

A SYMMETRY RESULT FOR COOPERATIVE ELLIPTIC SYSTEMS WITH SINGULARITIES

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ABSTRACT. We obtain symmetry results for solutions of an elliptic system of equation possessing a cooperative structure. The domain in which the problem is set may possess “holes” or “small vacancies” (measured in terms of capacity) along which the solution may diverge.

The method of proof relies on the moving plane technique, which needs to be suitably adapted here to take care of the complications arising from the vacancies in the domain and the analytic structure of the elliptic system.

1. INTRODUCTION AND MAIN RESULTS

The moving plane method was introduced in the pioneer works of Aleksandrov [1, 2] in order to characterize spheres as the only closed, smooth and connected surfaces having constant mean curvature. Afterwards, starting from the seminal paper of Serrin [26] concerning the overdetermined torsion problem, Gidas, Ni and Nirenberg [19] and Berestycki and Nirenberg [4] developed further this technique in order to establish some qualitative properties of solutions of elliptic partial differential equations such as symmetry and monotonicity. The method of proof is very elegant, it relies on a beautiful geometric intuition, and its essential ingredient is the appropriate use of the maximum principle in comparing the values of the solution of the equation at two different points after a suitable reflection, which is determined by a hyperplane which gets moved up to a critical position.

In this paper, we exploit the moving plane technique in order to obtain symmetry results in a setting which is not usually comprised by the classical method, since two difficulties will be taken into account. First of all, we will consider the case of general cooperative elliptic systems rather than that of a single equation, for which the moving plane technique has been settled by Troy [29]. This setting is also motivated by equations driven by polyharmonic operators with Navier boundary conditions (which, up to repeated substitutions, can be framed into elliptic systems of second order equations). Moreover, we take into account the case in which the domain presents “holes”, or “cuts”, or more general vacancies, along which the solution can become singular. This is an extension of our previous work [5] where we were dealing only with singularities made out of a single point, as studied in [6, 28] for the case of a single scalar equation.

Of course, one cannot expect a general treatment of these two situations without additional assumptions. Indeed, general elliptic systems do not satisfy the maximum principle and there is no natural order in the vectorial case, making the classical regularity theories fail in such a situation. Moreover, if the vacancies in the domain are too large, they can affect

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the geometry involved in the reflections and produce singularities that cannot be treated analytically in any convenient way.

To overcome these difficulties, inspired by the recent works [16, 25], we will restrict ourselves to the case of cooperative systems, in which an appropriate use of the maximum principle is possible, and consider domain vacancies that are “sufficiently small”, in terms of capacities.

The precise mathematical formulation in which we work is the following. Let $m \geq 2$ be a fixed natural number. Throughout the present paper, we shall be concerned with second-order cooperative (elliptic) systems of the following form

$$(1.1) \quad \begin{cases} -\Delta u_i = f_i(u_1, \dots, u_m), & \text{in } \Omega \setminus \Gamma, \\ u_i > 0, & \text{in } \Omega \setminus \Gamma, \\ u_i \equiv 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω , Γ and f_1, \dots, f_m satisfy assumptions (H.1)-to-(H.3) below:

(H.1) $\Omega \subseteq \mathbb{R}^n$ is a convex open set of class C^∞ which is bounded and symmetric with respect to the hyperplane $\Pi := \{x_1 = 0\}$;

(H.2) $\Gamma \subseteq \Omega \cap \Pi$ is a closed set consisting of a point, if $n = 2$, or verifying

$$(1.2) \quad \text{Cap}_2(\Gamma) = 0, \text{ if } n \geq 3.$$

(H.3) $f_1, \dots, f_m \in \text{Lip}(\mathbb{R}^m)$ and, for every $i, j \in \{1, \dots, m\}$ with $i \neq j$, the map

$$\mathbb{R} \ni t_j \mapsto f_i(t_1, \dots, t_{j-1}, t_j, t_{j+1}, \dots, t_m)$$

is non-decreasing on $(0, \infty)$ for every choice of $t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_n > 0$.

We refer to Definition 2.2 for the rigorous definition of solution used in this paper. See also Definition 2.1 for the precise meaning of *capacity of a set* and a detailed explanation of the assumption (H.2). We want to point out that the capacity assumption (H.2) cannot be removed nor replaced with the request that $\mathcal{L}^n(\Gamma) = 0$, see Remark 2.6.

We are ready to state the main result of this paper.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ and $\Gamma \subseteq \Omega$ fulfill, respectively, assumptions (H.1) and (H.2). Moreover, let f_1, \dots, f_m satisfy assumption (H.3) and let*

$$U = (u_1, \dots, u_m) \in H_{loc}^1(\Omega \setminus \Gamma; \mathbb{R}^m) \cap C(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m)$$

be a (vector-valued) solution of the elliptic system (1.1).

Then, u_1, \dots, u_m are symmetric with respect to the hyperplane Π and increasing in the x_1 -direction on $\Omega \cap \{x_1 < 0\}$. Furthermore, for every $i \in \{1, \dots, m\}$ one has

$$(1.3) \quad \frac{\partial u_i}{\partial x_1}(x) > 0 \quad \text{for every } x \in \Omega \cap \{x_1 < 0\}.$$

The proof of Theorem 1.1 is pretty much inspired by [16, 25]. The main idea in there relies in proving the symmetry (and monotonicity) of the solution through a clever use of integral estimates. To be more precise, given the function u and its reflection across a given hyperplane, one considers the positive part of their difference and shows that its gradient is actually 0. Passing to elliptic systems this technique becomes more involved because the presence of more equations naturally leads to *interactions* between the solutions which have to be carefully treated. Indeed, these interaction between the different components of the (vectorial) solution, causes an important loss of information on the single equations. To overcome this difficulty we will implement a sort of bootstrap procedure in which an estimate on a single component is reflected into the next one, thus producing an iterative procedure that eventually leads to a closed formula valid for all the components of the

solution. We also want to stress that our result extends our previous result in [5] and it is general enough to cover a bunch of polyharmonic semilinear problems with Navier boundary conditions, even allowing for possibly singular terms.

The literature concerning symmetry results for elliptic PDEs is pretty wide and this makes it hard for us to present here an exhaustive list of references. We already mentioned the seminal papers [4, 19, 26] for the introduction and the use of the moving planes method in the elliptic PDEs setting. More recently, there has been an increasing interest in the study of elliptic PDE's (in bounded domains $\Omega \subset \mathbb{R}^n$) allowing for possible singularities, namely PDE's of the form

$$\begin{cases} -\Delta u = \frac{1}{u^\gamma} + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\gamma > 0$. In this perspective, we want to mention [13], which is one of the first contributions dealing with singular nonlinearities, and then the more recent series of papers [7–10].

To the best of our knowledge, one of the first papers dealing with symmetry of positive solutions of elliptic PDE's in domains with *holes* given by a single point, dates back to [28], which was then extended to slightly more general operators and sets in [6]. The same kind of result, but with a necessary and delicate modification of the technique involved, can be also obtained in presence of a *bigger hole*. In this direction, we refer to [16, 25] where the authors allow (respectively) for a hole given by a $n - 2$ -dimensional smooth manifold and a set of null capacity. Their ideas have also been successfully applied in the non-local setting, see [22].

Let us now spend a few words concerning the case of (cooperative) elliptic systems, which can also include the case of higher order polyharmonic PDE's with Navier boundary conditions. The first result aiming at extending the results in [19] to the vectorial case is contained in [29]. Subsequently, there has been an impressive amount of contributions dealing with the validity of maximum principles (see e.g. [15, 27]). Let us finally mention [3, 5, 11, 12, 14, 18] (for symmetry results for semilinear polyharmonic problems and cooperative elliptic systems with or without singularities).

The paper is organized as follows. In Section 2 we fix the notation used throughout the paper and we recall and prove a few technical results needed for the proof of Theorem 1.1, which is the content of the final Section 3.

2. NOTATIONS AND AUXILIARY RESULTS

The aim of this section is to introduce the relevant notations we shall need in the sequel, and to state some auxiliary results on which we shall base the proof of Theorem 1.1. To begin with, we briefly review in this remark the precise meaning of assumption (H.2) (in the meaningful case $n \geq 3$).

Definition 2.1. If $U \subseteq \mathbb{R}^n$ is open and $E \subseteq U$ is *compact*, the 2-capacity of the condenser (E, U) is defined as

$$\text{Cap}_2(E, U) := \inf \left\{ \int_U \|\nabla u\|^2 dx : u \in C_0^\infty(U) \text{ and } u \geq 1 \text{ on } E \right\}.$$

We then say that E has vanishing 2-capacity (and we write $\text{Cap}_2(E) = 0$) if

$$\text{Cap}_2(E \cap U, U) = 0, \quad \text{for every open set } U \subseteq \mathbb{R}^n.$$

We recall that it can be easily proved that a compact set $E \subseteq \mathbb{R}^n$ has vanishing 2-capacity if and only if there exists a *bounded* open neighborhood U_0 of E such that

$$\text{Cap}_2(E \cap U_0, U_0) = 0.$$

For a demonstration of this fact we refer, e.g., to [21, Lemma 2.9].

We now specify what we mean by a solution of the system in (1.1).

Definition 2.2. Under the above assumptions (H.1)-to-(H.3), we say that a vector-valued function $U = (u_1, \dots, u_m) : \Omega \rightarrow \mathbb{R}^m$ is a **solution** of the system in (1.1) if

(1) $U \in H_{loc}^1(\Omega \setminus \Gamma; \mathbb{R}^m) \cap C(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m)$, that is,

$$u_i \in H_{loc}^1(\Omega \setminus \Gamma) \cap C(\overline{\Omega} \setminus \Gamma), \quad \text{for every } i = 1, \dots, m;$$

(2) for every $i \in \{1, \dots, m\}$ one has

$$(2.1) \quad \int_{\Omega} \langle \nabla u_i, \nabla \varphi \rangle dx = \int_{\Omega} f_i(u_1, \dots, u_m) \varphi dx, \quad \text{for every } \varphi \in C_0^\infty(\Omega \setminus \Gamma; \mathbb{R});$$

(3) for every $i \in \{1, \dots, m\}$ one has $u_i > 0$ a.e. on Ω and $u_i \equiv 0$ on $\partial\Omega$.

In this paper, if $U \subseteq \mathbb{R}^n$ is an arbitrary open set, the space $H_0^1(U)$ is intended as the closure of $C_0^\infty(U, \mathbb{R})$ (or, equivalently, of $\text{Lip}(U) \cap C_0(U, \mathbb{R})$) with respect to the norm

$$\|u\|_{H^1(U)} := \|u\|_{L^2(U)} + \left(\int_U \|\nabla u\|^2 dx \right)^{1/2}.$$

Remark 2.3. We point out that, on account of assumption (H.3), the right-hand side of any equation of the system in (1.1) is locally bounded; as a consequence, if $U = (u_1, \dots, u_m)$ is a solution of this system of PDEs, from standard elliptic regularity we infer that

$$u_1, \dots, u_m \in C_{loc}^{1,\alpha}(\Omega \setminus \Gamma; \mathbb{R}), \quad \text{for every } 0 < \alpha < 1.$$

As a consequence, by condition (3) in Definition 2.2 we have $u_i > 0$ for every $x \in \Omega$.

