SEMILINEAR ELLIPTIC EQUATIONS INVOLVING MIXED LOCAL AND NONLOCAL OPERATORS

STEFANO BIAGI, SERENA DIPIERRO, ENRICO VALDINOCI, AND EUGENIO VECCHI

Abstract. In this paper, we consider an elliptic operator obtained as the superposition of a classical second-order differential operator and a nonlocal operator of fractional type. Though the methods that we develop are quite general, for concreteness we focus on the case in which the operator takes the form $-\Delta + (-\Delta)^s$, with $s \in (0, 1)$. We focus here on symmetry properties of the solutions and we prove a radial symmetry result, based on the moving plane method, and a one-dimensional symmetry result, related to a classical conjecture by G.W. Gibbons.

1. Introduction

In this article we discuss some symmetry properties for the solutions of semilinear equations driven by a mixed operator. Specifically, we will consider operators that combine local and nonlocal features. For the sake of concreteness, we focus on operators of the form

$$\mathcal{L} := -\Delta + (-\Delta)^s \tag{1.1}$$

where $s \in (0, 1)$ and

$$(-\Delta)^s u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy.$$

The study of mixed operators has a consolidated interest in the recent literature, both in terms of theoretical studies and in view of real-world applications. The development of the theory includes, among the others, viscosity solutions methods (see [33, 34, 4, 10, 18, 2, 3]), parabolic equations (see [25]), Aubry-Mather theory (see [19]), Cahn-Hilliard equations (see [17]), porous medium equations (see [20]) phase transitions (see [15]), fractional damping effects (see [21]), Bernstein-type

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regularity results (see [14]), existence and non-existence results (see [1, 39]), regularity theory (see [8]). Concrete applications of mixed operators also arise naturally in plasma physics (see [12]) and population dynamics (see [23]), and numerical methods have been also developed to take into account the specifics of mixed operators (see [11]).

In this article, we provide two sets of symmetry results for solutions of semilinar equations driven by mixed operators: the first type of results deals with the radial symmetry of the solutions, and relies on the moving plane method; the second type of results is inspired by a classical conjecture by G.W. Gibbons and establishes the one-dimensional symmetry of the global solutions that attain uniformly their limit values at infinity.

In this spirit, the first symmetry result that we present is as follows:

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function, and let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with smooth boundary. We assume that Ω is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$.*

If $u \in C(\mathbb{R}^N)$ is any weak solution of

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

then u is symmetric with respect to $\{x_1 = 0\}$ and increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.

As usual, from Theorem 1.1 one deduces that if Ω is a ball, then the solutions of (1.2) are necessarily radial and radially decreasing.

The proof of Theorem 1.1 that we present combines the integral formulation of the moving plane method (see [38, 40]) with suitable adaptations of some results in [36], where the case of integral equations was taken into account by introducing a new small-volume maximum principle and a strong maximum principle for antisymmetric supersolutions. See also [26, 35, 22, 24, 6, 41] for related moving plane methods in the nonlocal setting.

In terms of one-dimensional symmetry for global solutions under uniform limit assumptions, we have the following result:

Theorem 1.2. *Let $f \in C^1(\mathbb{R})$ be such that*

$$\sup_{|r| \geq 1} f'(r) < 0. \tag{1.3}$$

Let $u \in C^3(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N)$ be a solution of the problem

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \mathbb{R}^N, \\ \lim\limits_{t \to \pm\infty} u(y,t) = \pm 1 & \text{uniformly for } y \in \mathbb{R}^{N-1}. \end{cases} \tag{1.4}$$

Then, there exists $u_0 : \mathbb{R} \to \mathbb{R}$ such that

$$u(y,t) = u_0(t) \qquad \text{for every } x = (y,t) \in \mathbb{R}^N. \tag{1.5}$$

The result in Theorem 1.2 is inspired by a classical conjecture by G.W. Gibbons, formulated when $\mathcal{L}$ was the classical Laplace operator and motivated by the cosmological problem of detecting the shape of the interfaces which "separate" the different regions of the universe after the big bang (see [32]).

The classical Gibbons conjecture was established, independently and with different methods, by [5, 7, 27]. See also [28, 29, 30] for related results.

The fractional version of Gibbons conjecture (i.e., the case in which the operator in (1.4) is the fractional Laplacian) has been established in [31, 16]. As a matter of fact, the method developed in [31] is very general and comprises a number of different operators in a unified way: for this, our proof of Theorem 1.2 will rely on the general structure provided in [31] by showing that the structural hypothesis of [31] are fulfilled in the case that we consider here.

In the rest of the paper we provide the proof of Theorem 1.1, which is contained in Section 2, and that of Theorem 1.2, which is contained in Section 3.

2. Radial symmetry and proof of Theorem 1.1

In this section, we prove Theorem 1.1. To this end, without loss of generality, we may assume that

$$\inf_{x\in\Omega} x_1 = -1.$$

We will combine the integral version of the moving plane method (see [38]) with a suitable generalization of a strong maximum principle for *antisymmetric supersolutions* (see [36]).

Let us now introduce and fix some notation needed in what follows. We define the bilinear form

$$\begin{aligned} B(u,v) := &\int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle \, dx \\ &+ \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy, \end{aligned} \tag{2.1}$$

and the function space

$$\mathcal{D}(\Omega) := \left\{ u \in H^1(\mathbb{R}^N) \text{ s.t. } u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}. \tag{2.2}$$

In this setting, we give the following definition of weak solution of (1.2):

Definition 2.1. We say that a function $u : \Omega \to \mathbb{R}$ is a weak solution of (1.2) if $u \in \mathcal{D}(\Omega)$, $u > 0$ in Ω, and

$$B(u,\varphi) = \int_{\mathbb{R}^N} f(u(x))\varphi(x)\, dx, \tag{2.3}$$

for any $\varphi \in \mathcal{D}(\Omega)$.

Also, given a set $U \subset \mathbb{R}^N$, we let

$$\rho(v,U) := \int_U |\nabla v|^2 + [v]^2_{H^s(U)}, \tag{2.4}$$

where

$$[v]^2_{H^s(U)} := \iint_{U\times U} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\, dx\, dy,$$

and

$$\mathcal{H}(U) := \left\{ v \in L^2(\mathbb{R}^N) \text{ s.t. } v \in H^1(U) \right\}. \tag{2.5}$$

As customary, for any $v \in L^2(\mathbb{R}^N)$ we define the positive and negative parts of v as follows

$$v^+ := \max\{v, 0\} \qquad \text{and} \qquad v^- := \max\{-v, 0\}.$$

As it is well known,

$$v(x) = v^+(x) - v^-(x), \quad \text{for a.e. } x \in \mathbb{R}^N \tag{2.6}$$

and

$$v^+(x)v^-(x) = 0, \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.7}$$

It is useful to observe that the functional introduced in (2.4) is monotone with respect to the operation of taking the positive and negative parts, as pointed out in the following result:

Lemma 2.2. *Let $U \subset \mathbb{R}^N$ be an open set and let $v \in \mathcal{H}(U)$. Then $v^\pm \in \mathcal{H}(U)$ and*

$$\rho(v^\pm, U) \le \rho(v, U). \tag{2.8}$$

Proof. Since $v \in \mathcal{H}(U) = L^2(\mathbb{R}^N) \cap H^1(U)$, it is easy to see that $v^\pm \in \mathcal{H}(U)$, in light of (2.6) and (2.7). We then focus on the proof of (2.8).

