On the deep connection between hyperstructures and quantum logic

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Four interconnected domains

- ⋆ **Quantum mechanics**: *physical framework* based on the notions of Hilbert space and projection operators.
- ⋆ **Lattice theory**: *logical framework* based on the notions of ordered set and propositional system.
- ⋆ **Projective geometry**: *geometrical framework* based on the notions of points and lines.
- ⋆ **Hypercompositional algebra**: both lattices and projective geometries can be interpreted in terms of structures with a *multivalued operation* or *hypercomposition*.

A short (hi)story

In the 1930s, Garrett Birkhoff and John Von Neumann proposed a non-classical logical system reflecting the nature of quantum phenomena and called *quantum logic*.

Between the 1960s and the 1970s, Joseph-Maria Jauch and Constantin Piron continued the investigation developing an approach [1], which is still studied nowadays.

Classical propositional logic	Quantum propositional logic
≡	≡
Boolean algebra	Non-distributive lattice
≡	≡
Random event structure	Quantum event structure
≡	≡
Classical probability	Quantum probability

The original attempt of Von Neumann was to translate the classical framework into a suitable quantum framework. However, the translation cannot be done trivially! At the end, Von Neumann chose to give up Hilbert spaces as mathematical framework of quantum mechanics!

I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of conserving the validity of all formal rules.

John von Neumann

The world of hyperstructures

A *canonical hypergroup* H has a hyperoperartion $\boxplus : H^2 \to H$ which is associative, commutative, has a neutral element 0, uniqueness of inverses and reversibility. If $x \boxplus x = \{0, x\}$ for all $x \in H$, then H is called a $\mathbf{K}$-vector space.

If on $H \setminus \{0\}$ we have an abelian group law which distributes over $\boxplus$, then H is a *hyperfield*.

Example. $\mathbf{K}$-vector space $\leftrightarrow x \boxplus y = \ell(x,y) \setminus \{x,y\}$.

Projective geometry $\leftrightarrow \ell(x,y) = x \boxplus y \cup \{x,y\}$.

Example. For a modular lattice with a bottom element $(L, \vee, \wedge, \perp)$ we have a canonical hypergroup with

$x \boxplus y := \{z \in L \mid x \vee y = x \vee z = y \vee z\}$ and $0 = \perp$.

Projective spaces over local fields

For a local field F, the n-dimensional projective space $\mathbb{P}^n_F$ has a canonical topology and corresponds to a unique $\mathbf{K}$-vector space $\mathbb{H}^n_F$.

Distributive and abelian group laws on $\mathbb{H} \setminus \{0\}$ correspond to incidence group structures on $\mathbb{P}$.

If F is $\mathbb{R}$ or $\mathbb{C}$, then the requirement that such a group is topological restricts the possibilities to the only case $n = 1$ and $F = \mathbb{R}$.

In the non-archimedean case ($\mathbb{Q}_p$, $\mathbb{F}_p((t))$, ...), we are able to prove that there are finitely many possibilities up to isomorphism of topological groups if $\mathrm{char}(F)$ does not divide $n+1$, and, in any case, countably many [2].

The infinite-dimensional case is still under investigation.

Big questions

Is it possible to define a hyperstructure associated to the lattice of closed subspace of a Hilbert space?
Can we use this new framework to describe quantum phenomena and quantum logic?
What about the probabilistic interpretation?

Back to quantum

The fundamental mathematical framework in quantum mechanics is the Hilbert space. Its relation with propositional systems is provided by the following statement.

Theorem. *Every irreducible propositional system with dimension greater than or equal to four is isomorphic to the lattice of all closed subspaces of a Hilbert space constructed on some division ring with involution.*

The best way to work with the closed subspace is to exploit the one-to-one correspondence between closed subspaces and *projector operators*.

One can prove that the set $\mathcal{P}(H)$, of closed subspaces of an Hilbert space H ordered by inclusion, is actually a propositional system.

Promising links

- The analogy with the mathematical journey, which led Von Neumann to introduce his algebras of bounded operators on a Hilbert space seems to provide a genuine inductive support to the conjecture of describing quantum phenomena allowing multivalued operations.
- Work on Quantum Gravity has revealed a connection between the category of manifolds and the category of Hilbert spaces and linear operators. This can be established via the concept of *span*, which is also closely related to multivalued functions [3].
- The article of Connes and Consani [4] established a fundamental connection between hyperstructures and projective geometries that can be naturally extended to the lattices of quantum logic.

Future studies

We are investigating a hypercompositional approach in quantum theories. The question whether a given hypergroup admits a (skew) hyperfield structure is, in the projective case, an algebraic formulation of the problem of symmetries of the space which preserve the incidence of lines with points. As a first step, we have analyzed finite-dimensional case projective spaces over local fields. The following step is to treat the infinite-dimensional case, especially over $\mathbb{R}$ and $\mathbb{C}$. Regarding the hypergroups associated to modular lattices, the problem of admissible (skew) hyperfield structures suggests a concept of symmetry in the quantum logical framework, preserving some fundamental properties. We remark that our approach becomes particularly natural once one becomes familiar with multivalued operations (or spans) and that the developments in general hypercompositional algebra may provide new and powerful tools in the study of quantum structures.

Some references

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