted sum of criteria, the single batch, flow shop, lot-streaming problem has been proposed (Kalir & Sarin, 2001) to minimize, among different objective functions considered, the weighted combination of measures involving, for example, makespan, mean flow time, average work-in-process, setup, and handling cost. Also, the minimization of the sum of the weighted sub-lot completion times has been addressed, for an m-machine single-lot flow shop scheduling problem (Topaloglu et al., 1994). The proposed approach applies to the weighted sum of the completion time of two jobs, but it requires that the sum of the weights is equal to one (Sarin & Jaiprakash, 2007). All the listed approaches consider regular objective functions (e.g., the total completion time or the makespan) or, when the weighted sum of completion times is considered, this is aimed at defining a multi-objective function, and the corresponding weights have to be positive or linked to each other.

This paper introduces an upper bound for the inter-exit time of two jobs in a permutation flow shop. Thus, dominance rules in scheduling are also relevant. In this context, there are two primary categories of theoretical findings:*job dominance* and *sequence dominance* rules (Emmons & Vairaktarakis, 2013).

Job dominance rules state that a job a dominates b if at least one schedule exists where a precedes b whose objective function value is better. Job dominance rules have been demonstrated to minimise the total completion time in a permutation flow shop with m stages (Dudek & Teuton, 1964; Szwarc, 1971). In particular, Szwarc bases his results on the calculation of $\Delta_k = C_{(\sigma\text{-}ab,k)} - C_{(\sigma\text{-}b,k)}$, i.e., the difference between the completion time on machine k of two partial schedules, σ-ab and σ-b, where a and b are two jobs and σ is a given partial schedule not including a and b. Δ_k is used to support the swap of jobs a and b with respect to its impact on the completion time of the obtained schedule.

This paper will address a similar measure, i.e., the difference between the completion times of two jobs. As the considered objective function is not regular, results related to job dominance rules cannot be exploited (Szwarc, 1971).

Another relevant set of theoretical results addresses sequence dominance rules, grounding on the idea that a sequence σ_1 dominates a sequence σ_2 if it entails better performance. Sequence dominance rules for the flow shop scheduling problem have been proposed (Johnson, 1954; Szwarc, 1983; Conway et al., 1967), considering the minimization of the makespan or other regular objective functions.

In addition, a comparison between preemptive and non-preemptive solutions has been discussed (Potts et al., 1991; Choi et al., 2007), considering the minimization of the total completion time as objective. A common feature of these works is imposing strong conditions on the processing times of operations, e.g., a decreasing order for the processing times of the operations in a stage of the flow shop (Panwalkar & Koulamas, 2017) or the presence of dominant machines (Ho & Gupta, 1995).

The findings presented in this paper pertain to the realm of sequence dominance rules. Specifically, given the inter-exit time between two jobs, a and b, the calculation of an upper bound for this value takes into account the sequence of jobs from a to b. The derived upper limit thus has the potential to establish dominance across varying alternative sequences. Nevertheless, since the considered inter-exit time is not regular, the previously mentioned dominance rules do not apply to these results.

Lastly, recent contributions related to dominance rules among permutation and non-permutation sequences in flow shop problems should also be acknowledged (Rossit et al., 2018, 2021). While considering the minimization of the total completion time in a flow shop with two jobs and m stages, the authors argue that permutation sequences dominate non-permutation ones without necessitating additional constraints on processing times. While the problem tackled is distinct, these results support the interest in this class of flow shop scheduling problems.

3 Problem definition

Let us consider a permutation flow shop problem, i.e., a processing system in which a set of n jobs must be processed on a set of m workstations or machines. A job represents a unit of work that needs to be completed and can be decomposed into a set of m tasks. A task is a single operation that needs to be performed and requires a specific machine/stage in the flow shop for its processing. In a permutation flow shop, all jobs visit the machines/stages in the

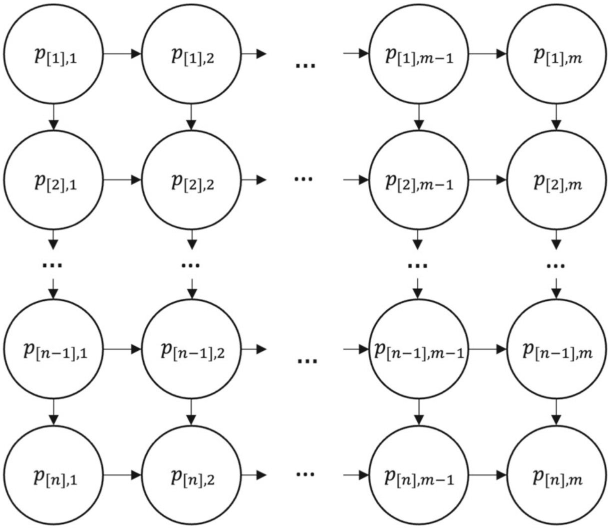


Fig. 1 *AoN* network representing a solution in a flow shop with n jobs and m machines

same order and, furthermore, all jobs must adhere to the same sequence across all machines or stages.

Each machine is perfectly reliable while *splitting* and *preemption* of the tasks are not allowed. We hypothesize that the transportation time between adjacent machines and possible set-up times are negligible (or included in the processing times of the operations). We limit the analysis to the class of static scheduling policies. Thus, all the jobs to be sequenced are known at the time of scheduling, and no release date exists.

A solution to this scheduling problem can be represented in terms of a network of activities with nodes representing operations, and arcs precedence constraints between operations (AoN). Given a general flow shop with n jobs and m machines, the associated network of activities for a given solution can be represented according to Fig. 1.

Given this solution schedule, let us consider two jobs sequenced in position h and k, with $k > h$. The processing times of these jobs on machine i are $p_{[h],i}$ and $p_{[k],i}$, respectively, while $C_{[h],i}$ and $C_{[k],i}$ are the completion times on the same machine.

We can define the *inter-exit time* between these two jobs as:

$$D_{k,h} = C_{[k],m} - C_{[h],m} \tag{1}$$

Generally speaking, the starting time of the operations of the job in position h depends on the execution of the previous jobs in the sequence, i.e., jobs in positions $h-1, h-2, \ldots, 1$.

Thus, to represent the execution of the jobs in position h to k through a general network of activities, a set of dummy operations is defined, with processing times $\{\delta_1, \delta_2, \ldots, \delta_m\}$, modelling the completion times at stage $1, 2, \ldots, m$ of the operations belonging to the job in position $h-1$ in the schedule (Fig. 2).

