THERMOELASTICITY OF MOORE-GIBSON-THOMPSON TYPE WITH HISTORY DEPENDENCE IN THE TEMPERATURE

MONICA CONTI, VITTORINO PATA AND RAMON QUINTANILLA

ABSTRACT. In this paper, we consider a thermoelastic model where heat conduction is described by the history dependent version of the Moore-Gibson-Thompson equation, arising via the introduction of a relaxation parameter in the Green-Naghdi type III theory. The well-posedness of the resulting integro-differential system is discussed. In the one-dimensional case, the exponential decay of the energy is proved.

1. INTRODUCTION

The Moore-Gibson-Thompson (MGT) equation

$$u_{ttt} + \alpha u_{tt} + \beta A u_t + \gamma A u = 0, \tag{1.1}$$

where A is a strictly positive operator on some Hilbert space H and $\alpha, \beta, \gamma > 0$ are given parameters, has deserved much attention in recent years, with several papers appeared in the literature on the argument (see [3, 11, 12, 13, 23, 25, 31, 39, 40], among others). The model has been originally introduced in connection with fluids mechanics [50]. In this work, instead, we propose a different interpretation within the theory of thermoelasticity, where the heat transfer is ruled by an integro-differential equation. We will see how the history dependent version of the MGT equation is obtained in a natural way from the Green-Naghdi heat conduction model, through the introduction of a relaxation parameter.

Notation. Along this work, we will denote a vector indifferently by $\boldsymbol{v}$ or by its generic i^{th}-component v_i. Given any function $f = f(\boldsymbol{x}, t)$, we will write $f_{,i}$ to mean its derivative with respect to the space variable x_i, and f_t to mean the derivative in time. Whenever confusion may occur, we will write $\partial_t f$ instead. We will also employ the Einstein notation, where $v_{i,i} = \operatorname{div} \boldsymbol{v}$.

1.1. **Classical heat conduction.** The classical theory of heat conduction, for a heat conductor occupying a volume Ω, is based on the Fourier law

$$q_i = k\theta_{,i},$$

where q_i is the flux vector and θ is the relative temperature, both depending on $\boldsymbol{x} \in \Omega$ and on time t, while $k > 0$ is the thermal conductivity of the material[1]. Substituting q_i into the energy equation

$$q_{i,i} = c\theta_t, \tag{1.2}$$

Key words and phrases. Moore-Gibson-Thompson equation, relaxation parameter, memory kernel, thermoelasticity, solution semigroup, exponential stability.

[1]Several authors include a sign "−" at the right hand side of the relation, however here we adopt the approach of [21] (see page 41).

where $c > 0$ is the thermal capacity, we obtain the classical heat equation

$$c\theta_t - k\Delta\theta = 0,$$

predicting the instantaneous propagation of thermal waves. A fact which is known to be incompatible with the causality principle. This is the reason why many scientists have suggested alternative approaches in the description of heat conduction. A well-established theory is the one introduced by Maxwell and Cattaneo [1], where the Fourier law is replaced by the constitutive equation, containing a relaxation parameter $\tau > 0$,

$$\tau\partial_t q_i + q_i = k\theta_{,i}. \tag{1.3}$$

The combination of (1.2) and (1.3) leads to the damped hyperbolic equation

$$\tau c\theta_{tt} + c\theta_t - k\Delta\theta = 0,$$

in which thermal waves propagate indeed with finite speed. This setting has been extended to cover thermoelasticity in the Lord-Shulman theory [29]. The system of equations obtained in this way has been widely investigated (see e.g. [2, 19, 20, 22, 46, 47]).

1.2. **Green-Naghdi heat conduction.** Three theories for heat conduction have been proposed by Green and Naghdi at the end of the last century [16, 17, 18], nowadays known as type I, II and III, respectively. Type I heat conduction is nothing but the Fourier law. Type II concerns with another hyperbolic equation where there is no dissipation. In this case, the heat flux vector is a linear expression of the thermal displacement α, satisfying the relation

$$\alpha_t(\boldsymbol{x}, t) = \theta(\boldsymbol{x}, t),$$

of the form

$$q_i = k^*\alpha_{,i},$$

where $k^* > 0$ is the conductivity rate parameter. The constitutive equation for the heat flux vector of type III theory reads

$$q_i = k^*\alpha_{,i} + k\theta_{,i}. \tag{1.4}$$

In particular, when $k^* = 0$ or $k = 0$ we boil down to type I or type II, respectively. Type III theory attracted a lot of interest, witnessed by a flurry of publications appeared on the argument (e.g. [5, 15, 24, 26, 27, 30, 32, 33, 34, 35, 36, 42, 43, 45, 49]). In the nontrivial case when both k and k^* are positive, and we substitute q_i into (1.2), we obtain a generalization of the Fourier classical heat equation, namely,

$$c\alpha_{tt} - k\Delta\alpha_t - k^*\Delta\alpha = 0.$$

Unfortunately, the equation above (sometimes called strongly damped wave equation) suffers from the same drawback of the Fourier one, as instantaneous propagation of thermal waves is still present (see [48, p.39]; see also [15]). To be more precise, in this model one observes an instantaneous regularization of the temperature $\theta = \alpha_t$. A natural way to overcome this problem is to modify the constitutive equation (1.4) by introducing a (small) relaxation parameter.

1.3. **MGT heat conduction.** Following the approach of Maxwell and Cattaneo, we correct the constitutive law (1.4) in the following manner:

$$\tau\partial_t q_i + q_i = k^*\alpha_{,i} + k\theta_{,i}, \tag{1.5}$$

where the relaxation parameter $\tau > 0$ is thought to be small compared to the other constants. Collecting (1.2) and (1.5), we find

$$c\theta_{tt} = \text{div } \mathbf{q}_t = \tau^{-1}(-\text{div } \mathbf{q} + k^*\Delta\alpha + k\Delta\theta) = \tau^{-1}(-c\theta_t + k^*\Delta\alpha + k\Delta\theta).$$

As a result, the corresponding heat equation becomes the MGT equation

$$\tau c\alpha_{ttt} + c\alpha_{tt} - k\Delta\alpha_t - k^*\Delta\alpha = 0. \tag{1.6}$$

It is well-known that the asymptotic behavior of the abstract MGT equation (1.1) strongly depends on the *stability number*

$$\chi = \beta - \frac{\gamma}{\alpha}.$$

In particular, the associated semigroup on the natural weak energy space is exponentially stable if and only if we are in the subcritical regime, corresponding to $\chi > 0$ (see e.g. [12, 23, 31, 39, 40]). Since, in absence of external heat sources, it is natural to assume that a reasonable heat model is exponentially stable, the new heat equation (1.6), understood to comply with the Dirichlet boundary conditions, is physically meaningful *if and only if* its stability number is strictly positive. In other words, it must be

$$\chi = \frac{1}{c}(\tau^{-1}k - k^*) > 0,$$

which is clearly implied by a choice of a sufficiently small relaxation parameter $\tau > 0$.