For this, recalling (2.4) and using again (2.6)-(2.7), we get

$$\begin{aligned}
\rho(v, U) &= \int_U |\nabla v|^2 + \iint_{U\times U} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&= \int_U |\nabla (v^+ - v^-)|^2 + \iint_{U\times U} \frac{|(v^+ - v^-)(x) - (v^+ - v^-)(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&= \int_U |\nabla v^+|^2 + \int_U |\nabla v^-|^2 \\
&\quad + \iint_{U\times U} \frac{|v^+(x) - v^+(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \iint_{U\times U} \frac{|v^-(x) - v^-(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad - 2\iint_{U\times U} \frac{\big(v^+(x) - v^+(y)\big)\big(v^-(x) - v^-(y)\big)}{|x-y|^{N+2s}}\, dx\, dy \\
&= \rho(v^+, U) + \rho(v^-, U) + 2\iint_{U\times U} \frac{v^+(x)v^-(y) + v^+(y)v^-(x)}{|x-y|^{N+2s}}\, dx\, dy \\
&\ge \rho(v^+, U) + \rho(v^-, U),
\end{aligned}$$

which gives the desired result in (2.8). □

Inspired by [36], we now deal with a linear problem associated to the reflection with respect to a given hyperplane. For this, with the notation in (2.1) and (2.2), for every open and bounded set $\Omega \subset \mathbb{R}^N$, we define the first (variational) eigenvalue of the operator $\mathcal{L}$ introduced in (1.1) as

$$\Lambda_1(\Omega) := \inf_{u \in \mathcal{D}(\Omega)} \frac{B(u,u)}{\|u\|^2_{L^2(\Omega)}}. \tag{2.9}$$

We point out that, since we can identify $\mathcal{D}(\Omega)$ with the space of functions in $H^1_0(\Omega)$ that vanish outside Ω, we see that

$$\Lambda_1(\Omega) \ge \Lambda_{-\Delta}(\Omega), \tag{2.10}$$

where $\Lambda_{-\Delta}(\Omega)$ stands for the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. Recalling that

$$\Lambda_{-\Delta}(\Omega) \to +\infty \quad \text{as } |\Omega| \to 0,$$

and setting

$$\Lambda_1(r) := \inf \big\{\Lambda_1(\Omega) \text{ with } \Omega \subset \mathbb{R}^n \text{ open with } |\Omega| = r\big\}, \quad r > 0,$$

it follows from (2.10) that

$$\Lambda_1(r) \to +\infty \quad \text{as } r \to 0^+. \tag{2.11}$$

Furthermore, let $H \subset \mathbb{R}^N$ be an open and affine halfspace. We denote by

$$Q : \mathbb{R}^N \to \mathbb{R}^N$$

the reflection with respect to ∂H. For convenience, we will sometimes denote with

$$\bar{x} := Q(x), \tag{2.12}$$

for every $x \in \mathbb{R}^N$. With this notation at hand, we say that a function $v : \mathbb{R}^N \to \mathbb{R}$ is *antisymmetric with respect to Q* if

$$v(\bar{x}) = -v(x), \quad \text{for every } x \in \mathbb{R}^N. \tag{2.13}$$

Moreover, we give the following definition of antisymmetric supersolutions:

Definition 2.3. Let $U \subset H$ be an open and bounded set. Let $c \in L^\infty(U)$. We say that a function $v : \mathbb{R}^N \to \mathbb{R}$ is an antisymmetric supersolution of

$$\begin{cases} \mathcal{L}v = cv & \text{in } U, \\ v \equiv 0 & \text{in } H \setminus U, \end{cases} \tag{2.14}$$

if it satisfies the following properties:

(i) v is antisymmetric,
(ii) $v \in \mathcal{H}(U')$ for some open set $U' \subset \mathbb{R}^N$ such that $Q(U') = U'$ and $\overline{U} \subset U'$,
(iii) $v \geq 0$ in $H \setminus U$ and, for every $\varphi \in \mathcal{D}(U)$ with $\varphi \geq 0$, one has

$$B(v, \varphi) \geq \int_U c(x)v(x)\varphi(x)\,dx. \tag{2.15}$$

The aim is now to provide a suitable maximum principle for antisymmetric supersolutions, as given in Definition 2.3.

We start with the following observation on the bilinear form introduced in (2.1):

Lemma 2.4. *Let $U' \subset \mathbb{R}^N$ be an open set such that $Q(U') = U'$. Let $v \in \mathcal{H}(U')$ be an antisymmetric function such that*

$$v \geq 0 \quad \text{in } H \setminus U, \tag{2.16}$$

for a certain open and bounded set $U \subset H$ with the property that

$$\overline{U} \subset H \cap U'. \tag{2.17}$$

Then, the function

$$w := \chi_H v^- \in \mathcal{D}(U) \tag{2.18}$$

and it holds that

$$B(w, w) \leq -B(v, w). \tag{2.19}$$

Proof. We first prove (2.18). To this end we first observe that, since $v \in L^2(\mathbb{R}^N)$, one obviously has $w \in L^2(\mathbb{R}^N)$. Also, recalling (2.5), we know that $v \in H^1(U')$, and therefore it is easy to see that $v^- \in H^1(U')$. In addition, in light of (2.16), we have that $v^- \equiv 0$ in $H \setminus U$. As a consequence of these observations and of (2.17), we have that there exists an open set W such that

$$\overline{U} \subset W \subset \overline{W} \subset U' \cap H \text{ and } v^- \in H^1_0(W).$$

Therefore, if we identify $w = \chi_H v^-$ with the zero extension of v^- outside of U, we get that $w \in H^1(\mathbb{R}^N)$. Moreover, we have that $w \equiv 0$ in $\mathbb{R}^N \setminus U$. These considerations imply (2.18).

Now we focus on the proof of (2.19). Recalling (2.1), we observe that

$$\begin{aligned} &B(w,w) + B(v,w) \\ &= \int_{\mathbb{R}^N} |\nabla w|^2\,dx + \iint_{\mathbb{R}^{2N}} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\qquad + \int_{\mathbb{R}^N} \langle \nabla v, \nabla w\rangle\,dx + \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dx\,dy. \end{aligned} \tag{2.20}$$

We notice that, thanks to (2.16),

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w|^2\,dx + \int_{\mathbb{R}^N} \langle \nabla v, \nabla w\rangle\,dx &= \int_U |\nabla v^-|^2\,dx + \int_U \langle \nabla v, \nabla v^-\rangle\,dx \\ &= \int_U |\nabla v^-|^2\,dx - \int_U \langle \nabla v^-, \nabla v^-\rangle\,dx = 0. \end{aligned} \tag{2.21}$$

Furthermore, we remark that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned} &w(x)\big(w(x)+v(x)\big) \\ &\quad = \chi_H(x)v^-(x)\big(\chi_H(x)v^-(x) + \chi_H(x)v(x) + \chi_{\mathbb{R}^N\setminus H}(x)v(x)\big) \\ &\quad = \chi_H(x)v^-(x)\big(\chi_H(x)v^+(x) + \chi_{\mathbb{R}^N\setminus H}(x)v(x)\big) = 0, \end{aligned}$$

and therefore

$$\begin{aligned} &(w(x)-w(y))^2 + (v(x)-v(y))(w(x)-w(y)) \\ &\qquad = (w(x)-w(y))\big((w(x)+v(x)) - (w(y)+v(y))\big) \\ &\qquad = -w(x)(w(y)+v(y)) - w(y)(w(x)+v(x)). \end{aligned}$$

As a consequence, using (2.13) and the change of variable $Y := \bar{y}$ (also recall the notation in (2.12)), we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{w(x)(w(y)+v(y)) + w(y)(w(x)+v(x))}{|x-y|^{N+2s}}\,dx\,dy \\
&= -2\iint_{\mathbb{R}^{2N}} \frac{w(x)(w(y)+v(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= -2\iint_{H\times\mathbb{R}^N} \frac{v^-(x)\big(\chi_H(y)v^-(y)+v(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
&= -2\iint_{H\times\mathbb{R}^N} \frac{v^-(x)\big(\chi_H(y)v^+(y)+\chi_{\mathbb{R}^N\setminus H}(y)v(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
&= -2\iint_{H\times H} \frac{v^-(x)v^+(y)}{|x-y|^{N+2s}}\,dx\,dy - 2\iint_{H\times(\mathbb{R}^N\setminus H)} \frac{v^-(x)v(y)}{|x-y|^{N+2s}}\,dx\,dy \\
&= -2\iint_{H\times H} \frac{v^-(x)v^+(y)}{|x-y|^{N+2s}}\,dx\,dy + 2\iint_{H\times(\mathbb{R}^N\setminus H)} \frac{v^-(x)v(\bar{y})}{|x-y|^{N+2s}}\,dx\,dy \\
&= -2\iint_{H\times H} \frac{v^-(x)v^+(y)}{|x-y|^{N+2s}}\,dx\,dy + 2\iint_{H\times H} \frac{v^-(x)v(Y)}{|x-\bar{Y}|^{N+2s}}\,dx\,dY \\
&= -2\iint_{H\times H} v^-(x)v^+(y)\left(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-\bar{y}|^{N+2s}}\right)dx\,dy \\
&\qquad - 2\iint_{H\times H} \frac{v^-(x)v^-(y)}{|x-\bar{y}|^{N+2s}}\,dx\,dy \\
&\le 0.
\end{aligned}$$

