Eigenvalue variation under moving mixed Dirichlet–Neumann boundary conditions and applications

L. Abatangelo[*], V. Felli[†], C. Léna[‡]

Revised version, March 27, 2019

Abstract

We deal with the sharp asymptotic behaviour of eigenvalues of elliptic operators with varying mixed Dirichlet–Neumann boundary conditions. In case of simple eigenvalues, we compute explicitly the constant appearing in front of the expansion's leading term. This allows inferring some remarkable consequences for Aharonov–Bohm eigenvalues when the singular part of the operator has two coalescing poles.

Keywords. Mixed boundary conditions, asymptotics of eigenvalues, Aharonov–Bohm eigenvalues.

MSC classification. Primary: 35P20; Secondary: 35P15, 35J25.

1 Introduction and main results

The present paper deals with elliptic operators with varying mixed Dirichlet–Neumann boundary conditions and their spectral stability under varying of the Dirichlet and Neumann boundary regions. More precisely, we study the behaviour of eigenvalues under a homogeneous Neumann condition on a portion of the boundary concentrating at a point and a homogeneous Dirichlet boundary condition on the complement.

Let Ω be a bounded open set in $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ having the following properties:

$$\Omega \text{ is Lipschitz,} \tag{1}$$

$$\text{there exists } \varepsilon_0 > 0 \text{ such that } \Gamma_{\varepsilon_0} := [-\varepsilon_0, \varepsilon_0] \times \{0\} \subset \partial\Omega. \tag{2}$$

We consider the eigenvalue problem for the Dirichlet Laplacian on the domain Ω

$$\begin{cases} -\Delta\, u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We denote by $(\lambda_j)_{j\geq 1}$ the eigenvalues of Problem (3), arranged in non-decreasing order and counted with multiplicities.

For each $\varepsilon \in (0, \varepsilon_0]$, we also consider the following eigenvalue problem with mixed boundary conditions:

$$\begin{cases} -\Delta\, u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_\varepsilon, \end{cases} \tag{4}$$

[*]Dipartimento di Matematica e Applicazioni, Università di Milano–Bicocca, Via Cozzi 55, 20125 Milano, Italy, laura.abatangelo@unimib.it

[†]Dipartimento di Scienza dei Materiali, Università di Milano–Bicocca, Via Cozzi 55, 20125 Milano, Italy, veronica.felli@unimib.it

[‡]Matematiska institutionen, Stockholms universitet, 106 91 Stockholm, Sweden, corentin@math.su.se

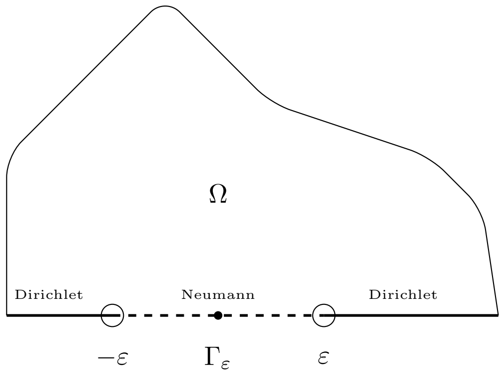

Figure 1: The mixed boundary condition problem in the domain Ω.

with $\Gamma_\varepsilon := [-\varepsilon, \varepsilon] \times \{0\}$, see Figure 1. We denote by $(\lambda_j(\varepsilon))_{j\geq 1}$ the eigenvalues of Problem (4), arranged in non-decreasing order and counted with multiplicities.

A rigorous weak formulation of the eigenvalue problems described above can be given as follows. For $\varepsilon \in (0, \varepsilon_0]$, we define

$$\mathcal{Q}_\varepsilon = \left\{ u \in H^1(\Omega) : \chi_{\partial\Omega\setminus\Gamma_\varepsilon}\gamma_0(u) = 0 \text{ in } L^2(\partial\Omega) \right\},$$

where γ_0 is the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$, which is a continuous linear mapping (see for instance [21, Definition 13.2]) and $\chi_{\partial\Omega\setminus\Gamma_\varepsilon}$ is the indicator function of $\partial\Omega \setminus \Gamma_\varepsilon$ in $\partial\Omega$. Furthermore, we define the quadratic form q on $H^1(\Omega)$ by

$$q(u) := \int_\Omega |\nabla u|^2 \, dx. \tag{5}$$

Let us denote by q_0 the restriction of q to $H^1_0(\Omega)$ and by q_ε the restriction of q to $\mathcal{Q}_\varepsilon$. The sequences $(\lambda_j)_{j\geq 1}$ and $(\lambda_j(\varepsilon))_{j\geq 1}$ for $\varepsilon \in (0, \varepsilon_0]$ can then be defined by the min-max principle:

$$\lambda_j := \min_{\substack{\mathcal{E}\subset H^1_0(\Omega) \text{ subspace} \\ \dim(\mathcal{E})=j}} \max_{u\in\mathcal{E}} \frac{q(u)}{\|u\|^2} \tag{6}$$

and

$$\lambda_j(\varepsilon) := \min_{\substack{\mathcal{E}\subset \mathcal{Q}_\varepsilon \text{ subspace} \\ \dim(\mathcal{E})=j}} \max_{u\in\mathcal{E}} \frac{q(u)}{\|u\|^2}, \tag{7}$$

where

$$\|u\|^2 = \int_\Omega u^2(x) \, dx.$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ (see e.g. [21, Lemma 18.4]), the eigenvalues of q_0, defined by Equation (6), and those of q_ε, defined by Equation (7), are of finite multiplicity, and form sequences tending to $+\infty$.

Remark 1.1. *Let us fix ε_1 and ε_2 in $(0, \varepsilon_0]$ such that $\varepsilon_1 > \varepsilon_2$. Since $H^1_0(\Omega) \subset \mathcal{Q}_{\varepsilon_2} \subset \mathcal{Q}_{\varepsilon_1}$, the definitions given by Formulas (6) and (7) immediately imply that $\lambda_j(\varepsilon_1) \leq \lambda_j(\varepsilon_2) \leq \lambda_j$ for each integer $j \geq 1$. The function $(0, \varepsilon_0] \ni \varepsilon \mapsto \lambda_j(\varepsilon)$ is therefore non-increasing and bounded by λ_j for each integer $j \geq 1$.*

Remark 1.2. *For the sake of simplicity, in the present paper we assume that the domain Ω satisfies assumption (2), i.e. that $\partial\Omega$ is straight in a neighborhood of 0. We observe that, since we are in dimension 2, this assumption is not restrictive. Indeed, starting from a general sufficiently regular* *domain* Ω*, a conformal transformation leads us to consider a new domain satisfying* (2)*, see e.g. [10]. The counterpart is the appearance of a conformal weight in the new eigenvalue problem; however, if* Ω *is sufficiently regular, the weighted problem presents no additional difficulties.*

The purpose of the present paper is to study the eigenvalue function $\varepsilon \mapsto \lambda_j(\varepsilon)$ as $\varepsilon \to 0^+$. The continuity of this map as well as some asymptotic expansions were obtained in [12] (see also Appendix C of the present paper for an alternative proof of some results of [12]). Here we mean to provide some explicit characterization of the leading terms in the expansion given in [12] and of the limit profiles arising from blowing-up of eigenfunctions.

Spectral stability and asymptotic expansion of the eigenvalue variation in a somehow complementary setting were obtained in [3]; indeed, if we consider the eigenvalue problem under homogeneous Dirichlet boundary conditions on a vanishing portion of a straight part of the boundary, Neumann conditions on the complement in the straight part and Dirichlet conditions elsewhere, by a reflection the problem becomes equivalent to the one studied in [3], i.e. a Dirichlet eigenvalue problem in a domain with a small segment removed.

Related spectral stability results were discussed in [8, Section 4] for the first eigenvalue under mixed Dirichlet-Neumann boundary conditions on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$), both for vanishing Dirichlet boundary portion and for vanishing Neumann boundary portion.

We also mention that some regularity results for solutions to second-order elliptic problems with mixed Dirichlet–Neumann type boundary conditions were obtained in [13, 20], see also the references therein, whereas asymptotic expansions at Dirichlet-Neumann boundary junctions were derived in [10].

Let us assume that

$$\lambda_N \text{ (i.e. the } N\text{-th eigenvalue of } q_0\text{) is simple.} \tag{8}$$

Let u_N be a normalized eigenfunction associated to λ_N, i.e. u_N satisfies

$$\begin{cases} -\Delta u_N = \lambda_N u_N, & \text{in } \Omega, \\ u_N = 0, & \text{on } \partial\Omega, \\ \int_\Omega u_N^2(x)\,dx = 1. \end{cases} \tag{9}$$

It is known (see [12]) that, under assumption (8), the rate of the convergence $\lambda_N^\varepsilon \to \lambda_N$ is strongly related to the order of vanishing of the Dirichlet eigenfunction u_N at 0. Moreover u_N has an integer order of vanishing $k \geq 1$ at $0 \in \partial\Omega$ and there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$r^{-k} u_N(r\cos t, r\sin t) \to \beta \sin(kt) \text{ as } r \to 0 \text{ in } C^{1,\tau}([0,\pi]) \tag{10}$$

for any $\tau \in (0,1)$, see e.g. [9, Theorem 1.1]. Actually, one can see that β is directly linked to the norm of the k-th differential of u_N at 0. More precisely, if we consider

$$\|d^j u(x)\|^2 := \sum_{i_1,\dots,i_j=1}^{2} \left| \frac{\partial^j u}{\partial x_{i_1} \dots \partial x_{i_j}}(x) \right|^2,$$

then

$$\beta^2 = \frac{\|d^k u_N(0)\|^2}{(k!)^2\, 2^{k-1}}.$$

This follows by differentiating the harmonic homogeneous functions $\beta r^k \sin(kt)$ and $\beta r^k \cos(kt)$ with respect to x_1 and x_2 and considering $d^k u_N(0)$.

Our main results provide sharp asymptotic estimates with explicit coefficients for the eigenvalue variation $\lambda_N - \lambda_N(\varepsilon)$ as $\varepsilon \to 0^+$ under assumption (8) (Theorem 1.3), as well as an explicit representation in elliptic coordinates of the limit blow-up profile for the corresponding eigenfunction u_N^ε (Theorem 1.4).

Theorem 1.3. *Let Ω be a bounded open set in $\mathbb{R}^2$ satisfying* (1) *and* (2)*. Let $N \geq 1$ be such that the N-th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunctions having in 0 a zero of order k with k as in* (10)*. For $\varepsilon \in (0, \varepsilon_0)$, let $\lambda_N(\varepsilon)$ be the N-th eigenvalue of q_ε on Ω. Then*

$$\lim_{\varepsilon\to 0^+} \frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}} = \beta^2 \, \frac{k\pi}{2^{2k-1}} \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}^2$$

with $\beta \neq 0$ being as in (9)*–*(10)*.*

Theorem 1.4. *Let Ω be a bounded open set in $\mathbb{R}^2$ satisfying* (1) *and* (2)*. Let $N \geq 1$ be such that the N-th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunctions having in 0 a zero of order k with k as in* (10)*. For $\varepsilon \in [0, \varepsilon_0)$, let $\lambda_N(\varepsilon)$ be the N-th eigenvalue of q_ε on Ω and u_N^ε be an associated eigenfunction satisfying $\int_\Omega |u_N^\varepsilon|^2\,dx = 1$ and $\int_\Omega u_N^\varepsilon\, u_N\,dx \geq 0$. Then*

$$\varepsilon^{-k} u_N^\varepsilon(\varepsilon x) \to \beta(\psi_k + W_k \circ F^{-1}) \quad \text{as } \varepsilon \to 0^+$$

in $H^1_{\rm loc}(\overline{\mathbb{R}^2_+})$, a.e. and in $C^2_{\rm loc}(\overline{\mathbb{R}^2_+} \setminus \{(1,0),(-1,0)\})$, where β is as in (9)*–*(10)*,*

$$\psi_k(r\cos t, r\sin t) = r^k \sin(kt), \quad \text{for } t \in [0,\pi] \text{ and } r > 0, \tag{11}$$

$$F(\xi,\eta) = (\cosh(\xi)\cos(\eta), \sinh(\xi)\sin(\eta)), \quad \text{for } \xi \geq 0,\ \eta \in [0, 2\pi), \tag{12}$$

and

$$W_k(\xi,\eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} \exp(-(k-2j)\xi)\sin((k-2j)\eta). \tag{13}$$

Actually, the fact that $\lim_{\varepsilon\to 0^+} \frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}}$ is finite and different from zero and the convergence of $\varepsilon^{-k} u_N^\varepsilon(\varepsilon x)$ to some nontrivial profile was proved in the paper [12] with a quite implicit description of the limits (see also Appendix C for an alternative proof). The original contribution of the present paper relies in the explicit characterization of the leading term of the expansion provided by [12] and in its applications to Aharonov–Bohm operators, see Section 2. The key tool allowing us to write explicitly the coefficients of the expansion consists in the use of elliptic coordinates, which turn out to be more suitable to our problem than radial ones, see Section 3.

2 Applications to Aharonov–Bohm operators

The present work is in part motivated by the study of Aharonov–Bohm eigenvalues. In this section we describe some applications of Theorem 1.3 to the problem of spectral stability of Aharonov–Bohm operators with two moving poles, referring to Section 4 for the proofs.

Let us first review some definitions and known results. For any point $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, we define the Aharonov–Bohm potential of circulation $1/2$ by

$$\mathbf{A}_\mathbf{a}(x) = \frac{1}{2}\left(\frac{-(x_2-a_2)}{(x_1-a_1)^2+(x_2-a_2)^2}, \frac{x_1-a_1}{(x_1-a_1)^2+(x_2-a_2)^2}\right).$$

Let us consider an open and bounded open set $\widehat{\Omega}$ with Lipschitz boundary, such that $0 \in \widehat{\Omega}$. For better readability, we denote by $\mathcal{H}$ the complex Hilbert space of complex-valued functions $L^2(\widehat{\Omega}, \mathbb{C})$, equipped with the scalar product defined, for all $u, v \in \mathcal{H}$, by

$$\langle u, v\rangle := \int_{\widehat{\Omega}} u\overline{v}\,dx.$$

We define, for $\mathbf{a} \in \widehat{\Omega}$,

$$\mathcal{Q}^{AB}_\mathbf{a} := \left\{ u \in H^1_0(\widehat{\Omega}, \mathbb{C})\,;\, \frac{|u|}{|x-\mathbf{a}|} \in L^2(\widehat{\Omega}) \right\}, \tag{14}$$

the quadratic form $q_{\mathbf{a}}^{AB}$ on $\mathcal{Q}_{\mathbf{a}}^{AB}$ by

$$q_{\mathbf{a}}^{AB}(u) := \int_{\widehat{\Omega}} |(i\nabla + \mathbf{A_a})u|^2 \, dx, \tag{15}$$

and the sequence of eigenvalues $\left(\lambda_j^{AB}(\mathbf{a})\right)_{j\geq 1}$ by the min-max principle

$$\lambda_j^{AB}(\mathbf{a}) := \min_{\substack{\mathcal{E}\subset\mathcal{Q}_{\mathbf{a}}^{AB} \text{ subspace} \\ \dim(\mathcal{E})=j}} \max_{u\in\mathcal{E}} \frac{q_{\mathbf{a}}^{AB}(u)}{\|u\|^2}. \tag{16}$$

It follows from the definition in Equation (14) that $\mathcal{Q}_{\mathbf{a}}^{AB}$ is compactly embedded in $\mathcal{H}$. The above eigenvalues are therefore of finite multiplicity and $\lambda_j^{AB}(\mathbf{a}) \to +\infty$ as $j \to +\infty$.