To support the definition of an upper bound for the inter-exit time between jobs in position h and k, we consider two cases:

a. The first job of the considered pair (job in position h) is the first in the sequence. Thus, $\delta_1, \delta_2, \ldots, \delta_m$ are equal to 0. In this case, we say that the group of jobs is processed *in*

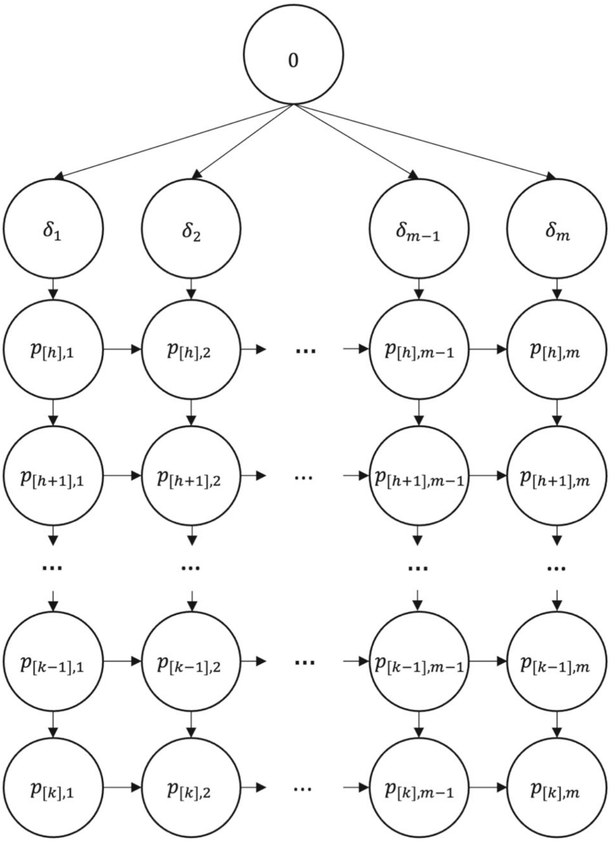


Fig. 2 *AoN* network representing the constraints affecting the execution of jobs in position h to k in a flow shop with m machines

isolation and use the notation $\hat{C}_{[h],j}$ for the completion time of the job in position h at stage j;

b. The first job of the considered pair (job in position h) is sequenced after an unknown number of jobs, and the completion times of its operations depend on the processing of the jobs sequenced before. Thus, $\{\delta_1, \delta_2, \ldots, \delta_m\}$ are non-negative values. In this case, we use the notation $C_{[h],j}$ for the completion time of the job in position h at stage j.

If we consider that jobs from h to k are executed in isolation, then the operations of job h can be processed without waiting for the completion of the ones belonging to the previous job in the sequence. This constitutes a special case that can support the estimation of the actual inter-exit time.

Thus, $\Delta D_{k,h}$ is defined as the difference between the inter-exit time between jobs h and k in the general case and the one obtained if they are executed in isolation:

$$\begin{aligned} \Delta D_{k,h} &= D_{k,h} - \hat{D}_{k,h} \\ &= \left(C_{[k],m} - C_{[h],m}\right) - \left(\hat{C}_{[k],m} - \hat{C}_{[h],m}\right) \\ &= \left(C_{[k],m} - \hat{C}_{[k],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \end{aligned} \tag{2}$$

Table 1 Alternative cases to demonstrate the base step of Theorem 1

$\delta_2 \geq \delta_1 + p_{[h],1}$	$\delta_2 + p_{[h],2} \geq \delta_1 + p_{[h],1} + p_{[h+1],1}$	Base case 1
	$\delta_2 + p_{[h],2} < \delta_1 + p_{[h],1} + p_{[h+1],1}$	Base case 2
$\delta_2 < \delta_1 + p_{[h],1}$	$p_{[h],2} \geq p_{[h+1],1}$	Base case 3
	$p_{[h],2} < p_{[h+1],1}$	Base case 4

4 An upper bound for the inter-exit time of subsequent jobs

This section aims to demonstrate that the inter-exit time for two subsequent jobs executed in isolation is an upper bound for the inter-exit time for the two jobs in a general position in a sequence.

Theorem 1 *In a permutation flow shop, the inter-exit time between two subsequent jobs in position $h + 1$ and h, executed in isolation, is an upper bound of their actual inter-exit time, i.e. $\Delta D_{h+1,h} \leq 0$.*

Proof The proof of Theorem 1 is provided by induction on the number of machines m. The base step verifies the statement of the theorem for a flow shop with two machines ($m = 2$). The induction step is operated by showing that, given the validity of the statement for a general number of machines m, it is also valid for $m + 1$ machines.

Base case Let us consider a two-machine flow shop and two subsequent jobs in positions h and $h + 1$ in a given sequence (Fig. 3). Referring to Eq. (2), the difference between the inter-exit times of these jobs is defined as:

$$\Delta D_{h+1,h} = D_{h+1,h} - \hat{D}_{h+1,h} \tag{3}$$

To simplify the notation, we will avoid making explicit reference to the indexes of the jobs, hence:

$$\Delta D_{h+1,h} \equiv \Delta D \tag{4}$$

$$D_{h+1,h} \equiv D \tag{5}$$

$$\hat{D}_{h+1,h} \equiv \hat{D} \tag{6}$$

The completion times of the two jobs can be calculated according to the following:

$$\begin{aligned}
\hat{C}_{[h],2} &= p_{[h],1} + p_{[h],2} \\
C_{[h],2} &= \begin{cases} \delta_2 + p_{[h],2} & \text{if } \delta_2 \geq \delta_1 + p_{[h],1} \\ \delta_1 + p_{[h],1} + p_{[h],2} & \text{otherwise} \end{cases} \\
\hat{C}_{[h+1],2} &= \begin{cases} \hat{C}_{[h],2} + p_{[h+1],2} & \text{if } \hat{C}_{[h],2} \geq p_{[h],1} + p_{[h+1],1} \\ p_{[h],1} + p_{[h+1],1} + p_{[h+1],2} & \text{otherwise} \end{cases} \\
C_{[h+1],2} &= \begin{cases} C_{[h],2} + p_{[h+1],2} & \text{if } C_{[h],2} \geq \delta_1 + p_{[h],1} + p_{[h+1],1} \\ \delta_1 + p_{[h],1} + p_{[h+1],1} + p_{[h+1],2} & \text{otherwise} \end{cases}
\end{aligned} \tag{7}$$

Starting from Eqs. 7 four alternative sub-cases can be identified, listed in Table 1. The demonstration of the base step of the theorem will be done for each of these cases.