The goal of the present paper is to develop a thermoelastic theory based on heat conduction of MGT type. This strategy has been first devised in [44]. However, we will consider here, rather than the pure MGT, an integro-differential equation of which the MGT heat law (1.6) is just a particular instance, corresponding to the choice of a (negative) exponential convolution kernel. Accordingly, following the lines of [12], we will view (1.6) as an integro-differential equation, sharing the same mathematical structure of the one of linear viscoelasticity (see e.g. [4, 6, 7, 8, 9, 10, 14, 37, 48]).

1.4. **Plan of the paper.** In the next Section 2, we introduce a nonhomogeneous version of the MGT equation, and we translate it into an integro-differential one. After that, in Section 3, we define the corresponding thermoelastic system. The well-posedness of the problem in space-dimension 3, and under quite general assumptions on the memory kernel, is studied in Section 4. In the final Section 5, we discuss the exponential decay of the solutions in the one-dimensional case.

2. The MGT Heat Equation with History

We consider a nonhomogeneous heat conductor occupying a bounded domain $\Omega \subset \mathbb{R}^3$. According to linear type III theory with relaxation parameter, the general constitutive equation for centrosymmetric materials is given by

$$\tau\partial_t q_i + q_i = k_{ij}\theta_{,j} + k^*_{ij}\alpha_{,j}, \tag{2.1}$$

where, as before,

$$\alpha_t = \theta.$$

Here, $\tau = \tau(\boldsymbol{x})$ is the relaxation function, supposed to be strictly positive and bounded, $k_{ij} = k_{ij}(\boldsymbol{x})$ is the thermal conductivity tensor, and $k^*_{ij} = k^*_{ij}(\boldsymbol{x})$ is the conductivity rate tensor. Both k_{ij} and k^*_{ij} are assumed to be symmetric. Note that (2.1) can be rewritten as

$$\partial_t\big[q_i e^{\tau^{-1}t}\big] = \tau^{-1}\big[k_{ij}\theta_{,j} + k^*_{ij}\alpha_{,j}\big]e^{\tau^{-1}t}.$$

We also make the reasonable assumptions that

$$\limsup_{s\to-\infty} q_i(\boldsymbol{x},s)e^{\tau^{-1}(\boldsymbol{x})s} = 0 \qquad \text{and} \qquad \limsup_{s\to-\infty} \alpha_{,i}(\boldsymbol{x},s)e^{\tau^{-1}(\boldsymbol{x})s} = 0. \tag{2.2}$$

Then, an integration on $(-\infty,t)$ yields, omitting the dependence on $\boldsymbol{x}$,

$$q_i(t) = \tau^{-1}\int_{-\infty}^{t} e^{-\tau^{-1}(t-s)}\big[k_{ij}\theta_{,j}(s) + k^*_{ij}\alpha_{,j}(s)\big]ds.$$

In light of the second limit in (2.2), we observe that

$$\int_{-\infty}^{t} e^{-\tau^{-1}(t-s)}\theta_{,j}(s)ds = \alpha_{,j}(t) - \tau^{-1}\int_{-\infty}^{t} e^{-\tau^{-1}(t-s)}\alpha_{,j}(s)ds.$$

Therefore, we end up with

$$q_i = \tau^{-1}k_{ij}\alpha_{,j}(t) - \tau^{-1}\int_{-\infty}^{t} e^{-\tau^{-1}(t-s)}\big[\tau^{-1}k_{ij} - k^*_{ij}\big]\alpha_{,j}(s)ds.$$

Defining the kernel

$$g_{ij}(\boldsymbol{x},s) = k^*_{ij}(\boldsymbol{x}) + e^{-\tau^{-1}(\boldsymbol{x})s}[\tau^{-1}(\boldsymbol{x})k_{ij}(\boldsymbol{x}) - k^*_{ij}(\boldsymbol{x})], \tag{2.3}$$

we are led to the constitutive equation (2.1) in the integro-differential form (again, omitting the dependence on $\boldsymbol{x}$)

$$q_i(t) = g_{ij}(0)\alpha_{,j}(t) + \int_0^\infty g'_{ij}(s)\alpha_{,j}(t-s)ds. \tag{2.4}$$

As a final step, in order to write the corresponding heat equation, we need the nonhomogeneous version of the energy equality. Assuming that the thermal capacity of the material depends on the point, we have

$$q_{i,i} = c\theta_t, \tag{2.5}$$

where $c = c(\boldsymbol{x})$ is strictly positive and bounded. Coupling now (2.4) and (2.5), we arrive at the following heat conduction equation:

$$c\partial_{tt}\alpha - \Big(g_{ij}(0)\alpha_{,j}(t) + \int_0^\infty g'_{ij}(s)\alpha_{,j}(t-s)ds\Big)_{,i} = 0, \tag{2.6}$$

having the form of an equation of viscoelasticity in the variable α, with a memory kernel g_{ij} that is allowed to depend on the spatial variable.

Remark 2.1. When τ is independent of $\boldsymbol{x}$, equation (2.6) can be rewritten in a generalized MGT form. Indeed, adding (2.6) and its time-derivative times τ, and keeping in mind the explicit form of g_{ij} given by (2.3), we obtain (see [44])

$$\tau c\partial_{ttt}\alpha + c\partial_{tt}\alpha + A\partial_t\alpha + A^*\alpha = 0,$$

where A and A^* are the second order differential operators defined by

$$Av = (k_{ij}v_{,j})_{,i}, \quad \text{and} \quad A^*v = (k^*_{ij}v_{,j})_{,i},$$

which are strictly positive whenever the same is true for the (symmetric) tensors k_{ij} and k^*_{ij}.

3. Thermoelasticity of MGT Type

Along the same lines of [44], we now propose a thermoelasticity model, where heat conduction is ruled by the MGT law (2.4). Actually, in more generality, we consider a kernel g_{ij} of which (2.4) is only a particular case, and whose properties will be specified shortly. We start from the constitutive equations

$$\begin{aligned} t_{ij} &= C_{ijkl}u_{k,l} - \beta_{ij}\theta, \\ e &= c\theta + \beta_{ij}u_{i,j}. \end{aligned}$$

Here, $u_i = u_i(\boldsymbol{x}, t)$ is the displacement vector, $t_{ij} = t_{ij}(\boldsymbol{x}, t)$ is the stress tensor, $e = e(\boldsymbol{x}, t)$ is the entropy, $C_{ijkl} = C_{ijkl}(\boldsymbol{x})$ is the elasticity tensor, satisfying the symmetry condition

$$C_{ijkl} = C_{klij},$$

and $\beta_{ij} = \beta_{ij}(\boldsymbol{x})$ is the coupling tensor. The evolution of the displacement and the entropy is described by the equations

$$\begin{aligned} \rho\partial_{tt}u_i &= t_{ij,j}, \\ T_0\partial_t e &= q_{i,i}, \end{aligned}$$

where T_0 is the reference temperature, assumed to be constant ($T_0 = 1$ in the sequel), and $\rho = \rho(\boldsymbol{x})$ is the mass density. Substituting the constitutive equations into the evolution, we find the system

$$\text{(3.1)} \qquad \begin{cases} \rho\partial_{tt}u_i(t) = \big(C_{ijkl}u_{k,l}(t) - \beta_{ij}\partial_t\alpha(t)\big)_{,j}, \\ c\partial_{tt}\alpha(t) = -\beta_{ij}\partial_t u_{i,j}(t) + \Big(g_{ij}(0)\alpha_{,j}(t) + \displaystyle\int_0^\infty g'_{ij}(s)\alpha_{,j}(t-s)ds\Big)_{,i}. \end{cases}$$