We are now ready to set the standing notation needed to perform the moving plane technique. If $\Omega \subseteq \mathbb{R}^n$ satisfies assumption (H.1), we set

$$\mathbf{a}_\Omega := \inf_{\Omega} x_1.$$

Moreover, for every fixed $\lambda \in \mathbb{R}$, we define

$$\Sigma_\lambda := \{x \in \Omega : x_1 < \lambda\},$$

and we denote by R_λ the symmetry with respect to the hyperplane $H_\lambda := \{x_1 = \lambda\}$, i.e.,

$$R_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad R_\lambda(x) = x_\lambda := (2\lambda - x_1, x_2, \dots, x_n).$$

We explicitly notice that, since Ω is open, then the same is true of $\Omega_\lambda := R_\lambda(\Omega)$; furthermore, since Ω is convex, we clearly have that Σ_λ is convex and $\Sigma_\lambda \subseteq \Omega \cap \Omega_\lambda$. We collect in the next Lemma 2.4 some topological facts we shall need in the sequel.

Lemma 2.4. *The following assertions hold true:*

- (1) if $E \subseteq \mathbb{R}^n$ is a compact set with vanishing 2-capacity and if $U \subseteq \mathbb{R}^n$ is a convex open set, then $U \setminus E$ is (path-)connected;
- (2) for every fixed $\lambda \in (\mathbf{a}_\Omega, 0)$ one has

$$\text{Cap}_2(\gamma_\lambda) = 0, \quad \text{where } \gamma_\lambda := \Omega \cap \{x_1 = \lambda\}.$$

Proof. (1) First of all we observe that, since the set E has vanishing 2-capacity, for every open neighborhood \mathcal{O} of E one has (see Remark 2.1)

$$(2.2) \quad \text{Cap}_2(E, \mathcal{O}) = 0.$$

Let then $x_0 \neq y_0 \in U$ be fixed, and let $\mathcal{O}_0 \subseteq \mathbb{R}^n$ be an open neighborhood of E such that $x_0, y_0 \notin \mathcal{O}_0$. Moreover, let $\rho > 0$ be so small that

$$(2.3) \quad y_0 \notin B(x_0, \rho), \quad B(x_0, \rho) \subseteq U \quad \text{and} \quad B(x_0, \rho) \cap \mathcal{O}_0 = \emptyset.$$

We claim that there exist a point $x \in B(x_0, \rho) \subseteq U$ such that

$$(2.4) \quad \text{the segment } [x, y_0] \text{ joining } x \text{ to } y_0 \text{ does not intersect } E.$$

Taking this claim for granted for a moment, we are able to complete the demonstration of this assertion: in fact, if $x \in B(x_0, \rho)$ is as in (2.4), the polygonal

$$c := [x_0, x] \cup [x, y_0]$$

connects x_0 to y_0 and it is contained in $U \setminus E$ (this is a straightforward consequence of (2.3), (2.4) and of the fact that, by assumption, U is convex).

We now turn to prove the above claim. To this end, we argue by contradiction and we assume that, for every fixed $x \in B(x_0, \rho)$, there exists $\bar{t} = \bar{t}_x \in (0, 1)$ such that

$$(2.5) \quad x + \bar{t}_x(y_0 - x) \in E.$$

If $u \in C_0^\infty(\mathcal{O}_0, \mathbb{R})$ is any smooth function satisfying $u \geq 1$ on E , by combining (2.3) with (2.5) we obtain the following estimate (note that $x \notin \mathcal{O}_0 \supset \text{supp}(u)$):

$$\begin{aligned} 1 &\leq u(x + \bar{t}_x(y_0 - x)) = u(x + \bar{t}_x(y_0 - x)) - u(x) \\ &= \int_0^{\bar{t}_x} \langle (\nabla u)(x + s(y_0 - x)), y_0 - x \rangle ds \\ (2.6) \quad &\leq \|y_0 - x\| \int_0^{\bar{t}_x} \|(\nabla u)(x + s(y_0 - x))\| ds \\ &\quad (\text{by Hölder's inequality, and setting } \kappa_0 := \|y_0 - x_0\| + \rho) \\ &\leq \kappa_0 \left(\int_0^{\bar{t}_x} \|(\nabla u)(x + s(y_0 - x))\|^2 ds \right)^{1/2}. \end{aligned}$$

Due to the arbitrariness of $x \in B(x_0, \rho)$, we are entitled to integrate both sides of (2.6) on $B(x_0, \rho)$ with respect to x : this gives (with $\omega_n := |B(0, 1)|$)

$$\begin{aligned} \omega_n \rho^n &\leq \kappa_0^2 \int_{B(x_0, \rho)} \left(\int_0^{\bar{t}_x} \|(\nabla u)(x + s(y_0 - x))\|^2 ds \right) dx \\ &\leq \kappa_0^2 \int_{\mathbb{R}^n} \left(\int_0^1 \|(\nabla u)(x + s(y_0 - x))\|^2 ds \right) dx \\ &= \kappa_0^2 \int_{\mathbb{R}^n} \|\nabla u\|^2 dx. \end{aligned}$$

Since the function u was arbitrary, the above estimate implies that

$$\inf \left\{ \int_{\mathbb{R}^n} \|\nabla u\|^2 dx : u \in C_0^\infty(\mathcal{O}_0, \mathbb{R}) \text{ and } u \geq 1 \text{ on } E \right\} \geq \frac{\omega_n \rho^n}{\kappa_0^2},$$

but this is in contradiction with (2.2). Thus, (2.4) holds.

(2) If $n = 2$, from the convexity of Ω (and the fact that, by assumption, $\lambda > \mathbf{a}_\Omega$) it readily follows that γ_λ consists exactly of two points; as a consequence,

$$\text{Cap}_2(\gamma_\lambda) = 0.$$

If, instead, $n \geq 3$, we claim that

$$(2.7) \quad \gamma_\lambda \text{ is a smooth } (n-2)\text{-dimensional manifold.}$$

Taking this claim for granted for a moment, we are able to complete the proof of the statement: indeed, on account of (2.7), it is readily seen that the Hausdorff dimension of γ_λ is precisely $n-2$; as a consequence, we have (see, e.g., [17])

$$\text{Cap}_2(\gamma_\lambda) = 0.$$

We then turn to prove (2.7). To this end, let $\xi \in \gamma_\lambda$ be fixed. Since Ω is an open set of class C^∞ (see assumption (H.1)), there exist an index $i \in \{1, \dots, n\}$, a number $\rho > 0$ and a map $\theta \in C^\infty(B(\xi', \rho), \mathbb{R})$ (where $\xi' = (\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_n)$) such that

$$\begin{aligned} \partial\Omega \cap ((\xi_i - \rho, \xi_i + \rho) \times B(\xi', \rho)) &= \\ &= \{x = (x_i, x') \in (\xi_i - \rho, \xi_i + \rho) \times B(\xi', \rho) : x_i = \theta(x')\}. \end{aligned}$$

Moreover, since Ω is convex and $\lambda > \mathbf{a}_\Omega$, it is quite easy to recognize that θ is either convex or concave on $B(\xi', \rho)$ and that, setting $g(x) = g(x_i, x') := x_i - \theta(x')$,

$$\nabla g(\xi) \text{ is not parallel to } e_1 = (1, 0, \dots, 0).$$

As a consequence, if we introduce the \mathbb{R}^2 -valued function

$$\alpha(x) = \alpha(x_i, x') := (x_i - \theta(x'), x_1 - \lambda) = (g(x), x_1 - \lambda)$$

(with $x = (x_i, x') \in (\xi_i - \rho, \xi_i + \rho) \times B(\xi', \rho)$), we clearly have that

- (a) α is smooth on $\mathcal{U} := (\xi_i - \rho, \xi_i + \rho) \times B(\xi', \rho)$;
- (b) the Jacobian matrix of α at ξ has full rank;
- (c) $\gamma_\lambda \cap \mathcal{U} := \{x \in \mathcal{U} : \alpha(x) = 0\}$.

Gathering together all these facts, we conclude that γ_λ is a smooth manifold of dimension $n-2$, and the proof is finally complete. \square

Remark 2.5. We explicitly observe that, on account of Lemma 2.4-(1), we have that

$$(2.8) \quad \Sigma_\lambda \setminus R_\lambda(\Gamma) \text{ is connected for every } \lambda \in (\mathbf{a}_\Omega, 0).$$

In fact, since Γ fulfills (H.2), we have that $R_\lambda(\Gamma)$ is compact and $\text{Cap}_2(R_\lambda(\Gamma)) = 0$ (for every $n \geq 2$); moreover, as Ω is convex, the same is true of $\Sigma_\lambda = \Omega \cap \{x_1 < \lambda\}$.

Actually, (2.8) can be proved in a more direct (and simpler) way by observing that

$$(2.9) \quad R_\lambda(\Gamma) \subseteq \{x_1 = 2\lambda\}.$$

In fact, since $R_\lambda(\Gamma)$ has vanishing 2-capacity, it is well-known that

$$\mathcal{H}_{\text{dim}}(R_\lambda(\Gamma)) \leq n-2,$$

where $\mathcal{H}_{\text{dim}}(R_\lambda(\Gamma))$ stands for the Hausdorff dimension of $R_\lambda(\Gamma)$ in \mathbb{R}^n (see, e.g., [21]); as a consequence, there necessarily exists (at least) one point

$$(2.10) \quad \bar{x} \in \Sigma_\lambda \cap \{x_1 = 2\lambda\} \text{ such that } x \notin R_\lambda(\Gamma).$$

By combining (2.9) with (2.10) it is very easy to recognize that, if $x_0 \neq y_0 \in \Sigma_\lambda \setminus R_\lambda(\Gamma)$ are arbitrary, the polygonal $c = [x_0, \bar{x}] \cup [\bar{x}, y_0]$ connects x_0 to y_0 and it lays in $\Sigma_\lambda \setminus R_\lambda(\Gamma)$.