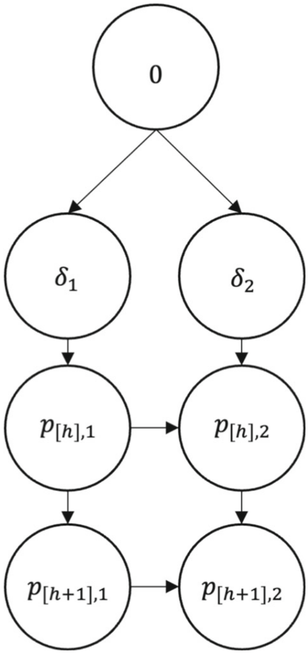


Fig. 3 *AoN* network representing the constraints affecting the execution of two subsequent jobs h and $h+1$ in a two-machine flow shop

Base case 1 *For this sub-case, the following assumptions apply:*

$$\delta_2 \geq \delta_1 + p_{[h],1} \tag{8}$$

$$\delta_2 + p_{[h],2} \geq \delta_1 + p_{[h],1} + p_{[h+1],1} \tag{9}$$

Thus, according to Eq. (7):

$$\hat{C}_{[h],2} = p_{[h],1} + p_{[h],2} \tag{10}$$

$$C_{[h],2} = \delta_2 + p_{[h],2} \tag{11}$$

$$\hat{C}_{[h+1],2} = p_{[h],1} + max\left(p_{[h],2}, p_{[h+1],1}\right) + p_{[h+1],2} \tag{12}$$

$$C_{[h+1],2} = \delta_2 + p_{[h],2} + p_{[h+1],2} \tag{13}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned} \Delta D &= \left(C_{[h+1],2} - \hat{C}_{[h+1],2}\right) - \left(C_{[h],2} - \hat{C}_{[h],2}\right) \\ &= \left(\delta_2 + p_{[h],2} - p_{[h],1} - max\left(p_{[h],2}, p_{[h+1],1}\right)\right) - \left(\delta_2 - p_{[h],1}\right) \\ &= p_{[h],2} - max\left(p_{[h],2}, p_{[h+1],1}\right) \leq 0 \end{aligned} \tag{14}$$

□

Base case 2 *For this sub-case, the following assumptions apply:*

$$\delta_2 \geq \delta_1 + p_{[h],1} \tag{15}$$

$$\delta_2 + p_{[h],2} < \delta_1 + p_{[h],1} + p_{[h+1],1} \tag{16}$$

Thus, according to Eq. (7):

$$\hat{C}_{[h],2} = p_{[h],1} + p_{[h],2} \tag{17}$$
$$C_{[h],2} = \delta_2 + p_{[h],2} \tag{18}$$
$$\hat{C}_{[h+1],2} = p_{[h],1} + max\left(p_{[h],2}, p_{[h+1],1}\right) + p_{[h+1],2} \tag{19}$$
$$C_{[h+1],2} = \delta_1 + p_{[h],1} + p_{[h+1],1} + p_{[h+1],2} \tag{20}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],2} - \hat{C}_{[h+1],2}\right) - \left(C_{[h],2} - \hat{C}_{[h],2}\right)\\ &= \left(\delta_1 + p_{[h],1} + p_{[h+1],1} - p_{[h],1} - max\left(p_{[h],2}, p_{[h+1],1}\right)\right) - \left(\delta_2 - p_{[h],1}\right)\\ &= \left(\delta_1 + p_{[h+1],1} - max\left(p_{[h],2}, p_{[h+1],1}\right)\right) - \left(\delta_2 - p_{[h],1}\right)\end{aligned} \tag{21}$$

Grounding on the assumption in Eq. (15), we have that:

$$\delta_1 + p_{[h],1} - \delta_2 \leq 0 \tag{22}$$

As a consequence, we obtain that:

$$\begin{aligned}&\Delta D = \delta_1 + p_{[h],1} - \delta_2 + p_{[h+1],1} - max\left(p_{[h],2}, p_{[h+1],1}\right)\\ &\Rightarrow \Delta D \leq 0 + p_{[h+1],1} - max\left(p_{[h],2}, p_{[h+1],1}\right) \leq 0\end{aligned} \tag{23}$$

□

Base case 3 *For this sub-case, the following assumptions apply:*

$$\delta_2 < \delta_1 + p_{[h],1} \tag{24}$$
$$p_{[h],2} \geq p_{[h+1],1} \tag{25}$$

Thus, according to Eq. (7):

$$\hat{C}_{[h],2} = p_{[h],1} + p_{[h],2} \tag{26}$$
$$C_{[h],2} = \delta_1 + p_{[h],1} + p_{[h],2} \tag{27}$$
$$\hat{C}_{[h+1],2} = p_{[h],1} + p_{[h],2} + p_{[h+1],2} \tag{28}$$
$$C_{[h+1],2} = \delta_1 + p_{[h],1} + p_{[h],2} + p_{[h+1],2} \tag{29}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],2} - \hat{C}_{[h+1],2}\right) - \left(C_{[h],2} - \hat{C}_{[h],2}\right)\\ &= \left(\delta_1 + p_{[h],1} + p_{[h],2} - p_{[h],1} - p_{[h],2}\right) - \left(\delta_1 + p_{[h],1} - p_{[h],1}\right) = 0\end{aligned} \tag{30}$$

□

Base case 4 *For this sub-case, the following assumptions apply:*

$$\delta_2 < \delta_1 + p_{[h],1} \tag{31}$$
$$p_{[h],2} < p_{[h+1],1} \tag{32}$$

Thus, according to Eq. (7):

$$\hat{C}_{[h],2} = p_{[h],1} + p_{[h],2} \tag{33}$$
$$C_{[h],2} = \delta_1 + p_{[h],1} + p_{[h],2} \tag{34}$$

$$\hat{C}_{[h+1],2} = p_{[h],1} + p_{[h+1],1} + p_{[h+1],2} \tag{35}$$

$$C_{[h+1],2} = \delta_1 + p_{[h],1} + p_{[h+1],1} + p_{[h+1],2} \tag{36}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned} \Delta D &= \left(C_{[h+1],2} - \hat{C}_{[h+1],2}\right) - \left(C_{[h],2} - \hat{C}_{[h],2}\right) \\ &= \left(\delta_1 + p_{[h],1} + p_{[h+1],1} - p_{[h],1} - p_{[h+1],1}\right) - \left(\delta_1 + p_{[h],1} - p_{[h],1}\right) = 0 \end{aligned} \tag{37}$$

□

This completes the proof of the base step. □

General case Here the induction step is operated. We assume Theorem 1 valid for m machines, thus:

$$\begin{aligned} &\left(C_{[h+1],m} - C_{[h],m}\right) - \left(\hat{C}_{[h+1],m} - \hat{C}_{[h],m}\right) \\ &= \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0 \end{aligned} \tag{38}$$

Let us consider a flow shop with $m + 1$ machines and a batch of dimension 2 containing jobs 1 and 2 (Fig. 4). In this case, the statement of Theorem 1 to be demonstrated is:

$$\begin{aligned} \Delta D_{h+1,h} &= D_{h+1,h} - \hat{D}_{h+1,h} \\ &= \left(C_{[h+1],m+1} - C_{[h],m+1}\right) - \left(\hat{C}_{[h+1],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \leq 0 \end{aligned} \tag{39}$$

To simplify the notation, we will avoid making explicit reference to the indexes of the jobs, hence:

$$\begin{aligned} \Delta D_{h+1,h} &\equiv \Delta D \\ D_{h+1,h} &\equiv D \\ \hat{D}_{h+1,h} &\equiv \hat{D} \end{aligned} \tag{40}$$

As in the previous case, we need to prove that $\Delta D \leq 0 \Rightarrow D \leq \hat{D}$.