Remark 2.1. *Let us note that, as shown in [16, Lemma 2.1], $\mathcal{Q}_{\mathbf{a}}^{AB}$ is the completion of the set of smooth functions supported in $\widehat{\Omega}\setminus\{\mathbf{a}\}$ for the norm $\|\cdot\|_{\mathbf{a}}$ defined by*

$$\|u\|_{\mathbf{a}}^2 = \|u\|^2 + q_{\mathbf{a}}^{AB}(u).$$

Let us point out that functions in $\mathcal{Q}_{\mathbf{a}}^{AB}$ satisfy a Dirichlet boundary condition, which is not the case in [16]. However, this difference is unimportant for the compact embedding.

Remark 2.2. *We could also consider the Friedrichs extension of the differential operator*

$$(i\nabla + \mathbf{A_a})^*(i\nabla + \mathbf{A_a})u = -\Delta u + 2i\mathbf{A_a}\cdot\nabla u + |\mathbf{A_a}|^2 u$$

acting on functions $u \in C_c^\infty(\widehat{\Omega}\setminus\{\mathbf{a}\},\mathbb{C})$. As shown for instance in [15, Section I] or [7, Section 2]), this defines a positive and self-adjoint operator with compact resolvent, whose eigenvalues, counted with multiplicities, are $\left(\lambda_j^{AB}(\mathbf{a})\right)_{j\geq 1}$. It is called the Aharonov-Bohm operator of pole $\mathbf{a}$ and circulation $1/2$.

In recent years, several authors have studied the dependence of eigenvalues on the position of the pole. It has been established in [7, Theorem 1.1], that the functions $\mathbf{a} \mapsto \lambda_j^{AB}(\mathbf{a})$ are continuous in $\overline{\Omega}$. In [1, 2], the two first authors obtained the precise rate of convergence $\lambda_j^{AB}(\mathbf{a}) \to \lambda_j^{AB}(0)$ as $\mathbf{a}$ converges to the interior point 0 for simple eigenvalues. In order to state the most complete result, given in [2, Theorem 1.2], we consider an L^2-normalized eigenfunction u_N^0 of q_0^{AB} associated with the eigenvalue $\lambda_N^{AB}(0)$. We additionally assume that $\lambda_N^{AB}(0)$ is simple. From [11, Section 7] it follows that there exists an odd positive integer k and a non-zero complex number β_0 such that, up to a rotation of the coordinate axes,

$$r^{-\frac{k}{2}} u_N^0(r\cos t, r\sin t) \to \beta_0 e^{i\frac{t}{2}} \sin\left(\frac{k}{2}t\right) \text{ in } C^{1,\tau}([0,2\pi],\mathbb{C})$$

as $r \to 0^+$, for all $\tau \in (0,1)$. The integer k has a simple geometric interpretation: it is the number of nodal lines of the function u_N^0 which meet at 0. We say that u_N^0 has a zero of order $k/2$ in 0. Our coordinate axes are chosen in such a way that one of these nodal lines is tangent to the positive x_1 semi-axis.

Theorem 2.3. *Let us define $\mathbf{a}_\varepsilon := \varepsilon(\cos(\alpha),\sin(\alpha))$, with $\varepsilon > 0$. We have, as $\varepsilon \to 0^+$,*

$$\lambda_N^{AB}(\mathbf{a}_\varepsilon) = \lambda_N^{AB}(0) - \frac{k\pi\beta_0^2}{2^{2k-1}} \begin{pmatrix} k-1 \\ \lfloor\frac{k-1}{2}\rfloor \end{pmatrix}^2 \cos(k\alpha)\varepsilon^k + o\left(\varepsilon^k\right).$$

Remark 2.4. *The expansion in [1, 2] involves a constant depending on k, defined as the minimal energy in a Dirichlet-type problem. We compute this constant in Appendix A in order to obtain the more explicit result in Theorem 2.3.*

Let us now consider, for any $\varepsilon > 0$, an Aharonov-Bohm potential with two poles $(\varepsilon, 0)$ and $(-\varepsilon, 0)$, of fluxes respectively $1/2$ and $-1/2$:

$$\mathbf{A}_\varepsilon := \mathbf{A}_{(\varepsilon,0)} - \mathbf{A}_{(-\varepsilon,0)}.$$

As in the case of one pole, we define the vector space $\mathcal{Q}_\varepsilon^{AB}$ by

$$\mathcal{Q}_\varepsilon^{AB} := \left\{ u \in H_0^1\big(\widehat{\Omega}, \mathbb{C}\big)\,;\, \frac{|u|}{|x \pm \varepsilon\, \mathbf{e}|} \in L^2\big(\widehat{\Omega}\big) \right\}, \tag{17}$$

where $\mathbf{e} = (1, 0)$, the quadratic form q_ε^{AB} on $\mathcal{Q}_\varepsilon^{AB}$ by

$$q_\varepsilon^{AB}(u) := \int_{\widehat{\Omega}} |(i\nabla + \mathbf{A}_\varepsilon)u|^2\, dx, \tag{18}$$

and the sequence of eigenvalue $\big(\lambda_j^{AB}(\varepsilon)\big)_{j\geq 1}$ by the min-max principle

$$\lambda_j^{AB}(\varepsilon) := \min_{\substack{\mathcal{E} \subset \mathcal{Q}_\varepsilon^{AB} \text{ subspace} \\ \dim(\mathcal{E}) = j}} \max_{u \in \mathcal{E}} \frac{q_\varepsilon^{AB}(u)}{\|u\|^2}. \tag{19}$$

It follows from [15, Corollary 3.5] that, for any $j \geq 1$, $\lambda_j^{AB}(\varepsilon)$ converges to the j-th eigenvalue of the Laplacian in $\widehat{\Omega}$ as $\varepsilon \to 0^+$. In [4, 3] the authors obtained in some cases a sharp rate of convergence. In order to state the result, let us introduce some notation. We denote by $\widehat{q}_0$ the quadratic form on $H_0^1(\widehat{\Omega})$ defined by Equation (5), replacing Ω with $\widehat{\Omega}$, and we denote by $\big(\widehat{\lambda}_j\big)_{j\geq 1}$ the sequence of eigenvalues defined by Equation (6), replacing Ω with $\widehat{\Omega}$ and q with $\widehat{q}_0$. We fix an integer $N \geq 1$ and assume that $\widehat{\lambda}_N$ is a simple eigenvalue. We denote by $\widehat{u}_N$ an associated eigenfunction, normalized in $L^2\big(\widehat{\Omega}\big)$.

Theorem 2.5. *[4, Theorem 1.2] If $\widehat{u}_N(0) \neq 0$, we have, as $\varepsilon \to 0$,*

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{2\pi}{|\log(\varepsilon)|}\widehat{u}_N^2(0) + o\left(\frac{1}{|\log(\varepsilon)|}\right).$$

In the case $\widehat{u}_N(0) = 0$, it is well known that there exist $k \in \mathbb{N} \setminus \{0\}$, $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$ and $\alpha \in [0, \pi)$ such that

$$r^{-k}\widehat{u}_N(r\cos t, r\sin t) \to \widehat{\beta}\sin(\alpha - kt) \text{ in } C^{1,\tau}([0, 2\pi], \mathbb{C})$$

as $r \to 0^+$ for all $\tau \in (0, 1)$. In particular, there is a nodal line whose tangent makes the angle α/k with the positive x_1 semi-axis. As above we can characterize $\widehat{\beta}$ as $|\widehat{\beta}|^2 = \frac{\|d^k\widehat{u}_N(0)\|^2}{(k!)^2\, 2^{k-1}}$.

Let us assume that

$$\widehat{\Omega} \text{ is symmetric with respect to the } x_1\text{-axis}.$$

Since $\widehat{\lambda}_N$ is simple, $\widehat{u}_N$ is either even or odd in the variable x_2 and α is either $\pi/2$ or 0 accordingly.

Theorem 2.6. *[3, Theorem 1.16] If $\widehat{u}_N$ is even in x_2, which corresponds to $\alpha = \pi/2$, we have, as $\varepsilon \to 0^+$,*

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}}\begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2 \varepsilon^{2k} + o\left(\varepsilon^{2k}\right).$$

Remark 2.7. *The statements in [3] contain a constant C_k which we put in a simpler form in Appendix A, in order to obtain Theorem 2.6.*

As a corollary of Theorem 1.3, we prove in Section 4 the following result, which complements the previous theorem.

Theorem 2.8. *If $\widehat{u}_N$ is odd in x_2, which corresponds to $\alpha = 0$, we have, as $\varepsilon \to 0^+$,*

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{k\pi\widehat{\beta}^2}{4^{k-1}}\begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2 \varepsilon^{2k} + o\left(\varepsilon^{2k}\right).$$

Remark 2.9. *As discussed in Section 4, the assumption that $\widehat{\lambda}_N$ is simple can be slightly relaxed, admitting, in some cases, also double eigenvalues.*

3 Sharp asymptotics for the eigenvalue variation

3.1 Related results from the literature

As already mentioned in the introduction, some asymptotic expansions for the eigenvalue variation $\lambda_N - \lambda_N(\varepsilon)$ were derived in [12]. Let us first recall the results from [12] which are the starting of our analysis.

Let $s := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1 \text{ or } x_1 \leq -1\}$. We denote as $\mathcal{Q}$ the completion of $C_c^\infty(\overline{\mathbb{R}^2_+} \setminus s)$ under the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx)^{1/2}$. From the Hardy type inequality proved in [14] and a change of gauge, it follows that functions in $\mathcal{Q}$ satisfy the Hardy type inequalities

$$\frac{1}{4} \int_{\mathbb{R}^2_+} \frac{|\varphi(x)|^2}{|x - \mathbf{e}|^2} \, dx \leq \int_{\mathbb{R}^2_+} |\nabla \varphi(x)|^2 \, dx, \quad \text{for all } \varphi \in \mathcal{Q}, \tag{20}$$

and

$$\frac{1}{4} \int_{\mathbb{R}^2_+} \frac{|\varphi(x)|^2}{|x + \mathbf{e}|^2} \, dx \leq \int_{\mathbb{R}^2_+} |\nabla \varphi(x)|^2 \, dx, \quad \text{for all } \varphi \in \mathcal{Q}, \tag{21}$$

where $\mathbf{e} = (1, 0)$. Inequalities (20) and (21) allow characterizing $\mathcal{Q}$ as the following concrete functional space:

$$\mathcal{Q} = \Big\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+) : \ \nabla u \in L^2(\mathbb{R}^2_+), \ \tfrac{u}{|x \pm \mathbf{e}|} \in L^2(\mathbb{R}^2_+), \text{ and } u = 0 \text{ on } s \Big\}.$$

We refer to the paper [12], where the following theorem can be found as a particular case of more general results.

Theorem 3.1 ([12])**.** *Let Ω be a bounded open set in $\mathbb{R}^2$ satisfying* (1) *and* (2)*. Let $N \geq 1$ be such that the N-th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunction u_N having in 0 a zero of order k with k as in* (10)*. For $\varepsilon \in (0, \varepsilon_0)$, let $\lambda_N(\varepsilon)$ be the N-th eigenvalue of q_ε on Ω and u_N^ε be its associated eigenfunction, normalized to satisfy $\int_\Omega |u_N^\varepsilon|^2 \, dx = 1$ and $\int_\Omega u_N^\varepsilon \, u_N \, dx \geq 0$. Then, as $\varepsilon \to 0^+$,*

$$\frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}} \to -\beta^2 \int_{-1}^{1} \frac{\partial w_k}{\partial x_2} \, w_k \, dx_1, \tag{22}$$

$$\varepsilon^{-k} u_N^\varepsilon(\varepsilon x) \to \beta(\psi_k + w_k) \quad \text{in } H^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+}), \tag{23}$$

with $\beta \neq 0$ being as in (9)–(10)*, ψ_k being defined in* (11)*, and w_k being the unique $\mathcal{Q}$-weak solution to the problem*

$$\begin{cases} -\Delta w_k = 0, & \text{in } \mathbb{R}^2_+, \\ w_k = 0, & \text{on } s, \\ \frac{\partial w_k}{\partial \nu} = -\frac{\partial \psi_k}{\partial \nu}, & \text{on } \Gamma_1. \end{cases} \tag{24}$$

Convergence (22) can be obtained combining [12, Equation (4.6)] for simple eigenvalues, [12, Equation (3.4)] together with [12, Lemma 3.3]. As well, (23) is given by [12, Equation (2.3)], which is a consequence of [12, Theorem 5.2], [12, Equation (4.10)], [12, Lemma 3.3]. For the sake of clarity and completeness, we present an alternative proof in Appendix C, which relies on energy estimates obtained by an Almgren type monotonicity argument and blow-up analysis.

We remark that in [12] the author describes the limit profile w_k solving (24) with polar coordinates. On the contrary, our contribution relies essentially on the use of elliptic coordinates in place of polar ones. This allows us to compute explicitly the right hand side of (22), thus obtaining the following result.

Proposition 3.2. *For any positive integer k,*

$$\int_{-1}^{1} \frac{\partial w_k}{\partial x_2} \, w_k \, dx_1 = -\frac{k\pi}{2^{2k-1}} \begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2.$$

The proof of Proposition 3.2 relies in an explicit construction of the limit profile w_k, using a parametrization of the upper half-plane $\mathbb{R}^2_+$ by elliptic coordinates, a finite trigonometric expansion, and the simplification of a sum involving binomial coefficients.

3.2 Computation of the limit profile w_k

Let us first compute w_k. By uniqueness, any function in the functional space $\mathcal{Q}$ that satisfies all the conditions of Problem (24) is equal to w_k. In order to find such a function, we use the elliptic coordinates (ξ, η) defined by

$$\begin{cases} x_1 = \cosh(\xi)\cos(\eta), \\ x_2 = \sinh(\xi)\sin(\eta). \end{cases} \tag{25}$$

More precisely, we consider the function $F : (\xi, \eta) \mapsto (x_1, x_1)$ defined by the equations (25). It is a C^∞ diffeomorphism from $D := (0, +\infty) \times (0, \pi)$ to $\mathbb{R}^2_+$. We note that F is actually a conformal mapping. Indeed, if we define the complex variables $z := x_1 + ix_2$ and $\zeta := \xi + i\eta$, we have $z = \cosh(\zeta)$, which proves the claim since cosh is an entire function. Let us denote by $h(\xi, \eta)$ the scale factor associated with F, expressed in elliptic coordinates. We have

$$h(\xi, \eta) = \left|\cosh'(\zeta)\right| = |\sinh(\zeta)| = |\sinh(\xi)\cos(\eta) + i\cosh(\xi)\sin(\eta)| = \sqrt{\cosh^2(\xi) - \cos^2(\eta)}.$$

For any function $u \in \mathcal{Q}$, let us define $U := u \circ F$. From the fact the F is conformal, it follows that $|\nabla U|$ is in $L^2(D)$ with

$$\int_D |\nabla U|^2 \, d\xi d\eta = \int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx.$$

We also have

$$\frac{\partial u}{\partial \nu}(x) = -\frac{1}{h(0,\eta)} \frac{\partial U}{\partial \xi}(0, \eta) \tag{26}$$

for any $x \in \Gamma_1$, where $\eta \in (0, \pi)$ satisfies $x = F(0, \eta) = (\cos(\eta), 0)$. Furthermore, U is harmonic in D if, and only if, u is harmonic in $\mathbb{R}^2_+$.

