Blow-up and global existence for solutions to the porous medium equation with reaction and slowly decaying density

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Abstract

We study existence of global solutions and finite time blow-up of solutions to the Cauchy problem for the porous medium equation with a variable density $\rho(x)$ and a power-like reaction term $\rho(x)u^p$ with $p > 1$; this is a mathematical model of a thermal evolution of a heated plasma (see [28]). The density decays slowly at infinity, in the sense that $\rho(x) \lesssim |x|^{-q}$ as $|x| \to +\infty$ with $q \in [0, 2)$. We show that for large enough initial data, solutions blow-up in finite time for any $p > 1$. On the other hand, if the initial datum is small enough and $p > \bar{p}$, for a suitable $\bar{p}$ depending on ρ, m, N, then global solutions exist. In addition, if $p < \underline{p}$, for a suitable $\underline{p} \leq \bar{p}$ depending on ρ, m, N, then the solution blows-up in finite time for any nontrivial initial datum; we need the extra hypotehsis that $q \in [0, \epsilon)$ for $\epsilon > 0$ small enough, when $m \leq p < \underline{p}$. Observe that $\underline{p} = \overline{p}$, if $\rho(x)$ is a multiple of $|x|^{-q}$ for $|x|$ large enough. Such results are in agreement with those established in [47], where $\rho(x) \equiv 1$, and are related to some results in [31, 32]. The case of fast decaying density at infinity, i.e. $q \geq 2$, is examined in [37].

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1 Introduction

We investigate global existence and blow-up of nonnegative solutions to the Cauchy parabolic problem

$$\begin{cases} \rho(x)u_t = \Delta(u^m) + \rho(x)u^p & \text{in } \mathbb{R}^N \times (0,\tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.1}$$

where $m > 1, p > 1, N \geq 3, \tau > 0$; furthermore, we always assume that

$$\begin{cases} \text{(i) } \rho \in C(\mathbb{R}^N), \, \rho > 0 \text{ in } \mathbb{R}^N\,; \\ \text{(ii) there exist } k_1, k_2 \in (0, +\infty) \text{ with } k_1 \leq k_2 \text{ and } 0 \leq q < 2 \text{ such that} \\ \quad k_1|x|^q \leq \dfrac{1}{\rho(x)} \leq k_2|x|^q \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0)\,; \\ \text{(iii) } u_0 \in L^\infty(\mathbb{R}^N), \, u_0 \geq 0 \text{ in } \mathbb{R}^N\,. \end{cases} \tag{H}$$

The parabolic equation in problem (1.1) is of the *porous medium* type, with a variable density $\rho(x)$ and a reaction term $\rho(x)u^p$. Clearly, such parabolic equation is degenerate, since $m > 1$. Moreover, the differential equation in (1.1) is equivalent to

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \qquad \text{in } \mathbb{R}^N \times (0,\tau);$$

therefore, the related diffusion operator is $\frac{1}{\rho(x)}\Delta$, and in view of (H), the coefficient $\dfrac{1}{\rho(x)}$ can positively diverge at infinity. Problem (1.1) has been introduced in [28] as a mathematical model of evolution of plasma temperature, where u is the temperature, $\rho(x)$ is the particle density, $\rho(x)u^p$ represents the volumetric heating of plasma. Indeed, in [28, Introduction] a more general source term of the type $A(x)u^p$ has also been considered; however, then the authors assume that $A \equiv 0$; only some remarks for the case $A(x) = \rho(x)$ are made in [28, Section 4], when the problem is set in a slab in one space dimension. Then in [26] and [27] problem (1.1) is dealt with in the case without the reaction term $\rho(x)u^p$.

We refer to $\rho(x)$ as a *slowly decaying density* at infinity because, in view of (H),

$$\frac{1}{k_2|x|^q} \leq \rho(x) \leq \frac{1}{k_1|x|^q} \quad \text{for all } |x| > 1\,,$$

with

$$0 \leq q < 2.$$

Global existence and blow-up of solutions for problem (1.1) with *fast decaying density* at infinity, i.e. $q \geq 2$, is investigated in [37]. We regard the value $q = 2$

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as the threshold one, indeed, the behavior of solutions is very different according to the fact that $q < 2$ or $q = 2$ or $q > 2$. Such important role played by the value $q = 2$ does not surprise. In fact, for problem (1.1) without the reaction term u^p, that is

$$\begin{cases} \rho u_t = \Delta(u^m) & \text{in } \mathbb{R}^N \times (0, \tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.2}$$

in [41], it is shown that for $q \leq 2$ there exists a unique bounded solution, whereas for $q > 2$, for any $u_0 \in L^\infty(\mathbb{R}^N)$ there exist infinitely many bounded solutions.

Let us briefly recall some results in the literature concerning well-posedness for problems related to (1.1). Problem (1.1) with $\rho \equiv 1$ and without the reaction term, that is

$$\begin{cases} u_t = \Delta(u^m) & \text{in } \mathbb{R}^N \times (0, \tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.3}$$

has been the object of detailed investigations. We refer the reader to the book [51] and references therein, for a comprehensive account of the main results. Also problem (1.1) with variable density, without reaction term, that is problem (1.2), has been widely examined. In particular, depending on the behaviour of $\rho(x)$ as $|x| \to \infty$, existence and uniqueness of solutions and the asymptotic behaviour of solutions for large times have been addressed (see, e.g., [8, 9, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 41, 43, 44, 45]).

For problem (1.1) with $m = 1$ and $\rho \equiv 1$, global existence and blow-up of solutions have been studied. To be specific, if

$$p \leq 1 + \frac{2}{N},$$

then finite time blow-up occurs, for all nontrivial nonnegative data, whereas, for

$$p > 1 + \frac{2}{N},$$

global existence prevails for sufficiently small initial conditions (see, e.g., [5, 6, 10, 11, 17, 29, 42, 46, 48, 52]). In addition, in [30] (see also [7]), problem (1.1) with $m = 1$ has been considered. Let assumption (H) be satisfied, and let

$$b := 2 - q. \tag{1.4}$$

Obviously, since $q \in [0, 2)$, we have that

$$b \in (0, 2].$$

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It is shown that if
$$p \leq 1 + \frac{b}{N - 2 + b},$$
then solutions blow-up in finite time, for all nontrivial nonnegative data, whereas, for
$$p > 1 + \frac{b}{N - 2 + b},$$
global in time solutions exist, provided that u_0 is small enough.

Now, let us recall some results established in [47] for problem (1.1) with $\rho \equiv 1, m > 1, p > 1$ (see also [12, 38]). We have:

- ([47, Theorem 1, p. 216]) For any $p > 1$, for all sufficiently large initial data, solutions blow-up in finite time;

- ([47, Theorem 2, p. 217]) if $p \in \left(1, m + \frac{2}{N}\right)$, for *all* initial data, solutions blow-up in finite time;

- ([47, Theorem 3, p. 220]) if $p > m + \frac{2}{N}$, for all sufficiently small initial data, solutions exist globally in time.

Similar results for quasilinear parabolic equations, also involving p-Laplace type operators or double-nonlinear operators, have been stated in [1], [2], [3], [31], [?], [?], [36], [37], [39], [49], [53] (see also [34] for the case of Riemannian manifolds); moreover, in [16] the same problem on Cartan-Hadamard manifolds has been investigated.

Let us observe that the results in [47] illustrated above have been proved by means of comparison principles and suitable sub– and supersolutions of the form

$$w(x, t) = C\zeta(t) \left[1 - \frac{|x|^2}{a} \eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x, t) \in \mathbb{R}^N \times [0, T),$$

for appropriate auxiliary functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0$.

In [31, 32] double-nonlinear operators, including in particular problem (1.1), are investigated. It is shown that (see [31, Theorem 1]) if $\rho(x) = |x|^{-q}$ with $q \in (0, 2)$, for any $x \in \mathbb{R}^N \setminus \{0\}$,

$$p > m + \frac{b}{N - 2 + b},$$

$u_0 \geq 0$ and

$$\int_{\mathbb{R}^N} \left\{u_0(x) + [u_0(x)]^{\bar{q}}\right\} \rho(x) dx < \delta, \tag{1.5}$$

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for some $\delta > 0$ small enough and $\bar{q} > \frac{N}{2}(p - m)$, then there exists a global solution of problem (1.1). In addition, a smoothing estimate holds. On the other hand, if $\rho(x) = |x|^{-q}$ or $\rho(x) = (1 + |x|)^{-q}$ with $q \in [0, 2)$, $u_0 \not\equiv 0$ and

$$p < m + \frac{b}{N - 2 + b},$$

then blow-up prevails, in the sense that there exist $\theta \in (0, 1), R > 0, T > 0$ such that

$$\int_{B_R} [u(x, t)]^\theta \rho(x) dx \to +\infty \quad \text{as } t \to T^-.$$

Such results have also been generalized to more general initial data, decaying at infinity with a certain rate (see [32]). We compare the results in [31] with ours below (see Remarks 2.3, 2.5 and 2.8).

1.1 Outline of our results

We prove the following results.

- (See Theorem 2.1). Suppose that

$$\frac{k_2}{k_1} < m + \frac{(m - 1)(N - 2)}{b}, \tag{1.6}$$

and define

$$\overline{p} := \frac{m(N - 2 + b) + \frac{b}{m-1}(m - \frac{k_2}{k_1})}{N - 2 + \frac{b}{m-1}\left(m - \frac{k_2}{k_1}\right)}. \tag{1.7}$$

If u_0 has compact support and is small enough,

$$p > \overline{p},$$

then global solutions exist.

Note that for $k_1 = k_2$,

$$\overline{p} = m + \frac{b}{N + 2 - b};$$

this is coherent with [31, Theorem 1] (see Remark 2.3 below for more details). If in addition $\rho \equiv 1$, and so $b = 2$, we have

$$\overline{p} = m + \frac{2}{N}.$$

Thus, our results are in accordance with those in [47]. Furthermore, for $m = 1$, they are in agreement with the results established in [30], and in [11, 17] when $\rho \equiv 1$.

- (See Theorem 2.4). For any $p > 1$, if u_0 is sufficiently large, then solutions to problem (1.1) blow-up in finite time.

- (see Theorem 2.6). If $1 < p < m$, then for any $u_0 \not\equiv 0$, solutions to problem (1.1) blow-up in finite time. In addition (see Theorem 2.7), if

$$m \leq p < m + \frac{b}{N - 2 + b}$$

  and $q \in [0, \epsilon)$ for $\epsilon > 0$ small enough, then for any $u_0 \not\equiv 0$, solutions to problem (1.1) blow-up in finite time.

It remains to be understood if the restriction $q \in [0, \epsilon)$ can be removed.

Actually, we obtain similar results to those described above, also when assumption (H) is fulfilled for general $0 < k_1 < k_2$. In that case, the blow-up result for large initial data can be stated exactly as in the previous case $k_1 = k_2$. Instead, in order to get global existence, the assumption on p changes, since it also depends on the parameters k_1 and k_2. More precisely, Indeed, also our blow-up results for any nontrivial initial datum holds when $0 < k_1 < k_2$. The case $1 < p < m$ is exactly as before. Moreover (see Theorem 2.7), if

$$m \leq p < \underline{p},$$

where

$$\underline{p} = \frac{m \left(N - 2 + b \right) + \frac{b}{m-1} \left(m - \frac{k_1}{k_2} \right)}{N - 2 + \frac{b}{m-1} \left(m - \frac{k_1}{k_2} \right)}, \tag{1.8}$$

then the solution blows-up for any nontrivial initial datum, under the extra hypothesis that $q \in [0, \epsilon)$ for $\epsilon > 0$ small enough. Note that in view of (1.6), it can be easily checked that

$$\underline{p} \leq \bar{p}.$$

In particular, $\underline{p} = \bar{p}$ whenever $k_1 = k_2$.