Looking at the network of activities in Fig. 4, the completion times of jobs in positions h and $h + 1$ are determined by the longest paths from the source node (0) to nodes $p_{[h],m+1}$ and $p_{[h+1],m+1}$ for the jobs in positions h and $h + 1$ respectively. The length of these paths can be calculated according to Eq. (41).

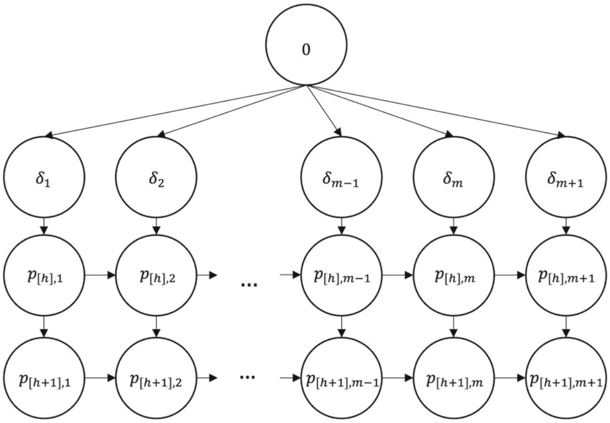


Fig. 4 *AoN* network representing the constraints affecting the execution of two subsequent jobs in an m-machine flow-shop

Combining the different options in Eq. (41), twelve different sub-cases can be identified, listed in Table 2. The general step of the theorem will be demonstrated for each of these sub-cases.

$$\begin{aligned}
&\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \\
&C_{[h],m+1} = \begin{cases} C_{[h],m} + p_{[h],m+1} & \text{if } C_{[h],m} \geq \delta_{m+1} \\ \delta_{m+1} + p_{[h],m+1} & \text{otherwise} \end{cases} \\
&\hat{C}_{[h+1],m+1} = \begin{cases} \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \\ \text{if } \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \\ \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \\ \text{otherwise} \end{cases} \\
&C_{[h+1],m+1} = \begin{cases} \delta_{m+1} + p_{[h],m+1} + p_{[h+1],m+1} \\ \text{if } \delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \\ \text{and } \delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m} \\ C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \\ \text{if } \delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \\ \text{and } C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m} \\ C_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \\ \text{if } C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \\ \text{and } \delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m} \end{cases}
\end{aligned} \tag{41}$$

General case 1 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} \geq \delta_{m+1} \tag{42}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{43}$$

$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{44}$$

$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{45}$$

Table 2 Alternative cases to demonstrate the general step of Theorem 1

$C_{[h],m} \geq \delta_{m+1}$	$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m}$	$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 1
		$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 2
		$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m}$	General case 3
	$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m}$	$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 4
		$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 5
		$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m}$	General case 6
$C_{[h],m} < \delta_{m+1}$	$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m}$	$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 7
		$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 8
		$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m}$	General case 9
	$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m}$	$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 10
		$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m}$	General case 11
		$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \wedge \delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m}$	General case 12

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{46}$$

$$C_{[h],m+1} = C_{[h],m} + p_{[h],m+1} \tag{47}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{48}$$

$$C_{[h+1],m+1} = \delta_{m+1} + p_{[h],m+1} + p_{[h+1],m+1} \tag{49}$$

The difference between the inter-exit times, also considering Eq. (42), can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(\delta_{m+1} + p_{[h],m+1} - \hat{C}_{[h],m} - max\left(p_{[h],m+1}, p_{[h+1],m}\right)\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \\ &= \left(\delta_{m+1} - C_{[h],m}\right) + \left(p_{[h],m+1} - max\left(p_{[h],m+1}, p_{[h+1],m}\right)\right) \leq 0\end{aligned} \tag{50}$$

□

General case 2 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} \geq \delta_{m+1} \tag{51}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{52}$$

$$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{53}$$

$$C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{54}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{55}$$

$$C_{[h],m+1} = C_{[h],m} + p_{[h],m+1} \tag{56}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{57}$$

$$C_{[h+1],m+1} = C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{58}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(C_{[h],m} - \hat{C}_{[h],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) = 0\end{aligned} \tag{59}$$

□

General case 3 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} \geq \delta_{m+1} \tag{60}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{61}$$

$$\delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m} \tag{62}$$

$$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{63}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{64}$$

$$C_{[h],m+1} = C_{[h],m} + p_{[h],m+1} \tag{65}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{66}$$

$$C_{[h+1],m+1} = C_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{67}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h],m} - max\left(p_{[h],m+1}, p_{[h+1],m}\right)\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right)\end{aligned} \tag{68}$$

If $p_{[h+1],m} \geq p_{[h],m+1}$, considering the definition of the path to $\hat{C}_{[h+1],m}$ determined by Eq. (61), we obtain:

$$\begin{aligned}&\hat{C}_{[h+1],m} = \hat{C}_{[h],m} + p_{[h+1],m}\\ &\Rightarrow \hat{C}_{[h],m} = \hat{C}_{[h+1],m} - p_{[h+1],m}\end{aligned} \tag{69}$$

Moreover, with respect to the path defining $C_{[h+1],m}$, from Eqs. (62) and (63), we have:

$$C_{[h+1],m} = C_{[h+1],m-1} + p_{[h+1],m} \tag{70}$$

Thus, grounding on the induction hypothesis (38), we have:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h],m} - p_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right)\\ &= \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0\end{aligned} \tag{71}$$

On the contrary, if $p_{[h],m+1} > p_{[h+1],m}$, the expression for $\hat{C}_{[h+1],m}$, due to Eqs. (61) and (62), is:

$$\hat{C}_{[h+1],m} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{72}$$

In addition, from Eq. (63), we obtain:

$$C_{[h+1],m} = C_{[h+1],m-1} + p_{[h+1],m} \tag{73}$$

Thus, considering the induction hypothesis (38), we have:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h],m} - p_{[h],m+1}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right)\\ &= \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0\end{aligned} \tag{74}$$

□

General case 4 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} \geq \delta_{m+1} \tag{75}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{76}$$

$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{77}$$

$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{78}$$

Considering Eq. (77), and assuming $p_{[h],m+1} \geq p_{[h+1],m}$, we have:

$$\delta_{m+1} \geq C_{[h],m} \tag{79}$$

If both Eqs. (79) and (75) are verified, then $\delta_{m+1} = C_{[h],m}$.