Plugging this information and (2.21) into (2.20) we obtain (2.19), as desired. $\square$

With the aid of Lemma 2.4, we now prove the following maximum principle:

Proposition 2.5. *Let $U \subset \mathbb{R}^N$ be an open and bounded set with $\overline{U} \subset H$. Moreover, let $c \in L^\infty(U)$ be such that*

$$\|c^+\|_{L^\infty(U)} < \Lambda_1(U), \tag{2.22}$$

where the notation in (2.9) *has been used.*

Then, every antisymmetric supersolution v of (2.14) *(in U) is nonnegative throughout H, that is, $v(x) \ge 0$ for a.e. $x \in H$.*

Proof. We consider the function w introduced in (2.18) and we claim that

$$w \equiv 0. \tag{2.23}$$

To prove it, we argue towards a contradiction, supposing that $\|w\|_{L^2(U)} \neq 0$. By Lemma 2.4, we know that $w \in \mathcal{D}(U)$, and hence it is an admissible test function in (2.15). Accordingly,

$$B(v,w) \ge \int_U c(x)v(x)w(x)\,dx.$$

From this, (2.9), (2.19) and (2.22), we conclude that

$$\begin{aligned}\Lambda_1(U)\|w\|^2_{L^2(U)} &\le B(w,w) \le -B(v,w) \le -\int_U c(x)v(x)w(x)\,dx\\ &= \int_U c(x)w^2(x)\,dx \le \|c^+\|_{L^\infty(U)}\|w\|^2_{L^2(U)} < \Lambda_1(U)\|w\|^2_{L^2(U)},\end{aligned}$$

which is a contradiction. This proves (2.23), we implies the desired result. □

We are now in the position of establishing a strong maximum principle for antisymmetric supersolutions (which is the counterpart in the setting of mixed local-nonlocal operators of [36, Proposition 3.6]):

Proposition 2.6. *Let $U \subset H$ be an open and bounded set. Let $c \in L^\infty(U)$ and let v be an antisymmetric supersolution of* (2.14) (*in U*). *Assume that*

$$v \ge 0 \quad a.e.\ in\ H. \tag{2.24}$$

Then, either $v \equiv 0$ in $\mathbb{R}^N$ or

$$\operatorname{ess\,inf}_K v > 0, \quad \textit{for every compact set } K \subset U.$$

Proof. If $v \equiv 0$ in $\mathbb{R}^N$, there is nothing to prove, so we assume that

$$v \not\equiv 0 \quad \text{in } \mathbb{R}^N. \tag{2.25}$$

In this case, it suffices to show that, for a fixed $x_0 \in U$, one has

$$\operatorname{ess\,inf}_{B_r(x_0)} v > 0, \tag{2.26}$$

for a some radius $r > 0$ small enough. We then prove (2.26).

First of all, in light of (2.24), (2.25) and the fact that v is antisymmetric, we can find a bounded set $M \subset H$, with positive measure, which does not contain a small neighborhood of x_0 and such that

$$\delta := \inf_M v > 0. \tag{2.27}$$

In addition, by (2.11), we find a radius

$$r \in \left(0, \frac{\operatorname{dist}(x_0; (\mathbb{R}^N \setminus H) \cup M)}{4}\right) \tag{2.28}$$

such that

$$\Lambda_1(B_{2r}(x_0)) > \|c\|_{L^\infty(U)}. \tag{2.29}$$

We now pick a function $g \in C^2_0(\mathbb{R}^N, [0,1])$ such that

$$g(x) := \begin{cases} 1, & \text{if } x \in B_r(x_0),\\ 0, & \text{if } x \in \mathbb{R}^N \setminus B_{2r}(x_0).\end{cases}$$

Moreover, for a given $a > 0$ to be chosen later, we define the function

$$h : \mathbb{R}^N \to \mathbb{R}, \quad h(x) := g(x) - g(\bar{x}) + a\left(\chi_M(x) - \chi_M(\bar{x})\right), \tag{2.30}$$

where we are using the notation in (2.12). We also define the sets $U_0 := B_{2r}(x_0)$ and $U_0' := B_{3r}(x_0) \cup Q(B_{3r}(x_0))$.

We observe that h is antisymmetric, and moreover

$$h \equiv 0 \ \text{ on } H \setminus (U_0 \cup M) \quad \text{and} \quad h \equiv a \ \text{ on } M, \tag{2.31}$$

thanks to (2.28). From (2.28) we also deduce that

$$(M \cup Q(M)) \cap U_0' = \varnothing. \tag{2.32}$$

This and the fact that M is bounded give that $h \in \mathcal{H}(U_0')$. We now claim that there exists a constant $C_1 > 0$, depending on g, such that

$$B(g,\varphi) \le C_1 \int_{U_0} \varphi(x)\,dx, \quad \text{for every } \varphi \in \mathcal{D}(U_0) \text{ with } \varphi \ge 0. \tag{2.33}$$

In fact, for any $\varphi \in \mathcal{D}(U_0)$ with $\varphi \ge 0$, by an integration by parts,

$$\begin{aligned} \int_{\mathbb{R}^N} \langle \nabla g, \nabla\varphi\rangle\,dx &= \int_{U_0} \langle \nabla g, \nabla\varphi\rangle\,dx \\ &= -\int_{U_0} \Delta g\,\varphi\,dx \le \|g\|_{C^2(\mathbb{R}^N)} \int_{U_0} \varphi(x)\,dx. \end{aligned} \tag{2.34}$$

Moreover, by Proposition 2.3-(ii) in [36] (applied here with $v := g$ and $u := \varphi$), we have that

$$\begin{aligned} \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(g(x)-g(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy &= \int_{\mathbb{R}^N} (-\Delta)^s g(x)\,\varphi(x)\,dx \\ = \int_{U_0} (-\Delta)^s g(x)\,\varphi(x)\,dx &\le \|(-\Delta)^s g\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,dx. \end{aligned}$$

Recalling (2.1), this and (2.34) imply (2.33). Similarly, one has that

$$B(g\circ Q,\varphi) \le C_2 \int_{U_0} \varphi(x)\,dx, \quad \text{for every } \varphi \in \mathcal{D}(U_0) \text{ with } \varphi \ge 0, \tag{2.35}$$

for some $C_2 > 0$. In addition, we see that, for any $\varphi \in \mathcal{D}(U_0)$ and any $x \in \mathbb{R}^N$, from (2.28) we infer that

$$(\chi_M(x) - \chi_M(\bar{x}))\varphi(x) = 0;$$

as a consequence,

$$\begin{aligned} &\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{\big((\chi_M(x)-\chi_M(\bar{x})) - (\chi_M(y)-\chi_M(\bar{y}))\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &= -\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(\chi_M(x)-\chi_M(\bar{x}))\varphi(y) + (\chi_M(y)-\chi_M(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,dx\,dy \\ &= -\iint_{U_0\times\mathbb{R}^N} \frac{(\chi_M(y)-\chi_M(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,dx\,dy \\ &= -\iint_{U_0\times\mathbb{R}^N} \frac{(\chi_M(y)-\chi_M(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,dx\,dy \\ &= -\int_{U_0} \varphi(x)\left(\int_M \frac{dy}{|x-y|^{N+2s}} - \int_{Q(M)} \frac{dy}{|x-y|^{N+2s}}\right)dx \\ &= -\int_{U_0} \varphi(x)\left(\int_M \frac{dy}{|x-y|^{N+2s}} - \int_M \frac{dy}{|x-\bar{y}|^{N+2s}}\right)dx \\ &\le -C_0 \int_{U_0} \varphi(x)\,dx, \end{aligned} \tag{2.36}$$

where

$$C_0 := \inf_{x\in U_0} \left(\int_M \frac{dy}{|x-y|^{N+2s}} - \int_M \frac{dy}{|x-\bar{y}|^{N+2s}}\right).$$

We stress on the fact that the constant C_0 is finite, thanks to (2.28).