The methods used in [7, 11, 17, 30] cannot work in the present situation, since they strongly require $m = 1$. Indeed, our proofs mainly relies on suitable comparison principles (see Propositions 3.6, 3.7) and properly constructed sub- and supersolutions. Let us mention that the arguments exploited in [47] cannot be directly used in our case, due to the presence of the coefficient $\rho(x)$. In fact, we construct appropriate sub– and supersolutions, which crucially depend on the behavior at infinity of the inhomogeneity term $\rho(x)$. More precisely, whenever $|x| > 1$, they are of the type

$$w(x, t) = C\zeta(t) \left[1 - \frac{|x|^b}{a} \eta(t) \right]_+^{\frac{1}{m-1}} \quad \text{for any } (x, t) \in \left[\mathbb{R}^N \setminus B_1(0) \right] \times [0, T),$$

for suitable functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0$. In view of the term $|x|^b$ with $b \in (0, 2]$, we cannot show that such functions are sub- and supersolutions in $B_1(0) \times (0, T)$. Thus we have to extend them in a suitable way in $B_1(0) \times (0, T)$. This is not only a technical aspect. In fact, in order to extend our sub– and supersolutions, we need to impose some extra conditions on $\zeta = \zeta(t)$, $\eta = \eta(t)$, C and a. Thus, it appears a sort of interplay between the behavior of the density $\rho(x)$ in compact sets, say $B_1(0)$, and its behavior for large values of $|x|$. Finally, let us comment about the proofs of the blow-up result for any nontrivial initial datum. For $1 < p < m$, the result follows by a direct application of Theorem 2.4. For $m < p < \underline{p}$, the proof is more involved. The corresponding result for the case $\rho \equiv 1$ established in [47] is proved by means of the Barenblatt solutions of the porous medium equation

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^N \times (0, +\infty).$$

In our situation, we do not have self-similar solutions, since our equation in (1.1) is not scaling invariant, in view of the presence of the term $\rho(x)$. Indeed, we construct a suitable subsolution z of equation

$$u_t = \frac{1}{\rho} \Delta u^m \quad \text{in } \mathbb{R}^N \times (0, +\infty).$$

By means of z, we can show that after a certain time, the solution u of problem (1.1) satisfies the hypotheses required by Theorem 2.4. Hence u blows-up in finite time.

The paper is organized as follows. In Section 2 we state our main results, in Section 3 we give the precise definitions of solutions, we establish a local in time existence result and some useful comparison principles. In Section 4 we prove the global existence theorem. The blow-up results are proved in Section 5 for sufficiently big initial data, and in Section 6 for any initial datum.

2 Statements of the main results

In view of (H)-(i), there exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 \leq \rho_2$ such that

$$\rho_1 \leq \frac{1}{\rho(x)} \leq \rho_2 \quad \text{for all } x \in \overline{B_1(0)}. \tag{2.1}$$

As a consequence of hypothesis (H) and (2.1), we can assume that

$$k_1 = \rho_1, \quad k_2 = \rho_2. \tag{2.2}$$

Let $\overline{p}$ be defined by (1.7). It is immediate to see that $\overline{p}$ is monotonically increasing with respect to the ratio $\frac{k_2}{k_1}$; furthermore,

$$\overline{p} > m \,.$$

Define

$$\mathfrak{r}(x) := \begin{cases} |x|^b & \text{if} \quad |x| \geq 1, \\ \frac{b|x|^b + 2 - b}{b} & \text{if} \quad |x| < 1 \,. \end{cases} \tag{2.3}$$

The first result concerns the global existence of solutions to problem (1.1) for $p > \overline{p}$.

Theorem 2.1. *Let assumptions* (H), *(1.6) and (2.2) be satisfied. Suppose that*

$$p > \overline{p},$$

where $\overline{p}$ *is given in (1.7), and that* u_0 *is small enough and has compact support. Then problem (1.1) admits a global solution* $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.

More precisely, if $C > 0$ *is small enough,* $T > 0$ *is big enough,* $a > 0$ *with*

$$\omega_0 \leq \frac{C^{m-1}}{a} \leq \omega_1,$$

for suitable $0 < \omega_0 < \omega_1$,

$$\alpha \in \left(\frac{1}{p-1}, \frac{1}{m-1} \right), \quad \beta = 1 - \alpha(m-1), \tag{2.4}$$

$$u_0(x) \leq CT^{-\alpha} \left[1 - \frac{\mathfrak{r}(x)}{a} T^{-\beta} \right]_+^{\frac{1}{m-1}} \quad \text{for any } x \in \mathbb{R}^N \,, \tag{2.5}$$

then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$. *Moreover,*

$$u(x,t) \leq C(T+t)^{-\alpha} \left[1 - \frac{\mathfrak{r}(x)}{a} (T+t)^{-\beta} \right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \mathbb{R}^N \times [0, +\infty) \,. \tag{2.6}$$

The precise choice of the parameters $C > 0, T > 0$ and $a > 0$ in Theorem 2.1 is discussed in Remark 4.2 below. Observe that if u_0 satisfies (2.5), then

$$\|u_0\|_\infty \leq CT^{-\alpha},$$

$$\operatorname{supp} u_0 \subseteq \{ x \in \mathbb{R}^N \,:\, \mathfrak{r}(x) \leq aT^\beta \} \,.$$

In view of the choice of C, T, a (see also Remark 4.2), $\|u_0\|_\infty$ is small enough, but $\operatorname{supp} u_0$ can be large, since we can select $aT^\beta > r_0$ for any fixed $r_0 > 0$.

Moreover, from (2.6) we can infer that

$$\operatorname{supp} u(\cdot, t) \subseteq \{ x \in \mathbb{R}^N \,:\, \mathfrak{r}(x) \leq a(T+t)^\beta \} \quad \text{for all } t > 0 \,. \tag{2.7}$$

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Remark 2.2. *Note that if $k_1 = k_2$, then*

$$\bar{p} = m + \frac{b}{N - 2 + b}.$$

In particular, for $q = 0$, i.e. $b = 2$, we obtain

$$\bar{p} = m + \frac{2}{N}.$$

Hence, Theorem 2.1 is coherent with the results in [47].

Remark 2.3. *In [31, Theorem 1] a similar global existence result is proved, for $\rho(x) = |x|^{-q}$ for any $x \in \mathbb{R}^N \setminus \{0\}$ with $q \in [0, 2)$ and for suitable u_0 not necessarily compactly supported. Clearly, such ρ does not satisfy assumption (H). Moreover, we can consider a more general behaviour of $\rho(x)$ for $|x|$ large; this affects the definition of $\bar{p}$, and consequently the choice of p. The smallness condition in Theorem 2.1 is different from that in [31], and it is not possible in general to say which is stronger. Moreover, since we consider u_0 with compact support, we can obtain the estimates (2.6) and (2.7), which do not have a counterpart in [31]. Finally, in [31] energy methods are used and a smoothing estimate is derived; hence the proof is completely different from our.*

The next result concerns the blow-up of solutions in finite time, for every $p > 1$ and $m > 1$, provided that the initial datum is sufficiently large.

Let

$$\mathfrak{s}(x) := \begin{cases} |x|^b & \text{if} \quad |x| > 1, \\ |x|^2 & \text{if} \quad |x| \leq 1. \end{cases}$$

Theorem 2.4. *Let assumptions (H) and (2.2) be satisfied. For any $p > 1, m > 1$ and for any $T > 0$, if the initial datum u_0 is large enough, then the solution u of problem (1.1) blows-up in a finite time $S \in (0, T]$, in the sense that*

$$\|u(t)\|_\infty \to +\infty \quad \text{as} \quad t \to S^-. \tag{2.8}$$

More precisely, we have the following three cases.

(a) *Let $p > m$. If $C > 0, a > 0$ are large enough, $T > 0$,*

$$u_0(x) \geq CT^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} T^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}}, \tag{2.9}$$

then the solution u of problem (1.1) blows-up and satisfies the bound from below

$$u(x,t) \geq C(T-t)^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a}(T-t)^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}} \quad \text{for any} \ (x,t) \in \mathbb{R}^N \times [0, S). \tag{2.10}$$

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(b) *Let $p < m$. If $\frac{C^{m-1}}{a} > 0$ and $a > 0$ are big enough, $T > 0$ and (2.9) holds, then the solution u of problem (1.1) blows-up and satisfies the bound from below (2.10).*

(c) *Let $p = m$. If $\frac{C^{m-1}}{a} > 0$ and $a > 0$ are big enough, $T > 0$ and (2.9) holds, then the solution u of problem (1.1) blows-up and satisfies the bound from below (2.10).*

Observe that if u_0 satisfies (2.9), then

$$\operatorname{supp} u_0 \supseteq \{x \in \mathbb{R}^N \ : \ \mathfrak{s}(x) < aT^{\frac{p-m}{p-1}}\}\,.$$

In all the cases $(a), (b), (c)$, from (2.10) we can infer that

$$\operatorname{supp} u(\cdot, t) \supseteq \{x \in \mathbb{R}^N \ : \ \mathfrak{s}(x) < a(T-t)^{\frac{p-m}{p-1}}\} \quad \text{for all } t \in [0, S)\,. \qquad (2.11)$$

The precise choice of parameters $C > 0$, $T > 0$, $a > 0$ in Theorem 2.4 is discussed in Remark 5.2 below.

Remark 2.5. *Let us mention that in [31], where some blow-up results are shown for problem (1.1), there is not a counterpart of Theorem 2.4, since our result concerns any $p > 1$ and sufficiently large initial data.*

2.1 Blow-up for any nontrivial initial datum

In this Subsection we discuss a further result concerning the blow-up of the solution to problem (1.1) for any initial datum $u_0 \in C(\mathbb{R}^N), u_0 \geq 0, u_0 \not\equiv 0$.

Let $\underline{p}$ and be defined by (1.8) and (1.7), respectively. Assume (1.6). It is direct to see that

$$\underline{p} \leq \bar{p}\,. \qquad (2.12)$$

In particular, $\underline{p} = \bar{p}$, whenever $k_1 = k_2$. We distinguish between two cases:

1) $1 < p < m\,,$

2) $m \leq p < \underline{p}\,.$

In case 2), we need an extra hypothesis. In fact, we assume that (H) holds with

$$q \in (0, \epsilon)\,, \qquad (2.13)$$

for some $\epsilon > 0$ to be fixed small enough later. Then, b defined by (1.4), satisfies

$$2 - \epsilon \ < \ b \ < \ 2\,. \qquad (2.14)$$

Theorem 2.6. *Let assumption* (H) *be satisfied. Suppose that*

$$1 < p < m \,,$$

and that $u_0 \in C(\mathbb{R}^N), u_0(x) \not\equiv 0$. *Then, for any sufficiently large* $T > 0$, *the solution* u *of problem* (1.1) *blows-up in a finite time* $S \in (0,T]$, *in the sense that*

$$\|u(t)\|_\infty \to +\infty \quad as \ t \to S^- \,.$$

More precisely, the bound from below (2.10) *holds, with* b, C, a, ζ, η *as in Theorem 2.4-(b).*

Theorem 2.7. *Let assumptions* (H) *and* (2.13) *be satisfied for* $\epsilon > 0$ *small enough. Let* $u_0 \in C^\infty(\mathbb{R}^N)$ *and* $u_0 \not\equiv 0$. *If*

$$m \le p < \underline{p}, \tag{2.15}$$

then there exist sufficiently large $t_1 > 0$ *and* $T > 0$ *such that the solution* u *of problem* (1.1) *blows-up in a finite time* $S \in (0, T + t_1]$, *in the sense that*

$$\|u(t)\|_\infty \to +\infty \quad as \ t \to S^- \,.$$

More precisely, when $S > t_1$, *we have the bound from below*

$$u(x,t) \ge C(T+t_1-t)^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a}(T + t_1 - t)^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}} \quad \textit{for any } (x,t) \in \mathbb{R}^N \times (t_1, S) \,,$$

$$\tag{2.16}$$

with C, a *as in Theorem 2.4-(a).*

Remark 2.8. *As it has been mentioned in the Introduction, in* [31, *Theorem 3] blow-up of solutions to problem* (1.1) *is shown when* $\rho(x) = |x|^{-q}$ *or* $\rho(x) = (1 + |x|)^{-q}$ *with* $q \in [0,2)$. *However, the results in* [31] *are different, in fact it is obtained an integral blow-up, that is, for some* $R > 0$, $\theta \in (0,1)$, $T > 0$, $\int_{B_R}[u(x,t)^\theta]\rho(x)dx \to +\infty$ *as* $t \to T^-$. *On the other hand, we should mention that the extra hypothesis* (2.13), *that we need in Theorem 2.7, in* [31] *is not used. Furthermore, the methods of proofs in* [31] *are completely different, since they are based on the choice of a special test function and integration by parts.*

3   Preliminaries

In this section we give the precise definitions of solution of all problems we address, then we state a local in time existence result for problem (1.1). Moreover, we recall some useful comparison principles.

Throughout the paper we deal with *very weak* solutions to problem (1.1) and to the same problem set in different domains, according to the following definitions.