In this case, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{80}$$

$$C_{[h],m+1} = C_{[h],m} + p_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{81}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{82}$$

$$C_{[h+1],m+1} = \delta_{m+1} + p_{[h],m+1} + p_{[h+1],m+1} \tag{83}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(\delta_{m+1} + p_{[h],m+1} - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right) \\ &= p_{[h],m+1} - \hat{C}_{[h+1],m-1} - p_{[h+1],m} + \hat{C}_{[h],m}\end{aligned} \tag{84}$$

From Eq. (76), we obtain that:

$$-\hat{C}_{[h+1],m-1} - p_{[h+1],m} < -\hat{C}_{[h],m} - p_{[h],m+1} \tag{85}$$

As a consequence, we have:

$$\Delta D < p_{[h],m+1} - \hat{C}_{[h],m} - p_{[h],m+1} + \hat{C}_{[h],m} = 0 \tag{86}$$

On the contrary, considering Eq. (77) and assuming $p_{[h+1],m} > p_{[h],m+1}$*, we have:*

$$\begin{aligned}&C_{[h],m} + p_{[h+1],m} \leq \delta_{m+1} + p_{[h],m+1} \\ &\Rightarrow C_{[h],m} + p_{[h+1],m} - \delta_{m+1} \leq p_{[h],m+1} < p_{[h+1],m} \\ &\Rightarrow C_{[h],m} + p_{[h+1],m} - \delta_{m+1} < p_{[h+1],m} \\ &\Rightarrow C_{[h],m} - \delta_{m+1} < 0\end{aligned} \tag{87}$$

Equation (87) contradicts Eq. (75) thus, this second sub-case is not possible. □

General case 5 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} \geq \delta_{m+1} \tag{88}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{89}$$

$$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{90}$$

$$C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{91}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{92}$$

$$C_{[h],m+1} = C_{[h],m} + p_{[h],m+1} \tag{93}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{94}$$

$$C_{[h+1],m+1} = C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{95}$$

The difference between the inter-exit times, also considering Eq. (89), can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right)\\ &= \left(\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right)\right) - \left(\hat{C}_{[h+1],m-1} + p_{[h+1],m}\right) < 0\end{aligned} \tag{96}$$

□

General case 6 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} \geq \delta_{m+1} \tag{97}$$
$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{98}$$
$$\delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m} \tag{99}$$
$$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{100}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{101}$$
$$C_{[h],m+1} = C_{[h],m} + p_{[h],m+1} \tag{102}$$
$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{103}$$
$$C_{[h+1],m+1} = C_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{104}$$

From Eq. (98), we obtain that:

$$\hat{C}_{[h+1],m} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{105}$$

From Eqs. (99) and (100):

$$C_{[h+1],m} = C_{[h+1],m-1} + p_{[h+1],m} \tag{106}$$

As a consequence, the difference between the inter-exit times, also considering the induction hypothesis (38), can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right)\\ &= \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0\end{aligned} \tag{107}$$

□

General case 7 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} < \delta_{m+1} \tag{108}$$
$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{109}$$
$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{110}$$
$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{111}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{112}$$
$$C_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{113}$$
$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{114}$$
$$C_{[h+1],m+1} = \delta_{m+1} + p_{[h],m+1} + p_{[h+1],m+1} \tag{115}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(\delta_{m+1} + p_{[h],m+1} - \hat{C}_{[h],m} - max\left(p_{[h],m+1}, p_{[h+1],m}\right)\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right)\\ &= p_{[h],m+1} - max\left(p_{[h],m+1}, p_{[h+1],m}\right) \leq 0\end{aligned} \tag{116}$$

□

General case 8 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} < \delta_{m+1} \tag{117}$$
$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{118}$$
$$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{119}$$
$$C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{120}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{121}$$
$$C_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{122}$$
$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{123}$$
$$C_{[h+1],m+1} = C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{124}$$

The difference between the inter-exit times, due to Eq. (117), can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(C_{[h],m} - \hat{C}_{[h],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right)\\ &= C_{[h],m} - \delta_{m+1} < 0\end{aligned} \tag{125}$$

□

General case 9 *For this sub-case the following assumptions apply:*

$$C_{[h],m} < \delta_{m+1} \tag{126}$$
$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{127}$$
$$\delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m} \tag{128}$$
$$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{129}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{130}$$

$$C_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{131}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{132}$$

$$C_{[h+1],m+1} = C_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{133}$$

If $p_{[h+1],m} \geq p_{[h],m+1}$, the difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h],m} - p_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right)\end{aligned} \tag{134}$$

Under this assumption, from Eq. (127):

$$\hat{C}_{[h+1],m} = \hat{C}_{[h],m} + p_{[h+1],m} \tag{135}$$

and from Eq. (129):

$$C_{[h+1],m} = C_{[h+1],m-1} + p_{[h+1],m} \tag{136}$$

Thus, the difference between the inter-exit times can be calculated as:

$$\Delta D = \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right) \tag{137}$$

Considering Eq. (126) together with the induction hypothesis (38):

$$\Delta D < \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0 \tag{138}$$

On the contrary, if $p_{[h+1],m} < p_{[h],m+1}$, the difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right)\\ &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h],m} - p_{[h],m+1}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right)\\ &= C_{[h+1],m-1} + p_{[h+1],m} - \delta_{m+1} - p_{[h],m+1}\end{aligned} \tag{139}$$

As a consequence:

$$\begin{aligned}&\Delta D \leq C_{[h+1],m-1} + p_{[h],m+1} - \delta_{m+1} - p_{[h],m+1}\\ &\Rightarrow \Delta D \leq C_{[h+1],m-1} - \delta_{m+1}\end{aligned} \tag{140}$$

Given Eqs. (128) and (129):

$$\begin{aligned}&C_{[h+1],m} = C_{[h+1],m-1} + p_{[h+1],m}\\ &\Rightarrow C_{[h+1],m-1} = C_{[h+1],m} - p_{[h+1],m}\end{aligned} \tag{141}$$

As a consequence, from Eq. (140):

$$\Delta D \leq C_{[h+1],m} - p_{[h+1],m} - \delta_{m+1} \tag{142}$$

Referring to the induction hypothesis (38):

$$\hat{C}_{[h+1],m} = max\left(\hat{C}_{[h],m} + p_{[h+1],m}, \hat{C}_{[h+1],m-1} + p_{[h+1],m}\right) \tag{143}$$

If $\hat{C}_{[h],m} + p_{[h+1],m} \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m}$:

$$\begin{aligned}&\hat{C}_{[h+1],m} = \hat{C}_{[h],m} + p_{[h+1],m}\\ &\Rightarrow p_{[h+1],m} = \hat{C}_{[h+1],m} - \hat{C}_{[h],m}\end{aligned} \tag{144}$$