Now, recalling (2.30), and using (2.33), (2.35) and (2.36), we conclude that, for any $\varphi \in \mathcal{D}(U_0)$, one has

$$
\begin{aligned}
B(h,\varphi) &= B(g,\varphi) + B(g\circ Q,\varphi) \\
&\quad + a \iint_{\mathbb{R}^{2N}} \frac{\big((\chi_M(x)-\chi_M(\bar{x})) - (\chi_M(y)-\chi_M(\bar{y}))\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\le C_a \int_{U_0} \varphi(x)\,dx,
\end{aligned} \tag{2.37}
$$

where

$$C_a := C_1 + C_2 - 2a\,C_0.$$

Now we perform our choice of the parameter a: we choose $a>0$ such that

$$C_a < -\|c\|_{L^\infty(U_0)}.$$

In particular, with this choice, (2.37) yields that

$$
\begin{aligned}
B(h,\varphi) &\le -\|c\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,dx \le -\|c^-\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,dx \\
&\le -\int_{U_0} c^-(x)\varphi(x)\,dx \le -\int_{U_0} c^-(x)h(x)\varphi(x)\,dx \\
&\le \int_{U_0} c^+(x)h(x)\varphi(x)\,dx - \int_{U_0} c^-(x)h(x)\varphi(x)\,dx \\
&= \int_{U_0} c(x)h(x)\varphi(x)\,dx,
\end{aligned} \tag{2.38}
$$

since $h(x) = g(x) \in [0,1]$ for every $x \in U_0$. Now, we recall (2.27), we define the function $\tilde{v}$ as

$$\tilde{v}(x) := v(x) - \frac{\delta}{a} h(x), \tag{2.39}$$

and we notice that $\tilde{v} \in \mathcal{H}(U_0')$ and it is antisymmetric, since both v and h are so. Furthermore, by (2.24), (2.27) and (2.31), we have that

$$\tilde{v} \ge 0 \quad \text{on } H \setminus U_0.$$

In addition, for any $\varphi \in \mathcal{D}(U_0)$ with $\varphi \ge 0$,

$$
\begin{aligned}
B(\tilde{v},\varphi) &= B(v,\varphi) - \frac{\delta}{a} B(h,\varphi) \\
&\ge \int_{U_0} c(x)v(x)\varphi(x)\,dx - \frac{\delta}{a}\int_{U_0} c(x)h(x)\varphi(x)\,dx \\
&= \int_{U_0} c(x)\tilde{v}(x)\varphi(x)\,dx,
\end{aligned}
$$

thanks to (2.15) and (2.38).

As a consequence, we have that $\tilde{v}$ is an antisymmetric supersolution of

$$
\begin{cases}
\mathcal{L}\tilde{v} = c\tilde{v} & \text{in } U_0, \\
\tilde{v} \equiv 0 & \text{in } H \setminus U_0.
\end{cases}
$$

Since $\|c\|_{L^\infty(U_0)} < \Lambda_1(U_0)$, thanks to (2.29), we are in the position to apply Proposition 2.5 to conclude that $\tilde{v} \geq 0$ a.e. on U_0. Recalling (2.39), this gives

$$v \geq \frac{\delta}{a} > 0 \quad \text{a.e. on } B_r(x_0).$$

This establishes (2.26), and the proof of Proposition 2.6 is thereby complete. $\square$

With this preliminary work, we now prove Theorem 1.1. For this, let $u \in C(\Omega)$ be a weak solution of (1.2). We fix the usual notation needed to implement the moving plane method. For every $\lambda \in (-1, 0]$ we define the following:

$$\begin{aligned} \Omega_\lambda &:= \{x \in \Omega : x_1 < \lambda\}, \\ \Sigma_\lambda &:= \{x \in \mathbb{R}^N : x_1 < \lambda\}, \\ Q_\lambda(x) = x_\lambda &:= (2\lambda - x_1, x_2, \dots, x_N), \\ \text{and} \quad u_\lambda(x) &:= u(x_\lambda). \end{aligned}$$

We also define the function

$$c(x) := \begin{cases} \dfrac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)}, & \text{if } u_\lambda(x) \neq u(x), \\ 0, & \text{if } u_\lambda(x) = u(x). \end{cases} \tag{2.40}$$

We observe that $c \in L^\infty(\Omega_\lambda)$, thanks to the Lipschitz assumption on f.

Furthermore, setting

$$v_\lambda := u_\lambda - u, \tag{2.41}$$

we point out the following observation:

Lemma 2.7. *Let u be a weak solution of* (1.2) *according to Definition 2.1.*

Then, the function v_λ in (2.41) *is an antisymmetric supersolution of* (2.14) *in Ω_λ, according to Definition 2.3, with c as in* (2.40).

Proof. We notice that $v_\lambda \in H^1(\mathbb{R}^N) \subset \mathcal{H}(U')$, for every open set $U' \subset \mathbb{R}^N$ such that $Q(U') = U'$ and $\overline{\Omega_\lambda} \subset U'$. Moreover, since $u \geq 0$ in $\mathbb{R}^N$ and $u \equiv 0$ on $\Sigma_\lambda \setminus \Omega_\lambda$, we have that $v_\lambda \geq 0$ on $\Sigma_\lambda \setminus \Omega_\lambda$. In addition, for any $\varphi \in \mathcal{D}(\Omega_\lambda)$ and for any $x \in \mathbb{R}^N$, we have

$$\begin{aligned} \langle \nabla u_\lambda(x), \nabla \varphi(x) \rangle &= \big(-\partial_1 u, \partial_2 u, \cdots, \partial_N u \big)(\bar{x}) \cdot \big(\partial_1 \varphi, \partial_2 \varphi, \cdots, \partial_N \varphi \big)(x) \\ &= \big(\partial_1 u, \partial_2 u, \cdots, \partial_N u \big)(X) \cdot \big(-\partial_1 \varphi, \partial_2 \varphi, \cdots, \partial_N \varphi \big)(\bar{X}) \\ &= \langle \nabla u(X), \nabla \varphi_\lambda(X) \rangle, \end{aligned} \tag{2.42}$$

where $X := \bar{x}$. Similarly, setting also $Y := \bar{y}$,

$$\begin{aligned} &\frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} = \frac{(u(\bar{x}) - u(\bar{y}))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \\ &\quad = \frac{(u(X) - u(Y))(\varphi(\bar{X}) - \varphi(\bar{Y}))}{|\bar{X} - \bar{Y}|^{N+2s}} = \frac{(u(X) - u(Y))(\varphi(\bar{X}) - \varphi(\bar{Y}))}{|X - Y|^{N+2s}} \\ &\quad = \frac{(u(X) - u(Y))(\varphi_\lambda(X) - \varphi_\lambda(Y))}{|X - Y|^{N+2s}}. \end{aligned}$$

From this and (2.42), we obtain that

$$\begin{aligned}
&B(u_\lambda, \varphi) \\
&= \int_{\mathbb{R}^N} \langle \nabla u_\lambda(x), \nabla \varphi(x) \rangle \, dx + \iint_{\mathbb{R}^{2N}} \frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy \\
&= \int_{\mathbb{R}^N} \langle \nabla u(X), \nabla \varphi_\lambda(X) \rangle \, dX + \iint_{\mathbb{R}^{2N}} \frac{(u(X) - u(Y))(\varphi_\lambda(X) - \varphi_\lambda(Y))}{|X-Y|^{N+2s}} \, dX \, dY \\
&= B(u, \varphi_\lambda).
\end{aligned}$$

As a consequence, since $\varphi_\lambda \in \mathcal{D}(Q_\lambda(\Omega_\lambda)) \subset \mathcal{D}(\Omega)$, we can use Definition 2.1 to find that

$$\begin{aligned}
B(u_\lambda, \varphi) &= \int_{\mathbb{R}^N} f(u(x)) \varphi_\lambda(x) \, dx \\
&= \int_{\mathbb{R}^N} f(u(\bar{X})) \varphi(X) \, dX = \int_{\mathbb{R}^N} f(u_\lambda(X)) \varphi(X) \, dX.
\end{aligned}$$