Definition 3.1. *Let $u_0 \in L^\infty(\mathbb{R}^N)$ with $u_0 \geq 0$. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$ is a solution of problem* (1.1) *if*

$$
\begin{aligned}
-\int_{\mathbb{R}^N} \int_0^\tau \rho(x) u \varphi_t \, dt \, dx = {} & \int_{\mathbb{R}^N} \rho(x) u_0(x) \varphi(x, 0) \, dx \\
& + \int_{\mathbb{R}^N} \int_0^\tau u^m \Delta\varphi \, dt \, dx \qquad (3.1) \\
& + \int_{\mathbb{R}^N} \int_0^\tau \rho(x) u^p \varphi \, dt \, dx
\end{aligned}
$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \tau)), \varphi \geq 0$. Moreover, we say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (3.1) *with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ " with $\varphi \geq 0$.*

For any $x_0 \in \mathbb{R}^N$ and $R > 0$ we set

$$
B_R(x_0) = \{x \in \mathbb{R}^N : \|x - x_0\| < R\}.
$$

When $x_0 = 0$, we write $B_R \equiv B_R(0)$. For every $R > 0$, we consider the auxiliary problem

$$
\begin{cases}
u_t = \frac{1}{\rho(x)} \Delta(u^m) + u^p & \text{in } B_R \times (0, \tau) \\
u = 0 & \text{on } \partial B_R \times (0, \tau) \\
u = u_0 & \text{in } B_R \times \{0\}.
\end{cases}
\qquad (3.2)
$$

Definition 3.2. *Let $u_0 \in L^\infty(B_R)$ with $u_0 \geq 0$. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(B_R \times (0, S))$ for any $S < \tau$ is a solution of problem* (3.2) *if*

$$
\begin{aligned}
-\int_{B_R} \int_0^\tau \rho(x) u \, \varphi_t \, dt \, dx = {} & \int_{B_R} \rho(x) u_0(x) \varphi(x, 0) \, dx \\
& + \int_{B_R} \int_0^\tau u^m \Delta\varphi \, dt \, dx \qquad (3.3) \\
& + \int_{B_R} \int_0^\tau \rho(x) u^p \varphi \, dt \, dx
\end{aligned}
$$

for any $\varphi \in C_c^\infty(\overline{B_R} \times [0, \tau))$ with $\varphi|_{\partial B_R} = 0$ for all $t \in [0, \tau)$. Moreover, we say that a nonnegative function $u \in L^\infty(B_R \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (3.3) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ ", with $\varphi \geq 0$.

Proposition 3.3. *Let hypothesis (H) be satisfied. Then there exists a solution u to problem (3.2) with*

$$\tau \geq \tau_R := \frac{1}{(p-1)\|u_0\|_{L^\infty(B_R)}^{p-1}}.$$

Proof. Note that $\underline{u} \equiv 0$ is a subsolution to (3.2). Moreover, let $\bar{u}_R(t)$ be the solution of the Cauchy problem

$$\begin{cases} \bar{u}'(t) = \bar{u}^p \\ \bar{u}(0) = \|u_0\|_{L^\infty(B_R)}, \end{cases}$$

that is

$$\bar{u}_R(t) = \frac{\|u_0\|_{L^\infty(B_R)}}{\left[1 - (p-1)t\|u_0\|_{L^\infty(B_R)}^{p-1}\right]^{\frac{1}{p-1}}} \quad \text{for all } t \in [0, \tau_R).$$

Clearly, for every $R > 0$, $\bar{u}_R$ is a supersolution of problem (3.2). Due to hypothesis (H),

$$0 < \min_{\bar{B}_R} \frac{1}{\rho} \leq \frac{1}{\rho(x)} \leq \max_{\bar{B}_R} \frac{1}{\rho} \quad \text{for all } x \in \overline{B_R}.$$

Hence, by standard results (see, e.g., [51]), problem (3.2) admits a nonnegative solution $u_R \in L^\infty(B_R \times (0, S))$ for any $S < \tau$, where $\tau \geq \tau_R$ is the maximal time of existence, i.e.

$$\|u_R(t)\|_\infty \to \infty \quad \text{as } t \to \tau_R^-.$$

$\square$

Moreover, the following comparison principle for problem (3.2) holds (see [4] for the proof).

Proposition 3.4. *Let assumption (H) hold. If u is a subsolution of problem (3.2) and v is a supersolution of (3.2), then*

$$u \leq v \quad \text{a.e. in } B_R \times (0, \tau).$$

Proposition 3.5. *Let hypothesis* (H) *be satisfied. Then there exists a solution u to problem* (1.1) *with*

$$\tau \geq \tau_0 := \frac{1}{(p-1)\|u_0\|_\infty^{p-1}}.$$

Moreover, u is the minimal solution*, in the sense that for any solution v to problem* (1.1) *there holds*

$$u \leq v \quad in \quad \mathbb{R}^N \times (0, \tau).$$

Proof. For every $R > 0$ let u_R be the unique solution of problem (3.2). It is easily seen that if $0 < R_1 < R_2$, then

$$u_{R_1} \leq u_{R_2} \quad \text{in} \quad B_{R_1} \times (0, \tau_0). \tag{3.4}$$

In fact, u_{R_2} is a supersolution, while u_{R_1} is a solution of problem (3.2) with $R = R_1$. Hence, by Proposition 3.4, (3.4) follows. Let $\bar{u}(t)$ be the solution of

$$\begin{cases} \bar{u}'(t) = \bar{u}^p \\ \bar{u}(0) = \|u_0\|_\infty, \end{cases}$$

that is

$$\bar{u}(t) = \frac{\|u_0\|_\infty}{\left[1 - (p-1)t\|u_0\|_\infty^{p-1}\right]^{\frac{1}{p-1}}} \quad \text{for all } t \in [0, \tau_0).$$

Clearly, for every $R > 0$, $\bar{u}$ is a supersolution of problem (3.2). Hence

$$0 \leq u_R(x, t) \leq \bar{u} \quad \text{in} \quad B_R \times (0, \tau_0). \tag{3.5}$$

In view of (3.4), the family $\{u_R\}_{R>0}$ is monotone increasing w.r.t. R. Moreover, (3.5) implies that the family $\{u_R\}$ is uniformly bounded. Hence $\{u_R\}_{R>0}$ converges point-wise to a function, say $u(x, t)$, as $R \to +\infty$, i.e.

$$\lim_{R \to +\infty} u_R(x, t) = u(x, t) \quad \text{a.e. in } \mathbb{R}^N \times (0, \tau_0).$$

Moreover, by the monotone convergence theorem, passing to the limit as $R \to +\infty$ in (3.3) we obtain

$$-\int_{\mathbb{R}^N} \int_0^{\tau_0} \rho(x) u \varphi_t \, dt \, dx = \int_{\mathbb{R}^N} \rho(x) u_0(x) \varphi(x, 0) \, dx$$
$$+ \int_{\mathbb{R}^N} \int_0^{\tau_0} u^m \Delta \varphi \, dt \, dx$$
$$+ \int_{\mathbb{R}^N} \int_0^{\tau_0} \rho(x) u^p \varphi \, dt \, dx$$

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for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \tau_0)), \varphi \geq 0$. Hence u is a solution of problem (1.1) $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$, where $\tau \geq \tau_0$ is the maximal time of existence, i.e.

$$\|u(t)\|_\infty \to \infty \quad \text{as } t \to \tau^-.$$

Let us now prove that u is the minimal nonnegative solution to problem (1.1). Let v be any other solution to problem (1.1). Note that, for every $R > 0$, v is a supersolution to problem (3.2). Hence, thanks to Proposition 3.4,

$$u_R \leq v \quad \text{in } B_R \times (0, \tau).$$

Then passing to the limit as $R \to \infty$, we get

$$u \leq v \text{ in } \mathbb{R}^N \times (0, \tau).$$

Therefore, u is the minimal nonnegative solution. $\qquad\square$

In conclusion, we can state the following two comparison results.

Proposition 3.6. *Let hypothesis (H) be satisfied. Let $\bar{u}$ be a supersolution to problem (1.1). Then, if u is the minimal solution to problem (1.1) given by Proposition 3.5, then*

$$u \leq \bar{u} \quad \text{a.e. in } \mathbb{R}^N \times (0, \tau). \tag{3.6}$$

In particular, if $\bar{u}$ exists until time τ, then also u exists at least until time τ.

Proof. Clearly, for any $R > 0$, $\bar{u}$ is a supersolution to problem 3.2. Hence, by Proposition 3.4,

$$u_R \leq \bar{u} \quad \text{in } B_R \times (0, \tau).$$

By passing to the limit as $R \to +\infty$, we easily obtain (3.6), which trivially ensures that u does exist at least up to τ, by the definition of maximal existence time. $\quad\square$

Proposition 3.7. *Let hypothesis (H) be satisfied. Let u be a solution to problem (1.1) for some time $\tau = \tau_1 > 0$ and $\underline{u}$ a subsolution to problem (1.1) for some time $\tau = \tau_2 > 0$. Suppose also that*

$$\operatorname{supp} \underline{u}|_{\mathbb{R}^N \times [0,S]} \text{ is compact for every } S \in (0, \tau_2).$$

Then

$$u \geq \underline{u} \quad \text{in } \mathbb{R}^N \times (0, \min\{\tau_1, \tau_2\}). \tag{3.7}$$

Proof. We fix any $S < \min\{\tau_1, \tau_2\}$. It $R > 0$ is so large that

$$\operatorname{supp} \underline{u}|_{\mathbb{R}^N \times [0,S]} \subseteq B_R \times [0, S],$$

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then u and $\underline{u}$ are a supersolution and a subsolution, respectively, to 3.2. Hence

$$u \geq \underline{u} \quad \text{in} \quad B_R \times (0, S).$$

Inequality (3.7) then just follows by letting $R \to +\infty$ and using the arbitrariness of S. $\quad\square$

Remark 3.8. *Note that by minor modifications in the proof of [?, Theorem] one could show that problem (1.1) admits at most one bounded solution.*

In what follows we also consider solutions of equations of the form

$$u_t = \frac{1}{\rho(x)} \Delta(u^m) + u^p \quad \text{in} \quad \Omega \times (0, \tau), \tag{3.8}$$

where $\Omega \subseteq \mathbb{R}^N$. Solutions are meant in the following sense.

Definition 3.9. *Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(\Omega \times (0, S))$ for any $S < \tau$ is a solution of problem (3.2) if*

$$
\begin{aligned}
-\int_\Omega \int_0^\tau \rho(x) u\, \varphi_t \, dt \, dx &= \int_\Omega \int_0^\tau u^m \Delta\varphi \, dt \, dx \\
&\quad + \int_\Omega \int_0^\tau \rho(x) u^p \varphi \, dt \, dx
\end{aligned}
\tag{3.9}
$$

for any $\varphi \in C_c^\infty(\overline{\Omega} \times [0, \tau))$ with $\varphi|_{\partial\Omega} = 0$ for all $t \in [0, \tau)$. Moreover, we say that a nonnegative function $u \in L^\infty(\Omega \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (3.3) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ ", with $\varphi \geq 0$.

Finally, let us recall the following well-known criterion, that will be used in the sequel; we reproduce it for reader's convenience. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Suppose that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$, and that $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$ is of class C^1. Let n be the unit outwards normal to Ω_1 at Σ. Let

$$
u = \begin{cases} u_1 & \text{in} \ \Omega_1 \times [0, T), \\ u_2 & \text{in} \ \Omega_2 \times [0, T), \end{cases}
\tag{3.10}
$$

where $\partial_t u \in C(\Omega_1 \times (0, T))$, $u_1^m \in C^2(\Omega_1 \times (0, T)) \cap C^1(\overline{\Omega}_1 \times (0, T))$, $\partial_t u_2 \in C(\Omega_2 \times (0, T))$, $)u_2^m \in C^2(\Omega_2 \times (0, T)) \cap C^1(\overline{\Omega}_2 \times (0, T))$.

Lemma 3.1. *Let assumption (H) be satisfied.*

(i) Suppose that

$$\partial_t u_1 \geq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \text{for any } (x,t) \in \Omega_1 \times (0,T),$$

$$\partial_t u_2 \geq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \text{for any } (x,t) \in \Omega_2 \times (0,T),$$

(3.11)

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \geq \frac{\partial u_2^m}{\partial n} \quad \text{for any } (x,t) \in \Sigma \times (0,T). \tag{3.12}$$

Then u, defined in (3.10), is a supersolution to equation (3.8), in the sense of Definition 3.9.

(ii) Suppose that

$$\partial_t u_1 \leq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \text{for any } (x,t) \in \Omega_1 \times (0,T),$$

$$\partial_t u_2 \leq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \text{for any } (x,t) \in \Omega_2 \times (0,T),$$

(3.13)

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \leq \frac{\partial u_2^m}{\partial n} \quad \text{for any } (x,t) \in \Sigma \times (0,T). \tag{3.14}$$

Then u, defined in (3.10), is a subsolution to equation (3.8), in the sense of Definition 3.9.

Proof. Take any $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\tau))$ with $\varphi|_{\partial\Omega} = 0$ for all $t \in [0,\tau), \varphi \geq 0$.