We now give an explicit formula for $w_k \circ F$.

Proposition 3.3. *For any positive integer k, $w_k \circ F = W_k$, where W_k is defined in* (13).

Proof. Let us begin by computing the function $\Psi_k := \psi_k \circ F$. We have $\psi_k(x) = \operatorname{Im}\left(z^k\right)$, so that $\Psi_k(\xi, \eta) = \operatorname{Im}\left((\cosh(\zeta))^k\right)$, where the complex variables z and ζ are defined as above. Using the binomial theorem, we find

$$\Psi_k(\xi, \eta) = \operatorname{Im}\left(\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} e^{(k-2j)\zeta}\right) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} e^{(k-2j)\xi}\sin\left((k-2j)\eta\right).$$

This can be written

$$\Psi_k(\xi, \eta) = \frac{1}{2^{k-1}}\sum_{j=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}\binom{k}{j}\sinh\left((k-2j)\xi\right)\sin\left((k-2j)\eta\right)$$

by grouping terms of the sum in pairs, starting from opposite extremities. In particular, for all $\eta \in (0, \pi)$,

$$\frac{\partial \Psi_k}{\partial \xi}(0, \eta) = \frac{1}{2^{k-1}}\sum_{j=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}(k-2j)\binom{k}{j}\sin\left((k-2j)\eta\right).$$

We now define

$$V(\xi, \eta) = \frac{1}{2^{k-1}}\sum_{j=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}\binom{k}{j} e^{-(k-2j)\xi}\sin\left((k-2j)\eta\right).$$

The function $|\nabla V|$ is in $L^2(D)$ and, for all $\eta \in (0, \pi)$,

$$\frac{\partial V}{\partial \xi}(0, \eta) = -\frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j} \sin((k-2j)\eta) .$$

Additionally, V vanishes on half-lines defined by $\eta = 0$ and $\eta = \pi$, which are the lower and upper boundary of D, respectively, and are mapped to $\mathbb{R} \times \{0\} \setminus \Gamma_1$ by F. It can be checked directly that $V \circ F^{-1} \in \mathcal{Q}$. Finally, V is harmonic in D, since it is a linear combination of functions of the type $(\xi, \eta) \mapsto e^{\pm n\xi} e^{\pm in\eta}$, which are harmonic. We conclude that $V \circ F^{-1}$ is a solution of Problem (24), and therefore $V = w_k \circ F$ by uniqueness. □

Proof of Theorem 1.4. Theorem 1.4 follows combining Theorem 3.1 and Proposition 3.3. □

Corollary 3.4. *For any positive integer $k \geq 1$,*

$$\int_{-1}^{1} \frac{\partial w_k}{\partial x_2} w_k \, dx_2 = -\frac{\pi}{2^{2k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j}^2 . \tag{27}$$

Proof. Using (13), a direct computation gives

$$\nabla W_k(\xi, \eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j} e^{-(k-2j)\xi} (-\sin((k-2j)\eta), \cos((k-2j)\eta) .$$

Recalling (26), we perform a standard change of variables in the left-hand side of (27) to elliptic coordinates and this yields the thesis. □

3.3 Simplification of the sum

We now prove the following result.

Lemma 3.5. *For every integer $k \geq 1$,*

$$\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \binom{k}{j}^2 = k \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}^2 .$$

Proof. We will use repeatedly the two following properties of binomial coefficients. First, the *Vandermonde identity*: for any non-negative integers m, n and r,

$$\sum_{j=0}^{r} \binom{m}{j} \binom{n}{r-j} = \binom{m+n}{r} ; \tag{28}$$

and second, the elementary identity

$$\binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1} \tag{29}$$

with n and r positive integers.

Let us now fix an integer $k \geq 1$. To simplify the notation, we write

$$s := \left\lfloor \frac{k-1}{2} \right\rfloor \qquad \text{and} \qquad S := \sum_{j=0}^{s} (k-2j) \binom{k}{j}^2 .$$

Next, we remark that

$$S = S_0 - \frac{2}{k} S_1 - \frac{2}{k} S_2,$$

with

$$S_0 := \sum_{j=0}^{s} k \begin{pmatrix} k \\ j \end{pmatrix}^2, \quad S_1 := \sum_{j=0}^{s} j(k-j) \begin{pmatrix} k \\ j \end{pmatrix}^2, \quad S_2 := \sum_{j=0}^{s} j^2 \begin{pmatrix} k \\ j \end{pmatrix}^2.$$

Let us compute the previous sums when $k = 2p+1$, with p a non-negative integer. We first have

$$\frac{S_0}{k} = \frac{1}{2} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix}^2 = \frac{1}{2} \begin{pmatrix} 2k \\ k \end{pmatrix},$$

where the last equality is a special case of identity (28). We then find

$$S_1 = \frac{1}{2} \sum_{j=0}^{k} j \begin{pmatrix} k \\ j \end{pmatrix} (k-j) \begin{pmatrix} k \\ k-j \end{pmatrix} = \frac{k^2}{2} \sum_{j=1}^{k-1} \begin{pmatrix} k-1 \\ j-1 \end{pmatrix} \begin{pmatrix} k-1 \\ k-j-1 \end{pmatrix}$$
$$= \frac{k^2}{2} \sum_{\ell=0}^{k-2} \begin{pmatrix} k-1 \\ \ell \end{pmatrix} \begin{pmatrix} k-1 \\ k-2-\ell \end{pmatrix} = \frac{k^2}{2} \begin{pmatrix} 2k-2 \\ k-2 \end{pmatrix}$$

by applying Identity (29) followed by (28). Finally, Identity (29) implies

$$S_2 = k^2 \sum_{j=1}^{p} \begin{pmatrix} k-1 \\ j-1 \end{pmatrix}^2 = k^2 \sum_{\ell=0}^{p-1} \begin{pmatrix} k-1 \\ \ell \end{pmatrix}^2$$
$$= \frac{k^2}{2} \left(\sum_{\ell=0}^{k-1} \begin{pmatrix} k-1 \\ \ell \end{pmatrix}^2 - \begin{pmatrix} k-1 \\ p \end{pmatrix}^2 \right) = \frac{k^2}{2} \begin{pmatrix} 2k-2 \\ k-1 \end{pmatrix} - \frac{k^2}{2} \begin{pmatrix} k-1 \\ p \end{pmatrix}^2.$$

We obtain

$$S = \frac{k}{2} \begin{pmatrix} 2k \\ k \end{pmatrix} - k \begin{pmatrix} 2k-2 \\ k-2 \end{pmatrix} - k \begin{pmatrix} 2k-2 \\ k-1 \end{pmatrix} + k \begin{pmatrix} k-1 \\ p \end{pmatrix}^2$$
$$= \frac{k}{2} \begin{pmatrix} 2k \\ k \end{pmatrix} - k \begin{pmatrix} 2k-1 \\ k-1 \end{pmatrix} + k \begin{pmatrix} k-1 \\ p \end{pmatrix}^2 = k \begin{pmatrix} k-1 \\ p \end{pmatrix}^2$$

where the second equality follows from Pascal's identity and the third from Identity (29).

Let us now treat the case $k = 2p$, with p a positive integer. In a similar way as before, we find

$$S_0 = \frac{k}{2} \left(\sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix}^2 - \begin{pmatrix} k \\ p \end{pmatrix}^2 \right) = \frac{k}{2} \begin{pmatrix} 2k \\ k \end{pmatrix} - \frac{k}{2} \begin{pmatrix} k \\ p \end{pmatrix}^2,$$

$$S_1 = \frac{1}{2} \left(\sum_{j=0}^{k} j \begin{pmatrix} k \\ j \end{pmatrix} (k-j) \begin{pmatrix} k \\ k-j \end{pmatrix} - p^2 \begin{pmatrix} k \\ p \end{pmatrix}^2 \right) = \frac{k^2}{2} \begin{pmatrix} 2k-2 \\ k-2 \end{pmatrix} - \frac{k^2}{8} \begin{pmatrix} k \\ p \end{pmatrix}^2$$

and

$$S_2 = k^2 \sum_{j=0}^{p-2} \begin{pmatrix} k-1 \\ j \end{pmatrix}^2 = \frac{k^2}{2} \left(\sum_{j=0}^{k-1} \begin{pmatrix} k-1 \\ j \end{pmatrix}^2 - 2 \begin{pmatrix} k-1 \\ p-1 \end{pmatrix}^2 \right) = \frac{k^2}{2} \begin{pmatrix} 2k-2 \\ k-1 \end{pmatrix} - k^2 \begin{pmatrix} k-1 \\ p-1 \end{pmatrix}^2.$$

We finally obtain, after simplifications,

$$S = k \begin{pmatrix} k-1 \\ p-1 \end{pmatrix}^2.$$

This completes the proof of Lemma 3.5. $\square$

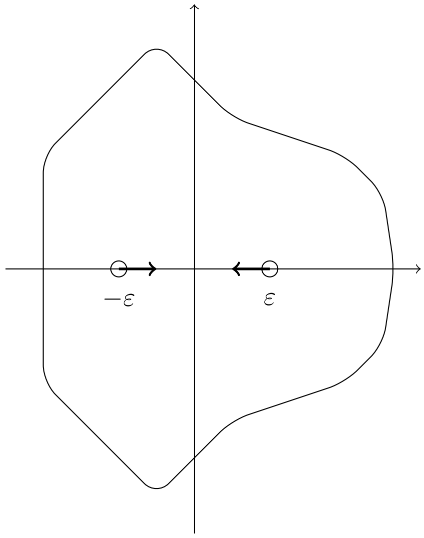

Figure 2: The domain considered for Aharonov–Bohm eigenvalues with collapsing symmetric poles.

3.4 Conclusions

By the results from the preceding subsections, we can now prove Proposition 3.2 and Theorem 1.3.

Proof of Proposition 3.2. It follows from Corollary 3.4 and Lemma 3.5. ☐

Combining the above results, we can now prove our main theorem.

Proof of Theorem 1.3. Theorem 1.3 follows from the combination of Theorem 3.1 and Proposition 3.2. ☐

4 Asymptotic estimates for Aharonov–Bohm eigenvalues

4.1 Symmetry for the Aharonov–Bohm operator

As in Section 2, we assume $\widehat{\Omega} \subset \mathbb{R}^2$ to be a bounded open set with a Lipschitz boundary, such that $0 \in \widehat{\Omega}$. We additionally assume that $\widehat{\Omega}$ is symmetric with respect to the x_1-axis and that $\Omega := \widehat{\Omega} \cap \mathbb{R}^2_+$ also has a Lipschitz boundary.

According to [18, Theorem VIII.15], there exists a unique Friedrichs extension H_ε of the quadratic form q_ε^{AB}, that is to say a self-adjoint operator whose domain $\mathcal{D}(H_\varepsilon)$ is contained in $\mathcal{Q}_\varepsilon^{AB}$ and which satisfies

$$\langle H_\varepsilon u, v\rangle = q_\varepsilon^{AB}(u,v) = \int_{\widehat{\Omega}} (i\nabla + \mathbf{A}_\varepsilon)u \cdot \overline{(i\nabla + \mathbf{A}_\varepsilon)v}\, dx \quad \text{for all } u, v \in \mathcal{D}(H_\varepsilon),$$

where we are denoting by q_ε^{AB} both the quadratic form defined in (18) and the associated bilinear form (see Figure 2). We recall in this section the results proved in [3] concerning the properties of H_ε, in particular the effect of the symmetry of the domain on its spectrum. Since most of the proofs in the present section reduce to a series of standard verifications, we generally only give an indication of them. We use gauge functions Φ_ε, for $\varepsilon \in (0, \varepsilon_0]$, whose existence is guaranteed by the following result. In the sequel the denote as σ the reflection through the x_1-axis, i.e. $\sigma(x_1, x_2) = (x_1, -x_2)$.

Lemma 4.1. *For each $\varepsilon > 0$, there exists a function Φ_ε in $C^\infty\left(\mathbb{R}^2 \setminus \Gamma_\varepsilon\right)$ satisfying*

(i) $\Phi_\varepsilon \circ \sigma = \overline{\Phi_\varepsilon}$ *in* $\mathbb{R}^2 \setminus \Gamma_\varepsilon$;

(ii) $|\Phi_\varepsilon| = 1$ *in* $\mathbb{R}^2 \setminus \Gamma_\varepsilon$*;*

(iii) $(i\nabla + \mathbf{A}_\varepsilon)\,\Phi_\varepsilon = 0$ *in* $\mathbb{R}^2 \setminus \Gamma_\varepsilon$*;*

(iv) $\Phi_\varepsilon = 1$ *on* $(\mathbb{R} \times \{0\}) \setminus \Gamma_\varepsilon$ *and* $\lim_{\delta \to 0^+} \Phi_\varepsilon(t, \pm\delta) = \pm i$ *for every* $t \in (-\varepsilon, \varepsilon)$*.*

We define the anti-unitary operators K_ε and Σ^c by $K_\varepsilon u := \Phi_\varepsilon^2 \overline{u}$ and $\Sigma^c u := \overline{u} \circ \sigma$. The subspace $\mathcal{D}(H_\varepsilon) \subset \mathcal{H}$ is preserved by K_ε and Σ^c. The operators K_ε, Σ^c and H_ε mutually commute. In particular, we can define the following subsets

$$\mathcal{H}_{K,\varepsilon} := \{u \in \mathcal{H} :\ K_\varepsilon u = u\};$$
$$\mathcal{D}(H_{K,\varepsilon}) := \{u \in \mathcal{D}(H_\varepsilon) :\ K_\varepsilon u = u\}.$$

The scalar product $\langle \cdot\,, \cdot \rangle$ gives $\mathcal{H}_{K,\varepsilon}$ the structure of a real Hilbert space. As suggested by the notation, we define $H_{K,\varepsilon}$ as the restriction of H_ε to $\mathcal{D}(H_{K,\varepsilon})$. It is a positive self-adjoint operator on $\mathcal{H}_{K,\varepsilon}$ of domain $\mathcal{D}(H_{K,\varepsilon})$, with compact resolvent. It has the same eigenvalues as H_ε, with the same multiplicities. The fact that K and Σ^c commute ensures that $\mathcal{H}_{K,\varepsilon}$ and $\mathcal{D}(H_{K,\varepsilon})$ are Σ^c-invariant. We can therefore define

$$\mathcal{H}^s_{K,\varepsilon} := \{u \in \mathcal{H}_{K,\varepsilon} : \Sigma^c u = u\};$$
$$\mathcal{D}(H^s_{K,\varepsilon}) := \{u \in \mathcal{D}(H_{K,\varepsilon}) : \Sigma^c u = u\};$$
$$\mathcal{H}^a_{K,\varepsilon} := \{u \in \mathcal{H}_{K,\varepsilon} : \Sigma^c u = -u\};$$
$$\mathcal{D}(H^a_{K,\varepsilon}) := \{u \in \mathcal{D}(H_{K,\varepsilon}) : \Sigma^c u = -u\}.$$

We have the following orthogonal decomposition of $\mathcal{H}_{K,\varepsilon}$ into spaces of symmetric and antisymmetric functions:

$$\mathcal{H}_{K,\varepsilon} = \mathcal{H}^s_{K,\varepsilon} \oplus \mathcal{H}^a_{K,\varepsilon}. \tag{30}$$

We also define $H^s_{K,\varepsilon}$ and $H^a_{K,\varepsilon}$ as the restrictions of $H_{K,\varepsilon}$ to $\mathcal{D}(H^s_{K,\varepsilon})$ and $\mathcal{D}(H^a_{K,\varepsilon})$ respectively. The operator $H^s_{K,\varepsilon}$ is positive and self-adjoint on $\mathcal{H}^s_{K,\varepsilon}$ of domain $\mathcal{D}(H^s_{K,\varepsilon})$, with compact resolvent. Similar conclusions hold for $\mathcal{H}^a_{K,\varepsilon}$. Decomposition (30) implies the following result.