Let now $\Gamma \subseteq \Omega$ satisfy assumption (H.2), and let f_1, \dots, f_m be as in assumption (H.3). If $U = (u_1, \dots, u_m) : \Omega \rightarrow \mathbb{R}^m$ is any solution of the elliptic system (1.1) (according to Definition 2.2), we then introduce the following functions (defined on $\Omega_\lambda \setminus R_\lambda(\Gamma)$):

$$(2.11) \quad u_i^{(\lambda)} := u_i \circ R_\lambda \quad \text{and} \quad U_\lambda := (u_1^{(\lambda)}, \dots, u_m^{(\lambda)}) = U \circ R_\lambda.$$

On account of Remark 2.3, we clearly have (for every $0 < \alpha < 1$)

$$(2.12) \quad U_\lambda \in C^{1,\alpha}(\Omega_\lambda \setminus R_\lambda(\Gamma); \mathbb{R}^m) \cap C(\overline{\Omega_\lambda} \setminus R_\lambda(\Gamma); \mathbb{R}^m).$$

Furthermore, since U solves (1.1), we have

$$(2.13) \quad \begin{cases} -\Delta u_i^{(\lambda)} = f_i(u_1^{(\lambda)}, \dots, u_m^{(\lambda)}), & \text{in } \Omega_\lambda \setminus R_\lambda(\Gamma); \\ u_i^{(\lambda)} > 0, & \text{in } \Omega_\lambda \setminus R_\lambda(\Gamma); \\ u_i^{(\lambda)} \equiv 0, & \text{on } \partial\Omega_\lambda. \end{cases}$$

We explicitly notice that, since U_λ is not of class C^2 , by saying that $u_1^{(\lambda)}, \dots, u_m^{(\lambda)}$ solve the (system of) PDEs in (2.13) we mean, precisely, that

$$(2.14) \quad \int_{\Omega_\lambda} \langle \nabla u_i^{(\lambda)}, \nabla \varphi \rangle dx = \int_{\Omega_\lambda} f_i(u_1^{(\lambda)}, \dots, u_m^{(\lambda)}) \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega_\lambda \setminus R_\lambda(\Gamma); \mathbb{R}).$$

Remark 2.6. As already mentioned in the Introduction, assumption (H.2) is somehow sharp. Let us clarify this fact with a couple of examples in the scalar case.

Example 1: In Euclidean space \mathbb{R}^n , let $\Omega := B(0, 1)$ and let $\Gamma := \overline{B(0, 1/2)}$. Since all the boundary points of the annulus $\mathcal{O} := \Omega \setminus \Gamma$ are regular for the Dirichlet problem for Δ , there exists a unique function $u \in C^\infty(\mathcal{O}, \mathbb{R}) \cap C(\overline{\mathcal{O}}, \mathbb{R})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{O} = \Omega \setminus \Gamma, \\ u \equiv 0 & \text{on } \partial\Omega, \\ u(x) = e^{x_1} & \text{for every } x \in \partial B(0, 1/2). \end{cases}$$

Owing to the classical weak and strong maximum principles, it is readily seen that $u > 0$ on $\mathcal{O} = \Omega \setminus \Gamma$; moreover, since u is continuous up to $\overline{\mathcal{O}}$ and since $x \mapsto e^{x_1}$ is not even in x_1 , we infer that u cannot be symmetric with respect to the hyperplane $\Pi = \{x_1 = 0\}$.

Summing up, the function u is a solution of (1.1) (with $m = 1$ and $f \equiv 0$) which is *not* symmetric with respect to the hyperplane $\{x_1 = 0\}$. Notice that both Ω and Γ are symmetric w.r.t. $\{x_1 = 0\}$, but Γ has not vanishing 2-capacity (since $|\Gamma| > 0$).

Example 2: In Euclidean space \mathbb{R}^2 , let $\Omega := B(0, 1)$ and let $\Gamma := \{0\} \times [-1/2, 1/2]$. Moreover, for every fixed $n \geq 2$, we consider the (closed) rectangle

$$R_n := [-1/n, 1/n] \times [-1/2, 1/2],$$

and we choose a function $\varphi_n \in \text{Lip}(R_n)$ such that

$$\varphi_n \equiv 1 \text{ on } \{1/n\} \times [-1/2, 1/2] \quad \text{and} \quad \varphi_n \equiv 2 \text{ on } \{-1/n\} \times [-1/2, 1/2].$$

Finally, we define $\Omega_n := \Omega \setminus R_n$. Since Ω_n is regular for the Dirichlet problem for Δ , it is possible to find a unique function $u_n \in C^\infty(\Omega_n, \mathbb{R}) \cap C(\overline{\Omega_n}, \mathbb{R})$ such that

$$\begin{cases} \Delta u_n = 0 & \text{in } \Omega_n, \\ u_n \equiv 0 & \text{on } \partial\Omega, \\ u_n \equiv \varphi_n & \text{on } \partial R_n. \end{cases}$$

Furthermore, by the classical weak and strong maximum principles we have

$$(2.15) \quad 0 \leq u_n \leq 2 \text{ on } \overline{\Omega_n} \quad \text{and} \quad u_n > 0 \text{ on } \Omega_n.$$

We claim that the sequence $\{u_n\}_n$ has a cluster point u_0 which is a solution of (1.1) (with $m = 1$ and $f \equiv 0$) but which is not symmetric with respect to the hyperplane $\{x_1 = 0\}$.

To prove the claim we first observe that, if $k \in \mathbb{N}$ is arbitrarily fixed and if

$$\mathcal{O}_k := \{x \in \Omega \setminus \Gamma : d(x, \partial(\Omega \setminus \Gamma)) > 1/k\},$$

there exists a natural $n_k \geq 2$ such that $\overline{\mathcal{O}_k} \subseteq \Omega_n$ for every $n \geq n_k$. As a consequence, since $\{u_n\}_{n \geq n_k}$ is a sequence of harmonic functions in \mathcal{O}_k which is uniformly bounded on \mathcal{O}_k , there exists a harmonic function u_{0k} on \mathcal{O}_k such that (up to a sub-sequence)

$$\lim_{n \rightarrow \infty} u_n = u_{0k}, \quad \text{uniformly on every compact set of } \mathcal{O}_k.$$

From this, by exploiting a suitable Cantor diagonal argument, it is then possible to find a sub-sequence $\{u_{n_j}\}_j$ of $\{u_n\}_n$ and a harmonic function u_0 on $\Omega \setminus \Gamma$ such that

$$\lim_{j \rightarrow \infty} u_{n_j} = u_0, \quad \text{uniformly on every compact set of } \Omega \setminus \Gamma.$$

In particular, since $u_n \equiv 0$ on $\partial\Omega$ and $u_n > 0$ on Ω_n for every $n \in \mathbb{N}$, we infer that

$$u_0 \equiv 0 \text{ on } \partial\Omega \quad \text{and} \quad u_0 \geq 0 \text{ on } \Omega \setminus \Gamma.$$

Let now $n \geq 2$ be arbitrarily fixed, let $P_n := (-1/n, 0)$ and let

$$B_n^- := B(P_n, 1/4) \cap \{x_1 < -1/n\} \subseteq \Omega_n.$$

Since φ_n is Lipschitz-continuous on R_n and since $B_n^- \cap \{x_1 = -1/n\}$ is a Lipschitz portion of ∂B_n^- , it follows from classical results (see, e.g., Theorem 4.11 in [20]) that

$$|u_n(x) - 2| = |u_n(x) - u_n(P_n)| \leq C|x_1 + 1/n|, \quad \text{for any } x = (x_1, 0) \in B_n^-,$$

where C is a suitable positive constant which is *independent of* n . From this, by letting $n \rightarrow \infty$ (and reminding that $u_{n_j} \rightarrow u_0$ as $j \rightarrow \infty$ point-wise on $\Omega \setminus \Gamma$) we get

$$|u_0(x) - 2| \leq C|x_1|, \quad \text{for every } x = (x_1, 0) \in \Omega \setminus \Gamma \text{ with } x_1 < 0.$$

As a consequence, we infer that

$$(2.16) \quad \exists \lim_{\substack{x \rightarrow 0 \\ x_1 < 0}} u_0(x) = 2.$$

On the other hand, if $Q_n := (1/n, 0)$ and if

$$B_n^+ := B(Q_n, 1/4) \cap \{x_1 > 1/n\} \subseteq \Omega_n,$$

by arguing exactly as before we get

$$|u_n(x) - 1| = |u_n(x) - u_n(Q_n)| \leq C'|x_1 - 1/n|, \quad \text{for any } x = (x_1, 0) \in B_n^+,$$

where C' is another positive constant which is *independent of* n . From this, by letting $n \rightarrow \infty$ and by taking the limit as $x \rightarrow 0$ with $x_1 > 0$, we obtain

$$(2.17) \quad \exists \lim_{\substack{x \rightarrow 0 \\ x_1 > 0}} u_0(x) = 1.$$

Gathering together (2.16) and (2.17) we readily see that u_0 *cannot be* symmetric with respect to the hyperplane $\{x_1 = 0\}$; moreover, since u_0 is harmonic and non-negative on $\Omega \setminus \Gamma$, by the strong maximum principle we conclude that $u_0 > 0$ on $\Omega \setminus \Gamma$.

Summing up, u_0 is a solution of (1.1) (with $m = 1$ and $f \equiv 0$) which is not symmetric with respect to the hyperplane $\{x_1 = 0\}$. Note that, even if $|\Gamma| = 0$, the set Γ cannot have vanishing 2-capacity: in fact, its Hausdorff dimension is strictly greater than $n - 2 = 0$.

After these preliminaries, we continue this section by constructing two sequences of functions which shall play a fundamental rôle in the proof of Theorem 1.1. In order to do this, we exploit some ideas contained in [16] (see, precisely, Section 2).

First of all we observe that, if $\lambda \in (\mathbf{a}_\Omega, 0)$ is arbitrarily fixed, on account of Lemma 2.4-(2) we have $\text{Cap}_2(R_\lambda(\Gamma)) = 0$ (both in the case $n = 2$ and in the case $n \geq 3$); as a consequence, if $\mathcal{O} \subseteq \mathbb{R}^n$ is any open neighborhood of $R_\lambda(\Gamma)$, we have

$$(2.18) \quad \text{Cap}_2(R_\lambda(\Gamma), \mathcal{O}) = 0.$$

On account of (2.18), for any $k \in \mathbb{N}$ it is possible to find a function $\tilde{\psi}_k \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ (also depending on the fixed λ) such that

- $\tilde{\psi}_k \geq 1$ on $R_\lambda(\Gamma)$ and $\text{supp}(\tilde{\psi}_k) \subseteq \mathcal{O}_k$, where

$$\mathcal{O}_k = \left\{ x \in \mathbb{R}^n : d(x, R_\lambda(\Gamma)) < 2^{-k} \right\};$$

- $\int_{\mathbb{R}^n} |\nabla \tilde{\psi}_k|^2 dx \leq 1/k$.