(i) We multiply by φ both sides of the two inequalities in (3.11), then integrating two times by parts we get

$$-\int_0^\tau \int_{\Omega_1} \rho(u_1\varphi_t + u_1^p\varphi)dxdt$$

$$\geq \int_0^\tau u_1^m \Delta\varphi dxdt - \int_0^\tau \int_\Sigma u_1^m \frac{\partial\varphi}{\partial n}d\sigma dt + \int_0^\tau \int_\Sigma \varphi\frac{\partial u_1^m}{\partial n}d\sigma dt,$$

$$-\int_0^\tau \int_{\Omega_2} \rho(u_2\varphi_t - u_2^p\varphi)dxdt$$

$$\geq \int_0^\tau u_2^m \Delta\varphi dxdt + \int_0^\tau \int_\Sigma u_2^m \frac{\partial\varphi}{\partial n}d\sigma dt - \int_0^\tau \int_\Sigma \varphi\frac{\partial u_2^m}{\partial n}d\sigma dt.$$

Summing up the previous two inequalities and using (3.12) we obtain

$$-\int_0^\tau \int_\Omega \rho(u\varphi_t + u^p\varphi)\,dxdt \geq \int_0^\tau u^m \Delta\varphi\,dxdt.$$

Hence the conclusion follows in this case. The statement (ii) can be obtained in the same way. This completes the proof. □

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4   Global existence: proofs

In what follows we set $r \equiv |x|$. We want to construct a suitable family of super-solutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{4.1}$$

To this purpose, we define, for all $(x, t) \in \left[\mathbb{R}^N \setminus B_1(0)\right] \times [0, +\infty)$,

$$u(x, t) \equiv u(r(x), t) := C\zeta(t)\left[1 - \frac{r^b}{a}\eta(t)\right]_+^{\frac{1}{m-1}}, \tag{4.2}$$

where $\eta, \zeta \in C^1([0, +\infty); [0, +\infty))$ and $C > 0$, $a > 0$.

Now, we compute

$$u_t - \frac{1}{\rho}\Delta(u^m) - u^p.$$

To do this, let us set

$$F(r, t) := 1 - \frac{r^b}{a}\eta(t)$$

and define

$$D_1 := \left\{(x, t) \in [\mathbb{R}^N \setminus B_1(0)] \times (0, +\infty) \mid 0 < F(r, t) < 1\right\}.$$

For any $(x, t) \in D_1$, we have:

$$u_t = C\zeta'F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}F^{\frac{1}{m-1}-1}\left(-\frac{r^b}{a}\eta'\right)$$

$$= C\zeta'F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\left(1 - \frac{r^b}{a}\eta\right)\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \tag{4.3}$$

$$= C\zeta'F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1};$$

$$(u^m)_r = -C^m\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{b}{a}\eta r^{b-1}; \tag{4.4}$$

$$(u^m)_{rr} = -C^m\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{b^2}{a}\eta r^{b-2}\left(1 - \frac{r^b}{a}\eta\right)$$

$$+ C^m\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{b^2}{a}\eta r^{b-2} \tag{4.5}$$

$$- C^m\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{b(b-1)}{a}\eta r^{b-2}.$$

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$$\Delta(u^m) = (u^m)_{rr} + \frac{(N-1)}{r}(u^m)_r$$

$$= C^m \zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1} \frac{b^2}{a} \eta r^{b-2}$$

$$- C^m \zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}} \frac{b^2}{a} \eta r^{b-2}$$

$$- C^m \zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}} \frac{b(b-1)}{a} \eta r^{b-2}$$

$$+ \frac{(N-1)}{r} \left(-C^m \zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}} \frac{b}{a} \eta r^{b-1} \right) \tag{4.6}$$

$$= C^m \zeta^m \frac{m}{(m-1)^2} \frac{b^2}{a} \eta F^{\frac{1}{m-1}-1} r^{b-2}$$

$$- C^m (N-2) \zeta^m \frac{m}{m-1} \frac{b}{a} \eta F^{\frac{1}{m-1}} r^{b-2}$$

$$- C^m \zeta^m \frac{m^2}{(m-1)^2} \frac{b^2}{a} \eta F^{\frac{1}{m-1}} r^{b-2} \,.$$

We set $\overline{u} \equiv u$,

$$\overline{w}(x,t) \equiv \overline{w}(r(x),t) := \begin{cases} \overline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_1(0)] \times [0,+\infty), \\ \overline{v}(x,t) & \text{in } B_1(0) \times [0,+\infty), \end{cases} \tag{4.7}$$

where

$$\overline{v}(x,t) \equiv \overline{v}(r(x),t) := C\zeta(t) \left[1 - \frac{(br^2+2-b)}{2} \frac{\eta(t)}{a} \right]_+^{\frac{1}{m-1}} . \tag{4.8}$$

We also define

$$K := \left(\frac{m-1}{p+m-2} \right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2} \right)^{\frac{p+m-2}{p-1}} > 0 \,,$$

$$\bar{\sigma}(t) := \zeta' + \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C^{m-1} \zeta^m \frac{m}{m-1} \frac{b}{a} \eta k_1 \left(N - 2 + \frac{bm}{m-1} \right) ,$$

$$\bar{\delta}(t) := \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C^{m-1} \zeta^m \frac{m}{(m-1)^2} \frac{b^2}{a} \eta k_2, \tag{4.9}$$

$$\bar{\gamma}(t) := C^{p-1} \zeta^p(t) \,,$$

$$\bar{\sigma}_0(t) := \zeta' + \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C^{m-1} \zeta^m \frac{m}{m-1} N b \, k_1 \frac{\eta}{a},$$

$$\bar{\delta}_0(t) := \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C^{m-1} b^2 k_2 \zeta^m \frac{m}{(m-1)^2} \frac{\eta^2}{a^2}.$$

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Proposition 4.1. *Let $\zeta = \zeta(t)$, $\eta = \eta(t) \in C^1([0, +\infty); [0, +\infty))$. Let $K, \bar{\sigma}, \bar{\delta}, \bar{\gamma}, \bar{\sigma}_0, \bar{\delta}_0$ be defined in (4.9). Assume (1.6), (2.2), and that, for all $t \in (0, +\infty)$,*

$$\eta(t) < a, \tag{4.10}$$

$$-\frac{\eta'}{\eta^2} \geq \frac{b^2}{a} C^{m-1} \zeta^{m-1}(t) \frac{m}{m-1} k_2, \tag{4.11}$$

$$\zeta' + C^{m-1} \zeta^m \frac{b}{a} \frac{m}{m-1} \eta \left[k_1 \left(N - 2 + \frac{bm}{m-1} \right) - \frac{k_2 b}{m-1} \right] - C^{p-1} \zeta^p \geq 0, \tag{4.12}$$

$$-\frac{\eta'}{\eta^3} \geq \frac{C^{m-1}}{a^2} k_2 \zeta^{m-1} \frac{m}{m-1}, \tag{4.13}$$

$$\zeta' + N \zeta^m \frac{C^{m-1}}{a} \frac{m}{m-1} \eta \, k_1 - N \zeta^m \frac{C^{m-1}}{a^2} \frac{m}{(m-1)^2} \eta^2 \, k_2 - C^{p-1} \zeta^p \geq 0. \tag{4.14}$$

Then w defined in (4.7) is a supersolution of equation (4.1).

Proof of Proposition 4.1. In view of (4.3), (4.4), (4.5) and (4.6), for any $(x, t) \in D_1$,

$$\begin{aligned}
&\bar{u}_t - \frac{1}{\rho} \Delta(\bar{u}^m) - \bar{u}^p \\
&= C \zeta' F^{\frac{1}{m-1}} + C \zeta \frac{1}{m-1} \frac{\eta'}{\eta} F^{\frac{1}{m-1}} - C \zeta \frac{1}{m-1} \frac{\eta'}{\eta} F^{\frac{1}{m-1}-1} \\
&\quad - \frac{r^{b-2}}{\rho} \left\{ C^m \zeta^m \frac{m}{(m-1)^2} \frac{b^2}{a} \eta F^{\frac{1}{m-1}-1} - C^m (N-2) \zeta^m \frac{m}{m-1} \frac{b}{a} \eta F^{\frac{1}{m-1}} \right. \\
&\quad \left. - C^m \zeta^m \frac{m^2}{(m-1)^2} \frac{b^2}{a} \eta F^{\frac{1}{m-1}} \right\} - C^p \zeta^p F^{\frac{p}{m-1}}.
\end{aligned} \tag{4.15}$$

Thanks to hypothesis (H), we have

$$\frac{r^{b-2}}{\rho} \geq k_1, \quad -\frac{r^{b-2}}{\rho} \geq -k_2 \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0). \tag{4.16}$$

From (4.15) and (4.16) we get

$$\begin{aligned}
&\bar{u}_t - \frac{1}{\rho} \Delta(\bar{u}^m) - \bar{u}^p \\
&\geq C F^{\frac{1}{m-1}-1} \left\{ F \left[\zeta' + \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C^{m-1} \zeta^m \frac{m}{m-1} \frac{b}{a} \eta k_1 \left(N - 2 + \frac{bm}{m-1} \right) \right] \right. \\
&\quad \left. - \zeta \frac{1}{m-1} \frac{\eta'}{\eta} - C^{m-1} \zeta^m \frac{m}{(m-1)^2} \frac{b^2}{a} \eta k_2 - C^{p-1} \zeta^p F^{\frac{p+m-2}{m-1}} \right\}.
\end{aligned} \tag{4.17}$$

From (4.17), taking advantage from $\bar{\sigma}(t)$, $\bar{\delta}(t)$ and $\bar{\gamma}(t)$ defined in (4.9), we have

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq CF^{\frac{1}{m-1}-1}\left[\bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}\right]. \qquad (4.18)$$

For each $t > 0$, set

$$\varphi(F) := \bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}, \quad F \in (0,1).$$

Now our goal is to find suitable C, a, ζ, η such that, for each $t > 0$,

$$\varphi(F) \geq 0 \quad \text{for any } F \in (0,1).$$

We observe that $\varphi(F)$ is concave in the variable F, hence it is sufficient to have $\varphi(F)$ positive in the extrema of the interval of definition $(0,1)$. This reduces to the system

$$\begin{cases} \varphi(0) \geq 0 \\ \varphi(1) \geq 0, \end{cases} \qquad (4.19)$$

for each $t > 0$. The system is equivalent to

$$\begin{cases} -\bar{\delta}(t) \geq 0 \\ \bar{\sigma}(t) - \bar{\delta}(t) - \bar{\gamma}(t) \geq 0, \end{cases}$$

that is

$$\begin{cases} -\frac{\eta'}{\eta^2} \geq \frac{b^2}{a}C^{m-1}\zeta^{m-1}\frac{m}{m-1}k_2 \\[2ex] \zeta' + C^{m-1}\zeta^m\frac{b}{a}\frac{m}{m-1}\eta\left[k_1\left(N-2+\frac{bm}{m-1}\right) - \frac{k_2 b}{m-1}\right] - C^{p-1}\zeta^p \geq 0, \end{cases}$$

which is guaranteed by (1.6), (4.11) and (4.12). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in} \quad D_1.$$

Since $\bar{u}^m \in C^1([\mathbb{R}^N \setminus B_1(0)] \times (0,T))$, in view of Lemma 3.1-(i) (applied with $\Omega_1 = D_1, \Omega_2 = \mathbb{R}^N \setminus [B_1(0) \cup D_1], u_1 = \bar{u}, u_2 = 0, u = \bar{u}$), we can deduce that $\bar{u}$ is a supersolution of equation

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p = 0 \quad \text{in} \quad [\mathbb{R}^N \setminus B_1(0)] \times (0,+\infty), \qquad (4.20)$$

in the sense of Definition 3.9. Now let v be as in (4.8). Set

$$G(r,t) := 1 - \frac{br^2 + 2 - b}{2}\frac{\eta(t)}{a}.$$

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Due to (4.10),

$$0 < G(r, t) < 1 \quad \text{for all } (x, t) \in B_1(0) \times (0, +\infty).$$

For any $(x, t) \in B_1(0) \times (0, +\infty)$, we have:

$$\bar{v}_t = C\zeta' G^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} \frac{\eta'}{\eta} G^{\frac{1}{m-1}} - C\zeta \frac{1}{m-1} \frac{\eta'}{\eta} G^{\frac{1}{m-1}-1}; \tag{4.21}$$

$$(\bar{v}^m)_r = -C^m b \zeta^m \frac{m}{m-1} G^{\frac{1}{m-1}} \frac{\eta}{a} r; \tag{4.22}$$

$$(\bar{v}^m)_{rr} = C^m \zeta^m \frac{m}{(m-1)^2} G^{\frac{1}{m-1}-1} \frac{\eta^2}{a^2} b^2 r^2 - C^m b \zeta^m \frac{m}{m-1} G^{\frac{1}{m-1}} \frac{\eta}{a}. \tag{4.23}$$