Lemma 4.2. *The spectrum of* $H_{K,\varepsilon}$ *is the union of the spectra of* $H^s_{K,\varepsilon}$ *and* $H^a_{K,\varepsilon}$*, counted with multiplicities.*

Remark 4.3. *Let us note that we can give an alternative description of the spectra of* $H^s_{K,\varepsilon}$ *and* $H^a_{K,\varepsilon}$*. One can check that they are the spectra of the quadratic form* q^{AB}_ε *restricted to* $\mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^s_{K,\varepsilon}$ *and* $\mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^a_{K,\varepsilon}$ *respectively. These spectra can therefore be obtained by the min-max principle.*

4.2 Isospectrality

In this subsection, we establish an isospectrality result between Aharonov-Bohm eigenvalue problems with symmetry and Laplacian eigenvalue problems with mixed boundary conditions, in the spirit of [6].

To this aim, we define an additional family of eigenvalue problems, similar to Problems (3) and (4). With the notation $\partial\Omega_+ := \partial\Omega \cap \mathbb{R}^2_+$ and $\partial\Omega_0 := \partial\Omega \cap (\mathbb{R} \times \{0\})$, we consider the eigenvalue problem

$$\begin{cases} -\Delta\, u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega_+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega_0. \end{cases} \tag{31}$$

We denote by $(\mu_j)_{j \geq 1}$ the eigenvalues of Problem (31). We also consider, for each $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{cases} -\Delta\, u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega_+ \cup \Gamma_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega_0 \setminus \Gamma_\varepsilon, \end{cases} \tag{32}$$

and denote by $(\mu_j(\varepsilon))_{j\geq 1}$ the corresponding eigenvalues. In order to give a rigorous definitions, we use a weak formulation. We define

$$\mathcal{R}_0 = \left\{u \in H^1(\Omega)\,;\, \chi_{\partial\Omega_+}\gamma_0 u = 0 \text{ in } L^2(\partial\Omega)\right\},$$

and, for $\varepsilon \in (0, \varepsilon_0]$,

$$\mathcal{R}_\varepsilon = \left\{u \in H^1(\Omega)\,;\, \chi_{\partial\Omega_+\cup\Gamma_\varepsilon}\gamma_0 u = 0 \text{ in } L^2(\partial\Omega)\right\}.$$

We denote by r_0 and r_ε the restriction of the quadratic form q, defined in Equation (5), to $\mathcal{R}_0$ and $\mathcal{R}_\varepsilon$ respectively. We then define $(\mu_j)_{j\geq 1}$ and $(\mu_j(\varepsilon))_{j\geq 1}$ as, respectively, the eigenvalues of the quadratic forms r_0 and r_ε; they are obtained by the min-max principle.

Remark 4.4. *We can give another interpretation of the eigenvalues $(\mu_j)_{j\geq 1}$ and $(\lambda_j)_{j\geq 1}$. Using the unitary operator $\Sigma : u \mapsto u \circ \sigma$, we obtain a orthogonal decomposition of $L^2\big(\widehat{\Omega}\big)$ into symmetric and antisymmetric functions:*

$$L^2\big(\widehat{\Omega}\big) = ker\,(I - \Sigma) \oplus ker\,(I + \Sigma)\,. \tag{33}$$

This decomposition is preserved by the action of the Dirichlet Laplacian $-\widehat{\Delta}$, and we can therefore define $-\Delta^s$ (resp. $-\Delta^a$) as the restriction of $-\widehat{\Delta}$ to symmetric (resp. antisymmetric) functions in the domain of $-\widehat{\Delta}$. One can then check that $(\mu_j)_{j\geq 1}$ is the spectrum of $-\Delta^s$ and $(\lambda_j)_{j\geq 1}$ is the spectrum of $-\Delta^a$.

It remains to connect the eigenvalues of Problems (32) and (4) to the eigenvalues of H_ε. To this end, we define the following linear operator, which performs a gauge transformation:

$$\begin{array}{rccl} U_\varepsilon : & \mathcal{H} & \to & L^2(\Omega, \mathbb{C}) \\ & u & \mapsto & \sqrt{2}\,\overline{\Phi}_\varepsilon u_{|\Omega}. \end{array}$$

We recall that $L^2(\Omega)$ denotes the real Hilbert space of real-valued L^2 functions in Ω. We have the following result.

Lemma 4.5. *The operator U_ε satisfies the following properties:*

- *(i) $U_\varepsilon\left(\mathcal{H}_{K,\varepsilon}\right) \subset L^2(\Omega)$ and $U_\varepsilon\left(\mathcal{Q}^{AB}_\varepsilon\right) \subset H^1(\Omega, \mathbb{C})$;*
- *(ii) U_ε induces a real-unitary bijective map from $\mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^s_{K,\varepsilon}$ to $\mathcal{R}_\varepsilon$ such that $q^{AB}_\varepsilon(u) = q\left(U_\varepsilon u\right)$ for all $u \in \mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^s_{K,\varepsilon}$;*
- *(iii) U_ε induces a real-unitary bijective map from $\mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^a_{K,\varepsilon}$ to $\mathcal{Q}_\varepsilon$ such that $q^{AB}_\varepsilon(u) = q\left(U_\varepsilon u\right)$ for all $u \in \mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^a_{K,\varepsilon}$.*

Proof. If $u \in \mathcal{H}_{K,\varepsilon}$, then $u = \Phi^2_\varepsilon \overline{u}$, so that $\overline{\Phi}_\varepsilon u = \overline{\overline{\Phi}_\varepsilon u}$, that is to say $\overline{\Phi}_\varepsilon u$ is real-valued. This proves the first half of (i). For the second half, let us assume that $u \in \mathcal{Q}^{AB}_\varepsilon$. Using the definition of $\mathcal{Q}^{AB}_\varepsilon$, given in Equation (17), and Property (iii) of Lemma 4.1, we find the following identity, in the sense of distributions in Ω:

$$\nabla\left(\overline{\Phi}_\varepsilon u\right) = \overline{\Phi}_\varepsilon \nabla u + \nabla\left(\overline{\Phi}_\varepsilon\right) u = \overline{\Phi}_\varepsilon\left(\nabla - iA_\varepsilon\right) u \quad \text{in } \Omega.$$

This proves that $\overline{\Phi}_\varepsilon u_{|\Omega} \in H^1(\Omega, \mathbb{C})$ and that

$$\int_\Omega \left|\left(\nabla - iA_\varepsilon\right) u\right|^2\, dx = \int_\Omega \left|\nabla\left(\overline{\Phi}_\varepsilon u\right)\right|^2\, dx.$$

Let us now additionally assume that $u \in \mathcal{Q}^{AB}_\varepsilon \cap \mathcal{H}^s_{K,\varepsilon}$. Since $\Sigma^c u = u$, Property (i) of Lemma 4.1 implies that $(\overline{\Phi}_\varepsilon u) \circ \sigma = \Phi_\varepsilon \overline{u}$. Therefore,

$$\int_{\widehat{\Omega}} |u|^2\, dx = 2\int_\Omega \left|\overline{\Phi}_\varepsilon u\right|^2\, dx = \int_\Omega \left|U_\varepsilon u\right|^2\, dx.$$

Furthermore, Property (iv) of Lemma 4.1 and the equation $\Sigma^c u = u$ imply that u vanishes on Γ_ε, hence $U_\varepsilon u \in \mathcal{R}_\varepsilon$. This implies that $\overline{\Phi}_\varepsilon u \in H^1(\widehat{\Omega})$ and

$$\int_{\widehat{\Omega}} |(\nabla - iA_\varepsilon)\, u|^2 \, dx = \int_{\widehat{\Omega}} \left|\nabla \left(\overline{\Phi}_\varepsilon u\right)\right|^2 \, dx = 2 \int_{\Omega} \left|\nabla \left(\overline{\Phi}_\varepsilon u\right)\right|^2 \, dx = \int_{\Omega} |\nabla \left(U_\varepsilon u\right)|^2 \, dx.$$

We conclude that the mapping $U_\varepsilon : \mathcal{Q}_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s \to \mathcal{R}_\varepsilon$ is well-defined, real-unitary, and that $q_\varepsilon^{AB}(u) = q\left(U_\varepsilon u\right)$. To show that the mapping is bijective, we consider the operator V_ε defined in the following way: given $v \in L^2(\Omega)$, we denote by $\widetilde{v}$ its extension by symmetry to $\widehat{\Omega}$ and we set

$$V_\varepsilon v := \frac{1}{\sqrt{2}} \Phi_\varepsilon \widetilde{v}.$$

It can be checked, in a way similar to what has been done for U_ε, that V_ε induces the inverse of U_ε, from $\mathcal{R}_\varepsilon$ to $Q_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^s$. This proves (ii). The proof of (iii) is similar, the difference being that we must check that $\overline{\Phi}_\varepsilon u$ vanishes on $(\mathbb{R} \times \{0\}) \setminus \Gamma_\varepsilon$ when $u \in Q_\varepsilon^{AB} \cap \mathcal{H}_{K,\varepsilon}^a$. □

Corollary 4.6. *The spectra of* $H_{K,\varepsilon}^s$ *and* $H_{K,\varepsilon}^a$ *are* $(\mu_j(\varepsilon))_{j\geq 1}$ *and* $(\lambda_j(\varepsilon))_{j \geq 1}$ *respectively.*

4.3 Eigenvalues variations

Let us first state some auxiliary results, which we prove in Appendix B.

Proposition 4.7. *For all* $N \in \mathbb{N}^*$, $\mu_N(\varepsilon) \to \mu_N$ *as* $\varepsilon \to 0$.

Proposition 4.8. *Let* μ_N *be a simple eigenvalue of* $-\Delta^s$ *(see Remark 4.4) and* u_N *be an associated eigenfunction, normalized in* $L^2(\widehat{\Omega})$. *If* $u_N(0) \neq 0$, *then*

$$\mu_N(\varepsilon) = \mu_N + \frac{2\pi}{|\log(\varepsilon)|} u_N^2(0) + o\left(\frac{1}{|\log(\varepsilon)|}\right) \quad \text{as } \varepsilon \to 0.$$

If

$$r^{-k} u_N(r\cos t, r\sin t) \to \widehat{\beta} \cos(kt) \;\; \text{in } C^{1,\tau}\left([0,\pi], \mathbb{R}\right)$$

as $r \to 0^+$ *for all* $\tau \in (0,1)$, *with* $k \in \mathbb{N}^*$ *and* $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$, *then*

$$\mu_N(\varepsilon) = \mu_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2 \varepsilon^{2k} + o\left(\varepsilon^{2k}\right) \quad \text{as } \varepsilon \to 0.$$

We now prove Theorem 2.8. Since $\widehat{u}_N$ is odd in x_2, $\widehat{\lambda}_N$ belongs to the spectrum of $-\Delta^a$. Since $\widehat{\lambda}_N$ is simple, it does not belong to the spectrum of $-\Delta^s$, according to the orthogonal decomposition (33). It follows from Remark 4.4 that there exists $K \in \mathbb{N}^*$ such that $\widehat{\lambda}_N = \lambda_K$ and that λ_K is a simple eigenvalue of q_0 in Ω. By continuity, $\lambda_K(\varepsilon) \to \lambda_K$ as $\varepsilon \to 0^+$.

From Corollary 4.6, Proposition 4.7 and the fact that $\widehat{\lambda}_N$ is simple, it follows that there exists $\varepsilon_1 > 0$ such that $\lambda_N^{AB}(\varepsilon) = \lambda_K(\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_1)$. The conclusion of Theorem 2.8 follows from Theorem 1.3, using the fact that λ_K is simple. Let us note that the eigenfunction $\widehat{u}_N$ in Theorem 2.8 is normalized in $L^2(\widehat{\Omega})$, while the eigenfunction u_N in Theorem 1.3 is normalized in $L^2(\Omega)$. We therefore have to apply Theorem 1.3 with $\beta = \sqrt{2}\,\widehat{\beta}$ to obtain the correct result.

We can use the results of the preceding sections to study some multiple eigenvalues. Let $\widehat{\lambda}_N$ be an eigenvalue of $-\Delta$ on $\widehat{\Omega}$, possibly multiple. We define

$$N_0 := \min\left\{M \in \mathbb{N}^* \,;\, \widehat{\lambda}_M = \widehat{\lambda}_N\right\} \text{ and } N_1 := \max\left\{M \in \mathbb{N}^* \,;\, \widehat{\lambda}_M = \widehat{\lambda}_N\right\}.$$

According to Remark 4.4, there exists $K \in \mathbb{N}^*$ such that $\widehat{\lambda}_N = \lambda_K$ or there exists $L \in \mathbb{N}^*$ such that $\widehat{\lambda}_N = \mu_L$.