In this section, we study (3.1) in a three-dimensional domain Ω, whose boundary $\partial\Omega$ is smooth enough to apply the divergence theorem. The system is supplemented with the Dirichlet boundary conditions

$$\text{(3.2)} \qquad u_i(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0 \qquad \text{and} \qquad \alpha(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0.$$

3.1. **General assumptions.** We assume that the constitutive tensors are bounded above, and that there exist strictly positive constants c_0, ρ_0, C_0 such that

$$c(\boldsymbol{x}) \geq c_0, \qquad \rho(\boldsymbol{x}) \geq \rho_0, \tag{3.3}$$

and

$$\int_\Omega C_{ijkl}\xi_{i,j}\xi_{k,l}d\boldsymbol{x} \geq C_0 \int_\Omega \xi_{i,j}\xi_{i,j}d\boldsymbol{x}, \tag{3.4}$$

for every vector ξ_i vanishing on $\partial\Omega$. The memory kernel

$$g_{ij} = g_{ij}(s)$$

is supposed to be independent of the variable $\boldsymbol{x}$. This assumption, albeit not essential, simplifies the exposition. The precise hypotheses on g_{ij} read as follows (cf. [28]):

- **(i)** The tensor g_{ij} is symmetric, i.e. $g_{ij} = g_{ji}$.
- **(ii)** The tensor g_{ij} is twice differentiable with respect to s, and g'_{ij} is summable on $\mathbb{R}^+$.
- **(iii)** There exists a positive constant k_0 such that

$$g_{ij}(\infty)\xi_i\xi_j \geq k_0\xi_i\xi_i,$$

  where $g_{ij}(\infty) = \lim_{s\to\infty} g_{ij}(s)$.
- **(iv)** There exists a positive, decreasing, scalar function $\mu \in L^1(\mathbb{R}^+) \cap \mathcal{C}([0,\infty))$ and a constant $k_1 \geq 1$ such that

$$\mu(s)\xi_i\xi_i \leq -g'_{ij}(s)\xi_i\xi_j \leq k_1\mu(s)\xi_i\xi_i, \quad \forall s > 0.$$

- **(v)** The tensor g''_{ij} is nonnegative definite, i.e.

$$g''_{ij}(s)\xi_i\xi_j \geq 0, \quad \forall s > 0.$$

In (iii)-(v) above, ξ_i is any vector of $\mathbb{R}^3$.

Remark 3.1. Condition (iii) is natural to guarantee the stability of the solutions. For the MGT kernel g_{ij} previously defined, that is

$$g_{ij}(s) = k^*_{ij} + e^{-\tau^{-1}s}(\tau^{-1}k_{ij} - k^*_{ij}), \tag{3.5}$$

this is the same as taking k^*_{ij} positive definite. The fact that the derivative of g_{ij} is negative definite, as in (iv), corresponds to assume that $\tau^{-1}k_{ij} - k^*_{ij}$ is positive definite, which, as we saw, arises in a natural way in connection with the (dissipative) MGT-equation. In particular it implies that k_{ij} is also positive definite, a consequence of the second principle of Thermodynamics (see [16]). Condition (3.4) falls in the realm of the elastic stability theory. The meaning of the conditions on the heat capacity and the mass density is obvious.

3.2. **Functional setting.** We denote by $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ the usual Hilbert space $L^2(\Omega)$, and by $(V, \langle\cdot,\cdot\rangle_1, \|\cdot\|_1)$ the standard Sobolev space $H_0^1(\Omega)$ of H^1-functions vanishing on $\partial\Omega$. We denote by

$$\boldsymbol{H} = [L^2(\Omega)]^3, \qquad \boldsymbol{V} = [H_0^1(\Omega)]^3$$

the corresponding vectorial versions, keeping the same scalar notation for their norms. In order to translate system (3.1) into an ODE in the so-called past history framework of Dafermos [9, 10], we introduce the Hilbert space

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; V)$$

of square summable function with respect to the measure $\mu(s)ds$, endowed with the product and norm

$$\langle \eta, \eta^* \rangle_{\mathcal{M}} = \int_0^\infty \mu(s) \langle \eta(s), \eta^*(s) \rangle_1 ds \qquad \text{and} \qquad \|\eta\|^2_{\mathcal{M}} = \int_0^\infty \mu(s) \|\eta(s)\|_1^2 ds.$$

Note that, in light of assumption (iv),

$$|\eta|^2_{\mathcal{M}} = -\int_0^\infty \int_\Omega g'_{ij}(s) \eta_{,i}(\boldsymbol{x}, s) \eta_{,j}(\boldsymbol{x}, s) d\boldsymbol{x} ds$$

is an equivalent norm on $\mathcal{M}$, with corresponding scalar product

$$(\eta, \eta^*)_{\mathcal{M}} = -\int_0^\infty \int_\Omega g'_{ij}(s) \eta_{,i}(\boldsymbol{x}, s) \eta^*_{,j}(\boldsymbol{x}, s) d\boldsymbol{x} ds.$$

Finally, we define the product Hilbert space

$$\mathcal{H} = \boldsymbol{V} \times \boldsymbol{H} \times V \times H \times \mathcal{M},$$

endowed with the norm

$$|\boldsymbol{U}|^2_{\mathcal{H}} = \int_\Omega C_{ijkl} u_{i,j} u_{k,l} d\boldsymbol{x} + \|\rho^{1/2} \boldsymbol{v}\|^2 + \int_\Omega g_{ij}(\infty) \alpha_{,i} \alpha_{,j} d\boldsymbol{x} + \|c^{1/2}\theta\|^2 + |\eta|^2_{\mathcal{M}},$$

where $\boldsymbol{U} = (u_i, v_i, \alpha, \theta, \eta)$. Thanks to (3.3), (3.4) and (iii), this is equivalent to the standard product norm

$$\|\boldsymbol{U}\|^2_{\mathcal{H}} = \|\boldsymbol{u}\|_1^2 + \|\boldsymbol{v}\|^2 + \|\alpha\|_1^2 + \|\theta\|^2 + \|\eta\|^2_{\mathcal{M}}.$$