As a consequence, from Eq. (142):

$$\Delta D \leq C_{[h+1],m} - \left(\hat{C}_{[h+1],m} - \hat{C}_{[h],m}\right) - \delta_{m+1} \tag{145}$$

Due to Eq. (126), together with the induction hypothesis (38):

$$\begin{aligned} &\Delta D \leq C_{[h+1],m} - \left(\hat{C}_{[h+1],m} - \hat{C}_{[h],m}\right) - C_{[h],m} \\ &\Rightarrow \Delta D \leq \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0 \end{aligned} \tag{146}$$

Instead, if $\hat{C}_{[h],m} + p_{[h+1],m} < \hat{C}_{[h+1],m-1} + p_{[h+1],m}$:

$$\begin{aligned} &\hat{C}_{[h+1],m} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} \\ &\Rightarrow p_{[h+1],m} = \hat{C}_{[h+1],m} - \hat{C}_{[h+1],m-1} \end{aligned} \tag{147}$$

Considering again Eqs. (139) and (141):

$$\begin{aligned} \Delta D &= C_{[h+1],m-1} + p_{[h+1],m} - \delta_{m+1} - p_{[h],m+1} = \\ &= C_{[h+1],m} - \delta_{m+1} - p_{[h],m+1} \end{aligned} \tag{148}$$

Due to Eqs. (127) and (147):

$$\begin{aligned} &\hat{C}_{[h],m} + p_{[h],m+1} \geq \hat{C}_{[h+1],m-1} + p_{[h+1],m} = \hat{C}_{[h+1],m} \\ &\Rightarrow p_{[h],m+1} \geq \hat{C}_{[h+1],m} - \hat{C}_{[h],m} \end{aligned} \tag{149}$$

As a consequence, from Eq. (148):

$$\Delta D \leq C_{[h+1],m} - \delta_{m+1} + \hat{C}_{[h],m} - \hat{C}_{[h+1],m} \tag{150}$$

Considering Eq. (126) and the induction hypothesis (38):

$$\begin{aligned} &\Delta D \leq C_{[h+1],m} - C_{[h],m} + \hat{C}_{[h],m} - \hat{C}_{[h+1],m} \\ &\Rightarrow \Delta D \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0 \end{aligned} \tag{151}$$

□

General case 10 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} < \delta_{m+1} \tag{152}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{153}$$

$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{154}$$

$$\delta_{m+1} + p_{[h],m+1} \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{155}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{156}$$

$$C_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{157}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{158}$$

$$C_{[h+1],m+1} = \delta_{m+1} + p_{[h],m+1} + p_{[h+1],m+1} \tag{159}$$

Due to Eq. (153), the difference between the inter-exit times can be calculated as:

$$\begin{aligned} \Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(\delta_{m+1} + p_{[h],m+1} - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right) \\ &= \left(\hat{C}_{[h],m} + p_{[h],m+1}\right) - \left(\hat{C}_{[h+1],m-1} + p_{[h+1],m}\right) < 0 \end{aligned} \tag{160}$$

□

General case 11 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} < \delta_{m+1} \tag{161}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{162}$$

$$\delta_{m+1} + p_{[h],m+1} < C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{163}$$

$$C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \geq C_{[h+1],m-1} + p_{[h+1],m} \tag{164}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{165}$$

$$C_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{166}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{167}$$

$$C_{[h+1],m+1} = C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) + p_{[h+1],m+1} \tag{168}$$

Due to Eq. (161), the difference between the inter-exit times can be calculated as:

$$\begin{aligned} \Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right) \\ &= -\left(\hat{C}_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h],m} - max\left(p_{[h],m+1}, p_{[h+1],m}\right)\right) + \left(C_{[h],m} - \delta_{m+1}\right) < 0 \end{aligned} \tag{169}$$

□

General case 12 *For this sub-case, the following assumptions apply:*

$$C_{[h],m} < \delta_{m+1} \tag{170}$$

$$\hat{C}_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) < \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{171}$$

$$\delta_{m+1} + p_{[h],m+1} < C_{[h+1],m-1} + p_{[h+1],m} \tag{172}$$

$$C_{[h+1],m-1} + p_{[h+1],m} > C_{[h],m} + max\left(p_{[h],m+1}, p_{[h+1],m}\right) \tag{173}$$

Thus, according to Eq. (41):

$$\hat{C}_{[h],m+1} = \hat{C}_{[h],m} + p_{[h],m+1} \tag{174}$$

$$C_{[h],m+1} = \delta_{m+1} + p_{[h],m+1} \tag{175}$$

$$\hat{C}_{[h+1],m+1} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{176}$$

$$C_{[h+1],m+1} = C_{[h+1],m-1} + p_{[h+1],m} + p_{[h+1],m+1} \tag{177}$$

The difference between the inter-exit times can be calculated as:

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m+1} - \hat{C}_{[h+1],m+1}\right) - \left(C_{[h],m+1} - \hat{C}_{[h],m+1}\right) \\ &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right)\end{aligned} \tag{178}$$

Due to Eq. (172) and (173):

$$C_{[h+1],m} = C_{[h+1],m-1} + p_{[h+1],m} \tag{179}$$

Considering Eq. (171):

$$\hat{C}_{[h+1],m} = \hat{C}_{[h+1],m-1} + p_{[h+1],m} \tag{180}$$

Thus, the difference between the inter-exit times can be written as::

$$\begin{aligned}\Delta D &= \left(C_{[h+1],m-1} + p_{[h+1],m} - \hat{C}_{[h+1],m-1} - p_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right) \\ &= \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(\delta_{m+1} - \hat{C}_{[h],m}\right)\end{aligned} \tag{181}$$

Considering Eq. (170) and the induction hypothesis (38):

$$\Delta D \leq \left(C_{[h+1],m} - \hat{C}_{[h+1],m}\right) - \left(C_{[h],m} - \hat{C}_{[h],m}\right) \leq 0 \qquad \square \tag{182}$$

This completes the proof of the induction step and the theorem 1. ∎

In the following, the results obtained for the Theorem 1 are extended to a more general case, where the two jobs whose inter-exit time has to be estimated are not subsequent.

Theorem 2 *In a permutation flow shop, the inter-exit time between two jobs in positions k and h, with $k > h$, executed in isolation, is an upper bound of their actual inter-exit time, i.e. $\Delta D_{k,h} \leq 0$.*

Proof The complete proof of Theorem 2 is provided considering a general number of machines m and a general number of jobs.