Proposition 4.9. *Let us assume that $\widehat{\lambda}_N = \lambda_K$ with $K \in \mathbb{N}^*$ and that λ_K is a simple eigenvalue of q_0. Let us denote by u_K an associated normalized eigenfunction for q_0, and let us assume that*

$$r^{-k} u_K(r\cos t, r\sin t) \to \beta \sin(kt) \text{ in } C^{1,\tau}([0,\pi],\mathbb{R})$$

as $r \to 0^+$ for all $\tau \in (0,1)$, with $k \in \mathbb{N}^*$ and $\beta \in \mathbb{R}\setminus\{0\}$. Then

$$\lambda_{N_0}^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{k\pi\beta^2}{2^{2k-1}} \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}^2 \varepsilon^{2k} + o\left(\varepsilon^{2k}\right) \quad \text{as } \varepsilon \to 0.$$

Proof. Let us set $m := N_1 - N_0 + 1$, the multiplicity of $\widehat{\lambda}_N$. If $m = 1$, the conclusion follows from Theorem 2.8. We therefore assume $m \geq 2$ in the rest of the proof. Remark 4.4 and the fact that λ_K is simple imply that there exists $L \in \mathbb{N}^*$ such that $\mu_L = \mu_{L+1} = \cdots = \mu_{L+m-2} = \widehat{\lambda}_N$. From Proposition 4.7, we deduce that there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1)$,

$$\left\{\lambda_{N_0}^{AB}(\varepsilon); \lambda_{N_0+1}^{AB}(\varepsilon), \dots, \lambda_{N_1}^{AB}(\varepsilon)\right\} = \left\{\lambda_K(\varepsilon), \mu_L(\varepsilon), \dots, \mu_{L+m-2}(\varepsilon)\right\}.$$

The function $\varepsilon \mapsto \lambda_K(\varepsilon)$ is non-increasing, and the function $\varepsilon \mapsto \mu_j(\varepsilon)$ is non-decreasing for every $j \in \{L, \dots, L+m-2\}$, therefore $\mu_j(\varepsilon) \geq \mu_j = \widehat{\lambda}_N = \lambda_K \geq \lambda_K(\varepsilon)$. In particular $\lambda_{N_0}^{AB}(\varepsilon) = \lambda_K(\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_1)$. The conclusion follows from Theorem 1.3. ☐

Proposition 4.10. *Let us assume that $\widehat{\lambda}_N = \mu_L$ with $L \in \mathbb{N}^*$ and that μ_L is a simple eigenvalue of $-\Delta^s$. Let us denote by u_L an associated eigenfunction for $-\Delta^s$, normalized in $L^2(\Omega)$. If $u_L(0) \neq 0$, then*

$$\lambda_{N_1}^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{2\pi}{|\log(\varepsilon)|} u_L^2(0) + o\left(\frac{1}{|\log(\varepsilon)|}\right) \quad \text{as } \varepsilon \to 0.$$

If

$$r^{-k} u_L(r\cos t, r\sin t) \to \widehat{\beta} \cos(kt) \text{ in } C^{1,\tau}([0,\pi],\mathbb{R})$$

as $r \to 0^+$ for all $\tau \in (0,1)$, with $k \in \mathbb{N}^*$ and $\widehat{\beta} \in \mathbb{R}\setminus\{0\}$, then

$$\lambda_{N_1}^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}^2 \varepsilon^{2k} + o\left(\varepsilon^{2k}\right) \quad \text{as } \varepsilon \to 0.$$

Proof. In a similar way as in the proof of Proposition 4.9, we show that there exists $\varepsilon_1 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_1)$, $\lambda_{N_1}^{AB}(\varepsilon) = \mu_L(\varepsilon)$. The conclusion then follows from Proposition 4.8. ☐

4.4 Example: the square

As an application of the preceding results, let us study the first four eigenvalues of the Dirichlet Laplacian for the square

$$\widehat{\Omega} := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^2. \tag{34}$$

The open set $\widehat{\Omega}$ is symmetric with respect to the x_1-axis. We define $\Omega := \widehat{\Omega} \cap \mathbb{R}^2_+$. We denote by $(\widehat{\lambda}_j)_{j\geq 1}$ the eigenvalues of the Dirichlet Laplacian on the square $\widehat{\Omega}$ and, for $\varepsilon \in (0, \pi/2)$, we consider the Aharonov-Bohm eigenvalues $\left(\lambda_j^{AB}(\varepsilon)\right)_{j\geq 1}$ defined in Section 2.

It is well known that the eigenvalues of the Dirichlet Laplacian on $\widehat{\Omega}$ are

$$\widehat{\lambda}_{m,n} := m^2 + n^2,$$

with m and n positive integers, and that an associated orthonormal family of eigenfunctions is given by

$$u_{m,n}(x_1, x_2) = \frac{2}{\pi} f_m(x_1) f_n(x_2),$$

where

$$f_k(x) = \begin{cases} \sin(kx), & \text{if } k \text{ is even,} \\ \cos(kx), & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 4.11. *Let us assume that $\widehat{\lambda}_N$ is simple. Then $\widehat{\lambda}_N = \widehat{\lambda}_{m,m} = 2m^2$ for some positive integer m, and $\widehat{\lambda}_N$ cannot be written in any other way as a sum of squares of positive integers. Then we have, as $\varepsilon \to 0^+$,*

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{8}{\pi|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right)$$

if m is odd and

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{m^4}{2\pi}\varepsilon^4 + o\left(\varepsilon^4\right)$$

if m is even.

Proof. In the case where m is odd, an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$, is

$$u_{m,m}(x_1, x_2) = \frac{2}{\pi}\cos(mx_1)\cos(mx_2).$$

The first asymptotic expansion then follows from Theorem 2.5.

In the case where m is even, an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$, is

$$u_{m,m}(x_1, x_2) = \frac{2}{\pi}\sin(mx_1)\sin(mx_2).$$

Then $\widehat{\lambda}_N = \lambda_K$, where λ_K is a simple eigenvalue of q_0. Furthermore,

$$r^{-2}u_{m,m}(r\cos t, r\sin t) \to \frac{m^2}{\pi}\sin(2t) \text{ in } C^{1,\tau}([0,\pi],\mathbb{R})$$

as $r \to 0^+$ for all $\tau \in (0,1)$. An application of Proposition 4.9, taking care of normalizing in $L^2(\Omega)$, gives the second asymptotic expansion. □

Proposition 4.12. *Assume that $\widehat{\lambda}_N = \widehat{\lambda}_{m,n} = m^2 + n^2$ with m even and n odd, and that $\widehat{\lambda}_N$ has no other representation as a sum of two squares of positive integers, up to the exchange of m and n. Then $\widehat{\lambda}_N$ has multiplicity two; up to replacing N with $N-1$, we can assume that $\widehat{\lambda}_N = \widehat{\lambda}_{N+1}$. Then, as $\varepsilon \to 0^+$,*

$$\lambda_N^{AB}(\varepsilon) = \widehat{\lambda}_N - \frac{4m^2}{\pi}\varepsilon^2 + o\left(\varepsilon^2\right);$$
$$\lambda_{N+1}^{AB}(\varepsilon) = \widehat{\lambda}_N + \frac{4m^2}{\pi}\varepsilon^2 + o\left(\varepsilon^2\right).$$

Proof. The associated eigenfunctions

$$u_{m,n}(x_1, x_2) = \frac{2}{\pi}\sin(mx_1)\cos(nx_2)$$

and

$$u_{n,m}(x_1, x_2) = \frac{2}{\pi}\cos(nx_1)\sin(mx_2)$$

are normalized in $L^2(\widehat{\Omega})$ and respectively symmetric and antisymmetric in the variable x_2. It follows that $\widehat{\lambda}_N = \mu_L = \lambda_K$, where μ_L is a simple eigenvalue of r_0 and λ_K a simple eigenvalue of q_0. Furthermore,

$$r^{-1}u_{m,n}(r\cos t, r\sin t) \to \frac{2m}{\pi}\cos(t) \text{ in } C^{1,\tau}([0,\pi],\mathbb{R})$$

and

$$r^{-1}u_{n,m}(r\cos t, r\sin t) \to \frac{2m}{\pi}\sin(t) \text{ in } C^{1,\tau}([0,\pi],\mathbb{R})$$

as $r \to 0^+$ for all $\tau \in (0,1)$. The asymptotic expansions then follow from Propositions 4.10 and 4.9 □

Remark 4.13. *We note that if $\widehat{\lambda}_N$ is even, in any representation $\widehat{\lambda}_N = m^2+n^2$, m and n have the same parity. Therefore, if $n \neq m$, $\widehat{\lambda}_N$ cannot be a simple eigenvalue either of r_0 or of q_0. On the other hand, if $\widehat{\lambda}_N$ is odd, in any representation $\widehat{\lambda}_N = m^2+n^2$, m and n have the opposite parity. Therefore, as soon as $\widehat{\lambda}_N$ can be written in at least two different ways as the sum of two squares, $\widehat{\lambda}_N$ cannot be a simple eigenvalue either of r_0 or of q_0. The cases described in Propositions 4.11 and 4.12 are thus the only ones in which we can apply the results of Section 4.3 for the square.*

The first four eigenvalues of the Dirichlet Laplacian on the square $\widehat{\Omega}$ satisfy the assumptions of either Proposition 4.11 or Proposition 4.12, so we can apply the previous results to derive the following asymptotic expansions of the Aharonov-Bohm eigenvalues $\lambda_j^{AB}(\varepsilon)$ for $j = 1,2,3,4$.

Corollary 4.14. *Let $\lambda_j^{AB}(\varepsilon)$ be the Aharonov-Bohm eigenvalues defined in* (18)–(19) *with $\widehat{\Omega}$ being the square defined in* (34). *Then we have, as $\varepsilon \to 0^+$,*

$$\begin{aligned}
\lambda_1^{AB}(\varepsilon) &= 2 + \frac{8}{\pi}\frac{1}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right);\\
\lambda_2^{AB}(\varepsilon) &= 5 - \frac{16}{\pi}\varepsilon^2 + o\left(\varepsilon^2\right);\\
\lambda_3^{AB}(\varepsilon) &= 5 + \frac{16}{\pi}\varepsilon^2 + o\left(\varepsilon^2\right);\\
\lambda_4^{AB}(\varepsilon) &= 8 - \frac{8}{\pi}\varepsilon^4 + o\left(\varepsilon^4\right).
\end{aligned}$$

4.5 Example: the disk

Let $(r,t) \in [0,1]\times[0,2\pi)$ be the polar coordinates of the disk. It is well known that the eigenvalues of the Dirichlet Laplacian on the disk are given by the sequences

$$\{j_{0,k}^2\}_{k\geq 1} \cup \{j_{n,k}^2\}_{n,k\geq 1},$$

where $j_{n,k}$ denotes the k-th zero of the Bessel function J_n for $n \geq 0$, $k \geq 1$. We recall that $j_{n,k} = j_{n',k'}$ if, and only if, $n = n'$ and $k = k'$ (see [22, Section 15.28]). The first set is therefore made of simple eigenvalues; their eigenfunctions are given by the Bessel functions

$$u_{0,k}(r\cos t, r\sin t) := \sqrt{\tfrac{1}{\pi}}\tfrac{1}{|J_0'(j_{0,k})|}J_0(j_{0,k}r) \quad \text{for } k\geq 1. \tag{35}$$

The second set is made of double eigenvalues whose eigenfunctions are spanned by

$$u_{n,k}^s(r\cos t, r\sin t) := \sqrt{\tfrac{2}{\pi}}\tfrac{1}{|J_n'(j_{n,k})|}J_n(j_{n,k}r)\cos nt, \tag{36}$$

$$u_{n,k}^a(r\cos t, r\sin t) := \sqrt{\tfrac{2}{\pi}}\tfrac{1}{|J_n'(j_{n,k})|}J_n(j_{n,k}r)\sin nt, \tag{37}$$

for $n,k \geq 1$. We stress that these eigenfunctions have L^2-norm equal to 1 on the disk. It is convenient to recall (see [22, Chapter III]) that for any $n \in \mathbb{N}\cup\{0\}$

$$J_n(z) = \sum_{k=0}^{+\infty}\frac{(-1)^k(\frac{1}{2}z)^{n+2k}}{k!\,\Gamma(n+k+1)}. \tag{38}$$

We denote by $\big(\widehat{\lambda}_j\big)_{j\geq 1}$ the eigenvalues of the Dirichlet Laplacian on the disk

$$D_1 = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2<1\}$$

and, for $\varepsilon \in (0,1/2)$, we consider the Aharonov-Bohm eigenvalues $\big(\lambda_j^{AB}(\varepsilon)\big)_{j\geq 1}$ defined in Section 2.

Proposition 4.15. *If $\widehat{\lambda}_N$ is simple, there exists an integer $k \geq 1$ such that $\widehat{\lambda}_N = j_{0,k}^2$. Then*

$$\lambda_N^{AB}(\varepsilon) = j_{0,k}^2 + \frac{2}{|J_0'(j_{0,k})|^2} \frac{1}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right) \tag{39}$$

as $\varepsilon \to 0^+$. If $\widehat{\lambda}_N$ is double, there exist integers $n \geq 1$ and $k \geq 1$ such that $\widehat{\lambda}_N = j_{n,k}^2$. Up to replacing N by $N-1$, we can assume that $\widehat{\lambda}_N = \widehat{\lambda}_{N+1}$. Then, as $\varepsilon \to 0^+$,

$$\lambda_N^{AB}(\varepsilon) = j_{n,k}^2 - \frac{2n j_{n,k}^{2n}}{(n!)^2 4^{2n-1} |J_n'(j_{n,k})|^2} \begin{pmatrix} n-1 \\ \lfloor \frac{n-1}{2} \rfloor \end{pmatrix}^2 \varepsilon^{2n} + o\left(\varepsilon^{2n}\right), \tag{40}$$

$$\lambda_{N+1}^{AB}(\varepsilon) = j_{n,k}^2 + \frac{2n j_{n,k}^{2n}}{(n!)^2 4^{2n-1} |J_n'(j_{n,k})|^2} \begin{pmatrix} n-1 \\ \lfloor \frac{n-1}{2} \rfloor \end{pmatrix}^2 \varepsilon^{2n} + o\left(\varepsilon^{2n}\right). \tag{41}$$

Proof. We first consider the case where the eigenvalue $\widehat{\lambda}_N = j_{0,k}^2$ is simple; then an associated eigenfunction, normalized in the disk, is $u_{0,k}$ defined by Equation (35). It follows from Equation (38) that

$$u_{0,k}(0) = \sqrt{\frac{1}{\pi}} \frac{1}{|J_0'(j_{0,k})|} > 0.$$

Theorem 2.5 gives us the asymptotic expansion (39).