We will also consider the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, i.e. the linear operator T given by

$$T\eta = -\eta' \qquad \text{with domain} \qquad \mathcal{D}(T) = \big\{ \eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0 \big\},$$

the *prime* standing for the distributional derivative with respect to the variable $s > 0$. In light of the assumptions on the kernel one has the dissipative estimate (see e.g. [37])

$$(T\eta, \eta)_{\mathcal{M}} = -\frac{1}{2} \int_0^\infty \int_\Omega g''_{ij}(s) \eta_{,i}(\boldsymbol{x}, s) \eta_{,j}(\boldsymbol{x}, s) d\boldsymbol{x} ds \le 0, \quad \forall \eta \in \mathcal{D}(T). \tag{3.6}$$

3.3. **The system in the past history framework.** In the same spirit of [9, 10], we consider the auxiliary variable $\eta = \eta^t(s)$, containing all the information on the past history of α, and formally defined as (omitting the dependence on $\boldsymbol{x}$)

$$\eta^t(s) = \alpha(t) - \alpha(t-s).$$

Then, (3.1) becomes the evolution system

$$\begin{cases} \rho\partial_{tt}u_i = (C_{ijkl}u_{k,l} - \beta_{ij}\partial_t\alpha)_{,j}, \\ c\partial_{tt}\alpha = -\beta_{ij}\partial_t u_{i,j} + \Big(g_{ij}(\infty)\alpha_{,j}(t) - \displaystyle\int_0^\infty g'_{ij}(s)\eta^t_{,j}(s)ds\Big)_{,i}, \\ \partial_t\eta = T\eta + \partial_t\alpha. \end{cases} \tag{3.7}$$

Introducing the state vector

$$\boldsymbol{U}(t) = (u_i(t), \partial_t u_i(t), \alpha(t), \partial_t\alpha(t), \eta^t),$$

we view (3.7) as the ODE in $\mathcal{H}$

$$\frac{d}{dt}\boldsymbol{U}(t) = \mathbb{A}\boldsymbol{U}(t), \tag{3.8}$$

where $\mathbb{A}$ is the linear operator given by

$$\mathbb{A}\begin{pmatrix} u_i \\ v_i \\ \alpha \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} v_i \\ \rho^{-1}(C_{ijkl}u_{k,l} - \beta_{ij}\theta)_{,j} \\ \theta \\ c^{-1}\big[-\beta_{ij}v_{i,j} + \big(g_{ij}(\infty)\alpha_{,j}(t) - \int_0^\infty g'_{ij}(s)\eta_{,j}(s)ds\big)_{,i}\big] \\ T\eta + \theta \end{pmatrix}, \tag{3.9}$$

with (dense) domain

$$\mathcal{D}(\mathbb{A}) = \left\{ \boldsymbol{U} \in \mathcal{H} \;\middle|\; \begin{array}{l} v_i \in V \\ \theta \in V \\ \eta \in \mathcal{D}(T) \\ \rho^{-1}(C_{ijkl}u_{k,l} - \beta_{ij}\theta)_{,j} \in H \\ c^{-1}\big[-\beta_{ij}v_{i,j} + \big(g_{ij}(\infty)\alpha_{,j}(t) - \int_0^\infty g'_{ij}(s)\eta_{,j}(s)ds\big)_{,i}\big] \in H \end{array} \right\}.$$

4. The Solution Semigroup

The main result of this section concerns with the generation of the solution semigroup.

Theorem 4.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous linear semigroup $S(t)$ on the phase space $\mathcal{H}$. Besides, $S(t)$ is a contraction with respect to the $|\cdot|_{\mathcal{H}}$-norm of $\mathcal{H}$.*

Accordingly, for every initial datum $\boldsymbol{U}_0 \in \mathcal{H}$, equation (3.8) admits a unique mild solution (in the sense of [38]) $\boldsymbol{U}(t)$ given by

$$\boldsymbol{U}(t) = S(t)\boldsymbol{U}_0, \quad \forall t \geq 0,$$

and

$$|S(t)\boldsymbol{U}_0|_{\mathcal{H}} \leq |\boldsymbol{U}_0|_{\mathcal{H}}.$$

The proof of Theorem 4.1 is carried out via the Lumer-Phillips theorem [38], which amounts to prove the next two lemmas.

Lemma 4.2. *The operator* $\mathbb{A}$ *is dissipative, that is,*

$$\mathrm{Re}(\mathbb{A}\boldsymbol{U},\boldsymbol{U})_{\mathcal{H}} \le 0, \quad \forall \boldsymbol{U} \in \mathcal{D}(\mathbb{A}).$$

Proof. By direct computations,

$$\mathrm{Re}(\mathbb{A}\boldsymbol{U},\boldsymbol{U})_{\mathcal{H}} = (T\eta,\eta)_{\mathcal{M}}.$$

The claim then follows from (3.6).

Lemma 4.3. *There exists* $\omega > 0$ *such that*

$$\mathrm{Range}(\mathbb{A} - \omega\mathbb{I}) = \mathcal{H}.$$

Proof. Given any $\boldsymbol{F} = (\boldsymbol{f}_1, \boldsymbol{f}_2, f_3, f_4, f_5) \in \mathcal{H}$, we look for a solution

$$(\boldsymbol{u},\boldsymbol{v},\alpha,\theta,\eta) \in \mathcal{D}(\mathbb{A})$$

to the system

$$\begin{cases} (1-\omega)\boldsymbol{v} = \boldsymbol{f}_1, \\ (C_{ijkl}u_{k,l} - \beta_{ij}\theta)_{,j} - \omega\boldsymbol{u} = \rho\boldsymbol{f}_2, \\ (1-\omega)\theta = f_3, \\ -\beta_{ij}v_{i,j} + \Big(g_{ij}(\infty)\alpha_{,j} - \displaystyle\int_0^\infty g'_{ij}(s)\eta_{,j}ds\Big)_{,i} - \omega\alpha = cf_4, \\ -\eta' - \omega\eta + \theta = f_5. \end{cases}$$

The solutions $\boldsymbol{v}$ and θ are obviously found whenever $\omega \ne 1$. By assumptions (3.3)-(3.4), the equation

$$(C_{ijkl}u_{k,l})_{,j} - \omega u_i = \rho f_{2i} + (1-\omega)^{-1}(\beta_{ij}f_3)_{,j}$$

is elliptic for ω small enough (independent of $\boldsymbol{F}$). Since the right-hand side belongs to $\boldsymbol{H}$, we obtain a unique solution $\boldsymbol{u} \in \boldsymbol{V}$. Let us come to the last equation

$$\eta' + \omega\eta = h, \qquad \text{with} \qquad h = -f_5 + (1-\omega)^{-1}f_3 \in \mathcal{M}.$$

A straightforward integration, along with the condition $\eta(0) = 0$, entail

$$\eta(s) = \int_0^s e^{-\omega(s-y)}h(y)dy.$$

In order to show that $\eta \in \mathcal{M}$, we compute as follows, exploiting the fact that μ is decreasing:

$$\sqrt{\mu(s)}\|\eta(s)\|_1 \le \sqrt{\mu(s)}\int_0^s e^{-\omega(s-y)}\|h(y)\|_1 dy \le \int_0^\infty e^{-\omega(s-y)}\sqrt{\mu(y)}\|h(y)\|_1 dy.$$