Therefore, for all $(x, t) \in B_1(0) \times (0, +\infty)$,

$$\bar{v}_t - \frac{1}{\rho} \Delta(\bar{v}^m) - \bar{v}^p$$

$$= CG^{\frac{1}{m-1}-1} \left\{ G \left[\zeta' + \frac{\zeta}{m-1} \frac{\eta'}{\eta} + b \frac{N-1}{r} C^{m-1} \zeta^m \frac{m}{m-1} \frac{r}{\rho} \frac{\eta}{a} + \frac{b}{\rho} C^{m-1} \zeta^m \frac{m}{m-1} \frac{\eta}{a} \right] \right.$$

$$\left. - \frac{\zeta}{m-1} \frac{\eta'}{\eta} - \frac{r^2}{\rho} b^2 C^{m-1} \frac{m}{(m-1)^2} \zeta^m \frac{\eta^2}{a^2} - C^{p-1} \zeta^p G^{\frac{p+m-2}{m-1}} \right\}. \tag{4.24}$$

Using (2.1) and the fact that $r \in (0, 1)$, (4.24) yields, for all $(x, t) \in B_1(0) \times (0, +\infty)$,

$$\bar{v}_t - \frac{1}{\rho} \Delta(\bar{v}^m) - \bar{v}^p$$

$$\geq CG^{\frac{1}{m-1}-1} \left\{ G \left[\zeta' + \frac{\zeta}{m-1} \frac{\eta'}{\eta} + N b \, k_1 C^{m-1} \zeta^m \frac{m}{m-1} \frac{\eta}{a} \right] \right.$$

$$\left. - \frac{\zeta}{m-1} \frac{\eta'}{\eta} - C^{m-1} b^2 \, k_2 \frac{m}{(m-1)^2} \frac{\eta^2}{a^2} - C^{p-1} \zeta^p G^{\frac{p+m-2}{m-1}} \right\} \tag{4.25}$$

$$= CG^{\frac{1}{m-1}-1} \left[\bar{\sigma}_0(t) G - \bar{\delta}_0(t) - \bar{\gamma}(t) G^{\frac{p+m-2}{m-1}} \right].$$

Hence, due to (4.25), we obtain for all $(x, t) \in B_1(0) \times (0, +\infty)$,

$$\bar{v}_t - \frac{1}{\rho} \Delta(\bar{v}^m) - \bar{v}^p \geq CG^{\frac{1}{m-1}-1} \left[\bar{\sigma}_0(t) G - \bar{\delta}_0(t) - \bar{\gamma}(t) G^{\frac{p+m-2}{m-1}} \right]. \tag{4.26}$$

For each $t > 0$, set

$$\psi(G) := \bar{\sigma}_0(t) G - \bar{\delta}_0(t) - \bar{\gamma}(t) G^{\frac{p+m-2}{m-1}}, \quad G \in (0, 1).$$

Now our goal is to verify that, for each $t > 0$,

$$\psi(G) \geq 0 \quad \text{for any } G \in (0,1).$$

We observe that $\psi(G)$ is concave in the variable G, hence it is sufficient to have $\psi(G)$ positive in the extrema of the interval of definition $(0,1)$. This reduces to the system

$$\begin{cases} \psi(0) \geq 0 \\ \psi(1) \geq 0, \end{cases} \tag{4.27}$$

for each $t > 0$. The system is equivalent to

$$\begin{cases} -\bar{\delta}_0(t) \geq 0 \\ \bar{\sigma}_0(t) - \bar{\delta}_0(t) - \bar{\gamma}(t) \geq 0, \end{cases}$$

that is

$$\begin{cases} -\dfrac{\eta'}{\eta^3} \geq b^2 \dfrac{C^{m-1}}{a^2} k_2 \zeta^{m-1} \dfrac{m}{m-1} \\[3mm] \zeta' + \dfrac{C^{m-1}}{a} bN\, k_1 \zeta^m \dfrac{m}{m-1} \eta - b^2 \dfrac{C^{m-1}}{a^2} k_2 \zeta^m \dfrac{m}{(m-1)^2}\eta^2 - C^{p-1}\zeta^p \geq 0, \end{cases}$$

which is guaranteed by (1.6), (4.13) and (4.14). Hence we have proved that

$$\bar{v}_t - \frac{1}{\rho}\Delta(\bar{v}^m) - \bar{v}^p \geq 0 \quad \text{for all } (x,t) \in B_1(0) \times (0,+\infty) \tag{4.28}$$

Now, observe that $\bar{w} \in C(\mathbb{R}^N \times [0,+\infty))$; indeed,

$$\bar{u} = \bar{v} = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_1(0) \times (0,+\infty).$$

Moreover, $\bar{w}^m \in C^1(\mathbb{R}^N \times [0,+\infty))$; indeed,

$$(\bar{u}^m)_r = (\bar{v}^m)_r = -C^m \zeta(t)^m \frac{m}{m-1} b \frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_1(0) \times (0,+\infty).$$
$$\tag{4.29}$$

In conclusion, by (4.20), (4.25), (4.29) and Lemma 3.1-(i) (applied with $\Omega_1 = \mathbb{R}^N \setminus B_1(0), \Omega_2 = B_1(0), u_1 = \bar{u}, u_2 = \bar{v}, u = \bar{w}$), we can infer that $\bar{w}$ is a supersolution to equation (4.1) in the sense of Definition 3.9. $\square$

Remark 4.2. *Let*

$$p > \bar{p},$$

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and assumptions (1.6) and (2.2) be satisfied. Let $\omega := \frac{C^{m-1}}{a}$. In Theorem 2.1, the precise hypotheses on parameters $\alpha, \beta, C > 0, \omega > 0, T > 0$ are the following:

condition (2.4),

$$\beta - b^2 \omega k_2 \frac{m}{m-1} \geq 0 \,, \tag{4.30}$$

$$-\alpha + b\omega \frac{m}{m-1} \left[k_1 \left(N - 2 + \frac{bm}{m-1} \right) - \frac{k_2 b}{m-1} \right] \geq C^{p-1} \,, \tag{4.31}$$

$$\beta T^\beta \geq b^2 \frac{\omega}{a} k_2 \frac{m}{m-1}, \tag{4.32}$$

$$T^\beta > \frac{r_0}{a} \quad (for \ r_0 > 1), \tag{4.33}$$

$$-\alpha + b\omega \frac{m}{m-1} \left(k_1 N - b \frac{T^{-\beta}}{(m-1)a} k_2 \right) \geq C^{p-1} \,. \tag{4.34}$$

Lemma 4.1. *All the conditions in Remark 4.2 can be satisfied simoultaneously.*

Proof. We take α satisfying (2.4) and

$$\alpha < \min \left\{ \frac{k_1 \left(N - 2 + \frac{bm}{m-1} \right) - \frac{k_2 b}{m-1}}{k_1 \left[m \left(N - 2 + b \right) - (N-2) \right]}, \frac{k_1 N}{bk_2 + (m-1)k_1 N}, \frac{1}{m-1} \right\} \,. \tag{4.35}$$

This is possible, since

$$p > \overline{p} > m + \frac{k_2 b}{k_1 N} > m \,.$$

In view of (4.35), (1.6) and the fact that $\beta = 1 - \alpha(m-1)$, we can take $\omega > 0$ so that (4.30) holds, the left-hand-side of (4.31) is positive, and

$$-\alpha + b\omega \frac{m}{m-1} (k_1 N - \epsilon) > 0 \,,$$

for some $\epsilon > 0$. Then, we choose $C > 0$ so small that (4.31) holds and

$$-\alpha + b\omega \frac{m}{m-1} (k_1 N - \epsilon) > C^{p-1}; \tag{4.36}$$

therefore, also $a > 0$ is properly fixed, in view of the definition of ω. We select $T > 0$ so big that (4.32), (4.33) are valid and

$$k_1 N - b \frac{T^{-\beta}}{(m-1)a} k_2 \geq \epsilon \,. \tag{4.37}$$

From (4.37) and (4.36) inequality (4.34) follows. □

Proof of Theorem 2.1. We prove Theorem 2.1 by means of Proposition 2.1. In view of Lemma 4.1, we can assume that all the conditions of Remark 4.2 are fulfilled.

Set

$$\zeta(t) = (T+t)^{-\alpha}, \quad \eta(t) = (T+t)^{-\beta}, \quad \text{for all } t > 0.$$

Observe that condition (4.33) implies (4.10). Moreover, consider conditions (4.11), (4.12) of Proposition 4.1 with this choice of $\zeta(t)$ and $\eta(t)$. Therefore we obtain

$$\beta - \frac{b^2}{a} C^{m-1} \frac{m}{m-1} k_2 (T+t)^{-\alpha(m-1)-\beta+1} \geq 0 \tag{4.38}$$

and

$$-\alpha(T+t)^{-\alpha-1} + \frac{C^{m-1}}{a} \frac{mb}{m-1} \left[k_1 \left(N - 2 + \frac{bm}{m-1} \right) - \frac{k_2 b}{m-1} \right] (T+t)^{-\alpha m - \beta}$$
$$- C^{p-1}(T+t)^{-\alpha p} \geq 0. \tag{4.39}$$

Since, $\beta = 1 - \alpha(m-1)$, (4.38) and (4.39) become

$$C^{m-1} \frac{m}{m-1} \frac{b}{a} \leq \frac{1 - \alpha(m-1)}{k_2 b}, \tag{4.40}$$

$$\left\{ -\alpha + b \frac{C^{m-1}}{a} \frac{m}{m-1} \left[k_1 \left(N - 2 + \frac{bm}{m-1} \right) - \frac{k_2 b}{m-1} \right] \right\} (T+t)^{-\alpha-1}$$
$$\geq C^{p-1}(T+t)^{-\alpha p}. \tag{4.41}$$

Due to assumption (2.4),

$$\beta > 0, \quad -\alpha - 1 \geq -p\alpha. \tag{4.42}$$

Thus (4.40) and (4.41) follow from (6.47), (4.30) and (4.31).

We now consider conditions (4.13) and (4.14) of Proposition 4.1. Substituting $\zeta(t)$, $\eta(t)$, α and β previously chosen, we get (4.32) and

$$\left[-\alpha + b \frac{C^{m-1}}{a} \frac{m}{m-1} \left(k_1 N - b \frac{(T+t)^{-\beta}}{(m-1)a} k_2 \right) \right] (T+t)^{-\alpha-1} \geq C^{p-1}(T+t)^{-p\alpha}. \tag{4.43}$$

Condition (4.43) follows from (6.47) and (4.34).

Hence, we can choose $\alpha, \beta, C > 0$, $a > 0$ and T so that (4.40), (4.41), (4.32) and (4.43) hold. Thus the conclusion follows by Propositions 4.1 and 3.6. $\square$

5 Blow-up: proofs

Let

$$\underline{w}(x,t) \equiv \underline{w}(r(x),t) := \begin{cases} \underline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_1(0)] \times [0,T), \\ \underline{v}(x,t) & \text{in } B_1(0) \times [0,T), \end{cases} \tag{5.1}$$

where $\underline{u} \equiv u$ is defined in (4.2) and $\underline{v}$ is defined as follows

$$\underline{v}(x,t) \equiv \underline{v}(r(x),t) := C\zeta(t)\left[1 - r^2\frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}}. \tag{5.2}$$

Observe that for any $(x,t) \in B_1(0) \times (0,T)$, we have:

$$\underline{v}_t = C\zeta' G^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1}; \tag{5.3}$$

$$(\underline{v}^m)_r = -2C^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{\eta}{a}r;$$

$$(\underline{v}^m)_{rr} = 4C^m\zeta^m\frac{m}{(m-1)^2}G^{\frac{1}{m-1}-1}\frac{\eta}{a} - 2C^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{\eta}{a}$$
$$- 4C^m\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}G^{\frac{1}{m-1}},$$

$$\Delta(\underline{v}^m) = 4C^m\zeta^m\frac{m}{(m-1)^2}G^{\frac{1}{m-1}-1}\frac{\eta}{a} - 4C^m\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}G^{\frac{1}{m-1}}$$
$$- 2NC^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{\eta}{a}. \tag{5.4}$$