We then consider the case where $\widehat{\lambda}_N$ is double, with $\widehat{\lambda}_N = \widehat{\lambda}_{N+1} = j_{n,k}^2$, $n, k \geq 1$. We note that $j_{n,k}^2$ is a simple eigenvalue of q_0, and that the restriction of $\sqrt{2} u_{n,k}^a$ to the upper half-disk is an associated normalized eigenfunction. It follows from Equation (38) that

$$r^{-n} u_{n,k}^a(r\cos t, r\sin t) \to \sqrt{\frac{2}{\pi}} \frac{1}{|J_n'(j_{n,k})|} \frac{1}{\Gamma(n+1)} \left(\frac{j_{n,k}}{2}\right)^n \sin nt \quad \text{in } C^{1,\tau}\left([0,\pi], \mathbb{R}\right)$$

as $r \to 0^+$. The asymptotic expansion (40) then follows from Proposition 4.9. In a similar way, $j_{n,k}^2$ is a simple eigenvalue of $-\Delta^s$, and $u_{n,k}^s$ is an associated normalized eigenfunction. It follows from Equation (38) that

$$r^{-n} u_{n,k}^s(r\cos t, r\sin t) \to \sqrt{\frac{2}{\pi}} \frac{1}{|J_n'(j_{n,k})|} \frac{1}{\Gamma(n+1)} \left(\frac{j_{n,k}}{2}\right)^n \cos nt \quad \text{in } C^{1,\tau}\left([0,\pi], \mathbb{R}\right)$$

as $r \to 0^+$. The asymptotic expansion (41) then follows from the second case of Proposition 4.10. □

Additionally, there exist relations between the zeros of Bessel functions (to this aim we refer to [22, Chapter XV.22]): in particular, the positive zeros of the Bessel function J_n are interlaced with those of the Bessel function J_{n+1} and by Porter's Theorem there is an odd number of zeros of J_{n+2} between two consecutive zeros of J_n. Then, we have,

$$0 < j_{0,1} < j_{1,1} < j_{2,1} < j_{0,2} < j_{1,2} < \dots$$

and hence, since $j_{3,1} > j_{2,1}$, the first three zeros of Bessel functions are, in order,

$$0 < j_{0,1} < j_{1,1} < j_{2,1}.$$

Combining this information with Proposition 4.15, we find for example the following asymptotic expansions for the first few Aharonov-Bohm eigenvalues $\lambda_j^{AB}(\varepsilon)$ on the disk D_1 as $\varepsilon \to 0^+$:

$$\lambda_1^{AB}(\varepsilon) = j_{0,1}^2 + \frac{2}{|J_0'(j_{0,1})|^2}\frac{1}{|\log(\varepsilon)|} + o\left(\frac{1}{|\log(\varepsilon)|}\right),$$
$$\lambda_2^{AB}(\varepsilon) = j_{1,1}^2 - \frac{1}{2}\frac{j_{1,1}^2}{|J_1'(j_{1,1})|^2}\,\varepsilon^2 + o(\varepsilon^2),$$
$$\lambda_3^{AB}(\varepsilon) = j_{1,1}^2 + \frac{1}{2}\frac{j_{1,1}^2}{|J_1'(j_{1,1})|^2}\,\varepsilon^2 + o(\varepsilon^2),$$
$$\lambda_4^{AB}(\varepsilon) = j_{2,1}^2 - \frac{1}{64}\frac{j_{2,1}^4}{|J_2'(j_{2,1})|^2}\,\varepsilon^4 + o(\varepsilon^4),$$
$$\lambda_5^{AB}(\varepsilon) = j_{2,1}^2 + \frac{1}{64}\frac{j_{2,1}^4}{|J_2'(j_{2,1})|^2}\,\varepsilon^4 + o(\varepsilon^4).$$

A Computation of the constants

A.1 The Neumann-Dirichlet case

In the present section, we use the above results to compute the quantities appearing in [1, Section 4]. In order to avoid a conflict of notation with the present paper, for any odd positive integer k, we denote here by ψ_k', $\mathfrak{m}_k'$ and w_k' what is denoted in [1] by ψ_k, $\mathfrak{m}_k$ and w_k respectively.

As in [1], we use the notation

$$s_0 := \{(x_1', 0)\ ;\ x_1' \geq 0\}\,;$$

and

$$s := \{(x_1', 0)\ ;\ x_1' \geq 1\}\,.$$

We now define the mapping $G : \mathbb{R}^2_+ \to \mathbb{R}^2 \setminus s_0$ by

$$G(x) := (x_1^2 - x_2^2, 2x_1x_2).$$

The mapping is conformal; indeed, if for $x \in \mathbb{R}^2_+$ we write $z := x_1 + ix_2$ and $z' := x_1' + ix_2'$, with $x' = (x_1', x_2') := G(x)$, we have $z' = z^2$. The scale factor associated with G is $h(x) = 2|z| = 2|x|$. Let u' be a function in $H^1\left(\mathbb{R}^2 \setminus s_0\right)$ and $u := u' \circ G$. Since G is conformal, $|\nabla u|$ is in $L^2\left(\mathbb{R}^2_+\right)$, with

$$\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx = \int_{\mathbb{R}^2\setminus s_0} |\nabla u'|^2\, dx'.$$

Furthermore, for any x' in the segment $(0,1)\times\{0\}$, which we write as $x' = (x_1', 0)$, we have

$$\frac{\partial u'}{\partial \nu_+}(x') = -\frac{1}{2\sqrt{x_1'}}\frac{\partial u}{\partial x_2}\left(\sqrt{x_1'}, 0\right) \text{ and } \frac{\partial u'}{\partial \nu_-}(x') = -\frac{1}{2\sqrt{x_1'}}\frac{\partial u}{\partial x_2}\left(-\sqrt{x_1'}, 0\right),$$

where $\frac{\partial u'}{\partial \nu_+}(x')$ and $\frac{\partial u'}{\partial \nu_-}(x')$ denote the normal derivative at x' respectively from above and from below. We also note that u is harmonic in $\mathbb{R}^2_+$ if, and only if, u' is harmonic in $\mathbb{R}^2 \setminus s_0$.

Let us now denote by $\widetilde{w}_k'$ the extension by reflexion to $\mathbb{R}^2 \setminus s_0$ of w_k', originally defined on $\mathbb{R}^2_+$. We recall that w_k' is the unique finite energy solution to the problem

$$\begin{cases} -\Delta w_k' = 0, & \text{in } \mathbb{R}^2_+, \\ w_k' = 0, & \text{on } s, \\ \frac{\partial w_k'}{\partial \nu} = -\frac{\partial \psi_k'}{\partial \nu}, & \text{on } \partial\mathbb{R}^2_+ \setminus s, \end{cases}$$

where $\psi_k'(r\cos t, r\sin t) = r^{k/2}\sin\left(\frac{k}{2}\,t\right)$.

Lemma A.1. *For any odd positive integer k, $w_k = \widetilde{w}'_k \circ G$.*

Proof. Let us write $v := \widetilde{w}'_k \circ G$. By uniqueness, it is enough to prove that v solves (24). From the remarks at the beginning of the present section, it follows that v is harmonic in $\mathbb{R}^2_+$. Let us now show that $\psi_k := \psi'_k \circ G$. Indeed, for $x' \in \mathbb{R}^2 \setminus s_0$, $\psi'_k(x') = \mathrm{Im}\left((z')^{k/2}\right)$, and therefore $f(x) = \mathrm{Im}\left((z^2)^{k/2}\right) = \mathrm{Im}\left(z^k\right) = \psi_k(x)$, where $x' = G(x)$, z and z' are defined as above, and where we use the determination of the square root on $\mathbb{C} \setminus s_0$ defined by G^{-1}. From this and the previous remarks, it follows that v satisfies the boundary conditions of Problem (24). □

As in [1] we define

$$\mathfrak{m}'_k = -\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w'_k|^2 \, dx$$

and

$$\mathfrak{m}_k = -\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w_k|^2 \, dx. \tag{42}$$

We note that the right hand side of (22) is equal to $-2\beta^2\mathfrak{m}_k$.

Corollary A.2. *For any odd positive integer k, $\mathfrak{m}'_k = \frac{1}{2}\mathfrak{m}_k$.*

Proof. We have

$$\mathfrak{m}'_k = -\frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla w'_k|^2 \, dx' = -\frac{1}{4}\int_{\mathbb{R}^2 \setminus s_0} |\nabla \widetilde{w}'_k|^2 \, dx'.$$

Using Lemma A.1 and the conformal invariance of the L^2-norm of the gradient, we find

$$\int_{\mathbb{R}^2 \setminus s_0} |\nabla \widetilde{w}'_k|^2 \, dx' = \int_{\mathbb{R}^2_+} |\nabla w_k|^2 \, dx = -2\mathfrak{m}_k.$$ □

In particular, Corollary A.2 and Proposition 3.2 imply that

$$\mathfrak{m}'_k = -\frac{k\pi}{4\, 2^{2k-1}} \begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2,$$

thus proving, in view of [1, Theorem 1.2], the explicit constant appearing in the asymptotic expansion of Theorem 2.3.

A.2 The u-capacities of segments

In this last section, we simplify the constant C_k occurring in [3, Lemma 2.3].

Proposition A.3. *For any positive integer k,*

$$C_k = \frac{k}{4^{k-1}} \begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2.$$

Proof. According to Equation (22) in [3, Lemma 2.3],

$$C_k = \sum_{j=1}^{k} j \, |A_{j,k}|^2 ,$$

where $A_{j,k}$ is the j-th cosine Fourier coefficient of the function $\eta \mapsto (\cos\eta)^k$. To be more explicit, let us expand $(\cos\eta)^k$ into a trigonometric polynomial. We write

$$(\cos\eta)^k = \left(\frac{e^{i\eta} + e^{-i\eta}}{2}\right)^k = \frac{1}{2^k}\sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} e^{(k-2j)i\eta}.$$

By grouping the terms of the sum in pairs starting from opposite extremities, we find

$$(\cos\eta)^k = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \begin{pmatrix} k \\ j \end{pmatrix} \cos((k-2j)\eta) + c_k$$

where

$$c_k = 0 \text{ if } k = 2p+1 \qquad \text{and} \qquad c_k = \frac{1}{2^k} \begin{pmatrix} k \\ p \end{pmatrix} \text{ if } k = 2p.$$

It follows that

$$C_k = \frac{1}{4^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2j) \begin{pmatrix} k \\ j \end{pmatrix}^2$$

and we conclude using Lemma 3.5. □

Proposition A.3 and [3, Theorem 1.16] provide the explicit constant appearing in the asymptotic expansion of Theorem 2.6.

B Auxiliary results for eigenvalues variations

This section is dedicated to the proof of Propositions 4.7 and 4.8. In order to make a connection to the results of [3], which we use, let us present an alternative characterization of the eigenvalues $(\mu_j)_{j\geq 1}$ and $(\mu_j(\varepsilon))_{j\geq 1}$. We define

$$\widehat{\mathcal{Q}}_\varepsilon^s := \left\{ u \in H_0^1(\widehat{\Omega} \setminus \Gamma_\varepsilon) \,:\, u \circ \sigma = u \right\},$$

and we denote by $\widehat{q}_\varepsilon^s$ the restriction of $\widehat{q}_0$ (see the paragraph preceding Theorem 2.5 for the notation) to $\widehat{\mathcal{Q}}_\varepsilon$. One can then check that we obtain the eigenvalues $(\mu_j(\varepsilon))_{j\geq 1}$ from $\widehat{q}_\varepsilon^s$ by the min-max principle. In the same way, we define

$$\widehat{\mathcal{Q}}^s := \left\{ u \in H_0^1(\widehat{\Omega}) \,:\, u \circ \sigma = u \right\},$$

we denote by $\widehat{q}^s$ the restriction of the quadratic form $\widehat{q}_0$, and one can check that we obtain the eigenvalues $(\mu_j)_{j\geq 1}$ from $\widehat{q}^s$ by the min-max principle. Let us note that $-\Delta^s$, defined in Remark 4.4 as a self-adjoint operator in $\ker(I-\Sigma)$, is the Friedrichs extension of $\widehat{q}^s$. We denote by $-\Delta_\varepsilon^s$ the Friedrichs extension of $\widehat{q}_\varepsilon^s$, which is also a self-adjoint operator in $\ker(I-\Sigma)$.

Let us first prove Proposition 4.7. Since $\mu_N(\varepsilon) \geq \mu_N$ for all $\varepsilon \in (0, \varepsilon_0]$ and since $\varepsilon \mapsto \mu_N(\varepsilon)$ is non-decreasing, we have existence of $\mu_N^* := \lim_{\varepsilon\to 0^+} \mu_N(\varepsilon)$, with $\mu_N^* \geq \mu_N$. It only remains to show that $\mu_N^* \leq \mu_N$. In order to do this, let us note that the space

$$\mathcal{D}^s := \left\{ u \in C_c^\infty(\widehat{\Omega} \setminus \{0\}) \,:\, u = u \circ \sigma \right\}$$

is dense in $\ker(I-\Sigma)$. Indeed, the space $C_c^\infty(\widehat{\Omega} \setminus \{0\})$ is dense in $L^2(\widehat{\Omega})$, since $\{0\}$ has measure 0. Therefore, if we fix $u \in \ker(I-\Sigma)$, there exists a sequence $(\varphi_n)_{n\geq 1}$ of elements of $C_c^\infty(\widehat{\Omega} \setminus \{0\})$ converging to u in $L^2(\widehat{\Omega})$. We now set $\widetilde{\varphi}_n := 1/2(\varphi_n + \varphi_n \circ \sigma)$. We have $\widetilde{\varphi}_n \in \mathcal{D}^s$ for every integer $n \geq 1$. Since $u = 1/2(u + u\circ\sigma)$, we have the inequality

$$\|\widetilde{\varphi}_n - u\|_{L^2(\widehat{\Omega})} \leq \frac{1}{2}\|\varphi_n - u\|_{L^2(\widehat{\Omega})} + \frac{1}{2}\|\varphi_n \circ \sigma - u\circ\sigma\|_{L^2(\widehat{\Omega})} = \|\varphi_n - u\|_{L^2(\widehat{\Omega})},$$

and this implies that the sequence $(\widetilde{\varphi}_n)_{n\geq 1}$ converges to u in $\ker(I-\Sigma)$.

According to the min-max characterization of eigenvalues and the previous density result,

$$\mu_N = \inf_{\substack{\mathcal{E}\subset\mathcal{D}^s \text{ subspace} \\ \dim(\mathcal{E})=N}} \max_{u\in\mathcal{E}} \frac{\widehat{q}_0(u)}{\|u\|^2}.$$

Let us now fix $\delta > 0$ and an N-dimensional subspace $\mathcal{E}_\delta \subset \mathcal{D}^s$ such that

$$\max_{u \in \mathcal{E}_\delta} \frac{\widehat{q}_0(u)}{\|u\|^2} \leq \mu_N + \delta.$$

There exists $\varepsilon_1 > 0$ such that $\mathcal{E}_\delta \subset \widehat{\mathcal{Q}}^s_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_1]$. This implies that, for every $\varepsilon \in (0, \varepsilon_1]$,

$$\mu_N(\varepsilon) = \min_{\substack{\mathcal{E} \subset \widehat{\mathcal{Q}}^s_\varepsilon \text{ subspace} \\ \dim(\mathcal{E}) = N}} \max_{u \in \mathcal{E}} \frac{\widehat{q}^s_\varepsilon(u)}{\|u\|^2} \leq \max_{u \in \mathcal{E}_\delta} \frac{\widehat{q}_0(u)}{\|u\|^2} \leq \mu_N + \delta.$$

Passing to the limit, we obtain first $\mu_N^* \leq \mu_N + \delta$, and then $\mu_N^* \leq \mu_N$, concluding the proof.

Let us finally prove Proposition 4.8. We recall that, as a corollary of Theorem 1.10 in [3], taking into account Proposition A.3 we have the following result.