Hence

$$\|\eta\|^2_{\mathcal{M}} \le \|E * \phi\|_{L^2(\mathbb{R}^+)},$$

having set

$$E(s) = e^{-\omega s} \qquad \text{and} \qquad \phi(s) = \sqrt{\mu(s)}\|h(s)\|_1.$$

the star standing for convolution on $\mathbb{R}^+$. But

$$\|E * \phi\|_{L^2(\mathbb{R}^+} \leq \|E\|_{L^1(\mathbb{R}^+)} \|\phi\|_{L^2(\mathbb{R}^+)} = \frac{1}{\omega}\|h\|_{\mathcal{M}},$$

proving that $\eta \in \mathcal{M}$. By comparison, $\eta' \in \mathcal{M}$ if and only if $\eta \in \mathcal{M}$, hence $\eta \in \mathcal{D}(T)$. Finally, after substitution we have

$$(g_{ij}(\infty)\alpha_{,j})_{,i} - \omega\alpha = \Big(\int_0^\infty g'_{ij}(s)\eta_{,j} ds\Big)_{,i} + cf_4 + \beta_{ij} f_{1i,j}.$$

In light of (ii), this is an elliptic equation provided that ω is small enough. We thus obtain a unique solution $\alpha \in V$, and the proof is complete.

5. Exponential Stability: the One-Dimensional Case

5.1. **Statement of the result.** For the one-dimensional case, where we take $\Omega = (0, \ell)$, and assuming that the constitutive tensors do not depend on the point $x \in (0, \ell)$, we find the system

$$\begin{cases} \rho u_{tt} - \varkappa u_{xx} + \beta\alpha_{tx} = 0, \\ c\alpha_{tt} - g(\infty)\alpha_{xx}(t) + \displaystyle\int_0^\infty g'(s)\big[\alpha_{xx}(t) - \alpha_{xx}(t-s)\big] ds + \beta u_{tx} = 0, \end{cases}$$

where

$$g(s) = k^* + (\tau^{-1}k - k^*)e^{-\tau^{-1}s}.$$

The parameters appearing in the equations are related to the properties of the material, and have to satisfy some thermomechanical restrictions. In particular, it is assumed that all the constant above are strictly positive, except β that is only required to be nonzero, and

$$\tau^{-1}k > k^*.$$

These assumptions are in agreement with the thermomechanical axioms and the empirical experiments. The assumptions concerning the mass density and the thermal capacity are obvious. The condition on $\varkappa$ can be understood by invoking the elastic stability. The conditions on k, k^* and τ are the natural ones to have dissipation. Finally, $\beta \neq 0$ is needed in order to guarantee the coupling between the mechanical and the thermal parts.

In this case, we adopt different boundary conditions, namely, the Neumann boundary conditions for u

$$u_x(0, t) = u_x(\ell, t) = 0,$$

and the Dirichlet ones for α

$$\alpha(0, t) = \alpha(\ell, t) = 0.$$

Setting for simplicity all the constants but $\varkappa$ equal to 1, and calling

$$\mu(s) = -g'(s),$$

we introduce as before the auxiliary variable $\eta = \eta^t(s)$, formally defined as

$$\eta^t(s) = \alpha(t) - \alpha(t - s),$$

and we rewrite the system above in the form

$$
\begin{cases}
u_{tt} - \varkappa u_{xx} + \alpha_{tx} = 0, \\
\alpha_{tt} - \alpha_{xx} - \displaystyle\int_0^\infty \mu(s)\eta_{xx}(s)ds + u_{tx} = 0, \\
\eta_t = T\eta + \alpha_t.
\end{cases} \tag{5.1}
$$

In what follows, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the standard inner product and norm on the Hilbert space $H = L^2(0, \ell)$. We define the Hilbert subspace

$$
H_* = \Big\{ v \in H : \int_0^\ell v(x)dx = 0 \Big\}
$$

of zero-mean functions, along with the Sobolev spaces

$$
V = H_0^1(0, \ell) \qquad \text{and} \qquad V_* = H^1(0, \ell) \cap H_*,
$$

both endowed with the gradient norm, due to the Poincaré inequality. Then, introducing the phase space

$$
\mathcal{H} = V_* \times H_* \times V \times H \times \mathcal{M},
$$

where

$$
\mathcal{M} = L^2_\mu(\mathbb{R}^+; V),
$$

by the same arguments of the previous sections, system (5.1) generates a (linear) contraction semigroup $S(t)$ on $\mathcal{H}$.

Remark 5.1. The choice of the spaces of zero-mean functions for the variable u and its derivative is consistent. Indeed, calling

$$
m(t) = \int_0^\ell u(x, t)dx,
$$

and integrating the first equation of system (5.1) on $(0, \ell)$, we obtain the differential equation $m_{tt}(t) = 0$. Hence, if $m(0) = m_t(0) = 0$ it follows that $m(t)$ is zero for all times.

In fact, as we saw, the generation of the semigroup holds true also if we consider a kernel μ of a more general form.

Assumptions on the memory kernel. In greater generality, we will assume in the sequel that $\mu : [0, \infty) \to \mathbb{R}^+$ is an absolutely continuous summable function satisfying the inequality

$$
\mu'(s) + \delta\mu(s) \le 0, \tag{5.2}
$$

for some $\delta > 0$ and almost every $s > 0$. In particular, μ' is negative. For further use, we denote

$$
\kappa = \int_0^\infty \mu(s)ds \in (0, \infty).
$$

Then the following theorem holds.

Theorem 5.2. *Let μ satisfy (5.2). Then, the semigroup $S(t)$ is exponentially stable. Namely,*

$$\|S(t)\| \leq Me^{-\omega t},$$

for some $M \geq 1$ and $\omega > 0$, the norm above being the operator one on the Hilbert space $\mathcal{H}$.

Although it is possible to prove the result via linear semigroup techniques (via the Prüss theorem [41]), we will choose here to provide a more direct proof via energy estimates. This approach has the advantage that can be successfully exported to study nonlinear versions of the problem. However, before going to the proof, some comments are in order.

- The choice of the spaces of zero-mean functions is essential, with the above boundary conditions. Indeed, if we relax this assumption, every triple $(u(t), \alpha(t), \eta^t)$ of the form $(k, 0, 0)$, for a fixed constant k, solves system (5.1). In this case, we clearly do not have any decay.
- In the assumptions above, we did not take $\varkappa = 1$ to stress that the exponential decay occurs independently of the respective wave speeds of the two equations. In fact, when $\varkappa = 1$ the dissipation is stronger (see the forthcoming Remark 5.5), and the proof of the exponential decay becomes simpler.
- Actually, Theorem 5.2 holds true also for different boundary conditions, such as the Dirichlet-Dirichlet considered in the previous part of this work. In that case, the semigroup $S(t)$ acts (and is exponentially stable) on the space $V \times H \times V \times H \times \mathcal{M}$. This can be shown by semigroup techniques, whereas an energy-estimate based proof seems to be more difficult to obtain.