Therefore,

$$\begin{aligned}
B(v_\lambda, \varphi) &= B(u_\lambda, \varphi) - B(u, \varphi) \\
&= \int_{\mathbb{R}^N} f(u_\lambda(x)) \varphi(x) \, dx - \int_{\mathbb{R}^N} f(u(x)) \varphi(x) \, dx \\
&= \int_{\mathbb{R}^N} c(x) v_\lambda(x) \varphi(x) \, dx,
\end{aligned}$$

which proves (2.15), and thereby completes the proof of Lemma 2.7. □

With these considerations, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For every $\lambda \in (-1, 0)$, we define the function

$$w_\lambda : \mathbb{R}^N \to \mathbb{R}, \quad w_\lambda(x) := \begin{cases} (u - u_\lambda)^+(x) & \text{in } \Sigma_\lambda, \\ (u - u_\lambda)^-(x) & \text{in } \mathbb{R}^N \setminus \Sigma_\lambda, \end{cases} \tag{2.43}$$

where, *differently from before*, we have set

$$(u - u_\lambda)^- := \min\{u - u_\lambda, 0\},$$

which is *nonpositive*. We claim that

$$w_\lambda \in H^1(\mathbb{R}^N). \tag{2.44}$$

Indeed, we know that $u \in H^1(\mathbb{R}^N)$ and thus $u - u_\lambda \in H^1(\mathbb{R}^N)$. Accordingly, we have that (see e.g. the Chain Rule on page 296 of [37])

$$(u - u_\lambda)^+ \in H^1(\mathbb{R}^N). \tag{2.45}$$

Moreover, $u \in C(\mathbb{R}^N)$, and consequently

$$(u - u_\lambda)^+ \in C(\mathbb{R}^N). \tag{2.46}$$

In addition, $u = u_\lambda$ along $\partial \Sigma_\lambda$. From this fact, (2.45) and (2.46), we obtain that

$$(u - u_\lambda)^+ \chi_{\Sigma_\lambda} \in H_0^1(\Sigma_\lambda) \subset H^1(\mathbb{R}^N), \tag{2.47}$$

see e.g. [13, Theorem 9.17]. Similarly,

$$(u - u_\lambda)^- \chi_{\mathbb{R}^N \setminus \Sigma_\lambda} \in H^1(\mathbb{R}^N). \tag{2.48}$$

We also observe that

$$w_\lambda = (u-u_\lambda)^+\chi_{\Sigma_\lambda} + (u-u_\lambda)^-\chi_{\mathbb{R}^N\setminus\Sigma_\lambda}.$$

From this, (2.47) and (2.48), we obtain (2.44), as desired.

Furthermore, we claim that

$$w_\lambda \equiv 0 \text{ in } \mathbb{R}^N \setminus (\Omega_\lambda \cup Q_\lambda(\Omega_\lambda)) \subset \mathbb{R}^N \setminus \Omega. \tag{2.49}$$

Indeed, if $x \in \Sigma_\lambda \setminus \Omega_\lambda$, then $w_\lambda(x) = (0-u_\lambda(x))^+ = 0$. If instead $x \in Q_\lambda(\Sigma_\lambda \setminus \Omega_\lambda)$, then $\bar{x} \in \Sigma_\lambda \setminus \Omega_\lambda$ and accordingly

$$0 = w_\lambda(\bar{x}) = (u(\bar{x}) - u_\lambda(\bar{x}))^+ = (u_\lambda(x) - u(x))^+.$$

This gives that $u_\lambda(x) \le u(x)$, and therefore $w_\lambda(x) = (u(x) - u_\lambda(x))^- = 0$.

From these observations, we obtain (2.49). Then, (2.44) and (2.49) give that we can take w_λ as an admissibile test function in (2.3). In this way, we obtain

$$B(u, w_\lambda) = \int_{\mathbb{R}^N} f(u(x))w_\lambda(x)\,dx. \tag{2.50}$$

Similarly,

$$B(u_\lambda, w_\lambda) = \int_{\mathbb{R}^N} f(u_\lambda(x))w_\lambda(x)\,dx. \tag{2.51}$$

Subtracting (2.51) to (2.50), and recalling (2.1), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \langle \nabla(u-u_\lambda), \nabla w_\lambda\rangle\,dx \\
&\qquad + \iint_{\mathbb{R}^{2N}} \frac{((u(x)-u_\lambda(x)) - (u(y)-u_\lambda(y)))(w_\lambda(x)-w_\lambda(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&= \int_{\mathbb{R}^N} \big(f(u(x)) - f(u_\lambda(x))\big)w_\lambda(x)\,dx.
\end{aligned} \tag{2.52}$$

Now, we use formula (3.9) in [38], which gives that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{\big((u(x)-u_\lambda(x)) - (u(y)-u_\lambda(y))\big)(w_\lambda(x)-w_\lambda(y))}{|x-y|^{N+2s}}\,dx\,dy \\
&\qquad \ge \iint_{\mathbb{R}^{2N}} \frac{|w_\lambda(x)-w_\lambda(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \ge 0.
\end{aligned}$$

Using this information into (2.52), and recalling (2.49), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^N} \langle \nabla(u-u_\lambda), \nabla w_\lambda\rangle\,dx &\le \int_{\mathbb{R}^N} \big(f(u(x)) - f(u_\lambda(x))\big)w_\lambda(x)\,dx \\
&= \int_{\mathbb{R}^N} \frac{f(u(x)) - f(u_\lambda(x))}{u(x)-u_\lambda(x)} \big(u(x)-u_\lambda(x)\big)w_\lambda(x)\,dx \\
&= \int_{\mathbb{R}^N} \frac{f(u(x)) - f(u_\lambda(x))}{u(x)-u_\lambda(x)} w_\lambda^2(x)\,dx \\
&= \int_{\Omega_\lambda\cup Q(\Omega_\lambda)} \frac{f(u(x)) - f(u_\lambda(x))}{u(x)-u_\lambda(x)} w_\lambda^2(x)\,dx.
\end{aligned} \tag{2.53}$$

We also notice that, thanks to (2.49),

$$\int_{\mathbb{R}^N} \langle \nabla(u-u_\lambda), \nabla w_\lambda\rangle\,dx = \int_{\mathbb{R}^N} |\nabla w_\lambda|^2\,dx = \int_{\Omega_\lambda\cup Q(\Omega_\lambda)} |\nabla w_\lambda|^2\,dx.$$

From this and (2.53), we deduce that

$$\begin{aligned}\int_{\Omega_\lambda\cup Q(\Omega_\lambda)}|\nabla w_\lambda|^2\,dx &\le \int_{\Omega_\lambda\cup Q(\Omega_\lambda)}\frac{f(u(x))-f(u_\lambda(x))}{u(x)-u_\lambda(x)}w_\lambda^2(x)\,dx\\ &\le C\int_{\Omega_\lambda\cup Q(\Omega_\lambda)}|w_\lambda|^2\,dx,\end{aligned}\tag{2.54}$$

for some constant $C>0$, depending on f and $\|u\|_{L^\infty(\Omega)}$.

Now, using Lemma 2.10 in [9], we obtain that

$$\int_{\Omega_\lambda\cup Q(\Omega_\lambda)}|\nabla w_\lambda|^2\,dx \le C|\Omega_\lambda\cup Q(\Omega_\lambda)|^{1/N}\int_{\Omega_\lambda\cup Q(\Omega_\lambda)}|\nabla w_\lambda|^2\,dx,\tag{2.55}$$

uo to renaming C, which possibly depends also on N. As a consequence, if λ is sufficiently close to -1, we see that

$$C|\Omega_\lambda\cup Q(\Omega_\lambda)|^{1/N}<\frac{1}{2},$$

which, combined with (2.55), gives that

$$\int_{\Omega_\lambda\cup Q(\Omega_\lambda)}|\nabla w_\lambda|^2\,dx=0,$$

provided that λ is sufficiently close to -1. From this and the Poincaré inequality we get that $w_\lambda\equiv 0$ in $\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)$ if λ is sufficiently close to -1, which, recalling (2.43), implies that

$$u\le u_\lambda \quad \text{in } \Omega_\lambda,\tag{2.56}$$

if λ is sufficiently close to -1. As a matter of fact, formula (2.49) also gives

$$u\le u_\lambda \quad \text{in } \Sigma_\lambda\setminus\Omega_\lambda.\tag{2.57}$$

Now, we define the set

$$\Lambda_0:=\big\{\lambda\in(-1,0):\ u\le u_t \text{ in } \Omega_t \text{ for every } t\in(-1,\lambda]\big\}.$$