Starting from the sequence $\{\tilde{\psi}_k\}_{k \in \mathbb{N}}$, we then define

$$(2.19) \quad \psi_k := T \circ \tilde{\psi}_k, \quad \text{where } T(s) := \begin{cases} 1, & \text{if } s < 0, \\ 1 - 2s, & \text{if } 0 \leq s \leq 1/2, \\ 0, & \text{if } s > 1/2. \end{cases}$$

Clearly, $\{\psi_k\}_{k \in \mathbb{N}} \subseteq \text{Lip}(\mathbb{R}^n)$ and, for every fixed $k \in \mathbb{N}$, one has

$$(2.20) \quad 0 \leq \psi_k \leq 1, \quad \psi_k \equiv 1 \text{ on } \mathbb{R}^n \setminus \mathcal{O}_k, \quad \psi_k \equiv 0 \text{ on a small neighborhood of } R_\lambda(\Gamma).$$

Furthermore, since $\nabla \psi_k = (T' \circ \tilde{\psi}_k) \cdot \nabla \tilde{\psi}_k$ a.e. on \mathbb{R}^n , we also have

$$(2.21) \quad \int_{\mathbb{R}^n} |\nabla \psi_k|^2 dx \leq \frac{4}{k}, \quad \text{for every } k \in \mathbb{N}.$$

Arguing analogously, we construct a second sequence of functions $\{\phi_h\}_{h \in \mathbb{N}}$ such that, for every $h \in \mathbb{N}$, the function ϕ_h is identically 0 near the set

$$\gamma_\lambda = \partial\Omega \cap \Pi_\lambda = \{x \in \partial\Omega : x_1 = \lambda\}.$$

To this we first remind that, by Lemma 2.4-(2), we have $\text{Cap}_2(\gamma_\lambda) = 0$; as a consequence, for every open neighborhood $\mathcal{V} \subseteq \mathbb{R}^n$ of γ_λ one has

$$\text{Cap}_2(\gamma_\lambda, \mathcal{V}) = 0.$$

On account of this last fact, in correspondence to every natural h it is possible to construct a function $\tilde{\phi}_h \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ (also depending on the fixed λ) such that

- $\tilde{\phi}_h \geq 1$ on γ_λ and $\text{supp}(\tilde{\phi}_h) \subseteq \mathcal{V}_h$, where

$$\mathcal{V}_h = \left\{ x \in \mathbb{R}^n : d(x, \gamma_\lambda) < 2^{-h} \right\};$$

- $\int_{\mathbb{R}^n} |\nabla \tilde{\phi}_h|^2 dx \leq 1/h$.

Starting from the sequence $\{\tilde{\phi}_h\}_{h \in \mathbb{N}}$, we define (as above)

$$(2.22) \quad \phi_h := T \circ \tilde{\phi}_h, \quad \text{where } T \text{ is as in (2.19)}.$$

Clearly, $\{\phi_h\}_{h \in \mathbb{N}} \subseteq \text{Lip}(\mathbb{R}^n)$ and, for every fixed $h \in \mathbb{N}$, one has

$$(2.23) \quad 0 \leq \phi_h \leq 1, \quad \phi_h \equiv 1 \text{ on } \mathbb{R}^n \setminus \mathcal{V}_h, \quad \phi_h \equiv 0 \text{ on a small neighborhood of } \gamma_\lambda,$$

Furthermore, since $\nabla\phi_h = (T' \circ \tilde{\phi}_h) \cdot \nabla\tilde{\phi}_h$ a.e. on \mathbb{R}^n , we also have

$$(2.24) \quad \int_{\mathbb{R}^n} |\nabla\phi_h|^2 dx \leq \frac{4}{h}, \quad \text{for every } h \in \mathbb{N}.$$

Having defined the sequences $\{\phi_h\}_{h \in \mathbb{N}}$ and $\{\psi_k\}_{k \in \mathbb{N}}$, we conclude this section by stating some auxiliary results which shall be used to prove Theorem 1.1. To begin with, we state the following Lemmas 2.7 and 2.8, which are resemblant of [16, Lemma 3.1].

Lemma 2.7. *Let $\lambda \in (\mathbf{a}_\Omega, 0)$ be such that $R_\lambda(\Gamma) \cap \overline{\Omega} = \emptyset$, and let $\{\phi_h\}_{h \in \mathbb{N}}$ be the sequence defined in (2.22). Moreover, let $g \in C^1(\Sigma_\lambda, \mathbb{R}) \cap C(\overline{\Sigma}_\lambda, \mathbb{R})$ be such that*

$$g \equiv 0 \text{ on } \partial\Sigma_\lambda \cap \Pi_\lambda \quad \text{and} \quad g < 0 \text{ on } \partial\Sigma_\lambda \setminus \Pi_\lambda.$$

Then, the sequence of functions $\{\varphi_h\}_{h \in \mathbb{N}}$ defined by

$$\varphi_h(x) := \begin{cases} g^+(x) \phi_h^2(x), & \text{if } x \in \Sigma_\lambda, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Sigma_\lambda, \end{cases}$$

(here, $g^+ = \max\{g, 0\}$ is the positive part of g) satisfies the following properties:

- (i) $\{\varphi_h\}_{h \in \mathbb{N}} \subseteq \text{Lip}(\mathbb{R}^n)$;
- (ii) $\text{supp}(\varphi_h) \subseteq (\Omega \setminus \Gamma) \cap (\Omega_\lambda \setminus R_\lambda(\Gamma)) \cap \overline{\Sigma}_\lambda$ (for every fixed $h \in \mathbb{N}$);
- (iii) for every $h \in \mathbb{N}$, and a.e. on $\Omega \cup \Omega_\lambda$, one has

$$(2.25) \quad \nabla\varphi_h = [\phi_h^2(\mathbf{1}_{\{g>0\}} \cdot \nabla g) + 2\phi_h g^+ \cdot \nabla\phi_h] \mathbf{1}_{\text{supp}(\varphi_h)}.$$

In particular, $\varphi_h \in \text{Lip}(\overline{\Sigma}_\lambda)$ and $\varphi_h \equiv 0$ on $\partial\Sigma_\lambda$, so that $\varphi_h \in H_0^1(\Sigma_\lambda)$.

Lemma 2.8. *Let $\lambda \in (\mathbf{a}_\Omega, 0)$ be such that $R_\lambda(\Gamma) \cap \overline{\Omega} \neq \emptyset$, and let $\{\psi_k\}_{k \in \mathbb{N}}$, $\{\phi_h\}_{h \in \mathbb{N}}$ be the sequences defined, respectively, in (2.19) and in (2.22). Moreover, let*

$$g \in C^1(\Sigma_\lambda \setminus R_\lambda(\Gamma), \mathbb{R}) \cap C(\overline{\Sigma}_\lambda \setminus R_\lambda(\Gamma), \mathbb{R}),$$

be such that

$$g \equiv 0 \text{ on } (\partial\Sigma_\lambda \cap \Pi_\lambda) \setminus R_\lambda(\Gamma) \quad \text{and} \quad g < 0 \text{ on } (\partial\Sigma_\lambda \setminus \Pi_\lambda) \setminus R_\lambda(\Gamma).$$

Then, the (double) sequence of functions $\{\varphi_{h,k}\}_{h,k \in \mathbb{N}}$ defined by

$$\varphi_{h,k}(x) := \begin{cases} g^+(x) \phi_h^2(x) \psi_k^2(x), & \text{if } x \in \Sigma_\lambda, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Sigma_\lambda, \end{cases}$$

satisfies the following properties:

- (i) $\{\varphi_{h,k}\}_{h,k \in \mathbb{N}} \subseteq \text{Lip}(\mathbb{R}^n)$;
- (ii) $\text{supp}(\varphi_{h,k}) \subseteq (\Omega \setminus \Gamma) \cap (\Omega_\lambda \setminus R_\lambda(\Gamma)) \cap \overline{\Sigma}_\lambda$ (for every fixed $h \in \mathbb{N}$);
- (iii) for every $h \in \mathbb{N}$, and a.e. on $\Omega \cup \Omega_\lambda$, one has

$$(2.26) \quad \nabla\varphi_{h,k} = [\phi_h^2 \psi_k^2(\mathbf{1}_{\{g>0\}} \cdot \nabla g) + 2g^+(\psi_k^2 \phi_h \cdot \nabla\phi_h + \phi_h^2 \psi_k \cdot \nabla\psi_k)] \mathbf{1}_{\text{supp}(\varphi_{h,k})}.$$

In particular, $\varphi_{h,k} \in \text{Lip}(\overline{\Sigma}_\lambda)$ and $\varphi_{h,k} \equiv 0$ on $\partial\Sigma_\lambda$, so that $\varphi_{h,k} \in H_0^1(\Sigma_\lambda)$.

We also have the following regularity result for the solutions of (1.1), which can be demonstrated by arguing essentially as in the proof of [16, Lemma 3.2].

Lemma 2.9. *Let $\lambda \in (\mathbf{a}_\Omega, 0)$ and $i \in \{1, \dots, m\}$ be fixed. Then,*

$$(2.27) \quad (u_i - u_i^{(\lambda)})^+ \in H_0^1(\Sigma_\lambda).$$

Furthermore, if λ is such that $R_\lambda(\Gamma) \cap \overline{\Omega} = \emptyset$, then

$$(2.28) \quad \lim_{h \rightarrow \infty} \varphi_h = (u_i - u_i^{(\lambda)})^+ \text{ in } H_0^1(\Sigma_\lambda), \quad \text{where } \varphi_h \text{ is as in Lemma 2.7,} \\ \text{with } g = u_i - u_i^{(\lambda)}.$$

If, instead, λ is such that $R_\lambda(\Gamma) \cap \overline{\Omega} \neq \emptyset$, then

$$(2.29) \quad \lim_{h,k \rightarrow \infty} \varphi_{h,k} = (u_i - u_i^{(\lambda)})^+ \text{ in } H_0^1(\Sigma_\lambda), \quad \begin{array}{l} \text{where } \varphi_{h,k} \text{ is as in Lemma 2.8,} \\ \text{with } g = u_i - u_i^{(\lambda)}. \end{array}$$

Finally, we prove a technical lemma which will be used in the proof of Theorem 1.1.

Lemma 2.10. *Let $n \geq 2$ and let $U \subseteq \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. There exists a real constant $\Theta = \Theta_n > 0$, independent of U , such that*

$$(2.30) \quad \|u\|_{L^2(U)} \leq \Theta |U|^{1/n} \|\nabla u\|_{L^2(U)}, \quad \text{for every } u \in H_0^1(U).$$

Proof. We first prove (2.30) for a function $v \in C_0^\infty(U)$ (not identically vanishing on U). Since, in particular, we can think of v as a function belonging to $C_0^\infty(\mathbb{R}^n)$, by applying the Nash inequality (see, e.g., [23]) and Hölder's inequality we get

$$\begin{aligned} \|v\|_{L^2(U)}^{1+2/n} &= \|v\|_{L^2(\mathbb{R}^n)}^{1+2/n} \leq \Theta \|v\|_{L^1(\mathbb{R}^n)} \|\nabla v\|_{L^2(\mathbb{R}^n)} \\ &\leq \Theta |U|^{1/n} \|v\|_{L^2(U)}^{2/n} \|\nabla v\|_{L^2(U)}, \end{aligned}$$

where $\Theta > 0$ is a real constant only depending on the dimension n . As a consequence, since we have assumed that $v \not\equiv 0$ on U , we obtain

$$\|v\|_{L^2(U)} \leq \Theta |U|^{1/n} \|\nabla v\|_{L^2(U)}.$$

The proof of (2.30) for a general $u \in H_0^1(U)$ follows by a density argument. \square

3. PROOF OF THEOREM 1.1

In the present section we give the proof of our Theorem 1.1. In doing this, we take for granted all the notations introduced in the preceding sections.