Therefore, from (5.3) and (5.4) we get, for all $(x,t) \in B_1(0) \times (0,T)$,

$$\underline{v}_t - \frac{1}{\rho}\Delta(\underline{v}^m) - \underline{v}^p$$
$$= CG^{\frac{1}{m-1}-1}\left\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + 2NC^{m-1}\zeta^m\frac{m}{m-1}\frac{1}{\rho}\frac{\eta}{a} + \frac{4}{\rho}C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}\right]\right.$$
$$\left. - \frac{\zeta}{m-1}\frac{\eta'}{\eta} - \frac{4}{\rho}C^{m-1}\frac{m}{(m-1)^2}\frac{\eta}{a} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\right\}. \tag{5.5}$$

We also define

$$\underline{\sigma}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a}\eta k_2\left(N-2+\frac{bm}{m-1}\right),$$

$$\underline{\delta}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_1,$$

$$\underline{\gamma}(t) := C^{p-1}\zeta^p,$$

$$\underline{\sigma_0}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + 2C^{m-1}\zeta^m\frac{m}{m-1}\left(N+\frac{2}{m-1}\right)\rho_2\frac{\eta}{a}, \qquad (5.6)$$

$$\underline{\delta_0}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + 4\frac{C^{m-1}}{a}\zeta^m\rho_1\frac{m}{(m-1)^2}\eta,$$

$$K := \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0.$$

Proposition 5.1. *Let $T \in (0,\infty)$, $\zeta, \eta \in C^1([0,T);[0,+\infty))$. Let $\underline{\sigma}, \underline{\delta}, \underline{\gamma}, \underline{\sigma_0}, \underline{\delta_0}, K$ be defined in (5.6). Assume (2.2) and that, for all $t \in (0,T)$,*

$$K[\underline{\sigma}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)[\underline{\gamma}(t)]^{\frac{m-1}{p-1}}, \qquad (5.7)$$

$$(m-1)\underline{\sigma}(t) \le (p+m-2)\underline{\gamma}(t), \qquad (5.8)$$

$$K[\underline{\sigma_0}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta_0}(t)[\underline{\gamma}(t)]^{\frac{m-1}{p-1}}, \qquad (5.9)$$

$$(m-1)\underline{\sigma_0}(t) \le (p+m-2)\underline{\gamma}(t). \qquad (5.10)$$

Then w defined in (5.1) is a weak subsolution of equation (4.1).

Proof of Proposition 5.1. In view of (4.3), (4.4), (4.5) and (4.6) we obtain

$$\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p$$

$$= C\zeta'F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}$$

$$- \frac{r^{b-2}}{\rho}\left\{C^m\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}-1} - C^m\zeta^m\frac{m}{m-1}\frac{b}{a}\eta F^{\frac{1}{m-1}} - C^m\zeta^m\frac{m^2}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}}\right\}$$

$$- C^p\zeta^p F^{\frac{p}{m-1}} \quad \text{for all } (x,t) \in D_1.$$

$$(5.11)$$

In view of hypothesis (H), we can infer that

$$\frac{r^{b-2}}{\rho} \le k_2, \quad -\frac{r^{b-2}}{\rho} \le -k_1 \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0). \qquad (5.12)$$

From (5.11) and (5.12) we have

$$
\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p
$$
$$
\leq CF^{\frac{1}{m-1}-1}\left\{ F\left[\zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m \frac{m}{m-1}\frac{b}{a}\eta k_2\left(N - 2 + \frac{bm}{m-1} \right) \right] \right.
$$
$$
\left. -\zeta\frac{1}{m-1}\frac{\eta'}{\eta} - C^{m-1}\zeta^m \frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_1 - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}} \right\}.
$$
(5.13)

Thanks to (5.6), (5.13) becomes

$$
\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \leq CF^{\frac{1}{m-1}-1}\varphi(F),
\tag{5.14}
$$

where, for each $t \in (0, T)$,

$$
\varphi(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{p+m-2}{m-1}}.
$$

Our goal is to find suitable C, a, ζ, η such that, for each $t \in (0, T)$,

$$
\varphi(F) \leq 0 \quad \text{for any } F \in (0, 1).
$$

To this aim, we impose that

$$
\sup_{F \in (0,1)} \varphi(F) = \max_{F \in (0,1)} \varphi(F) = \varphi(F_0) \leq 0,
$$

for some $F_0 \in (0, 1)$. We have

$$
\frac{d\varphi}{dF} = 0 \iff \underline{\sigma}(t) - \frac{p+m-2}{m-1}\underline{\gamma}(t)F^{\frac{p-1}{m-1}} = 0
$$
$$
\iff F = F_0 = \left[\frac{m-1}{p+m-2}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)} \right]^{\frac{m-1}{p-1}}.
$$

Then

$$
\varphi(F_0) = K\frac{\underline{\sigma}(t)^{\frac{p+m-2}{p-1}}}{\underline{\gamma}(t)^{\frac{m-1}{p-1}}} - \underline{\delta}(t),
$$

where the coefficient K depending on m and p has been defined in (5.6). By hypoteses (5.7) and (5.8), for each $t \in (0, T)$,

$$
\varphi(F_0) \leq 0, \quad F_0 \leq 1.
\tag{5.15}
$$

So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p \leq 0 \quad \text{in} \quad D_1. \tag{5.16}$$

Furthermore, since $\underline{u}^m \in C^1([\mathbb{R}^N \setminus B_1(0)] \times (0,T))$, due to Lemma 3.1 (applied with $\Omega_1 = D_1, \Omega_2 = \mathbb{R}^N \setminus [B_1(0) \cup D_1], u_1 = \underline{u}, u_2 = 0, u = \underline{u})$, it follows that $\underline{u}$ is a subsolution to equation

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p = 0 \quad \text{in} \quad [\mathbb{R}^N \setminus B_1(0)] \times (0,T),$$

in the sense of Definition 3.9.

Let
$$D_2 := \{(x,t) \in B_1(0) \times (0,T) \,:\, 0 < G(r,t) < 1\}.$$

Using (2.1), (5.5) yields, for all $(x,t) \in D_2$,

$$v_t - \frac{1}{\rho}\Delta(v^m) - v^p$$

$$\leq CG^{\frac{1}{m-1}-1}\left\{ G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + 2\left(N + \frac{2}{m-1}\right)k_2C^{m-1}\zeta^m\frac{m}{m-1}\frac{\eta}{a}\right]\right.$$

$$\left. - \frac{\zeta}{m-1}\frac{\eta'}{\eta} - 4C^{m-1}k_1\frac{m}{(m-1)^2}\frac{\eta}{a} - C^{p-1}\zeta^pG^{\frac{p+m-2}{m-1}}\right\}$$

$$= CG^{\frac{1}{m-1}-1}\left[\underline{\sigma}_0(t)G - \underline{\delta}_0(t) - \underline{\gamma}(t)G^{\frac{p+m-2}{m-1}}\right]. \tag{5.17}$$

Now, by the same arguments used to obtain (5.16), in view of (5.9) and (5.10) we can infer that

$$\underline{v}_t - \frac{1}{\rho}\Delta\underline{v}^m \leq \underline{v}^p \quad \text{for any} \ (x,t) \in D_2. \tag{5.18}$$

Moreover, since $\underline{v}^m \in C^1(B_1(0) \times (0,T))$, in view of Lemma 3.1 (applied with $\Omega_1 = D_2, \Omega_2 = B_1(0) \setminus D_2, u_1 = \underline{v}, u_2 = 0, u = \underline{v})$, we get that $\underline{v}$ is a subsolution to equation

$$\underline{v}_t - \frac{1}{\rho}\Delta\underline{v}^m = \underline{v}^p \quad \text{in} \ B_1(0) \times (0,T), \tag{5.19}$$

in the sense of Definition 3.9. Now, observe that $\underline{w} \in C(\mathbb{R}^N \times [0,T))$; indeed,

$$\underline{u} = \underline{v} = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in} \ \partial B_1(0) \times (0,T).$$

Moreover, since $b \in (0,2]$,

$$(\underline{u}^m)_r \geq (\underline{v}^m)_r = -2C^m\zeta(t)^m\frac{m}{m-1}\frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in} \ \partial B_1(0) \times (0,T). \tag{5.20}$$

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In conclusion, in view of (5.20) and Lemma 3.1 (applied with $\Omega_1 = B_1(0), \Omega_2 = \mathbb{R}^N \setminus B_1(0), u_1 = \underline{v}, u_2 = \underline{u}, u = \underline{w}$), we can infer that $\underline{w}$ is a subsolution to equation (4.1), in the sense of Definition 3.9. $\square$

Remark 5.2. *Let* $\omega := \frac{C^{m-1}}{a}$. *In Theorem 2.4 the precise choice of the parameters* $C > 0, a > 0, T > 0$ *are as follows.*

(a) Let $p > m$. *We require that*

$$
K \left\{ \frac{1}{m-1} + bk_2\omega \frac{m}{m-1} \left(\frac{bm}{m-1} + N - 2 \right) \right\}^{\frac{p+m-2}{p-1}}
$$
$$
\leq \frac{C^{m-1}}{m-1} \left[b^2 k_1 \omega \frac{m}{m-1} + \frac{p-m}{p-1} \right], \tag{5.21}
$$

$$
1 + \omega m b k_2 \left(N - 2 + \frac{bm}{m-1} \right) \leq (p+m-2) C^{p-1}, \tag{5.22}
$$

$$
K \left[\frac{1}{m-1} + 2\,k_2\omega \frac{m}{m-1} \left(N + \frac{2}{m-1} \right) \right]^{\frac{p+m-2}{p-1}}
$$
$$
\leq \frac{C^{m-1}}{m-1} \left[4\,k_1\omega \frac{m}{m-1} + \frac{p-m}{p-1} \right], \tag{5.23}
$$

$$
1 + k_2\omega \left(N + \frac{2}{m-1} \right) \leq (p+m-2)C^{p-1}; \tag{5.24}
$$

(b) Let $p < m$. *We require that*

$$
\omega > \frac{(m-p)(m-1)}{b^2(p-1)mk_1}, \tag{5.25}
$$

$$
a \geq \max \left\{ \frac{K \left\{ \frac{1}{m-1} + \omega k_2 \frac{m}{m-1} b \left(N - 2 + \frac{bm}{m-1} \right) \right\}^{\frac{p+m-2}{p-1}}}{\omega \frac{1}{m-1} \left[\omega \frac{m}{m-1} k_1 b^2 - \frac{m-p}{p-1} \right]}, \right.
$$
$$
\left. \frac{K \left\{ \frac{1}{m-1} + 2\omega\,k_2 \frac{m}{m-1} \left(N + \frac{2}{m-1} \right) \right\}^{\frac{p+m-2}{p-1}}}{\omega \frac{1}{m-1} \left[4\,k_1\omega \frac{m}{m-1} - \frac{m-p}{p-1} \right]} \right\}, \tag{5.26}
$$

$$
(p+m-2)\,(a\omega)^{\frac{p-1}{m-1}} \geq \max \left\{ 1 + \omega m\,b\,k_2 \left(\frac{bm}{m-1} + N - 2 \right), \right.
$$
$$
\left. 1 + \omega\,k_2 \left(N + \frac{2}{m-1} \right) \right\}. \tag{5.27}
$$

(c) Let $p = m$. We require that $\omega > 0$,

$$a \geq \max \left\{ \frac{K \left\{\frac{1}{m-1} + \omega k_2 \frac{m}{m-1} b \left(N - 2 + \frac{bm}{m-1}\right)\right\}^2}{b^2 k_1 \omega^2 \frac{m}{(m-1)^2}}, \right.$$
$$\frac{K \left\{\frac{1}{m-1} + 2\omega\, k_2 \frac{m}{m-1} \left(N + \frac{2}{m-1}\right)\right\}^2}{4\, k_1 \omega^2 \frac{m}{(m-1)^2}},$$
$$\frac{1}{2(m-1)\omega} \left[1 + \omega\, m\, b\, k_2 \left(\frac{bm}{m-1} + N - 2\right)\right],$$
$$\left. \frac{1}{2(m-1)\omega} \left[1 + \omega\, k_2 \left(N + \frac{2}{m-1}\right)\right]\right\}. \tag{5.28}$$

Lemma 5.1. *All the conditions of Remark 5.2 can hold simultaneously.*

Proof. (a) We take any $\omega > 0$, then we select $C > 0$ big enough (therefore, $a > 0$ is also fixed, due to the definition of ω) so that (5.21)-(5.24) hold.

(b) We can take $\omega > 0$ so that (5.25) holds, then we take $a > 0$ sufficiently large to guarantee (5.26) and (5.27) (therefore, $C > 0$ is also fixed).