Proposition B.1. *Let $\widehat{\lambda}_N$ be a simple eigenvalue of $-\widehat{\Delta}$ and u_N an associated eigenfunction normalized in $L^2(\widehat{\Omega})$. Let us assume that $u_N \in \widehat{\mathcal{Q}}^s$. For $\varepsilon > 0$ small, we denote as $\widehat{\lambda}_N(\varepsilon)$ the N-th eigenvalue of the Dirichlet Laplacian in $\widehat{\Omega} \setminus \Gamma_\varepsilon$. If $u_N(0) \neq 0$, then*

$$\widehat{\lambda}_N(\varepsilon) = \widehat{\lambda}_N + \frac{2\pi}{|\log(\varepsilon)|} u_N(0)^2 + o\left(\frac{1}{|\log(\varepsilon)|}\right) \quad \text{as } \varepsilon \to 0^+.$$

If

$$r^{-k} u_N(r\cos t, r\sin t) \to \widehat{\beta} \cos(kt) \ \text{ in } C^{1,\tau}\left([0,\pi], \mathbb{R}\right)$$

as $r \to 0^+$ for all $\tau \in (0,1)$, with $k \in \mathbb{N}^*$ and $\widehat{\beta} \in \mathbb{R} \setminus \{0\}$, then

$$\widehat{\lambda}_N(\varepsilon) = \widehat{\lambda}_N + \frac{k\pi\widehat{\beta}^2}{4^{k-1}} \begin{pmatrix} k-1 \\ \lfloor \frac{k-1}{2} \rfloor \end{pmatrix}^2 \varepsilon^{2k} + o\left(\varepsilon^{2k}\right) \quad \text{as } \varepsilon \to 0^+.$$

Let us note that if the hypotheses of Proposition B.1 are satisfied, $\widehat{\lambda}_N$ is a simple eigenvalue of $-\Delta^s$ and u_N an associated eigenfunction. But the converse is not true. Indeed, we have seen in Section 4.5, in the case of $\widehat{\lambda}_3$ for the unit disk that $\widehat{\lambda}_N$ can be simple for $-\Delta^s$ without being simple for $-\widehat{\Delta}$. Proposition B.1 is therefore weaker than Proposition 4.8. However, the proof of Theorem 1.10 in [3] can be adapted to prove Proposition 4.8. Let us sketch the changes to be made. The proof in [3] mainly relies on Theorem 1.4 of [3], and uses the u-capacity and the associated potential defined in [3, Equations (6), (7), and (8)]. The following Lemma gives an alternative expression when both u and the compact set K are symmetric; it follows easily from Steiner symmetrization arguments.

Lemma B.2. *If $u \in \widehat{Q}^s$ and $K \subset \widehat{\Omega}$ is a compact set such that $\sigma(K) = K$, then*

$$\mathit{Cap}_{\widehat{\Omega}}(K, u) = \min\left\{\widehat{q}^s(V) \,:\, V \in \widehat{\mathcal{Q}}^s \text{ and } u - V \in H^1_0(\widehat{\Omega} \setminus K)\right\}$$

and the potential $V_{K,u}$ attaining the above minimum belongs to $\widehat{\mathcal{Q}}^s$.

Our proof of Proposition 4.8 relies on the following analog to [3, Theorem 1.4].

Proposition B.3. *Let μ_L be a simple eigenvalue of $-\Delta^s$ and u_L an associated eigenfunction, normalized in $L^2(\widehat{\Omega})$. Then*

$$\mu_L(\varepsilon) = \mu_L + \mathit{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u_L) + o\left(\mathit{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u_L)\right) \quad \text{as } \varepsilon \to 0^+.$$

In order to prove Proposition B.3, we note that Lemma B.2 implies in particular that $u_L - V_{\Gamma_\varepsilon, u_L}$ is the orthogonal projection of u_L on $H^1_0(\widehat{\Omega} \setminus \Gamma_\varepsilon) \cap \widehat{\mathcal{Q}}^s$ and $\mathrm{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u_L)$ the square of the distance of u_L from $H^1_0(\widehat{\Omega} \setminus \Gamma_\varepsilon) \cap \widehat{\mathcal{Q}}^s$, both defined with respect to the scalar product induced by $\widehat{q}^s$ on $\widehat{\mathcal{Q}}^s$. We also note that we can use the estimates of $V_{\Gamma_\varepsilon, u_L}$ given in Lemma A.1 and Corollary A.2 of [3]. We can therefore repeat step by step the proof of Theorem 1.4 in Appendix A of [3], replacing $L^2(\widehat{\Omega})$ by $\ker(I - \Sigma)$, $H^1_0(\widehat{\Omega})$ with $\widehat{\mathcal{Q}}^s$, $H^1_0(\widehat{\Omega} \setminus \Gamma_\varepsilon)$ by $H^1_0(\widehat{\Omega} \setminus \Gamma_\varepsilon) \cap \widehat{\mathcal{Q}}^s$, $\widehat{q}$ and $\widehat{q}_\varepsilon$ by $\widehat{q}^s$ and $\widehat{q}^s_\varepsilon$, $-\widehat{\Delta}$ and $-\widehat{\Delta}_\varepsilon$ by $-\Delta^s$ and $-\Delta^s_\varepsilon$, $\widehat{\lambda}_N$ by μ_L and $u_N \in H^1_0(\widehat{\Omega})$ by $u_L \in \widehat{\mathcal{Q}}^s$. We obtain Proposition B.3. The estimates of $\mathrm{Cap}_{\widehat{\Omega}}(\Gamma_\varepsilon, u)$ proved in [3, Section 2] then give us Proposition 4.8.

C Alternative proof of Theorem 3.1

We find useful to show an alternative proof of Theorem 3.1. This proof is based on sharp estimates from above and below of the Rayleigh quotients for the eigenvalues λ_N and $\lambda_N(\varepsilon)$. Such estimates require energy bounds on eigenfunctions obtained by an Almgren type monotonicity argument and blow-up analysis for scaled eigenfunctions. We mention that such a strategy was first developed in [1, 2, 5, 17] for eigenvalues of Aharonov–Bohm operators with a moving pole. On the other hand, the implementation of this procedure for our problem requires a quite different technique with respect to the case of Aharonov–Bohm operators with a single pole, when estimating a singular term appearing in the derivate of the Almgren frequency function (i.e. the term (58)). Indeed, in the single pole case estimates can be derived by rewriting the problem as a Laplace equation on the twofold covering, whereas in this case the singular term (58) turns out to have a negative sign and this is enough to proceed with the monotonicity argument (see Subsection C.2).

In this argument, an important step is a blow-up result for scaled eigenfunctions.

In what follows, we aim at pointing out the main steps of the proof, together with a more deepened analysis at the crucial points. We list below some notation used throughout this appendix.

- For $r > 0$ and $a \in \mathbb{R}^2$, $D_r(a) = \{x \in \mathbb{R}^2 : |x - a| < r\}$ denotes the disk of center a and radius r. We also denote the corresponding upper half-disk as $D_r^+(a) = \{(x_1, x_2) \in D_r(a) : x_2 > 0\}$.
- For all $r > 0$, $D_r = D_r(0)$ is the disk of center 0 and radius r; $D_r^+ = \{(x_1, x_2) \in D_r : x_2 > 0\}$ denotes the corresponding upper half-disk.
- For $r > 0$ and $a \in \mathbb{R}^2$, $S_r^+(a) = \{(x_1, x_2) \in \partial D_r(a) : x_2 > 0\}$ denotes the upper half-circle of center a and radius r. We also denote $S_r^+ := S_r^+(0)$.

C.1 Limit profile

This section contains a variational construction of the limit profile which will be used to describe the limit of the blow-up sequence.

Let us consider the functional $J_k : \mathcal{Q} \to \mathbb{R}$ (see Subsection 3.1 for the definition of $\mathcal{Q}$)

$$J_k(u) = \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla u(x)|^2 \, dx - \int_{-1}^{1} u(x_1, 0) \frac{\partial \psi_k}{\partial x_2}(x_1, 0) \, dx_1, \tag{43}$$

with ψ_k defined in (11). We observe that $\frac{\partial \psi_k}{\partial x_2}(x_1, 0) = k {x_1}^{k-1}$ and J_k is well-defined on $\mathcal{Q}$.

Lemma C.1. *For all $k \in \mathbb{N}$, $k \geq 1$, let $w_k \in \mathcal{Q}$ be the unique weak solution to* (24) *and let $\mathfrak{m}_k = -\frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla w_k|^2 \, dx$ be as in* (42)*. Then*

$$\mathfrak{m}_k = \min_{u \in \mathcal{Q}} J_k(u) = J_k(w_k) < 0. \tag{44}$$

Furthermore, $w_k(x) = O\big(\frac{1}{|x|}\big)$ as $|x| \to +\infty$.

Proof. The proof follows from standard minimization methods, Hardy Inequality and Kelvin Transform. ☐

Lemma C.2. *For every $k \in \mathbb{N}$, $k \geq 1$, there exists a unique $\Phi_k \in \bigcap_{R>0} H^1(D_R^+)$ such that*

$$\begin{cases} \Phi_k - \psi_k \in \mathcal{Q}, \\ -\Delta \Phi_k = 0, & \text{in } \mathbb{R}^2_+ \text{ in a distributional sense}, \\ \Phi_k = 0 & \text{on } s, \\ \frac{\partial \Phi_k}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases} \tag{45}$$

where $\nu = (0, -1)$ *is the outer normal unit vector on* $\partial\mathbb{R}^2_+$*. Furthermore, the unique solution to* (45) *is given by*

$$\Phi_k = \psi_k + w_k,$$

where w_k *is as in Lemma C.1 and* ψ_k *is defined in* (11)*.*

Proof. The existence part is proved by taking $\Phi_k = \psi_k + w_k$. To prove uniqueness, one can argue by contradiction exploiting the Hardy Inequality (see[1, Proposition 4.3] for a detailed proof in a similar problem). □

For future convenience, we state and prove here the following lemma, which relates the limit profile Φ_k (more precisely, its k-th Fourier coefficient) to the minimum $\mathfrak{m}_k$.

Lemma C.3. *Let* Φ_k *be as in Lemma C.2. Then*

$$\int_0^\pi \Phi_k(\cos t, \sin t)\, \sin(kt)\, dt = -\frac{\mathfrak{m}_k}{k} + \frac{\pi}{2}.$$

Proof. Let us define the function

$$\omega(r) := \int_0^\pi w_k(r\cos t, r\sin t)\, \sin(kt)\, dt, \quad r > 0,$$

where w_k is as in Lemma C.1. Then, recalling that $\Phi_k = w_k + \psi_k$, we have that

$$\omega(1) = \int_0^\pi \Phi_k(\cos t, \sin t)\, \sin(kt)\, dt - \frac{\pi}{2}. \tag{46}$$

Since ω is the k-th Fourier coefficient of the harmonic function w_k, it satisfies the differential equation $\omega'' + \frac{1}{r}\omega' - \frac{k^2}{r^2}\omega = 0$ in $(1, +\infty)$, i.e. $(r^{1+2k}(r^{-k}\omega)')' = 0$. Hence there exists $C_\omega \in \mathbb{R}$ such that $\left(r^{-k}\omega(r)\right)' = C_\omega r^{-(1+2k)}$, for $r > 1$. Integrating the previous equation over $(1, r)$ we obtain that

$$\frac{\omega(r)}{r^k} - \omega(1) = \frac{C_\omega}{2k}\left(1 - \frac{1}{r^{2k}}\right), \quad \text{for all } r \geq 1.$$

Lemma C.1 provides that $\omega(r) = O(r^{-1})$ as $r \to +\infty$, hence, letting $r \to +\infty$ in the previous identity, we obtain that necessarily $C_\omega = -2k\omega(1)$ and then

$$\omega(r) = \omega(1)r^{-k}, \quad \omega'(r) = -k\omega(1)r^{-k-1}, \quad \text{for all } r \geq 1. \tag{47}$$

On the other hand, by definition

$$\omega'(r) = r^{-k-1} \int_{S_r^+} \frac{\partial w_k}{\partial \nu}\, \psi_k\, ds, \tag{48}$$

with ν being the outer unit vector to ∂D_r^+. Combining (47) and (48) we obtain that

$$\omega(1) = -\frac{1}{k} \int_{S_1^+} \frac{\partial w_k}{\partial \nu}\, \psi_k\, ds.$$

Multiplying the equation $-\Delta w_k = 0$ by ψ_k, integrating by parts on D_1^+, and recalling that $\psi_k \equiv 0$ on Γ_1, we obtain that

$$\int_{D_1^+} \nabla w_k \cdot \nabla \psi_k\, dx = \int_{\partial D_1^+} \frac{\partial w_k}{\partial \nu}\, \psi_k\, ds = \int_{S_1^+} \frac{\partial w_k}{\partial \nu}\, \psi_k\, ds,$$

whereas multiplying $-\Delta \psi_k = 0$ by w_k and integrating by parts on D_1^+ we obtain that

$$\int_{D_1^+} \nabla w_k \cdot \nabla \psi_k\, dx = \int_{\partial D_1^+} \frac{\partial \psi_k}{\partial \nu}\, w_k\, ds.$$

Taking into account the boundary data, we obtain that

$$\int_{S_1^+} \frac{\partial w_k}{\partial \nu}\, \psi_k = \int_{S_1^+} \frac{\partial \psi_k}{\partial \nu}\, w_k - \int_{\Gamma_1} \frac{\partial \psi_k}{\partial x_2}\, w_k,$$

so that

$$\omega(1) = -\frac{1}{k} \int_{S_1^+} \frac{\partial \psi_k}{\partial \nu}\, w_k + \frac{1}{k} \int_{\Gamma_1} \frac{\partial \psi_k}{\partial x_2}\, w_k. \tag{49}$$

Since $\frac{\partial \psi_k}{\partial \nu} = k\psi_k$ on S_1^+, it results that $k\omega(1) = \int_{S_1^+} \frac{\partial \psi_k}{\partial \nu} w_k$, so that (49) can be rewritten as $\omega(1) = -\omega(1) + \frac{1}{k} \int_{\Gamma_1} \frac{\partial \psi_k}{\partial x_2}\, w_k$ and thus

$$\omega(1) = \frac{1}{2k} \int_{\Gamma_1} \frac{\partial \psi_k}{\partial x_2}\, w_k.$$

From (44) we deduce that $\omega(1) = -\frac{1}{k}\mathfrak{m}_k$, and recalling (46) the proof is concluded. □

C.2 Monotonicity argument

In order to prove convergence of blow-up eigenfunctions, energy estimates in small neighborhoods of the Dirichlet-Neumann junctions are needed; such estimates are obtained via an Almgren type monotonicity argument which is sketched here.