Proof (of Theorem 1.1). For every $\lambda \in (\mathbf{a}_\Omega, 0)$, we consider the functions

$$w_i^{(\lambda)} := u_i - u_i^{(\lambda)}, \quad W_\lambda := (w_1^{(\lambda)}, \dots, w_n^{(\lambda)}) = U - U_\lambda.$$

Taking into account the regularity of U and of U_λ (see, respectively, Definition 2.2 and (2.12)), and reminding that $\Sigma_\lambda \subseteq \Omega \cap \Omega_\lambda$, it is readily seen that (for any $0 < \alpha < 1$)

$$(3.1) \quad W_\lambda \in C^{1,\alpha}(\Sigma_\lambda \setminus R_\lambda(\Gamma); \mathbb{R}^m) \cap C(\overline{\Sigma_\lambda} \setminus R_\lambda(\Gamma); \mathbb{R}^m).$$

Furthermore, since U solves (1.1) and U_λ solves (2.14) we have (note that, as Ω is convex, the reflection of $\partial\Sigma_\lambda \setminus \Pi_\lambda$ with respect to Π_λ is entirely contained in Ω)

$$(3.2) \quad \begin{cases} -\Delta w_i^{(\lambda)} = \sum_{j=1}^m c_{ij}(x; \lambda) w_j^{(\lambda)}, & \text{on } \Sigma_\lambda \setminus R_\lambda(\Gamma), \\ w_i^{(\lambda)} < 0, & \text{on } (\partial\Sigma_\lambda \setminus \Pi_\lambda) \setminus R_\lambda(\Gamma), \\ w_i^{(\lambda)} \equiv 0, & \text{on } \partial\Sigma_\lambda \cap \Pi_\lambda, \end{cases}$$

where $c_{i1}(\cdot; \lambda), \dots, c_{im}(\cdot; \lambda) : \Sigma_\lambda \setminus R_\lambda(\Gamma) \rightarrow \mathbb{R}$ are defined as follows:

$$(3.3) \quad c_{ij}(x; \lambda) := \begin{cases} \frac{f_i(U(x)) - f_i(U_\lambda(x))}{u_j(x) - u_j^{(\lambda)}(x)}, & \text{if } u_i(x) \neq u_i^{(\lambda)}(x), \\ 0, & \text{otherwise.} \end{cases}$$

As for the case of U and U_λ , since W_λ is not of class C^2 on Σ_λ , by saying that $w_1^{(\lambda)}, \dots, w_m^{(\lambda)}$ solve the system of PDEs in (3.2) we mean, precisely, that

$$(3.4) \quad \int_{\Sigma_\lambda} \langle \nabla w_i^{(\lambda)}, \nabla \varphi \rangle dx = \sum_{j=1}^m \int_{\Sigma_\lambda} c_{ij}(\cdot; \lambda) w_j^{(\lambda)} \varphi dx, \quad \forall \varphi \in C_0^\infty(\Sigma_\lambda \setminus R_\lambda(\Gamma), \mathbb{R}).$$

Moreover, on account of assumption (H.3), we see that

- (i) $c_{ij}(\cdot; \lambda) \geq 0$ for every $i \in \{1, \dots, m\}$ and every $j \neq i$;
- (ii) there exists a real constant $\mathbf{c}_f > 0$ such that

$$(3.5) \quad |c_{ij}(\cdot; \lambda)| \leq \mathbf{c}_f, \quad \text{for every } i, j \in \{1, \dots, m\} \text{ and every } \lambda \in (\mathbf{a}_\Omega, 0).$$

According to the well-established moving planes technique, we now define

$$(3.6) \quad \mathcal{I} := \{ \lambda \in (\mathbf{a}_\Omega, 0) : w_i^{(t)} < 0 \text{ on } \Sigma_t \setminus R_t(\Gamma) \forall t \in (\mathbf{a}_\Omega, \lambda) \text{ and } \forall i \in \{1, \dots, m\} \},$$

and $\lambda_0 := \sup \mathcal{I}$.

Our aim is to demonstrate that $\mathcal{I} \neq \emptyset$ and that $\lambda_0 = 0$. From now on, in order to ease the readability, we split the proof into some different steps.

STEP I: In this step we prove that $\mathcal{I} \neq \emptyset$ and that $\lambda_0 > \mathbf{a}_\Omega$. We fix $t_0 \in (\mathbf{a}_\Omega, 0)$ such that $R_{t_0}(\Gamma) \subset \Omega^c$. Necessarily, we have that $R_t(\Gamma) \subset \Omega^c$ for every $t \in (\mathbf{a}_\Omega, t_0)$. Now, for every $i = 1, \dots, m$ we consider the function $\varphi_{i,h} : \Omega \rightarrow \mathbb{R}$ defined as

$$\varphi_{i,h} := (w_i^{(t)})^+ \phi_h^2 \mathbf{1}_{\Sigma_t},$$

where $\{\phi_h\}_{h \in \mathbb{N}}$ is as in (2.22). By density, we can use $\varphi_{i,h}$ as a test function, finding

$$\frac{1}{2} \int_{\Sigma_t} |\nabla (w_i^{(t)})^+|^2 \phi_h^2 dx \leq \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_t} (w_j^{(t)})^+ (w_i^{(t)})^+ \phi_h^2 dx + 2 \int_{\Sigma_t} (w_i^{(t)})^2 |\nabla \phi_h|^2 dx.$$

By Fatou Lemma, sending $h \rightarrow 0^+$ we get

$$\frac{1}{2} \int_{\Sigma_t} |\nabla (w_i^{(t)})^+|^2 dx \leq \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_t} (w_j^{(t)})^+ (w_i^{(t)})^+ dx.$$

By Hölder inequality on every term on the right hand side, we get

$$\frac{1}{2} \|\nabla (w_i^{(t)})^+\|_{L^2(\Sigma_t)}^2 dx \leq \mathbf{c}_f \sum_{j=1}^m \|(w_j^{(t)})^+\|_{L^2(\Sigma_t)} \|(w_i^{(t)})^+\|_{L^2(\Sigma_t)}.$$

From this, by using (2.30) (on every term on the right hand side), for every $t \in (\mathbf{a}_\Omega, t_0)$ and every index $i \in \{1, \dots, m\}$ we get

$$\frac{1}{2} \|\nabla (w_i^{(t)})^+\|_{L^2(\Sigma_t)}^2 dx \leq \mathbf{c}_f \theta_n^2(\Sigma_t) \sum_{j=1}^m \|\nabla (w_j^{(t)})^+\|_{L^2(\Sigma_t)} \|\nabla (w_i^{(t)})^+\|_{L^2(\Sigma_t)},$$

where we have introduced the notation (repeatedly used in the sequel)

$$\theta_n(\Sigma_t) := \Theta |\Sigma_t|^{1/n} \quad (\text{with } \Theta > 0 \text{ is as in Lemma 2.10}).$$

Now, if $\|\nabla (w_i^{(t)})^+\|_{L^2(\Sigma_t)} \neq 0$, from the above inequality we immediately get

$$(3.7) \quad \frac{1}{2} \|\nabla (w_i^{(t)})^+\|_{L^2(\Sigma_t)} \leq \mathbf{c}_f \theta_n^2(\Sigma_t) \sum_{j=1}^m \|\nabla (w_j^{(t)})^+\|_{L^2(\Sigma_t)}.$$

On the other hand, since (3.7) is trivially satisfied when $\|\nabla (w_i^{(t)})^+\|_{L^2(\Sigma_t)} = 0$, we conclude that such an inequality holds true for every $i \in \{1, \dots, m\}$ and every $t \in (\mathbf{a}_\Omega, t_0)$.

We now aim at proving the following assertion: *for every fixed $k \in \{1, \dots, m-1\}$ there exist $t_k \in (\mathbf{a}_\Omega, t_0)$ and a real constant $C_k = C_k(m, \mathbf{c}_f) > 0$ such that*

$$(3.8) \quad \|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} \leq C_k \theta_n^2(\Sigma_t) \sum_{j \geq i+1} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)}, \quad \begin{array}{l} \text{for all } 1 \leq i \leq k \text{ and} \\ \text{every } t \in (\mathbf{a}_\Omega, t_k). \end{array}$$

To prove (3.8) we argue by (finite) induction and we start with $k = 1$. By (3.7) we have

$$\left(\frac{1}{2} - \mathbf{c}_f \theta_n^2(\Sigma_t)\right) \|\nabla(w_1^{(t)})^+\|_{L^2(\Sigma_t)} \leq \mathbf{c}_f \theta_n^2(\Sigma_t) \sum_{j \geq 2} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)}.$$

Since $\theta_n(\Sigma_t) \rightarrow 0$ as $t \rightarrow \mathbf{a}_\Omega$, it is possible to find $t_1 \in (\mathbf{a}_\Omega, t_0)$ such that

$$\frac{1}{2} - \mathbf{c}_f \theta_n^2(\Sigma_t) > \frac{1}{4} \quad \text{for every } t \in (\mathbf{a}_\Omega, t_1).$$

As a consequence, we obtain

$$\|\nabla(w_1^{(t)})^+\|_{L^2(\Sigma_t)} \leq 4\mathbf{c}_0 \theta_n^2(\Sigma_t) \sum_{j \geq 2} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)},$$

which is precisely (3.8) for $i = 1$ (with $C_1 = 4\mathbf{c}_0$). Let us now suppose that (3.8) holds for a certain index $k \in \{1, \dots, m-2\}$ and, by shrinking t_k if necessary, let us also assume that $\theta_n(\Sigma_t) < 1$ for all $t \in (\mathbf{a}_\Omega, t_k)$. Owing to (3.7) (with $i = k+1$), we then have

$$\begin{aligned} & \left(\frac{1}{2} - \mathbf{c}_f \theta_n^2(\Sigma_t)\right) \|\nabla(w_{k+1}^{(t)})^+\|_{L^2(\Sigma_t)} \leq \mathbf{c}_f \theta_n^2(\Sigma_t) \sum_{i \neq k+1} \|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} \\ & = \mathbf{c}_f \theta_n^2(\Sigma_t) \left(\sum_{i=1}^k \|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} + \sum_{i \geq k+2} \|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} \right) \\ & \quad \text{(by (3.8), which we are assuming to hold for the index } k) \\ (3.9) \quad & \leq \mathbf{c}_f \theta_n^2(\Sigma_t) \left(C_k \sum_{i=1}^k \sum_{j \geq i+1} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)} + \sum_{i \geq k+2} \|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} \right) \\ & \leq \mathbf{c}_f \theta_n^2(\Sigma_t) \left(m C_k \sum_{j=2}^{k+1} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)} + \right. \\ & \quad \left. + (k C_k + 1) \sum_{i \geq k+2} \|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} \right) =: (\star), \end{aligned}$$