In light of (2.56), the following quantity is well defined:

$$\overline{\lambda}:=\sup\Lambda_0.\tag{2.58}$$

The goal is now to prove that

$$\overline{\lambda}=0.\tag{2.59}$$

For this, we recall the definition of v_λ in (2.41), and we observe that, since u is continuous in Ω,

$$v_{\overline{\lambda}}\ge 0, \quad \text{in } \Omega_{\overline{\lambda}}.$$

This and (2.57) imply that

$$v_{\overline{\lambda}}\ge 0, \quad \text{in } \Sigma_{\overline{\lambda}}.$$

As a consequence, by Lemma 2.7 and Proposition 2.6 (applied here with $H:=\Sigma_{\overline{\lambda}}$, $U:=\Omega_{\overline{\lambda}}$ and $v:=v_{\overline{\lambda}}$), we have that

$$v_{\overline{\lambda}}>0, \quad \text{in } \Omega_{\overline{\lambda}}.$$

Now, we consider a compact set $K\subset\Omega_{\overline{\lambda}}$ (to be chosen later on), and we notice that for $\overline{\tau}>0$ small enough, we have that

$$v_{\overline{\lambda}+\tau}>0 \text{ in } K \qquad (\text{for all } \tau\in(0,\overline{\tau})).\tag{2.60}$$

Now, for every fixed $\tau \in (0, \overline{\tau})$, we consider the function $w_{\overline{\lambda}+\tau}$ defined as in (2.43) (with $\lambda := \overline{\lambda}+\tau$). We notice that, thanks to (2.44) and (2.49), we can take $w_{\overline{\lambda}+\tau}$ as an admissibile test function in (2.3), obtaining that

$$B(u, w_{\overline{\lambda}+\tau}) = \int_{\mathbb{R}^N} f(u(x)) w_{\overline{\lambda}+\tau}(x)\, dx \qquad \text{and}$$
$$B(u_{\overline{\lambda}+\tau}, w_{\overline{\lambda}+\tau}) = \int_{\mathbb{R}^N} f(u_{\overline{\lambda}+\tau}(x)) w_{\overline{\lambda}+\tau}(x)\, dx.$$

From here, we repeat the same argument in (2.52)–(2.54) to find that

$$\int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\, dx \leq C \int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |w_{\overline{\lambda}+\tau}|^2\, dx,$$

for some constant $C > 0$, depending on f and $\|u\|_{L^\infty(\Omega)}$.

From this, recalling (2.60), we obtain that

$$\int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\, dx \leq C \int_{(\Omega_{\overline{\lambda}+\tau} \setminus K) \cup Q(\Omega_{\overline{\lambda}+\tau} \setminus K)} |w_{\overline{\lambda}+\tau}|^2\, dx.$$

Hence, making again use of Lemma 2.10 in [9], we get

$$\begin{aligned} &\int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\, dx \\ &\qquad \leq C\, |(\Omega_{\overline{\lambda}+\tau} \setminus K) \cup Q(\Omega_{\overline{\lambda}+\tau} \setminus K)|^{1/N} \int_{(\Omega_{\overline{\lambda}+\tau} \setminus K) \cup Q(\Omega_{\overline{\lambda}+\tau} \setminus K)} |\nabla v_{\overline{\lambda}+\tau}|^2\, dx, \end{aligned} \tag{2.61}$$

up to relabeling $C > 0$ (which may also depend on N). Now we choose the compact K big enough and the number $\overline{\tau}$ small enough such that

$$C\, |(\Omega_{\overline{\lambda}+\tau} \setminus K) \cup Q(\Omega_{\overline{\lambda}+\tau} \setminus K)|^{1/N} < 1.$$

Using this information into (2.61), we conclude that

$$\int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\, dx = 0.$$

From this and the Poincaré inequality, we find that $w_{\overline{\lambda}+\tau} \equiv 0$ in $\Omega_{\overline{\lambda}+\tau}$, hence

$$u \leq u_{\overline{\lambda}+\tau} \quad \text{in } \Omega_{\overline{\lambda}+\tau},$$

for every $\tau \in (0, \overline{\tau})$, provided $\overline{\tau} > 0$ is small enough. This yields a contradiction with (2.58), from which we conclude that (2.59) holds true, as desired.

In particular, from (2.59) we see that, for all $\lambda \in (-1, 0)$ and all $x \in \Omega_\lambda$,

$$u(x) \leq u_\lambda(x) = u(2\lambda - x_1, x_2, \cdots, x_N). \tag{2.62}$$

Consequently,

$$u(x) \leq u(-x_1, x_2, \cdots, x_N), \tag{2.63}$$

for all $x \in \Omega \cap \{x_1 < 0\}$. In the same way, sliding the moving plane from right to left, one sees that, for all $x \in \Omega \cap \{x_1 > 0\}$, one has

$$u(x) \leq u(-x_1, x_2, \cdots, x_N).$$

This implies that

$$u(-x_1, x_2, \cdots, x_N) \leq u(x),$$

for all $x \in \Omega \cap \{x_1 < 0\}$. From this and (2.63), we conclude that

$$u(x) = u(-x_1, x_2, \cdots, x_N),$$

for all $x \in \Omega$, which says that u is symmetric with respect to $\{x_1 = 0\}$.

Furthermore, from (2.62) it plainly follows that u is increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$. The proof of Theorem 1.1 is thereby complete. □

3. One-dimensional symmetry and proof of Theorem 1.2

In this section we provide the proof of Theorem 1.2. For this, we indicate the points $x \in \mathbb{R}^N$ by

$$(y, t), \text{ with } y \in \mathbb{R}^{N-1} \text{ and } t \in \mathbb{R}.$$

Moreover, we consider the functional space

$$\mathbb{X} := C^3(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N). \tag{3.1}$$

We point out that, if $u \in \mathbb{X}$, it is possible to compute $\mathcal{L}u$ in the classical sense, that is, $\mathcal{L}u(x)$ is well-defined for all $x \in \mathbb{R}^N$.

We shall derive Theorem 1.2 from the abstract approach developed in [31]. To this end, we check that the assumptions introduced in [31] are satisfied in our setting. We list these assumptions here for the convenience of the reader:

(H1): if $\varphi \in \mathbb{X}$ satisfies $\mathcal{L}\varphi = f(\varphi)$ in $\mathbb{R}^N$, then there exists an operator $\tilde{\mathcal{L}}$, acting on a suitable space of functions $\tilde{\mathbb{X}} \subseteq C(\mathbb{R}^N)$ which is translation-invariant[1], such that $\partial_\nu \varphi \in \tilde{\mathbb{X}}$ for any unit vector $\nu \in \mathbb{R}^N$ and

$$\tilde{\mathcal{L}}(\partial_\nu \varphi) = f'(\varphi)\, \partial_\nu \varphi \qquad \text{on } \mathbb{R}^N;$$

(H2): if $\varphi \in \mathbb{X}$ is a solution of (1.4), if $\{z_k\}_{k=1}^{\infty}$ is an arbitrary sequence of points in $\mathbb{R}^N$ (possibly unbounded) and if

$$\varphi_k := \varphi(\cdot + z_k) \qquad \text{for any } k \in \mathbb{N},$$

then there exists a function $\varphi_0 \in \mathbb{X}$ such that, up to a sub-sequence,

$$\begin{aligned} &\lim_{k\to\infty} \varphi_k(x) = \varphi_0(x), \\ &\lim_{k\to\infty} \nabla\varphi_k(x) = \nabla\varphi_0(x) \\ \text{and} \qquad &\lim_{k\to\infty} \mathcal{L}\varphi_k(x) = \varphi_0(x), \end{aligned}$$

for all $x \in \mathbb{R}^N$;

(H3): if $w \in \tilde{\mathbb{X}}$ satisfies $\tilde{\mathcal{L}}w + c(x)w = 0$ in $\mathbb{R}^N$, with

$$w(y,t) \geq 0 \text{ if } |t| \leq M \text{ and } c(y,t) \geq \kappa \text{ if } |t| \geq M$$

for some constants M, $\kappa > 0$, then

$$w(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}^N;$$

(H4): if $\varphi \in \mathbb{X}$ and if $w \in \tilde{\mathbb{X}}$ satisfies $\tilde{\mathcal{L}}w = f'(\varphi)w$ in $\mathbb{R}^N$, then

$$\begin{cases} w \geq 0 \text{ in } \mathbb{R}^N, \\ w(0) = 0, \end{cases} \quad \Longrightarrow \; w \equiv 0 \text{ on } \mathbb{R}^N;$$

[1] A (non-void) set $V \subseteq C(\mathbb{R}^N)$ is *translation-invariant* if, for every function $\varphi \in V$ and every point $y \in \mathbb{R}^N$, the 'translated' function $x \mapsto \varphi(x + y)$ belongs to V.