(c) For any $\omega > 0$, we take $a > 0$ sufficiently large to guarantee (5.28) (thus, $C > 0$ is also fixed). $\qquad\square$

Proof of Theorem 2.4. We now prove Theorem 2.4, by means of Proposition 5.1. In view of Lemma 5.1, we can assume that all the conditions in Remark 5.2 are fulfilled. Set
$$\zeta(t) = (T - t)^{-\alpha}, \quad \eta(t) = (T - t)^{\beta}$$
and
$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{m - p}{p - 1}.$$
Then

$$\underline{\sigma}(t) = \left[\frac{1}{m-1} + C^{m-1} \frac{m}{m-1} \frac{b}{a} k_2 \left(N - 2 + \frac{bm}{m-1}\right)\right] (T - t)^{\frac{-p}{p-1}},$$

$$\underline{\delta}(t) := \left[\frac{m-p}{(m-1)(p-1)} + C^{m-1} \frac{m}{(m-1)^2} \frac{b^2}{a} k_1\right] (T - t)^{\frac{-p}{p-1}},$$

$$\underline{\gamma}(t) := C^{p-1} (T - t)^{\frac{-p}{p-1}}.$$

Let $p > m$. Conditions (5.21) and (5.22) imply (5.7) and (5.8), whereas (5.23) and (5.24) imply (5.9) and (5.10). Hence, by Propositions 5.1 and 3.7 the thesis follows in this case.

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Let $p < m$. Conditions (5.26) and (5.27) imply (5.7) and (5.8), whereas conditions (5.23) and (5.24) imply (5.9) and (5.10). Hence, by Propositions 5.1 and 3.7 the thesis follows in this case, too.

Finally, let $p = m$. Condition (5.28) implies (5.7), (5.8), (5.9) and (5.10). Hence, by Propositions 5.1 and 3.7 the thesis follows in this case, too. The proof is complete.

<div style="text-align:right">□</div>

6 Blow-up for any nontrivial initial datum: proofs

Proof of Theorem 2.6. Since $u_0 \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, there exist $\varepsilon > 0, r_0 > 0$ and $x_0 \in \mathbb{R}^N$ such that

$$u_0(x) \geq \varepsilon, \quad \text{for all } x \in B_{r_0}(x_0).$$

Without loss of generality, we can assume that $x_0 = 0$. Let $\underline{w}$ be the subsolution of problem (1.1) considered in Theorem 2.4 (with $a > 0$ and $C > 0$ properly fixed). We can find $T > 0$ sufficiently big in such a way that

$$C\,T^{-\frac{1}{p-1}} \leq \varepsilon, \quad a\,T^{-\frac{m-p}{p-1}} \leq \min\{r_0^b, r_0^2\}. \tag{6.1}$$

From inequalities in (6.1), we can deduce that

$$\underline{w}(x,0) \leq u_0(x) \quad \text{for any } x \in \mathbb{R}.$$

Hence, by Theorem 2.4 and the comparison principle, the thesis follows. □

Let us explain the strategy of the proof of Theorem 2.7. Let u be a solution to problem (1.1) and let $\underline{w}$ be the subsolution to problem (1.1) given by Theorem 2.4. We look for a subsolution z to the equation

$$z_t = \frac{1}{\rho(x)}\Delta(z^m) \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{6.2}$$

such that

$$z(x,0) \leq u_0(x) \quad \text{for any } x \in \mathbb{R}^N, \tag{6.3}$$

and

$$z(x,t_1) \geq \underline{w}(x,0) \quad \text{for any } x \in \mathbb{R}^N \tag{6.4}$$

for $t_1 > 0$ and $T > 0$ large enough. Let $\tau > 0$ be the maximal existence time of u. If $\tau \leq t_1$, then nothing has to be proved, and $u(x,t)$ blows-up at a certain time

$S \in (0, t_1]$. Suppose that $\tau > t_1$. Since z is also a subsolution to problem (1.1), due to (6.3) and the comparison principle,

$$z(x,t) \le u(x,t) \quad \text{for any } (x,t) \in \mathbb{R}^N \times (0, \tau). \tag{6.5}$$

From (6.4) and (6.5),

$$u(x, t_1) \ge z(x, t_1) \ge \underline{w}(x, 0) \quad \text{for any } x \in \mathbb{R}^N.$$

Thus $u(x, t + t_1)$ is a supersolution, whereas $\underline{w}(x,t)$ is a subsolution of problem

$$\begin{cases} u_t = \frac{1}{\rho} \Delta(u^m) + u^p & \text{in } \mathbb{R}^N \times (0, +\infty) \\ u(x, t_1) = \underline{w}(x, 0) & \text{in } \mathbb{R}^N \times \{0\}. \end{cases}$$

Hence by Theorem 2.4, $u(x,t)$ blows-up in a finite time $S \in (t_1, t_1 + T)$.

In order to construct a suitable family of subsolutions of equation (6.2), let us consider two functions $\eta(t), \zeta(t) \in C^1([0, +\infty); [0, +\infty))$ and two constants $C_1 > 0$, $a_1 > 0$. Define

$$z(x,t) \equiv z(r(x), t) := \begin{cases} \xi(x,t) & \text{in } [\mathbb{R}^N \setminus B_1(0)] \times (0, +\infty) \\ \mu(x,t) & \text{in } B_1(0) \times (0, +\infty), \end{cases} \tag{6.6}$$

where

$$\xi(x,t) \equiv \xi(r(x), t) := C_1 \zeta(t) \left[1 - \frac{r^b}{a_1} \eta(t) \right]_+^{\frac{1}{m-1}} \tag{6.7}$$

and

$$\mu(x,t) \equiv \xi(r(x), t) := C_1 \zeta(t) \left[1 - \frac{br^2 + 2 - b}{2a_1} \eta(t) \right]_+^{\frac{1}{m-1}}. \tag{6.8}$$

Let us set

$$F(r,t) := 1 - \frac{r^b}{a_1} \eta(t), \quad G(r,t) := 1 - \frac{br^2 + 2 - b}{2a_1} \eta(t)$$

and define

$$D_1 := \left\{ (x,t) \in [\mathbb{R}^N \setminus B_1(0)] \times (0, +\infty) \mid 0 < F(r,t) < 1 \right\},$$

$$D_2 := \left\{ (x,t) \in B_1(0) \times (0, +\infty) \mid 0 < G(r,t) < 1 \right\}.$$

Furthermore, for $\epsilon_0 > 0$ small enough, let

$$\beta_0 = \frac{b \dfrac{k_1}{k_2}}{(m-1)(N-2) + bm}, \tag{6.9}$$

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$$\alpha_0 := \frac{1 - \beta_0}{m - 1} = \frac{N - 2 + \dfrac{b}{m - 1}\left(m - \dfrac{k_1}{k_2}\right)}{(m - 1)(N - 2) + bm}, \tag{6.10}$$

$$\tilde{\beta}_0 = \frac{2\dfrac{k_1}{k_2} - \epsilon_0}{N(m - 1) + 2}, \tag{6.11}$$

$$\tilde{\alpha}_0 := \frac{1 - \tilde{\beta}_0}{m - 1} = \frac{N(m - 1) + 2 - 2\dfrac{k_1}{k_2} + \epsilon_0}{(m - 1)\left[N(m - 1) + 2\right]}, \tag{6.12}$$

Observe that

$$0 < \beta_0 < 1, \qquad 0 < \tilde{\beta}_0 < 1. \tag{6.13}$$

Note that, if $\epsilon_0 > 0$ is small enough, then

$$0 < \beta_0 < \tilde{\beta}_0. \tag{6.14}$$

Proposition 6.1. *Let assumption (H) be satisfied. Assume that (2.13) holds, for $\epsilon > 0$ small enough. Let*

$$\bar{\beta} \in (0, \beta_0), \tag{6.15}$$

$$\bar{\alpha} := \frac{1 - \bar{\beta}}{m - 1}. \tag{6.16}$$

Suppose that

$$1 < p < m + \frac{\bar{\beta}}{\bar{\alpha}}. \tag{6.17}$$

Let $T_1 \in (0, \infty)$,

$$\zeta(t) = (T_1 + t)^{-\bar{\alpha}}, \quad \eta(t) = (T_1 + t)^{-\bar{\beta}}.$$

Then there exist $\omega_1 := \dfrac{C_1^{m-1}}{a_1} > 0$, $t_1 > 0$ and $T > 0$ such that z defined in (6.6) is a subsolution of equation (6.2) and satisfies (6.3) and (6.4).

Proof. We can argue as we have done to obtain (5.13), in order to get

$$\xi_t - \frac{1}{\rho}\Delta(\xi^m)$$

$$\leq C_1 F^{\frac{1}{m-1} - 1}\left\{F\left[\zeta' + \zeta\frac{1}{m - 1}\frac{\eta'}{\eta} + C_1^{m-1}\zeta^m\frac{m}{m - 1}\frac{b}{a_1}\eta k_2\left(N - 2 + \frac{bm}{m - 1}\right)\right]\right.$$

$$\left. - \zeta\frac{1}{m - 1}\frac{\eta'}{\eta} - C_1^{m-1}\zeta^m\frac{m}{(m - 1)^2}\frac{b^2}{a_1}\eta k_1\right\} \quad \text{for all } (x, t) \in D_1.$$

$$\tag{6.18}$$

We now define

$$\sigma(t) := \zeta' + \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C_1^{m-1} \zeta^m \frac{m}{m-1} \frac{b}{a} \eta k_2 \left(N - 2 + \frac{bm}{m-1} \right),$$

$$\delta(t) := \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + C_1^{m-1} \zeta^m \frac{m}{(m-1)^2} \frac{b^2}{a} \eta k_1. \tag{6.19}$$

Hence, (6.18) becomes

$$\xi_t - \frac{1}{\rho} \Delta(\xi^m) \le C_1 F^{\frac{1}{m-1}-1} \bar\varphi(F) \quad \text{in } D_1, \tag{6.20}$$

where

$$\bar\varphi(F) := \sigma(t) F - \delta(t). \tag{6.21}$$

Observe that ξ is a subsolution to equation

$$\xi_t - \frac{1}{\rho} \Delta(\xi^m) = 0 \quad \text{in } D_1, \tag{6.22}$$

whenever, for any $t > 0$

$$\bar\varphi(F) \le 0,$$

that is

$$\begin{cases} \sigma(t) > 0 \\ \delta(t) > 0 \qquad\qquad \text{for any } t > 0 \\ \sigma(t) - \delta(t) \le 0. \end{cases} \tag{6.23}$$

By using the very definition of ζ and η, we get

$$\sigma(t) = -\bar\alpha(T_1+t)^{-\bar\alpha-1} - \frac{\bar\beta}{m-1}(T_1+t)^{-\bar\alpha-1} + \frac{C_1^{m-1}}{a_1} k_2 \frac{m}{m-1} b \left(N - 2 + \frac{bm}{m-1} \right) (T_1+t)^{-\bar\alpha m - \bar\beta},$$

$$\delta(t) = -\frac{\bar\beta}{m-1}(T_1+t)^{-\bar\alpha-1} + \frac{C_1^{m-1}}{a_1} k_1 \frac{m}{(m-1)^2} b^2 (T_1+t)^{-\bar\alpha m - \bar\beta}.$$

By (6.13), (6.15) and (6.16),

$$0 < \bar\beta < 1, \quad \bar\alpha > 0. \tag{6.24}$$

Due to (6.16), (6.23) becomes

$$\begin{cases} -1 + \dfrac{C_1^{m-1}}{a_1} k_2 \, m \, b \left(N - 2 + \dfrac{bm}{m-1} \right) > 0, \\ -\bar\beta + \dfrac{C_1^{m-1}}{a_1} k_1 \dfrac{m}{m-1} b^2 > 0, \\ \bar\beta - 1 + \dfrac{C_1^{m-1}}{a_1} b \, m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1} \right) - k_1 \dfrac{b}{m-1} \right] \le 0, \end{cases} \tag{6.25}$$

which reduces to

$$\frac{C_1^{m-1}}{a_1} \geq \max \left\{ \frac{1}{b\,m\,k_2 \left(N - 2 + \dfrac{bm}{m-1} \right)}, \frac{\bar{\beta}(m-1)}{b^2\,m\,k_1} \right\}, \tag{6.26}$$

$$\frac{C_1^{m-1}}{a_1} \leq \frac{1 - \bar{\beta}}{b\,m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1} \right) - k_1 \dfrac{b}{m-1} \right]}. \tag{6.27}$$

If (6.26) and (6.27) are verified, then ξ is a subsolution to equation (6.22). We now show that it is possible to find $\omega_1 := \frac{C_1^{m-1}}{a_1}$ such that (6.26) (6.27) hold. Such ω_1 can be selected, if

$$\frac{1}{b\,m\,k_2 \left(N - 2 + \dfrac{bm}{m-1} \right)} < \frac{1 - \bar{\beta}}{b\,m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1} \right) - k_1 \dfrac{b}{m-1} \right]}, \tag{6.28}$$

and

$$\frac{\bar{\beta}(m-1)}{b^2\,m\,k_1} < \frac{1 - \bar{\beta}}{b\,m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1} \right) - k_1 \dfrac{b}{m-1} \right]}. \tag{6.29}$$