We now perform a backward induction argument to show that, as a consequence of the validity of (3.8) for the index k , the following fact holds: *for every fixed $j \in \{1, \dots, k\}$, it is possible to find a real constant $\mathcal{C}_j = \mathcal{C}_j(m, k, \mathbf{c}_f) > 0$ such that*

$$(3.10) \quad \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)} \leq \mathcal{C}_j \theta_n^2(\Sigma_t) \sum_{r \geq k+1} \|\nabla(w_r^{(t)})^+\|_{L^2(\Sigma_t)}, \quad \begin{array}{l} \text{for all } 1 \leq j \leq k \text{ and} \\ \text{every } t \in (\mathbf{a}_\Omega, t_k). \end{array}$$

For $j = k$, (3.10) follows immediately from (3.8) by taking $i = k$ (with $\mathcal{C}_k := C_k$). We then suppose the existence of an index $j \in \{2, \dots, k\}$ such that (3.10) holds for every $j \leq r \leq k$,

and we exploit once again (3.8) (with $i = j - 1 \leq k - 1$): this gives

$$\begin{aligned} \|\nabla(w_{j-1}^{(t)})^+\|_{L^2(\Sigma_t)} &\leq C_k \theta_n^2(\Sigma_t) \sum_{r \geq j} \|\nabla(w_r^{(t)})^+\|_{L^2(\Sigma_t)} \\ &= C_k \theta_n^2(\Sigma_t) \left(\sum_{r=j}^k \|\nabla(w_r^{(t)})^+\|_{L^2(\Sigma_t)} + \sum_{r \geq k+1} \|\nabla(w_r^{(t)})^+\|_{L^2(\Sigma_t)} \right) \\ &\text{(since (3.10) holds for } j \leq r \leq k, \text{ and } \theta_n(\Sigma_t) < 1) \\ &\leq C_k \left(m \max_{j \leq r \leq k} (C_r) + 1 \right) \theta_n^2(\Sigma_t) \sum_{r \geq k+1} \|\nabla(w_r^{(t)})^+\|_{L^2(\Sigma_t)}, \end{aligned}$$

so that (3.10) holds true also for $j - 1$. By the Induction Principle, we then conclude that estimate (3.10) is valid for every $j = 1, \dots, k$, as claimed.

With (3.10) at hand, we now continue the estimate (3.9): reminding that, by the choice of t_k , we have $\theta_n(\Sigma_t) < 1$ for every $t \in (\mathbf{a}_\Omega, t_k)$, we have

$$(3.11) \quad (\star) \leq M_k \theta_n^2(\Sigma_t) \sum_{j \geq k+1} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)},$$

where $M_k = M_k(m, \mathbf{c}_f) > 0$ is a suitable constant. As a consequence, we obtain

$$\left(\frac{1}{2} - \mathbf{c}_f \theta_n^2(\Sigma_t) - M_k \theta_n^2(\Sigma_t) \right) \|\nabla(w_{k+1}^{(t)})^+\|_{L^2(\Sigma_t)} \leq M_k \theta_n^2(\Sigma_t) \sum_{j \geq k+2} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)}.$$

Finally, since $\theta_n(\Sigma_t) \rightarrow 0$ as $t \rightarrow \mathbf{a}_\Omega$, we infer the existence of $\bar{t} \in (\mathbf{a}_\Omega, t_0)$ such that

$$\frac{1}{2} - \mathbf{c}_f \theta_n^2(\Sigma_t) - M_k \theta_n^2(\Sigma_t) > \frac{1}{4} \quad \text{for every } t \in (\mathbf{a}_\Omega, \bar{t});$$

from this, we obviously derive the estimate (valid for $t \in (\mathbf{a}_\Omega, \bar{t})$)

$$\|\nabla(w_{k+1}^{(t)})^+\|_{L^2(\Sigma_t)} \leq 4M_k \theta_n^2(\Sigma_t) \sum_{j \geq k+2} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)}.$$

Taking as $t_{k+1} := \min\{t_k, \bar{t}\}$, and setting $C_{k+1} := \max\{C_k, 4M_k\}$, we then obtain

$$\|\nabla(w_i^{(t)})^+\|_{L^2(\Sigma_t)} \leq C_{k+1} \theta_n^2(\Sigma_t) \sum_{j \geq i+1} \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)} \quad \begin{array}{l} \text{for all } 1 \leq i \leq k+1 \text{ and} \\ \text{every } t \in (\mathbf{a}_\Omega, t_{k+1}), \end{array}$$

so that (3.8) holds true also for $k+1$. By the Induction Principle, we conclude that estimate (3.8) is valid for every $k = 1, \dots, m-2$, as claimed.

Now we have established (3.8), we are able to complete the proof this step. In fact, since the cited (3.8) holds true for $k = m-1$, a (finite) backward induction argument shows the existence of a real constant $\mathcal{C}_m = \mathcal{C}_m(\mathbf{c}_0) > 0$ such that

$$(3.12) \quad \|\nabla(w_j^{(t)})^+\|_{L^2(\Sigma_t)} \leq \mathcal{C}_m \theta_n^2(\Sigma_t) \|\nabla(w_m^{(t)})^+\|_{L^2(\Sigma_t)}, \quad \begin{array}{l} \text{for all } 1 \leq j \leq m-1 \text{ and} \\ \text{every } t \in (\mathbf{a}_\Omega, t_{m-1}); \end{array}$$

gathering together (3.12) and (3.7) (with $i = m$), for any $t \in (\mathbf{a}_\Omega, t_{m-1})$ we get

$$\frac{1}{2} \|\nabla(w_m^{(t)})^+\|_{L^2(\Sigma_t)} \leq \mathbf{c}_f \theta_n^2(\Sigma_t) (m \mathcal{C}_m \theta_n^2(\Sigma_t) + 1) \|\nabla(w_m^{(t)})^+\|_{L^2(\Sigma_t)}.$$

Since $\theta_n(\Sigma_t) \rightarrow 0$ as $t \rightarrow \mathbf{a}_\Omega$, there exists $\tau_0 \in (\mathbf{a}_\Omega, t_{m-1})$ such that

$$\mathbf{c}_f \theta_n^2(\Sigma_t) (m \mathcal{C}_m \theta_n^2(\Sigma_t) + 1) < \frac{1}{4}, \quad \text{for every } t \in (\mathbf{a}_\Omega, \tau_0);$$

as a consequence, we obtain

$$\|\nabla(w_m^{(t)})\|_{L^2(\Sigma_t)} = 0, \quad \text{for every } t \in (\mathbf{a}_\Omega, \tau_0).$$

On account of (3.8), this proves that

$$\|\nabla(w_1^{(t)})\|_{L^2(\Sigma_t)} = \cdots = \|\nabla(w_m^{(t)})\|_{L^2(\Sigma_t)} = 0, \quad \text{for every } t \in (\mathbf{a}_\Omega, \tau_0);$$

as a consequence, by Lemma 2.10 (and since W_t is continuous on Σ_t) we get

$$(3.13) \quad u_i - u_i^{(t)} = w_i^{(t)} \leq 0 \text{ on } \Sigma_t \quad (\text{for every } i = 1, \dots, m \text{ and every } t \in (\mathbf{a}_\Omega, \tau_0)).$$

We finally claim that, by the Strong Maximum Principle for C^1 -subsolutions, we have

$$(3.14) \quad u_i < u_i^{(t)} \text{ on } \Sigma_t, \quad \text{for every } i = 1, \dots, m \text{ and every } t \in (\mathbf{a}_\Omega, \tau_0).$$

Indeed, let $i \in \{1, \dots, m\}$ and $t \in (\mathbf{a}_\Omega, \tau_0)$ be arbitrarily fixed. Clearly, the set Σ_t is (open and) connected; moreover, since the (vector-valued) map $W_t = U - U_t$ solves (3.2) and $c_{ij}(\cdot; t) \geq 0$ for every $j \neq i$, we have

$$-\Delta w_i^{(t)} = \sum_{j=1}^m c_{ij}(\cdot; t) w_j^{(t)} \leq c_{ii}(\cdot; t) w_i^{(t)} \quad (\text{as } u_i \leq u_i^{(t)}).$$

We explicitly point out that the above inequality has to be intended in the weak sense of distributions on Σ_t : this means, precisely, that

$$\int_{\Sigma_t} \langle \nabla w_i^{(t)}, \nabla \varphi \rangle dx \leq \int_{\Sigma_t} c_{ii}(\cdot; t) w_i^{(t)} dx, \quad \forall \varphi \in C_0^\infty(\Sigma_t, \mathbb{R}) \text{ with } \varphi \geq 0 \text{ on } \Sigma_t.$$

From this, taking into account (3.5) we get

$$-\Delta w_i^{(t)} + (\mathbf{c}_f - c_{ii}(\cdot; t)) w_i^{(t)} \leq \mathbf{c}_f w_i^{(t)} \leq 0,$$

and $\mathbf{c}_f - c_{ii}(\cdot; t) \geq 0$ on Σ_t . Gathering together all these facts, we can invoke the Strong Maximum Principle for C^1 -subsolutions (see, e.g., [20]), ensuring that

$$\text{either } w_i^{(t)} < 0 \text{ or } w_i^{(t)} \equiv 0 \text{ on } \Sigma_t.$$

Since, by (3.2), we know that the function $w_i^{(t)}$ is (strictly) negative on the set $\partial\Sigma_t \setminus \Pi_t$ (notice $t < \tau_0 < 0$), we then conclude that (3.14) holds true.

Finally, on account of (3.14) (and taking into account the very definition of \mathcal{I}), we see that $(\mathbf{a}_\Omega, \tau_0) \subseteq \mathcal{I}$, whence $\mathcal{I} \neq \emptyset$, and that $\lambda_0 = \sup \mathcal{I} \geq \tau_0 > \mathbf{a}_\Omega$.

STEP II: We now turn to demonstrate that $\lambda_0 = 0$. To this end, following [16], we argue by contradiction and we assume that $\lambda_0 \in (\mathbf{a}_\Omega, 0)$. Since W_{λ_0} is continuous on $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$, from the very definition of λ_0 we deduce that, for every $i \in \{1, \dots, m\}$,

$$(3.15) \quad w_i^{(\lambda_0)} \leq 0 \text{ on } \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma), \text{ that is, } u_i \leq u_{\lambda_0} \text{ on } \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma).$$

As a consequence, by the Strong Maximum Principle (for C^1 -subsolutions) we get

$$(3.16) \quad w_i^{(\lambda_0)} < 0 \text{ on } \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma), \quad \text{for every } i \in \{1, \dots, m\}.$$

In fact, taking into account that W_{λ_0} solves (3.2) and arguing exactly as in the last part of the previous step, we have the following family of inequalities (which has to be intended in the weak sense of distributions on $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$):

$$(3.17) \quad -\Delta w_i^{(\lambda_0)} + (\mathbf{c}_f - c_{ii}(\cdot; \lambda_0)) w_i^{(\lambda_0)} \leq \mathbf{c}_f w_i^{(\lambda_0)} \leq 0 \quad \text{for every } i \in \{1, \dots, m\}.$$

Moreover, since $\mathbf{c}_f - c_{ii}(\cdot; \lambda_0) \geq 0$ on $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ (see (3.5)) and since, by Lemma 2.4-(1), the set $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ is open and connected (see also Remark 2.5), we are entitled to apply the Strong Maximum Principle for C^1 -subsolution: this gives

$$\text{either } w_i^{(\lambda_0)} < 0 \text{ or } w_i^{(\lambda_0)} \equiv 0 \text{ on } \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) \quad (\text{for any } i \in \{1, \dots, m\}).$$

Finally, since we know that the functions $w_1^{(\lambda_0)}, \dots, w_m^{(\lambda_0)}$ are (strictly) negative on the set $(\partial\Sigma_{\lambda_0} \setminus \Pi_{\lambda_0}) \setminus R_{\lambda_0}(\Gamma)$ (as $\lambda_0 < 0$, see (3.2)), we conclude that (3.16) holds true.