(H5): given $\mu_- < \mu_+ \in \mathbb{R}$, if $U \subseteq \mathbb{R}^N$ is an open set contained in

$$\mathcal{S} := \{x = (y, t) \in \mathbb{R}^N : t \le \mu_- \text{ or } t \ge \mu_+\}$$

and if $v \in \mathbb{X}$ satisfies $\mathcal{L}v + c(x)v = 0$ in $\mathbb{R}^N$, with

$$v(x) \ge \text{ in } \mathbb{R}^N \setminus U \text{ and } c(x) \ge \kappa \text{ on } U$$

for some constant $\kappa > 0$, then

$$v(x) \ge 0 \qquad \text{for all } x \in \mathbb{R}^N;$$

(H6): if $\varphi \in \mathbb{X}$ and if $v \in \mathbb{X}$ satisfies $\mathcal{L}v = f(v + \varphi) - f(v)$ in $\mathbb{R}^N$, then

$$\begin{cases} v \ge 0 \text{ in } \mathbb{R}^N, \\ v(0) = 0, \end{cases} \quad \Longrightarrow \; v \equiv 0 \text{ on } \mathbb{R}^N.$$

The next lemmata establish the validity of (H1)—(H6) in our setting.

Lemma 3.1 (Validity of (H1))**.** *For every $\varphi \in \mathbb{X}$ and every unit vector $\nu \in \mathbb{R}^N$, one has*

$$\mathcal{L}(\partial_\nu \varphi) = \partial_\nu\big(\mathcal{L}\varphi\big). \tag{3.2}$$

In particular, assumption (H1) *is fulfilled with the choices*

$$\tilde{\mathcal{L}} := \mathcal{L} \tag{3.3}$$

and

$$\tilde{\mathbb{X}} := C^2(\mathbb{R}^N) \cap W^{3,\infty}(\mathbb{R}^N). \tag{3.4}$$

Proof. First of all, if $\mathbb{X}$ is as in (3.1) and $\tilde{\mathbb{X}}$ is as in (3.4), we obviously have that, for every $\varphi \in \mathbb{X}$ and every unit vector $\nu \in \mathbb{R}^N$,

$$\partial_\nu \varphi \in \tilde{\mathbb{X}} \qquad \text{and} \qquad -\Delta(\partial_\nu \varphi) = \partial_\nu(-\Delta\varphi).$$

Moreover, since $\mathbb{X} \subseteq W^{3,\infty}(\mathbb{R}^N)$, we can use formula (4.1) in [31], obtaining that

$$(-\Delta)^s(\partial_\nu \varphi) = \partial_\nu\big((-\Delta)^s \varphi\big).$$

Gathering together these facts, we obtain (3.2), as desired. As a result, with the choices in (3.3) and (3.4), assumption (H1) is obviously satisfied. $\square$

We point out that the space $\tilde{\mathbb{X}} \supseteq \mathbb{X}$ is 'good' for dealing with $\mathcal{L}$. Indeed, since any function $u \in \tilde{\mathbb{X}}$ has bounded derivatives up to second order, we can compute $\mathcal{L}u$ pointwise in $\mathbb{R}^N$.

Lemma 3.2 (Validity of (H2))**.** *Let $\tilde{\mathbb{X}}$ be as in* (3.4)*. Let $\varphi \in \tilde{\mathbb{X}}$ and $\{z_k\}_{k=1}^\infty$ be a sequence of points in $\mathbb{R}^N$* (*possibly unbounded*)*. Let also*

$$\varphi_k := \varphi(\cdot + z_k) \quad \textit{for any } k \in \mathbb{N}. \tag{3.5}$$

Then, there exists a function $\varphi_0 \in \mathbb{X}$ such that, up to a sub-sequence,

$$\lim_{k \to \infty} \varphi_k(x) = \varphi_0(x), \tag{3.6}$$

$$\lim_{k \to \infty} \nabla \varphi_k(x) = \nabla \varphi_0(x) \tag{3.7}$$

$$\textit{and} \quad \lim_{k \to \infty} \mathcal{L}\varphi_k(x) = \mathcal{L}\varphi_0(x), \tag{3.8}$$

for all $x \in \mathbb{R}^N$. In particular, assumption (H2) *is fulfilled.*

Proof. We observe that, since $\varphi \in \tilde{\mathbb{X}}$, the sequences

$$\{D^\alpha \varphi_k\}_{k=1}^\infty$$

are equi-continuous and equi-bounded on $\mathbb{R}^N$, for every multi-index $\alpha \in \mathbb{N}^N$ satisfying $0 \le |\alpha| \le 2$. As a consequence, Arzelà-Ascoli's Theorem ensures the existence of some function $\varphi_0 \in \mathbb{X}$ such that (up to a sub-sequence)

$$\lim_{k\to\infty} D^\alpha \varphi_k = D^\alpha \varphi_0 \quad \text{locally uniformly in } \mathbb{R}^N, \tag{3.9}$$

for every $\alpha \in \mathbb{N}^N$ with $|\alpha| \le 2$. Hence, (3.6) and (3.7) plainly follows from (3.9). We also deduce from (3.9) that

$$\lim_{k\to\infty} \Delta\varphi_k(x) = \Delta\varphi_0(x) \quad \text{locally uniformly in } \mathbb{R}^N. \tag{3.10}$$

We now claim that

$$\lim_{k\to\infty} (-\Delta)^s \varphi_k(x) = (-\Delta)^s \varphi_0(x) \qquad \text{for every } x \in \mathbb{R}^N. \tag{3.11}$$

To prove it, for any $x \in \mathbb{R}^N$ and for any $k \in \mathbb{N}$, we set

$$\mathcal{I}_k(z) := \frac{\varphi_k(x+z) - \varphi_k(x-z) - 2\varphi_k(x)}{|z|^{N+2s}} \qquad \text{for any } z \neq 0.$$

On account of (3.9), we have that

$$\lim_{k\to\infty} \mathcal{I}_k(z) = \frac{\varphi_0(x+z) - \varphi_0(x-z) - 2\varphi_0(x)}{|z|^{N+2s}} \qquad \text{for all } z \neq 0. \tag{3.12}$$

Moreover, recalling the definition of φ_k in (3.5), we see that, for every $z \neq 0$,

$$\begin{aligned}
|\mathcal{I}_k(z)| &= \frac{\big|\varphi_k(x+z) + \varphi_k(x-z)| - 2\varphi_k(x)\big|}{|z|^{N+2s}} \\
&\le \max_{|\alpha|=2} \|D^\alpha \varphi_k\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s-2}} \chi_{\{0<|z|\le 1\}} \\
&\qquad + 4\|\varphi_k\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s}} \chi_{\{|z|>1\}} \\
&= \max_{|\alpha|=2} \|D^\alpha \varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s-2}} \chi_{\{0<|z|\le 1\}} \\
&\qquad + 4\|\varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s}} \chi_{\{|z|>1\}}.
\end{aligned} \tag{3.13}$$

Now, since $\varphi \in \tilde{\mathbb{X}}$, we have that

$$\begin{aligned}
g(z) := &\max_{|\alpha|=2} \|D^\alpha \varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s-2}} \chi_{\{0<|z|\le 1\}} \\
&+ 4\|\varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s}} \chi_{\{|z|>1\}} \in L^1(\mathbb{R}^N).
\end{aligned}$$

From this, (3.12) and (3.13) we deduce that we can apply the Dominated Convergence Theorem to conclude that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned}
&\lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{\varphi_k(x+z) - \varphi_k(x-z) - 2\varphi_k(x)}{|z|^{N+2s}}\, dz \\
&\qquad = \int_{\mathbb{R}^N} \frac{\varphi_0(x+z) - \varphi_0(x-z) - 2\varphi_0(x)}{|z|^{N+2s}}\, dz.
\end{aligned}$$