Referring to Eq. (2), the difference between the inter-exit time of any pair of two jobs in position h and k, with $k > h$, under the hypothesis of execution in isolation, is defined as:

$$\Delta D_{k,h} = D_{k,h} - \hat{D}_{k,h} \tag{183}$$

Using the same approach exploited in the first part of this section, Eq. (183) is discussed as follows.

$$\begin{aligned}\Delta D_{k,h} &= D_{k,h} - \hat{D}_{k,h} \\ &= \left(C_{[k],m} - C_{[h],m}\right) - \left(\hat{C}_{[k],m} - \hat{C}_{[h],m}\right) \\ &= \left(C_{[k],m} - C_{[h],m} + C_{[k-1],m} - C_{[k-1],m} + \ldots + C_{[h+1],m} - C_{[h+1],m}\right) \\ &\quad - \left(\hat{C}_{[k],m} - \hat{C}_{[h],m} + \hat{C}_{[k-1],m} - \hat{C}_{[k-1],m} + \ldots + \hat{C}_{[h+1],m} - \hat{C}_{[h+1],m}\right) \\ &= \left[\left(C_{[k],m} - C_{[k-1],m}\right) + \left(C_{[k-1],m} - C_{[k-2],m}\right) + (\ldots) + \left(C_{[h+1],m} - C_{[h],m}\right)\right] \\ &\quad - \left[\left(\hat{C}_{[k],m} - \hat{C}_{[k-1],m}\right) + \left(\hat{C}_{[k-1],m} - \hat{C}_{[k-2],m}\right) + (\ldots) + \left(\hat{C}_{[h+1],m} - \hat{C}_{[h],m}\right)\right] \\ &= \left(D_{k,k-1} + D_{k-1,k-2} + \ldots + D_{h+1,h}\right) - \left(\hat{D}_{k,k-1} + \hat{D}_{k-1,k-2} + \ldots + \hat{D}_{h+1,h}\right) \\ &= \Delta D_{k,k-1} + \Delta D_{k-1,k-2} + \ldots + \Delta D_{h+1,h}\end{aligned} \tag{184}$$

Considering Theorem 1, we have:

$$\Delta D_{j+1,j} \leq 0, \forall j \in [1, n-1] \tag{185}$$

As a consequence, we obtain that

$$\Delta D_{k,h} = D_{k,h} - \hat{D}_{k,h} = \Delta D_{k,k-1} + \Delta D_{k-1,k-2} + \ldots + \Delta D_{h+1,h} \leq 0 \quad \square \tag{186}$$

This completes the proof of the Theorem 2. ■

5 Analysis of the upper bound

A preliminary assessment of the quality of the proposed upper bound has been carried out to investigate the overall tightness and possible dependencies on the characteristics of the scheduling problem. The analysis has been operated on the flow shop scheduling instances proposed in Vallada et al. (2015b) and the associated optimal schedule. For each instance, two positions h and k, with $h < k$, are selected in the optimal schedule. Hence, the schedule is evaluated to estimate the completion times C_h and C_k for the two jobs and, consequently, the real inter-exit time. Then, jobs from position h to k in the original sequence are scheduled in isolation, and their inter-exit time is assessed, being this an upper bound of the real value.

The approach is operated on the subset of 240 instances labelled as small Vallada et al. (2015a), containing scheduling problems with 10 to 60 jobs. Multiple experiments are executed for each instance, considering different distances between the jobs h and k, whose inter-exit time has to be estimated. Specifically, values of 3, 5 and 7 jobs are used, including jobs h and k. Furthermore, two additional options are considered by selecting jobs h and k in the middle of the schedule or at the end.

Considering the different instances and the combination of the factors, 1440 experiments have been executed, and for each of them, the tightness of the upper bound for the inter-exit time has been assessed through ΔUB, defined in Eq. 187.

$$\Delta UB = \frac{-\Delta D_{k,h}}{D_{k,h}} \cdot 100 \tag{187}$$

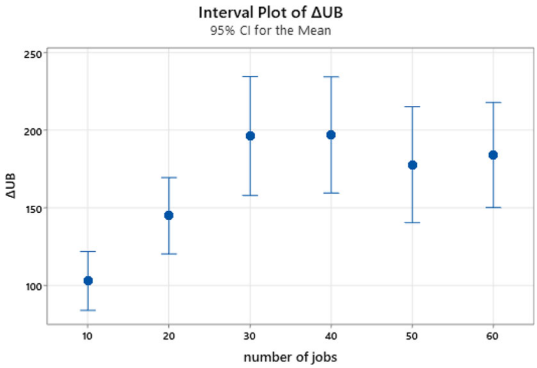
On average, the calculated upper bound for the inter-exit time between the considered pair of jobs was 167% larger than the real value (see last row in Table 3). Furthermore, the results show a very high variance, corresponding to a coefficient of variation of 398. Thus, the tightness of the bound is significantly variable, ranging from a minimum of 0% to a maximum of 2627%. Further analyses have been conducted to investigate the possible influence of the number of jobs in the problem instances. Figure 5 shows that the tightness of the bound is better for instances with a small number of jobs (10 or 20). In contrast, as this number increases, the average tightness decreases and seems to line up between 180% and 200% (see Table 3).

The reasonable explanation for this behaviour is that, for instances with a small number of jobs, the job in position h, thus the first executed job whose inter-exit time has to be assessed, is not far from the first job in the schedule. Thus, the impact of the preceding jobs on its execution is limited, and consequently, the execution of the jobs from h to k in isolation leads to a result similar to the execution according to the original schedule. In contrast, when a larger number of jobs is considered, the set of jobs from h to k can be far away from the first job in the schedule. Thus, the impact of the preceding jobs on their execution (i.e., the impact of $\delta_1, \delta_2, \ldots, \delta_m$) could be relevant, leading to different execution in isolation. Nevertheless,

Table 3 Tighteness of the inter-exit bound with respect to the number of jobs in the schedule

# jobs	ΔUB			
	Mean	Variance	Min	Max
10	103	22,212.9	0	1140
20	145	37,265.6	0	1771
30	196	90,903.9	0	1867
40	197	86,737.1	0	2158
50	178	86,133.7	0	2627
60	184	70,800.8	0	2350
Total	167	66,574.7	0	2627

Fig. 5 Interval plot for ΔUB with respect to the number of jobs in the schedule



it must be noticed that this seems highly influenced by non-controllable factors, namely the characteristics of the instance and the selected schedule. The variance of the tightness is still very high, and the possible values range from 0% (i.e., the upper bound of the inter-exit time exactly matches the real value) to 1771% (i.e., the upper bound is about 18 times the real value).

Hence, additional investigations have been carried out on instances with a number of jobs greater than 20 to check the influence on the tightness of the bound of other factors, specifically the number of machines in the flow shop, the distance between the jobs whose inter-exit time is estimated, and the position of this group of jobs in the sequence, namely in the middle or at the end of the schedule.