Now we have established (3.16), we then turn to prove the following assertion: *in correspondence to every compact set $K \subseteq \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ with Lipschitz boundary ∂K , it is possible to find a small $\epsilon = \epsilon(K, \lambda_0) \in (0, |\lambda_0|/2)$ such that*

- (a) $K \subseteq \Sigma_\lambda \setminus R_\lambda(\Gamma)$ for every $\lambda \in [\lambda_0, \lambda_0 + \epsilon]$;
- (b) $(w_i^{(\lambda)})^+ \equiv 0$ on K for every $i \in \{1, \dots, m\}$ and every $\lambda \in (\lambda_0, \lambda_0 + \epsilon]$;
- (c) for every $i \in \{1, \dots, m\}$ and every $\lambda \in (\lambda_0, \lambda_0 + \epsilon]$ we have

$$(3.18) \quad \|\nabla(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)} \leq \mathbf{c}_f \theta_n^2(\Sigma_\lambda \setminus K) \sum_{j=1}^m \|\nabla(w_j^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)},$$

where $\theta_n(\Sigma_\lambda \setminus K) = \Theta |\Sigma_\lambda \setminus K|^{1/n}$ (see Lemma 2.10).

We explicitly observe that, if $\epsilon < |\lambda_0|/2$, we have

$$[\lambda_0, \lambda_0 + \epsilon] \subseteq (\mathbf{a}_\Omega, 0).$$

Let now $K \subseteq \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ be an arbitrarily fixed compact set. Since both K and $R_{\lambda_0}(\Gamma)$ are closed, it is very easy to recognize that there exists a suitable $\nu = \nu(K, \lambda_0) > 0$, which we can assume to be smaller than $|\lambda_0|/2$, such that

$$(3.19) \quad K \subseteq \Sigma_\lambda \setminus R_\lambda(\Gamma), \quad \text{for every } \lambda \in [\lambda_0, \lambda_0 + \nu].$$

Moreover, on account of (3.16) (and remembering that W_{λ_0} is continuous on $\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$), it is possible to find a real constant $M_0 < 0$ such that

$$(3.20) \quad w_i^{(\lambda_0)} \leq M_0 < 0 \text{ on } \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma), \quad \text{for every } i \in \{1, \dots, m\}.$$

Since, for every fixed $i \in \{1, \dots, m\}$ and every $\lambda \in [\lambda_0, \lambda_0 + \nu]$, the function

$$(x, \lambda) \mapsto w_i^{(\lambda)}(x) = u_i(x) - u_i^{(\lambda)}(x),$$

is (well-defined and) uniformly continuous on $K \times [\lambda_0, \lambda_0 + \nu]$ (as it follows from (3.19)), there exists a real $\epsilon = \epsilon(K, \lambda_0) \in (0, \nu)$ (hence, $\epsilon < |\lambda_0|/2$) such that

$$(3.21) \quad w_i^{(\lambda)}(x) < w_i^{(\lambda_0)}(x) + \frac{|M_0|}{2} \stackrel{(3.20)}{\leq} \frac{M_0}{2} < 0, \quad \forall x \in K \text{ and } \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon].$$

Summing up, if $\lambda \in [\lambda_0, \lambda_0 + \epsilon]$, we have $K \subseteq \Sigma_\lambda \setminus R_\lambda(\Gamma)$ and $(w_i^{(\lambda)})^+ \equiv 0$ on K .

We then turn to prove (3.18). To this end, let $i \in \{1, \dots, m\}$ and let $\lambda \in (\lambda_0, \lambda_0 + \epsilon]$ be arbitrarily fixed. We consider the (double) sequence of functions defined by

$$\varphi_{h,k} := \begin{cases} (w_i^{(\lambda)})^+ \phi_h^2 \psi_k^2, & \text{on } \Sigma_\lambda, \\ 0, & \text{on } \mathbb{R}^n \setminus \Sigma_\lambda, \end{cases}$$

where $\{\phi_h\}_{h \in \mathbb{N}}$ is the sequence defined in (2.22) and associated with $\gamma_\lambda = \partial\Omega \cap \Pi_\lambda$, whilst $\{\psi_k\}_{k \in \mathbb{N}}$ is the sequence defined in (2.19) and associated with $R_\lambda(\Gamma)$ (actually, the functions

$\varphi_{h,k}$ also depend on the fixed i and λ ; however, in order to avoid cumbersome notations, we prefer to not keep trace of this dependence in the sequel).

By Lemma 2.8, for every $h, k \in \mathbb{N}$ we have $\varphi_{h,k} \in \text{Lip}(\overline{\Sigma_\lambda})$ and $\varphi_{h,k} \equiv 0$ on $\partial\Sigma_\lambda$; moreover, by (3.21), there exists an open neighborhood $\mathcal{U} \subseteq \Sigma_\lambda \setminus R_\lambda(\Gamma)$ of K such that

$$(3.22) \quad (w_i^{(\lambda)})^+ \equiv 0 \text{ on } \mathcal{U}, \quad \text{whence } \varphi_{h,k} \equiv 0 \text{ on } \mathcal{U} \text{ for every } h, k \in \mathbb{N}.$$

Gathering together all these facts, we deduce that

$$(3.23) \quad \varphi_{h,k} \in H_0^1(\Sigma_\lambda \setminus K),$$

Furthermore, since $\varphi_{h,k} \rightarrow (w_i^{(\lambda)})^+$ in $H_0^1(\Sigma_\lambda)$ as $h, k \rightarrow \infty$ (see Lemma 2.9), we also get

$$(3.24) \quad (w_i^{(\lambda)})^+ \in H_0^1(\Sigma_\lambda \setminus K).$$

Owing to (3.23), and by a standard density argument, we are entitled to use the function $\varphi_{h,k}$ (for every fixed $h, k \in \mathbb{N}$) as test function in (3.4), obtaining (see also (2.26))

$$\begin{aligned} & \int_{\Sigma_\lambda} |\nabla(w_i^{(\lambda)})^+|^2 \phi_h^2 \psi_k^2 dx + 2 \int_{\Sigma_\lambda} (w_i^{(\lambda)})^+ \psi_k^2 \phi_h \langle \nabla w_i^{(\lambda)}, \nabla \phi_h \rangle dx + \\ & \quad + 2 \int_{\Sigma_\lambda} (w_i^{(\lambda)})^+ \phi_h^2 \psi_k \langle \nabla w_i^{(\lambda)}, \nabla \psi_k \rangle dx \\ & = \int_{\Sigma_\lambda} \langle \nabla w_i^{(\lambda)}, \nabla \varphi_{h,k} \rangle dx = \sum_{j=1}^m \int_{\Sigma_\lambda} c_{ij}(\cdot; \lambda) w_j^{(\lambda)} \varphi_{h,k} dx \\ & = \sum_{j=1}^m \int_{\Sigma_\lambda} c_{ij}(\cdot; \lambda) (w_i^{(\lambda)})^+ w_j^{(\lambda)} \phi_h^2 \psi_k^2 dx. \end{aligned}$$

From this, by (3.22), (3.5) and the fact that $c_{ij}(\cdot; \lambda) \geq 0$ if $j \neq i$, we get

$$\begin{aligned} & \int_{\Sigma_\lambda \setminus K} |\nabla(w_i^{(\lambda)})^+|^2 \phi_h^2 \psi_k^2 dx = \int_{\Sigma_\lambda} |\nabla(w_i^{(\lambda)})^+|^2 \phi_h^2 \psi_k^2 dx \\ & \leq 2 \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ \psi_k^2 \phi_h |\nabla w_i^{(\lambda)}| |\nabla \phi_h| dx + 2 \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ \phi_h^2 \psi_k |\nabla w_i^{(\lambda)}| |\nabla \psi_k| dx + \\ & \quad + \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ (w_j^{(\lambda)})^+ \phi_h^2 \psi_k^2 dx. \end{aligned}$$

We now observe that, since $\nabla w_i^{(\lambda)} = \nabla(w_i^{(\lambda)})^+$ almost everywhere on the set $\{w_i^{(\lambda)} > 0\}$, the above inequality can be re-written as follows:

$$\begin{aligned} & \int_{\Sigma_\lambda \setminus K} |\nabla(w_i^{(\lambda)})^+|^2 \phi_h^2 \psi_k^2 dx \leq \int_{\Sigma_\lambda \setminus K} 2((w_i^{(\lambda)})^+ \psi_k |\nabla \phi_h|)(\psi_k \phi_h |\nabla(w_i^{(\lambda)})^+|) dx + \\ (3.25) \quad & \quad + \int_{\Sigma_\lambda \setminus K} 2((w_i^{(\lambda)})^+ \phi_h |\nabla \psi_k|)(\phi_h \psi_k |\nabla(w_i^{(\lambda)})^+|) dx + \\ & \quad + \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ (w_j^{(\lambda)})^+ \phi_h^2 \psi_k^2 dx. \end{aligned}$$

From this, by using the classical Young's inequality

$$2ab \leq 4a^2 + \frac{1}{4}b^2 \quad (\text{holding true for every } a, b \geq 0)$$

on the integrands of the first two integrals in the right-hand side of (3.25), we get

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_\lambda \setminus K} |\nabla(w_i^{(\lambda)})^+|^2 \phi_h^2 \psi_k^2 dx &\leq 4 \int_{\Sigma_\lambda \setminus K} [(w_i^{(\lambda)})^+]^2 \psi_k^2 |\nabla \phi_h|^2 dx + \\ &+ 4 \int_{\Sigma_\lambda \setminus K} [(w_i^{(\lambda)})^+]^2 \phi_h^2 |\nabla \psi_k|^2 dx + \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ (w_j^{(\lambda)})^+ \phi_h^2 \psi_k^2 dx. \end{aligned}$$