5.2. **Proof of Theorem 5.2.** Along the proof, $C > 0$ will stand for a *generic* constant. We will use several times the estimate, obtained via the Hölder inequality,

$$\int_0^\infty \mu(s)\|\eta_x(s)\|ds \; = \int_0^\infty \sqrt{\mu(s)}\sqrt{\mu(s)}\|\eta_x(s)\|ds \leq \sqrt{\kappa}\,\|\eta\|^2_{\mathcal{M}}. \tag{5.3}$$

Also, we will often apply without mention the Young inequality.

By density, it is clearly enough to obtain the desired decay for initial data in the domain of the infinitesimal generator $\mathbb{A}$ of $S(t)$. For any given such initial datum, let us define (twice) the energy of the system as

$$\mathsf{E}(t) = \varkappa\|u_x(t)\|^2 + \|u_t(t)\|^2 + \|\alpha_x(t)\|^2 + \|\alpha_t(t)\|^2 + \|\eta^t\|^2_{\mathcal{M}}.$$

As before, the basic multiplication in (5.1) yields

$$\frac{d}{dt}\mathsf{E} = \int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds \leq 0. \tag{5.4}$$

We now need to reconstruct the energy with the "good" sign. This will be done through the introduction of suitable energy functionals.

Lemma 5.3. *The functional*

$$\Phi(t) = 2\langle u_t(t), u(t)\rangle + 2\langle \alpha_t(t), \alpha(t)\rangle + 2\langle u_x(t), \alpha(t)\rangle$$

fulfills the estimate

$$\frac{d}{dt}\Phi + \varkappa\|u_x\|^2 + \|\alpha_x\|^2 + \|\eta^t\|^2_{\mathcal{M}} \leq C_\Phi\|u_t\|^2 + C_\Phi\|\alpha_t\|^2 - C_\Phi\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds,$$

for some $C_\Phi > 0$.

Proof. The time-derivative of Φ fulfills the equality

$$\begin{aligned}&\frac{d}{dt}\Phi + 2\varkappa\|u_x\|^2 + 2\|\alpha_x\|^2 + \|\eta^t\|^2_{\mathcal{M}}\\ &= 2\|u_t\|^2 + 2\|\alpha_t\|^2 + 4\langle\alpha_t, u_x\rangle + \|\eta^t\|^2_{\mathcal{M}} - 2\int_0^\infty \mu(s)\langle\eta_x(s), \alpha_x\rangle ds.\end{aligned}$$

Clearly,

$$4\langle\alpha_t, u_x\rangle \leq \varkappa\|u_x\|^2 + C\|\alpha_t\|^2,$$

and, using (5.3),

$$\begin{aligned}\|\eta^t\|^2_{\mathcal{M}} - 2\int_0^\infty \mu(s)\langle\eta_x(s), \alpha_x\rangle ds &\leq \|\eta^t\|^2_{\mathcal{M}} + \|\alpha_x\|\int_0^\infty \mu(s)\|\eta_x(s)\| ds\\ &\leq \|\alpha_x\|^2 + C\|\eta\|^2_{\mathcal{M}}.\end{aligned}$$

Since by (5.2)

$$\|\eta\|^2_{\mathcal{M}} \leq -\frac{1}{\delta}\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds,$$

we are done.

In order to introduce the next functional, we define

$$\hat{u}(x,t) = \int_0^x u(y,t)dy, \quad x \in (0,\ell).$$

Note that $\hat{u} \in V$ and

$$\|\hat{u}_t\| \leq \sqrt{\ell}\,\|u_t\|.$$

Lemma 5.4. *The functional*

$$\Psi(t) = -2\langle\alpha_t(t), \hat{u}_t(t)\rangle - 2\langle\alpha_x(t), u(t)\rangle$$

fulfills the estimate

$$\frac{d}{dt}\Psi + \|u_t\|^2 \leq \varepsilon_1\|u_x\|^2 + \frac{C_\Psi}{\varepsilon_1}\|\alpha_t\|^2 - C_\Psi\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds,$$

for every $\varepsilon_1 > 0$ *small and some* $C_\Psi > 0$.

Proof. The derivative of the first term in the right-hand side reads

$$-2\frac{d}{dt}\langle\alpha_t, \hat{u}_t\rangle = -2\langle\alpha_{tt}, \hat{u}_t\rangle - 2\langle\alpha_t, \hat{u}_{tt}\rangle.$$

Integrating on $(0,x)$ the first equation of (5.1), and using the boundary conditions, we find the relation

$$\hat{u}_{tt} = \varkappa u_x - \alpha_t.$$

Hence,

$$-2\frac{d}{dt}\langle \alpha_t, \hat{u}_t\rangle = 2\langle \alpha_x, u_t\rangle + 2\int_0^\infty \mu(s)\langle \eta_x(s), u_t\rangle ds - 2\|u_t\|^2 - 2\varkappa\langle \alpha_t, u_x\rangle + 2\|\alpha_t\|^2.$$

Concerning the second term, we readily obtain

$$-2\frac{d}{dt}\langle \alpha_x, u\rangle = 2\langle \alpha_t, u_x\rangle - 2\langle \alpha_x, u_t\rangle.$$

Adding the two identities gives

$$\frac{d}{dt}\Psi + 2\|u_t\|^2 = 2\|\alpha_t\|^2 + 2(1-\varkappa)\langle \alpha_t, u_x\rangle + 2\int_0^\infty \mu(s)\langle \eta_x(s), u_t\rangle ds.$$

Estimating

$$2(1-\varkappa)\langle \alpha_t, u_x\rangle \leq \varepsilon_1\|u_x\|^2 + \frac{C}{\varepsilon_1}\|\alpha_t\|^2,$$

and the latter integral via (5.3) as

$$2\int_0^\infty \mu(s)\langle \eta_x(s), u_t\rangle ds \leq \|u_t\|^2 + C\|\eta\|^2_{\mathcal{M}},$$

the claim follows by (5.2).

Remark 5.5. It is clear from the proof above that, in the particular case when $\varkappa = 1$, one obtains the better estimate

$$\frac{d}{dt}\Psi + \|u_t\|^2 \leq 2\|\alpha_t\|^2 - C_\Psi \int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds.$$

This would simplify the final argument, providing also a faster decay rate.

Lemma 5.6. *The functional*

$$\Theta(t) = -\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta^t(s), \alpha_t(t)\rangle ds.$$

fulfills the estimate

$$\frac{d}{dt}\Theta + \|\alpha_t\|^2 \leq \varepsilon_2\|\alpha_x\|^2 + \varepsilon_2\|u_t\|^2 - \frac{C_\Theta}{\varepsilon_2}\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds,$$

for every $\varepsilon_2 > 0$ small and some $C_\Theta > 0$.