The results are reported in Table 4, showing a clear influence of the considered factors on the tightness of the upper bound for the inter-exit time. Specifically, the tightness seems reduced as the number of machines increases (see Fig. 6). In contrast, a higher tightness is experienced as the distance between the jobs whose inter-exit time has to be calculated increases (Fig. 7a). Finally, the position of the set of jobs linked to the inter-exit time within the schedule also seems to play a role. The tightness is better for a set of jobs placed in the middle rather than at the end of the schedule (Fig. 7b).

Although the experiments report a clear dependence on these factors, it must be noticed that the dispersion of the results is extremely high. The variance is extremely high even if the mean values show these dependencies from the factors. In many cases, the minimum value for ΔUB is 0, while the maximum is extremely large. Thus, the tightness of the proposed upper bound for the inter-exit time reasonably also depends on the specific problem instance and schedule. Further investigations in this direction will be the subjects of future works.

Table 4 Tighteness of the inter-exit bound with respect to the considered factors

# machines	$k-h+1$	Position	ΔUB Mean	Variance	Min	Max
5	3	mid	55	3717	0.00	266.10
5	3	end	134	44,967	0.00	1000.00
5	5	mid	36	763	0.00	95.86
5	5	end	101	7039	0.00	403.85
5	7	mid	32	686	0.00	140.68
5	7	end	84	3120	0.00	250.85
10	3	mid	83	5355	0.00	412.33
10	3	end	423	214,112	0.00	2033.33
10	5	mid	80	3381	0.00	238.46
10	5	end	157	13,877	2.16	458.06
10	7	mid	62	757	0.00	126.62
10	7	end	114	5708	9.06	335.23
15	3	mid	128	14,888	0.00	583.33
15	3	end	585	333,549	5.41	2626.67
15	5	mid	97	2849	0.00	191.63
15	5	end	297	112,087	32.65	2157.89
15	7	mid	77	1296	12.94	175.42
15	7	end	177	10,456	54.49	506.17
20	3	mid	208	54,927	0.00	1108.70
20	3	end	743	401,247	0.00	2350.00
20	5	mid	151	14,305	2.44	571.93
20	5	end	370	32,244	49.68	751.28
20	7	mid	106	4655	3.93	339.15
20	7	end	232	15,210	41.00	515.60

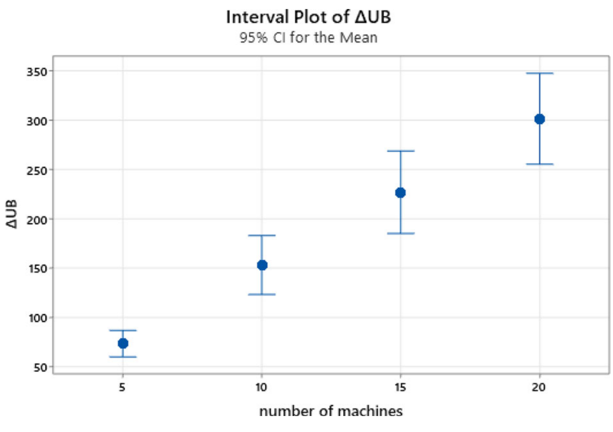


Fig. 6 Interval plot for ΔUB with respect to the number of machines in the problem instances

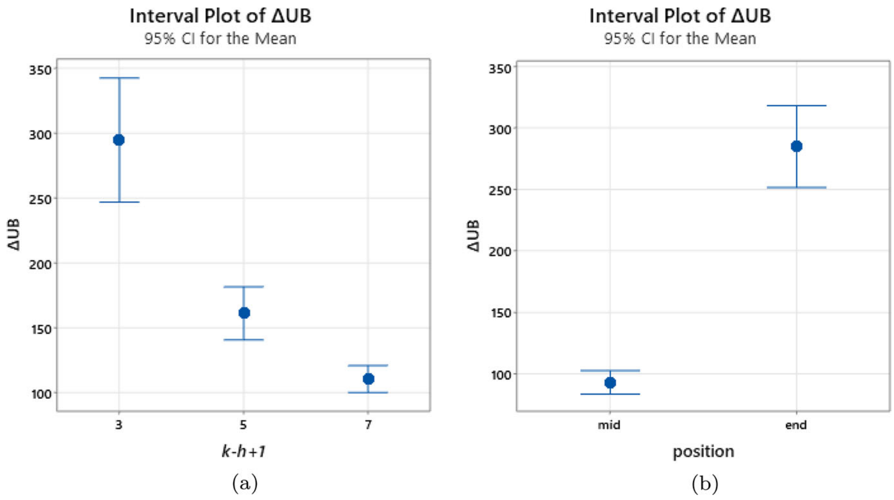


Fig. 7 Interval plot for ΔUB with respect to the number of jobs involved in the assessment of the inter-exit time (**a**) and the position of these jobs in the schedules (**b**)

6 Relevance of the proposed bound

The described theoretical results provide an upper bound for the inter-exit time between any two jobs in a schedule. Although the results showed a very variable tightness of the upper bound, it could be used to support flow shop scheduling approaches, possibly with a lower bound for the same performance measure. A lower bound for the inter-exit time can be easily obtained by just considering the sum of the processing times of the operations on the last machine of the flow shop.

Leveraging upper and lower bounds can support decisions related to the definition of batches (e.g., deciding the jobs that will stay in the same batch) or their sequencing, e.g., to minimize a function of the batch completion times, i.e., their sum, maximum value or variance.

Referring to the industrial relevance of the inter-exit time, as described in Sect. 1, the time needed to form a batch could incur specific costs in manufacturing systems. Examples are the need to load the completed parts forming a batch on a transporter (e.g., a truck), causing it to wait until the loading is completed. This is relevant for large parts that cannot be just laid on the shop floor due to the need for specific fixtures or space constraints (Urgo et al., 2018; Buergin et al., 2019). Thus, this class of operations can benefit from minimising the inter-exit time between jobs.

7 Conclusions

This paper addresses the scheduling in a permutation flow shop and, specifically, the assessment of the time between the completion times of any two jobs, i.e., their inter-exit time. An upper bound for this performance measure is provided for a general permutation flow shop problem without any constraints on the number of machines, the number of jobs, and the selection of the jobs whose inter-exit time has to be estimated. This result can support scheduling approaches for this class of flow shop problems, aiming to optimise the definition of batches and their sequencing. A preliminary assessment of the tightness of the proposed upper bound showed erratic performance, possibly influenced by the characteristics of the problem instance and the schedule. Further investigations will be in the direction of a thorough evaluation of the tightness of the proposed bound. Furthermore, future developments will pursue extending the obtained theoretical results to stochastic scheduling problems (Urgo, 2019; Liu & Urgo, 2023, 2024), possibly leveraging the structural properties of a schedule and its links with different values of the processing times (Rossit et al., 2021).

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Declarations

Conflict of interest The authors have no conflict of interest.

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