To proceed further towards the proof of (3.18) we observe that, since u_1, \dots, u_m are positive on $\Omega \setminus \Gamma$ and $R_\lambda(\Sigma_\lambda) \subseteq \Omega$ (by convexity), we have

$$0 \leq (w_i^{(\lambda)})^+ = (u_i - u_i^{(\lambda)})^+ \leq u_i, \quad \text{on } \Sigma_\lambda \setminus R_\lambda(\Gamma).$$

As a consequence, since u_i is continuous on the set $\overline{\Sigma_\lambda} \subseteq \overline{\Omega} \setminus \Gamma$ (remember that, by assumption $\lambda \leq \lambda_0 + \epsilon < 0$ and $\Gamma \subseteq \{x_1 = 0\}$), we get

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_\lambda \setminus K} |\nabla(w_i^{(\lambda)})^+|^2 \phi_h^2 \psi_k^2 dx &\leq 4 \|u\|_{L^\infty(\Sigma_{\lambda_0+\epsilon})}^2 \int_{\Sigma_\lambda \setminus K} \psi_k^2 |\nabla \phi_h|^2 dx + \\ &+ 4 \|u\|_{L^\infty(\Sigma_{\lambda_0+\epsilon})}^2 \int_{\Sigma_\lambda \setminus K} \phi_h^2 |\nabla \psi_k|^2 dx \\ &+ \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ (w_j^{(\lambda)})^+ \phi_h^2 \psi_k^2 dx \end{aligned}$$

(by (2.20), (2.21), (2.23) and (2.24))

$$\leq 16 \|u\|_{L^\infty(\Sigma_{\lambda_0+\epsilon})}^2 \left(\frac{1}{h} + \frac{1}{k} \right) + \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ (w_j^{(\lambda)})^+ dx.$$

Letting $h, k \rightarrow \infty$ (and using Fatou's lemma, see (2.20) and (2.23)), we then obtain

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_\lambda \setminus K} |\nabla(w_i^{(\lambda)})^+|^2 dx &\leq \mathbf{c}_f \sum_{j=1}^m \int_{\Sigma_\lambda \setminus K} (w_i^{(\lambda)})^+ (w_j^{(\lambda)})^+ dx \\ (3.26) \quad &\leq \mathbf{c}_f \|(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)} \sum_{j=1}^m \|(w_j^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)}. \end{aligned}$$

Now, by exploiting (3.24), we can apply (2.30) (for the Sobolev space $H_0^1(\Sigma_\lambda \setminus K)$) in the right-hand side of (3.26): this gives

$$\begin{aligned} \frac{1}{2} \|\nabla(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)}^2 &= \int_{\Sigma_\lambda \setminus K} |\nabla(w_i^{(\lambda)})^+|^2 dx \\ (3.27) \quad &\leq \theta_n^2(\Sigma_\lambda \setminus K) \mathbf{c}_f \|\nabla(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)} \sum_{j=1}^m \|\nabla(w_j^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)}, \end{aligned}$$

Finally, to complete the demonstration of assertion (c) we observe that, if

$$(3.28) \quad \|\nabla(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)} = 0,$$

then (3.18) is trivially satisfied. If, instead, (3.28) does not hold, by (3.27) one has

$$\frac{1}{2} \|\nabla(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)} \leq \theta_n^2(\Sigma_\lambda \setminus K) \mathbf{c}_f \sum_{j=1}^m \|\nabla(w_j^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K)},$$

and this is precisely the desired (3.18).

Now that we have proved (3.18), we are ready to complete the proof of the present step. To begin with, let $\delta_0 > 0$ be a fixed real number such that

$$K_\delta := \{x \in \Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) : \text{dist}(x, \partial(\Sigma_{\lambda_0} \setminus R_{\lambda_0}(\Gamma))) \geq \delta\} \neq \emptyset, \quad \forall \delta \in (0, \delta_0].$$

Moreover, given any $\delta \in (0, \delta_0]$, let $\epsilon_\delta = \epsilon(K_\delta, \lambda_0) \in (0, |\lambda_0|/2)$ be such that assertions (a)-to-(c) hold true for every $\lambda \in [\lambda_0, \lambda_0 + \epsilon_\delta]$ (note that K_δ has Lipschitz boundary).

Since $|R_\lambda(\Gamma)| = 0$ for every $\lambda \in \mathbb{R}$ (both in the case $n = 2$ and in the case $n \geq 3$, see assumption (H.2) and, e.g, [17, Sec. 4.7]), it is very easy to recognize that

$$(3.29) \quad \begin{array}{l} \text{for every } \eta > 0 \text{ there exists} \\ \delta_\eta \in (0, \delta_0) \text{ such that} \end{array} \quad \theta_n(\Sigma_\lambda \setminus K_\delta) < \eta \quad \begin{array}{l} \text{for every } 0 < \delta < \delta_\eta \text{ and} \\ \text{every } \lambda \in [\lambda_0, \lambda_0 + \epsilon_\delta]. \end{array}$$

Starting from (3.18) and performing an induction argument analogous to that in Step I (in which the information $\theta_n(\Sigma_t) \rightarrow 0$ as $t \rightarrow \mathbf{a}_\Omega$ is replaced by (3.29)), we infer the existence of a small $\sigma \in (0, \delta_0)$ and of a real $C_m = C_m(\mathbf{c}_f) > 0$ such that

$$(3.30) \quad \|\nabla(w_i^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K_\sigma)} \leq C_m \theta_n^2(\Sigma_\lambda \setminus K_\sigma) \sum_{j \geq i+1} \|\nabla(w_j^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K_\sigma)},$$

for every $i = 1, \dots, m-1$ and every $\lambda \in [\lambda_0, \lambda_0 + \epsilon_\sigma]$. From this, again by arguing exactly as in Step I, we can use a backward induction argument to prove that

$$(3.31) \quad \|\nabla(w_j^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K_\sigma)} \leq C_m \theta_n^2(\Sigma_\lambda \setminus K_\sigma) \|\nabla(w_m^{(\lambda)})^+\|_{L^2(\Sigma_\lambda \setminus K_\sigma)},$$

for all $j \in \{1, \dots, m-1\}$ and every $\lambda \in [\lambda_0, \lambda_0 + \epsilon_\sigma]$ (here, as usual, $C_m > 0$ is a real constant only depending on \mathbf{c}_f). By combining (3.30) with (3.31), and by possibly shrinking σ if necessary, we obtain (see also (3.13) in the last part of Step I and remember that the vector-valued map W_λ is continuous out of $R_\lambda(\Gamma)$, see (3.1))

$$(3.32) \quad w_i^{(\lambda)} \leq 0 \text{ on } \Sigma_\lambda \setminus (K_\sigma \cup R_\lambda(\Gamma)) \quad \forall i \in \{1, \dots, m\} \text{ and } \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon_\sigma].$$

Gathering together (3.32) and assertion (b), we then conclude that

$$w_i^{(\lambda)} \leq 0 \text{ on } \Sigma_\lambda \setminus R_\lambda(\Gamma) \quad \text{for every } i \in \{1, \dots, m\} \text{ and every } \lambda \in [\lambda_0, \lambda_0 + \epsilon_\sigma].$$

From this, a last application of the Strong Maximum Principle gives (as $\lambda_0 < 0$)

$$u_i - u_i^{(\lambda)} = w_i^{(\lambda)} < 0 \text{ on } \Sigma_\lambda \setminus R_\lambda(\Gamma) \quad \forall i \in \{1, \dots, m\} \text{ and } \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon_\sigma],$$

but this is contradiction with the definition of λ_0 . Hence, $\lambda_0 = 0$.

STEP III: In this step we prove that all the functions u_1, \dots, u_m are symmetric with respect to the hyperplane $\Pi = \{x_1 = 0\}$. To this end we first observe that, since we know from Step II that $\lambda_0 = \sup \mathcal{I} = 0$ and since W_λ is continuous out of $R_\lambda(\Gamma)$, one has

$$(3.33) \quad u_i(x_1, x_2, \dots, x_n) \leq u_i(-x_1, x_2, \dots, x_n) = u_i^{(0)}(x_1, \dots, x_n),$$

for every $i \in \{1, \dots, m\}$ and every $x \in \Omega_0 = \Omega \cap \{x_1 < 0\}$. By applying this result to the vector-valued function $\hat{U} : \Omega \rightarrow \mathbb{R}^m$ defined by

$$\hat{U}(x) := U(-x_1, x_2, \dots, x_n)$$

(which has the same regularity of U and is a solution (1.1)), we obtain

$$(3.34) \quad u_i(-x_1, \dots, x_n) = \hat{u}_i(x_1, x_2, \dots, x_n) \leq \hat{u}_i(-x_1, x_2, \dots, x_n) = u_i(x_1, \dots, x_n),$$

for every $i \in \{1, \dots, m\}$ and every $x \in \Omega_0$. By combining (3.33) with (3.34) we get

$$u_i(-x_1, \dots, x_n) = u_i(x_1, \dots, x_n), \quad \forall i \in \{1, \dots, m\} \text{ and } \forall x \in \Omega \cap \{x_1 < 0\},$$

and this proves that u_1, \dots, u_m are symmetric with respect to Π .

STEP IV: In this last step we prove (1.3), which clearly implies the monotonicity of the functions u_1, \dots, u_n in the x_1 -direction on $\Omega \cap \{x_1 < 0\}$. To this we first observe that, again from the fact that $\lambda_0 = \mathcal{I} = 0$ (see Step II), we have

$$w_i^{(\lambda)} = u_i - u_i^{(\lambda)} < 0 \text{ on } \Sigma_\lambda \setminus R_\lambda(\Gamma), \quad \forall i \in \{1, \dots, m\} \text{ and } \forall \lambda \in (\mathbf{a}_\Omega, 0).$$

Moreover, $w_i^{(\lambda)} \equiv 0$ on the hyperplane $\Pi_\lambda = \{x_1 = \lambda\}$ and, by (3.17),

$$-\Delta w_i^{(\lambda)} + (\mathbf{c}_f - c_{ii}(\cdot; \lambda))w_i^{(\lambda)} \leq 0, \quad \text{on } \Sigma_\lambda \setminus R_\lambda(\Gamma)$$

(where \mathbf{c}_f is as in assumption (H.3) and the $c_{ij}(\cdot; \lambda)$'s are defined in (3.3)). Since, by the choice of \mathbf{c}_f , we have $\mathbf{c}_f - c_{ii}(\cdot; \lambda) \geq 0$ on $\Sigma_\lambda \setminus R_\lambda(\Gamma)$, we are entitled to apply the Hopf lemma for C^1 -subolutions in [24] (see (3.1) and note that $\Sigma_\lambda \setminus R_\lambda(\Gamma)$ certainly satisfies the interior ball condition at any point of $\Pi_\lambda \cap \Omega$): this gives

$$0 < \frac{\partial w_i^{(\lambda)}}{\partial x_1}(x) = 2 \frac{\partial u_i}{\partial x_1}(x), \quad \forall i \in \{1, \dots, m\} \text{ and every } x \in \Pi_\lambda \cap \Omega,$$

which clearly implies the desired (1.3). Hence, the proof of Theorem 1.1 is complete. \square

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