Proof. By direct calculations we have

$$\frac{d}{dt}\Theta = -\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta_t(s), \alpha_t\rangle ds - \frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta(s), \alpha_{tt}\rangle ds.$$

Concerning the first term in the right-hand side,

$$-\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta_t(s), \alpha_t\rangle ds = -\frac{2}{\kappa}\int_0^\infty \mu(s)\langle T\eta(s), \alpha_t\rangle ds - 2\|\alpha_t\|^2.$$

Integrating by parts with respect to s, in light of the decay of μ and of the equality $\eta(0)=0$, and using the Poincaré inequality, we get

$$
\begin{aligned}
-\frac{2}{\kappa}\int_0^\infty \mu(s)\langle T\eta(s),\alpha_t\rangle ds &= -\frac{2}{\kappa}\int_0^\infty \mu'(s)\langle \eta(s),\alpha_t\rangle ds\\
&\leq \frac{2}{\kappa}\|\alpha_t\|\int_0^\infty \sqrt{-\mu'(s)}\sqrt{-\mu'(s)}\|\eta(s)\|ds\\
&\leq \frac{2\sqrt{\mu(0)}}{\kappa}\|\alpha_t\|\Big(-\int_0^\infty \mu'(s)\|\eta(s)\|^2 ds\Big)^{1/2}\\
&\leq \|\alpha_t\|^2 - C\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds.
\end{aligned}
$$

Here, we treated the integral as in (5.3). In summary,

$$
-\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta_t(s),\alpha_t\rangle ds = -\|\alpha_t\|^2 - C\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds.
$$

Coming to the second term, we write

$$
\begin{aligned}
&-\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta(s),\alpha_{tt}\rangle ds\\
&=\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta_x(s),\alpha_x\rangle ds + \frac{2}{\kappa}\Big\|\int_0^\infty \mu(s)\eta_x(s)ds\Big\|^2 - \frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta_x(s),u_t\rangle ds.
\end{aligned}
$$

By means of (5.3), for every $\varepsilon_2>0$ small we easily find

$$
-\frac{2}{\kappa}\int_0^\infty \mu(s)\langle \eta(s),\alpha_{tt}\rangle ds \leq \varepsilon_2\|\alpha_x\|^2 + \varepsilon_2\|u_t\|^2 + \frac{C}{\varepsilon_2}\|\eta\|^2_{\mathcal{M}}.
$$

Recalling (5.2), the conclusion follows.

At this point, for $a,b>0$ to be fixed (as well as $\varepsilon_1,\varepsilon_2$), we define

$$
\Lambda(t)=\Phi(t)+a\Psi(t)+b\Theta(t).
$$

Collecting the estimates of the three previous lemmas, we get

$$
\begin{aligned}
&\frac{d}{dt}\Lambda + (\varkappa - a\varepsilon_1)\|u_x\|^2 + (1-b\varepsilon_2)\|\alpha_x\|^2 + (a - C_\Phi - b\varepsilon_2)\|u_t\|^2\\
&\quad + \Big(b - C_\Phi - \frac{aC_\Psi}{\varepsilon_1}\Big)\|\alpha_t\|^2 + \|\eta^t\|^2_{\mathcal{M}}\\
&\leq -C_\Lambda\int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds,
\end{aligned}
$$

where

$$
C_\Lambda = C_\Phi + aC_\Psi + \frac{bC_\Theta}{\varepsilon_2}.
$$

Now it is necessary a subtle balance of the constants. We first fix

$$
a = 1 + C_\Phi.
$$

Then we choose ε_1 small enough that

$$\varkappa - a\varepsilon_1 \geq \frac{\varkappa}{2}.$$

Once a and ε_1 are chosen, we take b large enough that

$$b - C_\Phi - \frac{aC_\Psi}{\varepsilon_1} \geq \frac{1}{2}.$$

We are left to fix ε_2. We choose it small enough in such a way that

$$1 - b\varepsilon_2 \geq \frac{1}{2}.$$

Accordingly,

$$a - C_\Phi - b\varepsilon_2 = 1 - b\varepsilon_2 \geq \frac{1}{2}.$$

With this selection of the constants (which fixes also C_Λ), we arrive at the inequality

$$\frac{d}{dt}\Lambda + \frac{1}{2}\mathsf{E} \leq -C_\Lambda \int_0^\infty \mu'(s)\|\eta_x(s)\|^2 ds. \tag{5.5}$$

Finally, for $\varepsilon > 0$, we introduce the last energy functional

$$\mathsf{W}(t) = \mathsf{E}(t) + 2\varepsilon\Lambda(t).$$

Up to choosing ε sufficiently small (in particular, $\varepsilon \leq 1/2C_\Lambda$), it is clear that

$$\frac{1}{2}\mathsf{E}(t) \leq \mathsf{W}(t) \leq 2\mathsf{E}(t),$$

and, adding (5.4)-(5.5),

$$\frac{d}{dt}\mathsf{W} + \varepsilon\mathsf{E} \leq 0.$$

An application of the standard Gronwall lemma completes the argument. The proof of Theorem 5.2 is finished.

Remark 5.7. As a final comment, we dwell on the hypotheses on μ adopted in this section, as well as in the previous one. In particular, we assumed that $\mu(0) < \infty$. This assumption is actually not needed in Section 4, where the kernel μ can be (weakly) singular at zero. For instance, we can handle a kernel of the form

$$\mu(s) = \frac{e^{-s}}{\sqrt{s}}.$$

The restriction $\mu(0) < \infty$ is instead used in connection with the estimate of the functional Θ in Lemma 5.6. In fact, also in this case, the problem could be circumvented via a suitable cut-off technique, which however would render the computations much more involved (see e.g. [37]).

Acknowledgments. The authors thank the anonymous referee her/his useful comments. Work supported by the Project "Análisis Matemático de Problemas de la Termomecánica" (MTM2016-74934-P) (AEI/FEDER, UE) of the Spanish Ministry of Economy and Competitiveness.

References

[1] C. Cattaneo, *On a form of heat equation which eliminates the paradox of instantaneous propagation*, C. R. Acad. Sci. Paris **247** (1958), 431–433.

[2] D. S. Chandrasekharaiah, *Hyperbolic thermoelasticity: a review of recent literature*, Appl. Mech. Rev. **51** (1998), 705–729.

[3] J.A. Conejero, C. Lizama, F. Ródenas, *Chaotic behaviour of the solutions of the Moore-Gibson-Thompson equation*, Appl. Math. Inf. Sci. **9** (2015), 2233–2238.

[4] M. Conti, V. Danese, C. Giorgi, V. Pata, *A model of viscoelasticity with time-dependent memory kernels*, Amer. J. Math. **140** (2019), 349–389.

[5] M. Conti, S. Gatti, A. Miranville, R. Quintanilla, *On a Caginalp phase-field system with two temperatures and memory*, Milan J. Math. **85** (2017), 1–27.

[6] M. Conti, S. Gatti, V. Pata, *Uniform decay properties of linear Volterra integro-differential equations*, Math. Models Methods Appl. Sci. **18** (2008), 21–45.