This proves (3.11). From (3.10) and (3.11), recalling (1.1), we obtain (3.8). Finally, since $\mathbb{X} \subset \tilde{\mathbb{X}}$, we deduce that assumption (H2) is fulfilled, thus completing the proof of Lemma 3.2. $\square$

Lemma 3.3 (Validity of (H3) and (H5))**.** *Let $\tilde{\mathbb{X}}$ be as in* (3.4)*. Let $w \in \tilde{\mathbb{X}}$ satisfy*

$$\mathcal{L}w + c(x)w = 0 \quad \text{in } \mathbb{R}^N, \tag{3.14}$$

with

$$w(x) \geq 0 \text{ in } \mathbb{R}^N \setminus U \text{ and } c(x) \geq \kappa \text{ on } U \tag{3.15}$$

for some open set $U \subseteq \mathbb{R}^N$ and some constant $\kappa > 0$. Then

$$w(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}^N. \tag{3.16}$$

In particular, assumptions (H3) *and* (H5) *are fulfilled with the choice in* (3.3)*.*

Proof. Arguing by contradiction, we suppose that $m := \inf_{\mathbb{R}^N} w < 0$, and we choose a sequence of points $\{z_k\}_{k=1}^{\infty}$ in $\mathbb{R}^N$ satisfying

$$\lim_{k\to\infty} w(z_k) = m. \tag{3.17}$$

Since $m < 0$, it is not restrictive to assume that

$$w(z_k) \leq \frac{m}{2} < 0 \qquad \text{for all } k \in \mathbb{N}. \tag{3.18}$$

As a consequence, also in light of (3.15), for every $k \in \mathbb{N}$ we have

$$z_k \in U \qquad \text{and} \qquad c(z_k) \geq \kappa > 0. \tag{3.19}$$

Now, thanks to (3.14), from (3.18) and (3.19) we deduce that

$$\mathcal{L}w(z_k) = -c(z_k)w(z_k) \geq -\frac{m\,\kappa}{2} > 0, \qquad \text{for all } k \in \mathbb{N}.$$

In particular, setting $w_k := w(\cdot + z_k)$, we obtain

$$\mathcal{L}w_k(0) \geq -\frac{m\,\kappa}{2} > 0, \qquad \text{for all } k \in \mathbb{N}. \tag{3.20}$$

On the other hand, since $w \in \tilde{\mathbb{X}}$, from Lemma 3.2 we infer the existence of some function $w_0 \in \tilde{\mathbb{X}}$ such that (up to a sub-sequence)

$$\lim_{k\to\infty} w_k(x) = w_0(x) \qquad \text{and} \qquad \lim_{k\to\infty} \mathcal{L}w_k(x) = \mathcal{L}w_0(x), \tag{3.21}$$

for every fixed $x \in \mathbb{R}^N$. By taking the limit as $k \to \infty$ in (3.20), we then get

$$\mathcal{L}w_0(0) \geq -\frac{m\,\kappa}{2} > 0. \tag{3.22}$$

Now, we observe that, on account of (3.17) and (3.21), one has

$$w_0(0) = \lim_{k\to\infty} w_k(0) = \lim_{k\to\infty} w(z_k) = m = \inf_{\mathbb{R}^N} w \leq w(x + z_k) = w_k(x),$$

for every $x \in \mathbb{R}^N$ and every $k \in \mathbb{N}$. As a consequence,

$$w_0(0) \leq w_0(x) \qquad \text{for every } x \in \mathbb{R}^N,$$

and thus $x = 0$ is a minimum point for w_0 in $\mathbb{R}^N$. In particular,

$$\Delta w_0(0) \geq 0 \qquad \text{and} \qquad -(-\Delta)^s w_0(0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{w_0(x) - w_0(0)}{|x|^{N+2s}}\, dx \geq 0.$$

Therefore, recalling (1.1), this implies that $\mathcal{L}w_0(0) \leq 0$, which is is in contradiction with (3.22). This completes the proof of (3.16).

We point out that, with the choice in (3.3), from the first part of Lemma 3.3 we obtain the validity of assumption (H3). Indeed, for this, it is enough to apply the first part of Lemma 3.3 with

$$U := \big\{x = (y,t) \in \mathbb{R}^N \text{ s.t. } |t| \geq M\big\},$$

for some $M > 0$. Furthermore, from the first part of Lemma 3.3 we also obtain the validity of assumption (H5), by simply observing that $\mathbb{X} \subset \tilde{\mathbb{X}}$. $\square$

Lemma 3.4 (Validity of (H4) and (H6))**.** *Let $\tilde{\mathbb{X}}$ be as in* (3.4). *Let $c : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be any function satisfying*

$$c(x,0) = 0 \qquad \text{for every } x \in \mathbb{R}^N. \tag{3.23}$$

Let $w \in \tilde{\mathbb{X}}$ satisfy

$$\mathcal{L}w + c(x,w) = 0 \qquad \text{in } \mathbb{R}^N. \tag{3.24}$$

Then

$$\begin{cases} w \geq 0 \text{ in } \mathbb{R}^N, \\ w(0) = 0, \end{cases} \implies w \equiv 0 \text{ on } \mathbb{R}^N. \tag{3.25}$$

In particular, assumptions (H4) *and* (H6) *are fulfilled with the choices in* (3.3) *and* (3.4)*.*

Proof. We observe that, thanks to the assumptions in (3.25), $x = 0$ is a minimum point for w in $\mathbb{R}^N$. As a consequence, we have that

$$\Delta w(0) \geq 0 \qquad \text{and} \qquad -(-\Delta)^s w(0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x)}{|x|^{N+2s}}\,dx \geq 0. \tag{3.26}$$

On the other hand, by (3.23) and (3.24), and recalling also that $w(0) = 0$, we get

$$0 = c(0,0) = c(0,w(0)) = -\mathcal{L}w(0) = \Delta w(0) - (-\Delta)^s w(0) \geq -(-\Delta)^s w(0).$$

Gathering together this and (3.26), we conclude that

$$0 = -(-\Delta)^s w(0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x)}{|x|^{N+2s}}\,dx.$$

Since $w \geq 0$ in $\mathbb{R}^N$, we deduce that $w \equiv 0$ on the whole of $\mathbb{R}^N$, which completes the proof of the claim in (3.25).

Now, we check the validity of assumption (H4). For this, recalling (3.3) and (3.4), we take $\varphi \in \mathbb{X}$ and we define

$$c(x,w) := -f'(\varphi(x))\,w.$$

We observe that c satisfies (3.23). Hence, we can apply the first part of Lemma 3.4 to obtain that (H4) is satisfied. Finally, in order to show the validity of assumption (H6), given $\varphi \in \mathbb{X}$, we define

$$c(x,w) := f(\varphi(x) + w) - f(\varphi(x)).$$

This function satisfies (3.23). As a consequence of this and of the inclusion $\mathbb{X} \subseteq \tilde{\mathbb{X}}$, we deduce (H6) from the first part of Lemma 3.4. $\square$

Thanks to these statements, we can now prove Theorem 1.2:

Proof of Theorem 1.2. On account of Lemmata 3.1, 3.2, 3.3 and 3.4, we know that the assumptions in (H1)—(H6) are fulfilled in the setting of Theorem 1.2. Moreover, since $u \in \mathbb{X}$, we have that

$$\|u\|_{C^{1,\beta}(\mathbb{R}^N)} \text{ is finite for all } \beta \in (0,1).$$

From these considerations and (1.3), we have that the assumptions of Theorem 1.1 in [31] are satisfied. Hence, from Theorem 1.1 in [31] we have that there exists some function $u_0 : \mathbb{R} \to \mathbb{R}$ such that (1.5) holds true. □

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(S. Biagi) Politecnico di Milano - Dipartimento di Matematica
Via Bonardi 9, 20133 Milano, Italy
E-mail address: `stefano.biagi@polimi.it`

(S. Dipierro) Department of Mathematics and Statistics
University of Western Australia
35 Stirling Highway, WA 6009 Crawley, Australia
E-mail address: serena.dipierro@uwa.edu.au

(E. Valdinoci) Department of Mathematics and Statistics
University of Western Australia
35 Stirling Highway, WA 6009 Crawley, Australia
E-mail address: enrico.valdinoci@uwa.edu.au

(E. Vecchi) Politecnico di Milano - Dipartimento di Matematica
Via Bonardi 9, 20133 Milano, Italy
E-mail address: eugenio.vecchi@polimi.it