Conditions (6.28) and (6.29) are satisfied, if

$$\bar{\beta} < \beta_0. \tag{6.30}$$

Finally, condition (6.30) is guaranteed by hypothesis (6.15). Moreover, by Lemma 3.1, ξ is a subsolution to equation

$$\xi_t - \frac{1}{\rho(x)} \Delta \xi^m = 0 \quad \text{in } [\mathbb{R}^N \setminus B_1(0)] \times (0, T). \tag{6.31}$$

in the sense of Definition 3.9. We can argue as we have done to obtain (5.17), in order to get

$$\mu_t - \frac{1}{\rho} \Delta(\mu^m)$$

$$\leq C_1 G^{\frac{1}{m-1}-1} \left\{ G \left[\zeta' + \frac{\zeta}{m-1} \frac{\eta'}{\eta} + b\,k_2 \frac{m}{m-1} \frac{C_1^{m-1}}{a_1} \zeta^m \eta \left(N + \frac{2}{m-1} \right) \right] \right.$$

$$\left. - \frac{\zeta}{m-1} \frac{\eta'}{\eta} - 2\,k_1\,b \frac{C_1^{m-1}}{a_1} \frac{m}{(m-1)^2} \zeta^m \eta + (2-b)\,k_2\,b \frac{C_1^{m-1}}{a_1^2} \frac{m}{(m-1)^2} \zeta^m \eta^2 \right\}$$

for any $(x, t) \in D_2$.

$$\tag{6.32}$$

We now define

$$\underline{\sigma}_0(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + b\,k_2\,\frac{C_1^{m-1}}{a_1}\zeta^m\frac{m}{m-1}\left(N + \frac{2}{m-1}\right)\eta,$$

$$\underline{\delta}_0(t) := -\frac{\zeta}{m-1}\frac{\eta'}{\eta} + 2\,k_1\,b\,\frac{C_1^{m-1}}{a_1}\frac{m}{(m-1)^2}\zeta^m\,\eta - (2-b)\,k_2\,b\,\frac{C_1^{m-1}}{a_1^2}\frac{m}{(m-1)^2}\,\zeta^m\,\eta^2\,.$$

Hence, (6.32) becomes,

$$\mu_t - \frac{1}{\rho}\Delta(\mu^m) \le C_1\,G^{\frac{1}{m-1}-1}\,\phi(G)\ \text{ in } D_2\,, \qquad (6.33)$$

where

$$\phi(G) := \underline{\sigma}_0(t)G - \underline{\delta}_0(t)\,.$$

By arguing as above, we can infer that

$$\mu_t - \frac{1}{\rho}\Delta(\mu^m) \le 0 \quad \text{in } D_2\,, \qquad (6.34)$$

provided that

$$\begin{cases} \sigma_0(t) > 0 \\ \delta_0(t) > 0 \\ \sigma_0(t) - \delta_0(t) \le 0\,. \end{cases} \qquad \text{for any } t \in (0, T_1) \qquad (6.35)$$

By using the very definition of ζ and η, (6.35) becomes

$$-1 + b\,k_2\,\frac{C_1^{m-1}}{a_1}\,m\left(N + \frac{2}{m-1}\right) > 0,$$

$$-\bar\beta + 2\,b\,k_1\,\frac{C_1^{m-1}}{a_1}\frac{m}{m-1} - (2-b)\,b\,k_2\,\frac{C_1^{m-1}}{a_1^2}\frac{m}{m-1}(T_1+t)^{-\bar\beta} > 0,$$

$$\bar\beta - 1 + b\,k_2\,m\,\frac{C_1^{m-1}}{a_1}\,N + \frac{2}{m-1}\left(1 - \frac{k_1}{k_2}\right) + (2-b)\,k_2\,b\,\frac{C_1^{m-1}}{a_1^2}\frac{m}{m-1}(T_1+t)^{-\bar\beta} \le 0\,, \qquad (6.36)$$

which reduces to

$$\frac{C_1^{m-1}}{a_1} > \max\left\{ \frac{1}{b\,m\,k_2\left(N + \dfrac{2}{m-1}\right)}\,,\, \frac{\bar\beta(m-1)}{b\,m\,k_2\left[2\dfrac{k_1}{k_2} - \dfrac{2-b}{a_1}(T_1+t)^{-\bar\beta}\right]} \right\}\,, \qquad (6.37)$$

$$\frac{C_1^{m-1}}{a_1} \le \frac{1-\bar\beta}{b\,m\,k_2\left[N + \dfrac{2}{m-1}\left(1 - \dfrac{k_1}{k_2}\right) + \dfrac{2-b}{a_1}\dfrac{(T_1+t)^{-\bar\beta}}{m-1}\right]}\,. \qquad (6.38)$$

If (6.37) and (6.38) are verified then μ is a subsolution to equation

$$\mu_t - \frac{1}{\rho}\Delta\mu^m = 0 \quad \text{in } D_2.$$

In order to find $\omega_1 = \frac{C_1^{m-1}}{a_1}$ satisfying (6.37) and (6.38), we need

$$\frac{1}{bmk_2\left(N + \dfrac{2}{m-1}\right)} < \frac{1-\bar\beta}{bmk_2\left[N + \dfrac{2}{m-1}\left(1 - \dfrac{k_1}{k_2}\right) + \dfrac{2-b}{a_1}\dfrac{(T_1+t)^{-\bar\beta}}{m-1}\right]},$$

(6.39)

and

$$\frac{\bar\beta(m-1)}{bmk_2\left[2\dfrac{k_1}{k_2} - \dfrac{2-b}{a_1}(T_1+t)^{-\bar\beta}\right]} < \frac{1-\bar\beta}{bmk_2\left[N + \dfrac{2}{m-1}\left(1 - \dfrac{k_1}{k_2}\right) + \dfrac{2-b}{a_1}\dfrac{(T_1+t)^{-\bar\beta}}{m-1}\right]}.$$

(6.40)

Now we choose in (2.13) $\epsilon = \epsilon(a_1, T_1) > 0$ so that

$$\frac{\epsilon}{a_1}T_1^{-\bar\beta} \le \epsilon_0,$$

(6.41)

with ϵ_0 used in (6.11) and (6.12) to be appropriately fixed. By (2.13), (2.14) and (6.41),

$$\frac{2-b}{a_1}(T_1+t)^{-\bar\beta} < \frac{\epsilon}{a_1}T_1^{-\bar\beta} \le \epsilon_0.$$

So, conditions (6.39) and (6.40) are fulfilled, if

$$\frac{1}{b\,m\,k_2\left(N + \dfrac{2}{m-1}\right)} < \frac{1-\bar\beta}{b\,m\,k_2\left[N + \dfrac{2}{m-1}\left(1 - \dfrac{k_1}{k_2}\right) + \dfrac{\epsilon_0}{m-1}\right]},$$

(6.42)

and

$$\frac{\bar\beta(m-1)}{b\,m\,k_2\left[2\dfrac{k_1}{k_2} - \epsilon\right]} < \frac{1-\bar\beta}{b\,m\,k_2\left[N + \dfrac{2}{m-1}\left(1 - \dfrac{k_1}{k_2}\right) + \dfrac{\epsilon_0}{m-1}\right]}.$$

(6.43)

Finally, conditions (6.42) and (6.43) are satisfied, if

$$\bar\beta < \tilde\beta_0,$$

(6.44)

provided that $\epsilon_0 > 0$ is small enough. Observe that (6.44) is guaranteed due to hypothesis (6.14) and (6.15). Moreover, since $\mu^m \in C^1(B_1(0) \times (0, T_1))$, by Lemma 3.1, μ is a subsolution to

$$\mu_t - \frac{1}{\rho}\Delta(\mu^m) = 0 \quad \text{in } B_1(0) \times (0, T_1), \tag{6.45}$$

in the sense of Definition 3.9. Hence z is a subsolution of equation (6.2).

Since $u_0 \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, there exist $r_0 > 0$ and $\varepsilon > 0$ such that

$$u_0(x) > \varepsilon \quad \text{in } B_{r_0}(0).$$

Hence, if

$$\operatorname{supp} z(\cdot, 0) \subset B_{r_0}(0), \tag{6.46}$$

and

$$z(x, 0) \leq \varepsilon \quad \text{in } B_{r_0}(0), \tag{6.47}$$

then (6.3) follows. Moreover, if

$$\operatorname{supp} \underline{w}(\cdot, 0) \subset \operatorname{supp} z(\cdot, t_1), \tag{6.48}$$

and

$$\underline{w}(x, 0) \leq z(x, t_1) \text{ for all } x \in \mathbb{R}^N, \tag{6.49}$$

then (6.4) follows.

We first verify that z satisfies condition (6.46) and (6.47). If we require that

$$a_1 T_1^{\bar{\beta}} \leq \frac{r_0^2}{2}. \tag{6.50}$$

then

$$\operatorname{supp} z(\cdot, 0) \cap B_1(0) \subset B_{r_0}(0),$$

and

$$\operatorname{supp} z(\cdot, 0) \cap [\mathbb{R}^N \setminus B_1(0)] \subset B_{r_0}(0),$$

therefore (6.46) holds. Moreover, if

$$(a_1 \omega)^{\frac{1}{m-1}} \leq \varepsilon T_1^{\bar{\alpha}}, \tag{6.51}$$

then (6.47) holds. Obviously, for any $T_1 > 0$ we can choose $a_1 = a_1(T_1) > 0$ such that (6.50) and (6.51) are valid. On the other hand,

$$\operatorname{supp} \underline{w}(\cdot, 0) \cap B_1(0) \subset \operatorname{supp} z(\cdot, t_1) \cap B_1(0),$$

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and if
$$a_1 \left(T_1 + t_1\right)^{\bar{\beta}} \geq a\, T^{\frac{p-m}{p-1}} \tag{6.52}$$

then,
$$\text{supp } \underline{w}(\cdot, 0) \cap [\mathbb{R}^N \setminus B_1(0)] \subset \text{supp } z(\cdot, t_1) \cap [\mathbb{R}^N \setminus B_1(0)].$$

Hence, (6.48) holds. If
$$C_1 \left(T_1 + t_1\right)^{-\bar{\alpha}} \geq C\, T^{-\frac{1}{p-1}}, \tag{6.53}$$

then (6.49) holds. If we choose the equality in (6.53),
$$T_1 + t_1 = \left(\frac{C}{C_1}\right)^{-\frac{1}{\bar{\alpha}}} T^{\frac{1}{(p-1)\bar{\alpha}}},$$

then (6.52) becomes
$$\left(\frac{C}{C_1}\right)^{-\frac{\bar{\beta}}{\bar{\alpha}}} a_1\, T^{\frac{\bar{\beta}}{\bar{\alpha}}\frac{1}{(p-1)}} \geq a\, T^{\frac{p-m}{p-1}} .$$

The latter holds, if
$$T^{\frac{p-m-\frac{\bar{\beta}}{\bar{\alpha}}}{p-1}} \leq \left(\frac{C}{C_1}\right)^{-\frac{\bar{\beta}}{\bar{\alpha}}} \frac{a_1}{a} . \tag{6.54}$$

Condition (6.54) is satisfied thanks to (6.17), for $T > 0$ sufficiently large. This completes the proof. $\square$

Proof of Theorem 2.7. Let $\tau > 0$ be the maximal existence time of u. If $\tau \leq t_1$, then nothing has to be showed, and u blows-up at a certain time $S \in (0, t_1]$. Suppose $\tau > t_1$. Let us consider the subsolution z of equation (6.2) as defined in (6.6). Since $p < \underline{p}$, we can find $\bar{\beta}$ (and so $\bar{\alpha}$) such that (6.15), (6.16) and (6.17) hold. By Proposition 6.1, z satisfies (6.3) and (6.4). Thanks to condition (6.3) and the comparison principle, we have (6.5). From (6.4) and (6.5),
$$u(x, t_1) \geq z(x, t_1) \geq \underline{w}(x, 0) \quad \text{for any } x \in \mathbb{R}^N.$$

Thus $u(x, t + t_1)$ is a supersolution, whereas $\underline{w}(x, t)$ is a subsolution of problem
$$\begin{cases} u_t = \frac{1}{\rho}\Delta(u^m) + u^p & \text{in } \mathbb{R}^N \times (0, +\infty) \\ u = \underline{w} & \text{in } \mathbb{R}^N \times \{0\} . \end{cases}$$

Hence by Theorem 2.4, $u(x, t)$ blows-up in a finite time $S \in (t_1, t_1 + T)$. This completes the proof. $\square$

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