[7] M. Conti, E.M. Marchini, V. Pata, *Semilinear wave equations of viscoelasticity in the minimal state framework*, Discrete Contin. Dyn. Syst., **27** (2010), 1535–1552.

[8] M. Conti, V. Pata, *Weakly dissipative semilinear equations of viscoelasticity*, Commun. Pure Appl. Anal. **4** (2005), 705.

[9] C.M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations **7** (1970), 554–569.

[10] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 554–569.

[11] F. Dell'Oro, I. Lasiecka, V. Pata, *The Moore-Gibson-Thompson equation with memory in the critical case*, J. Differential Equations **261** (4188–4222), 2016.

[12] F. Dell'Oro, V. Pata, *On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity*, Appl. Math. Optim. **76** (2017), 641–655.

[13] F. Dell'Oro, V. Pata, *On a fourth-order equation of Moore-Gibson-Thompson type*, Milan J. Math. **85** (215–234), 2017.

[14] A.D. Drozdov, V.B. Kolmanovskii, *Stability in viscoelasticity*, North-Holland, Amsterdam, 1994.

[15] C. Giorgi, D. Grandi, V. Pata, *On the Green-Naghdi type III heat conduction model*, Discrete Contin. Dyn. Syst. Ser. B **19** (2143), 2133.

[16] A.E. Green, P.M. Naghdi, *On undamped heat waves in an elastic solid*, J. Thermal Stresses **15** (1992), 253–264.

[17] A.E. Green, P.M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elasticity **31** (1993), 189–208.

[18] A.E. Green, P.M. Naghdi, *A unified procedure for contruction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua*, Proc. Royal Society London A **448** (1995), 335–356, 357–377, 379–388.

[19] R.B. Hetnarski, J. Ignaczak, *Generalized thermoelasticity*, J. Thermal Stresses **22** (1999), 451–470.

[20] R.B. Hetnarski, J. Ignaczak, *Nonclassical dynamical thermoelasticity*, Internat. J. Solids Structures **37** (2000), 215–224.

[21] D. Iesan, A. Scalia, *Thermoelastic deformations*, Kluwer Academic Publishers, Dordrecht, 1996.

[22] J. Ignaczak, M. Ostoja-Starzewski, *Thermoelasticity with finite wave speeds*, Oxford Mathematical Monographs, Oxford, 2010.

[23] B. Kaltenbacher, I. Lasiecka, R. Marchand, *Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Control Cybernet. **40** (2011), 971–988.

[24] S. Kothari, S. Mukhopadhyay, *Study of harmonic plane waves in rotating thermoelastic media of tyep III*, Math. Mech. Solids **17** (2012), 824–839.

[25] I. Lasiecka, X. Wang *Moore-Gibson-Thompson equation with memory, part II: General decay of energy*, J. Differential Equations **259** (2015), 7610–7635.

[26] Y.Y. Liu, Y. Lin, Y. Li, *Convergence result for the thermoelasticity of type III*, Appl. Math. Letters **26** (2013), 97–102.

[27] Z. Liu, R. Quintanilla, *Analiticity of solutions in type III thermoelastic plates*, IMA J. Appl. Math. **75** (2010), 637–646.

[28] Z. Liu, S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 398, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[29] H.W. Lord, Y. Shulman, *A generalized dynamical theory of thermoelasticity*, J. Mech. Phys. Solids **15** (1967), 299–309.

[30] A. Magaña, R. Quintanilla, *Exponential stability in type III thermoelasticity with microtemperatures*, Z. Angew. Math. Phys. (ZAMP) **69** (2018), (5), 129(1)–129(8).

[31] R. Marchand, T. McDevitt, R. Triggiani *An abstract semigroup approach to the third order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability*, Math. Methods Appl. Sci. **35** (2012), 1896–1929.

[32] A. Miranville, R. Quintanilla, *Some generalizations of the Caginalp phase-field system*, Appl. Anal. **88** (2009), 877–894.

[33] A. Miranville, R. Quintanilla, *A type III phase-field system with logarithmic potential*, Appl. Math. Letters **24** (2011), 1003–1008.

[34] A. Miranville, R. Quintanilla, *Exponential stability in type III thermoelasticity with voids*, Appl. Math. Letters **94** (2019), 30–37.

[35] S. Mukhopadhyay, R. Kumar, *A problem of thermoelastic interactions in an infinite medium with a cylindrical hole in generalized thermoelasticity III*, J. Thermal Stress **31** (2008), 455–475.

[36] S. Mukhopadhyay, R. Prasad, R. Kumar, *Variational and reciprocal principles in linear theory of type-III thermoelasticity.* Math. Mech. Solids **16** (2011), 435–444.

[37] V. Pata, *Stability and exponential stability in linear viscoelasticity*, Milan J. Math. **77** (2009), 333–360.

[38] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[39] M. Pellicer, B. Said-Houari, *Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Appl. Math. Optim. (2017), 1–32.

[40] M. Pellicer, J. Sola-Morales, *Optimal scalar products in the Moore-Gibson-Thompson equation*, Evol. Equ. Control Theory **8** (2019), 203–220.

[41] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc. **284** (1984), 847–857.

[42] R. Quintanilla, *Structural stability and continuous dependence of solutions in thermoelasticity of type III*, Discrete Contin. Dyn. Syst. Ser. B **1** (2001), 463–470.

[43] R. Quintanilla, *Convergence and structural stability in thermoelasticity*, Appl. Math. Comput. **135** (2003), 287–300.

[44] R. Quintanilla, *Moore-Gibson-Thompson thermoelasticity*, Math. Mech. Solids, 2019 (to appear).

[45] R. Quintanilla, B. Straughan, *Growth and uniqueness in thermoelasticity*, Proc. Roy. Soc. London A **455** (2000), 1429–1429.

[46] R. Racke, *Asymptotic behavior of solutions in linear 2- or 3-D thermoelasticity with second sound*, Quart. Appl. Math. **61** (2003), 315–328.

[47] R. Racke, *Thermoelasticity with second sound–exponential stability in linear and non-linear 1-d*, Math. Meth. Appl. Sci. **25** (2002), 409–441.

[48] M. Renardy, W.J. Hrusa, J.A. Nohel, *Mathematical Problems in Viscoelasticity*, John Wiley & Sons, Inc., New York, 1987.

[49] B. Straughan, *Heat Waves*, Springer, New York, 2011.

[50] P.A. Thompson, *Compressible-fluid dynamics*, McGraw-Hill, New York, 1972.

Politecnico di Milano - Dipartimento di Matematica
Via Bonardi 9, 20133 Milano, Italy
E-mail address: `monica.conti@polimi.it` (M. Conti)
E-mail address: `vittorino.pata@polimi.it` (V. Pata)

Universitat Politècnica de Catalunya - Departament de Matemàtiques
C. Colom 11, 08222 Terrassa, Barcelona, Spain
E-mail address: ramon.quintanilla@upc.edu (R. Quintanilla)