For $\lambda \in \mathbb{R}$, $u \in H^1(\Omega)$ and $r \in (0, \varepsilon_0)$ such that $D_r^+ \subset \Omega$, the Almgren frequency function is defined as

$$\mathcal{N}(u, r, \lambda) = \frac{E(u, r, \lambda)}{H(u, r)},$$

where

$$E(u, r, \lambda) = \int_{D_r^+} \Big(|\nabla u(x)|^2 - \lambda u^2(x)\Big)\, dx, \quad H(u, r) = \frac{1}{r} \int_{S_r^+} u^2\, ds.$$

In the following, we assume that assumption (8) is satisfied, i.e. the N-th eigenvalue λ_N of q_0 is simple, and we fix an associated normalized eigenfunction u_N, so that u_N satisfies (9). For all $1 \le n < N$, let $u_n \in H_0^1(\Omega)$ be an eigenfunction of q_0 associated to the eigenvalue λ_n such that

$$\int_\Omega |u_n(x)|^2\, dx = 1 \quad \text{for all } 1 \le n < N$$

and

$$\int_\Omega u_n(x) u_m(x)\, dx = 0 \quad \text{if } 1 \le n, m \le N \text{ and } n \ne m.$$

For every $\varepsilon \in (0, \varepsilon_0]$, let u_N^ε be an eigenfunction of q_ε associated with $\lambda_N(\varepsilon)$, i.e. solving

$$\begin{cases} -\Delta\, u_N^\varepsilon = \lambda_N(\varepsilon)\, u_N^\varepsilon, & \text{in } \Omega, \\ u_N^\varepsilon = 0, & \text{on } \partial\Omega \setminus \Gamma_\varepsilon, \\ \dfrac{\partial u_N^\varepsilon}{\partial \nu} = 0, & \text{on } \Gamma_\varepsilon, \end{cases} \tag{50}$$

such that

$$\int_\Omega |u_N^\varepsilon(x)|^2\, dx = 1 \quad \text{and} \quad \int_\Omega u_N^\varepsilon(x)\, u_N(x)\, dx \ge 0. \tag{51}$$

For all $1 \le n < N$ and $\varepsilon \in (0, \varepsilon_0]$, let $u_n^\varepsilon \in \mathcal{Q}_\varepsilon$ be an eigenfunction of problem (4) associated to the eigenvalue $\lambda = \lambda_n(\varepsilon)$, i.e. solving

$$\begin{cases} -\Delta\, u_n^\varepsilon = \lambda_n(\varepsilon)\, u_n^\varepsilon, & \text{in } \Omega, \\ u_n^\varepsilon = 0, & \text{on } \partial\Omega \setminus \Gamma_\varepsilon, \\ \dfrac{\partial u_n^\varepsilon}{\partial \nu} = 0, & \text{on } \Gamma_\varepsilon, \end{cases} \tag{52}$$

such that

$$\int_\Omega |u_n^\varepsilon(x)|^2\,dx = 1 \quad \text{for all } 1 \le n < N \tag{53}$$

and

$$\int_\Omega u_n^\varepsilon(x) u_m^\varepsilon(x)\,dx = 0 \quad \text{if } 1 \le n, m \le N \text{ and } n \neq m. \tag{54}$$

We observe that, in view of Remark 1.1,

$$\lambda_n(\varepsilon) \le \lambda_N \quad \text{for all } \varepsilon \in (0, \varepsilon_0] \text{ and } 1 \le n \le N. \tag{55}$$

Arguing as in [1, Lemma 5.2], it is possible to prove the following properties:

(i) there exists $R_0 \in \big(0, \min\big\{\varepsilon_0, \frac{1}{2\sqrt{\lambda_N}}\big\}\big)$ such that $D^+_{R_0} \subset \Omega$ and

$$H(u_n^\varepsilon, r) > 0 \quad \text{for all } \varepsilon \in (0, R_0),\ r \in (\varepsilon, R_0) \text{ and } 1 \le n \le N;$$

(ii) for every $r \in (0, R_0]$, there exist $C_r > 0$ and $\alpha_r \in (0, r)$ such that $H(u_n^\varepsilon, r) \ge C_r$ for all $\varepsilon \in (0, \alpha_r)$ and $1 \le n \le N$.

By direct calculations it follows that, for all $\varepsilon \in (0, R_0)$, $\varepsilon < r < R_0$, and $n \in \{1, 2, \ldots, N\}$,

$$\frac{d}{dr} H(u_n^\varepsilon, r) = \frac{2}{r} \int_{S_r^+} u_n^\varepsilon \frac{\partial u_n^\varepsilon}{\partial \nu}\,ds = \frac{2}{r} E(u_n^\varepsilon, r, \lambda_n(\varepsilon)), \tag{56}$$

$$\frac{d}{dr} E(u_n^\varepsilon, r, \lambda_n(\varepsilon)) == 2 \int_{S_r^+} \left|\frac{\partial u_n^\varepsilon}{\partial \nu}\right|^2 ds - \frac{2}{r}\left(M(\varepsilon, u_n^\varepsilon, \lambda_n(\varepsilon)) + \lambda_n(\varepsilon) \int_{D_r^+} (u_n^\varepsilon(x))^2\,dx \right) \tag{57}$$

where ν denotes the exterior normal unit vector to D_r^+ and

$$M(\varepsilon, u, \lambda) = \lim_{\delta \to 0^+} \int_{\mathbb{R}^2_+ \cap \partial A^\varepsilon_\delta} \left(\frac{1}{2} |\nabla u|^2 x \cdot \boldsymbol{n} - \frac{\partial u}{\partial \boldsymbol{n}} (x \cdot \nabla u) - \frac{\lambda}{2} u^2 x \cdot \boldsymbol{n} \right) ds, \tag{58}$$

being $A^\varepsilon_\delta := D^+_\delta(-\varepsilon, 0) \cup D^+_\delta(\varepsilon, 0)$ and $\boldsymbol{n}$ denoting the exterior normal unit vector to $D_r^+ \setminus A^\varepsilon_\delta$. For details in a similar problem see [17, Lemma 5.5 and 5.6]. A crucial step in the monotonicity argument is the possibility of recognizing the sign of the quantity $M(\varepsilon, u, \lambda)$.

To this aim, we first state the following result describing the behaviour of solutions to (4) at Dirichlet-Neumann boundary junctions.

Proposition C.4. *Let $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in \mathbb{R}$, and $u \in \mathcal{Q}_\varepsilon \setminus \{0\}$ be a nontrivial solution to problem* (4)*. Then there exist two odd natural numbers $j_L = j_L(\varepsilon, u, \lambda), j_R = j_R(\varepsilon, u, \lambda) \in \mathbb{N}$ and two nonzero real numbers $\beta_L = \beta_L(\varepsilon, u, \lambda), \beta_R = \beta_R(\varepsilon, u, \lambda) \in \mathbb{R} \setminus \{0\}$ such that*

$$\delta^{-j_L/2} u((-\varepsilon, 0) + \delta \boldsymbol{\theta}(t)) \to \beta_L \cos\left(\tfrac{j_L}{2} t\right) \quad \text{in } C^{1,\sigma}([0, \pi]), \tag{59}$$

$$\delta^{-j_R/2} u((\varepsilon, 0) + \delta \boldsymbol{\theta}(t)) \to \beta_R \sin\left(\tfrac{j_R}{2} t\right) \quad \text{in } C^{1,\sigma}([0, \pi]), \tag{60}$$

as $\delta \to 0^+$ for any $\sigma \in (0, 1)$, where $\boldsymbol{\theta}(t) = (\cos t, \sin t)$. Moreover,

$$\delta^{-j_L/2+1} \nabla u((-\varepsilon, 0) + \delta \boldsymbol{\theta}(t)) \to \tfrac{j_L \beta_L}{2} \left(\cos\left(\tfrac{j_L}{2} t\right) \boldsymbol{\theta}(t) - \sin\left(\tfrac{j_L}{2} t\right) \boldsymbol{\tau}(t) \right) \tag{61}$$

$$\delta^{-j_R/2+1} \nabla u((\varepsilon, 0) + \delta \boldsymbol{\theta}(t)) \to \tfrac{j_R \beta_R}{2} \left(\sin\left(\tfrac{j_R}{2} t\right) \boldsymbol{\theta}(t) + \cos\left(\tfrac{j_R}{2} t\right) \boldsymbol{\tau}(t) \right) \tag{62}$$

in $C^{0,\sigma}([0, \pi])$ as $\delta \to 0^+$ for any $\sigma \in (0, 1)$, where $\boldsymbol{\tau}(t) = (-\sin t, \cos t)$.

Proof. Through a gauge transformation, in a neighbourhood of each junction $(\pm\varepsilon, 0)$ the problem can be rewritten as an elliptic equation with an Aharonov–Bohm vector potential with pole located at the junction; then the asymptotics follows from [11, Theorem 1.3]. ∎

Lemma C.5. *Let $\varepsilon \in (0,\varepsilon_0]$ and $u \in \mathcal{Q}_\varepsilon$ be a solution to (4) for some $\lambda \in \mathbb{R}$. Moreover, let $j_L = j_L(\varepsilon,u,\lambda)$, $j_R = j_R(\varepsilon,u,\lambda) \in \mathbb{N}$ odd and $\beta_L = \beta_L(\varepsilon,u,\lambda)$, $\beta_R = \beta_R(\varepsilon,u,\lambda) \in \mathbb{R}\setminus\{0\}$ be as in Proposition C.4 and let $M(\varepsilon,u,\lambda)$ be as in (58). Then*

$$M(\varepsilon,u,\lambda) = \begin{cases} 0, & \text{if } j_L > 1 \text{ and } j_R > 1,\\ -\varepsilon\frac{\pi}{8}\beta_L^2, & \text{if } j_L = 1 \text{ and } j_R > 1,\\ -\varepsilon\frac{\pi}{8}\beta_R^2, & \text{if } j_L > 1 \text{ and } j_R = 1,\\ -\varepsilon\frac{\pi}{8}\big(\beta_L^2+\beta_R^2\big), & \text{if } j_L = 1 \text{ and } j_R = 1. \end{cases}$$

In particular, $M(\varepsilon,u,\lambda) \le 0$.

Proof. Since $\partial A_\delta^\varepsilon \cap \mathbb{R}^2_+ = S_\delta^+(-\varepsilon,0) \cup S_\delta^+(\varepsilon,0)$, we split (58) into the corresponding two contributions.

Negligible terms. On $S_\delta^+(-\varepsilon,0)$, we have that $x = (-\varepsilon,0)+\delta\boldsymbol{\theta}(t)$ for some $t\in[0,\pi]$ and $\boldsymbol{n} = -\boldsymbol{\theta}$, where $\boldsymbol{\theta}(t) = (\cos t,\sin t)$; hence $x\cdot\boldsymbol{n} = \varepsilon\cos t - \delta$. From (59) and (61) we have that $u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t)) \to 0$ and $|\nabla u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))|^2 = \frac{j_L^2\beta_L^2}{4}\delta^{j_L-2}(1+o(1))$ uniformly on $[0,\pi]$ as $\delta\to 0$. From the Dominated Convergence Theorem we then obtain

$$\begin{aligned}
&\int_{S_\delta^+(-\varepsilon,0)} (|\nabla u|^2 - \lambda u^2)\, x\cdot\boldsymbol{n}\, ds\\
&\quad = \delta\int_0^\pi (|\nabla u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))|^2 - \lambda|u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))|^2)\,(\varepsilon\cos t-\delta)\,dt\\
&\quad \to \begin{cases} 0, & \text{if } j_L>1,\\ \frac{\beta_L^2\varepsilon}{4}\int_0^\pi \cos t\,dt = 0, & \text{if } j_L = 1,\end{cases}
\end{aligned}$$

as $\delta\to 0$.

Leading term. We now look at the last term

$$-\int_{S_\delta^+(-\varepsilon,0)} \frac{\partial u}{\partial\boldsymbol{n}}(x\cdot\nabla u)\,ds = \int_{S_\delta^+(-\varepsilon,0)} (\boldsymbol{\theta}\cdot\nabla u)(x\cdot\nabla u)\,ds,$$

since $\boldsymbol{\theta} = -\boldsymbol{n}$ on $S_\delta^+(-\varepsilon,0)$. From (61) we have

$$\delta^{-j_L/2+1}\nabla u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))\cdot\boldsymbol{\theta}(t) \to \frac{j_L}{2}\beta_L\cos\left(\frac{j_L}{2}t\right)$$

in $C^0([0,\pi])$ as $\delta\to 0$. On the other hand, for $x = (-\varepsilon,0)+\delta\boldsymbol{\theta}(t)$ we have that

$$\delta^{-\frac{j_L}{2}+1}\nabla u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))\cdot x \to -\varepsilon\frac{j_L}{2}\beta_L\left(\cos\left(\frac{j_L}{2}t\right)\cos t + \sin\left(\frac{j_L}{2}t\right)\sin t\right)$$

in $C^0([0,\pi])$ as $\delta\to 0$. Thus, by the Dominated Convergence Theorem, we have

$$\begin{aligned}
&-\int_{S_\delta^+(-\varepsilon,0)} \frac{\partial u}{\partial\boldsymbol{n}}(x\cdot\nabla u)\,ds\\
&= \delta\int_0^\pi (\nabla u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))\cdot((-\varepsilon,0)+\delta\boldsymbol{\theta}(t)))\,(\nabla u((-\varepsilon,0)+\delta\boldsymbol{\theta}(t))\cdot\boldsymbol{\theta}(t))\,dt\\
&\to \begin{cases} 0, & \text{if } j_L>1,\\ -\frac{\varepsilon}{4}(\beta_L)^2\int_0^\pi \left(\cos^2\left(\frac{t}{2}\right)\cos t + \cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\sin t\right)dt, & \text{if } j_L = 1,\end{cases}\\
&= \begin{cases} 0, & \text{if } j_L>1,\\ -\frac{\varepsilon}{4}\beta_L^2\int_0^\pi \cos^2\left(\frac{t}{2}\right)dt, & \text{if } j_L = 1,\end{cases}\\
&= \begin{cases} 0, & \text{if } j_L>1,\\ -\frac{\varepsilon}{8}\beta_L^2\pi, & \text{if } j_L = 1,\end{cases}
\end{aligned}$$

as $\delta \to 0$. One can follow the same argument to compute the contribution coming from $S_\delta^+(\varepsilon, 0)$. Putting together the two contributions we obtain the thesis. $\square$

This turns out to be sufficient to prove the following:

Lemma C.6. *For any $n \in \{1, \dots, N\}$, $\varepsilon \in (0, R_0)$, and r, R such that $\varepsilon < r < R \leq R_0$ we have that*

$$\mathcal{N}(u_n^\varepsilon, r, \lambda_n(\varepsilon)) + 1 \leq (\mathcal{N}(u_n^\varepsilon, R, \lambda_n(\varepsilon)) + 1)\, e^{2\lambda_N R^2}.$$

In particular, for every $\delta \in (0,1)$ there exists $r_\delta \in (0, R_0)$ such that, for any $\varepsilon \in (0, r_\delta)$ and $r \in (\varepsilon, r_\delta)$, $\mathcal{N}(u_N^\varepsilon, r, \lambda_N(\varepsilon)) \leq k + \delta$, k being as in (10).

Proof. Once the negative sign of $M(\varepsilon, u, \lambda)$ is established (Lemma C.5) the proof proceeds as in [1, Section 5]. $\square$

Lemma C.6 is the key point for a priori estimates on energy of the blow up sequence in half-disks. These estimates are in turn fundamental to deduce estimates on the difference of eigenvalues, as it appears in the following subsection.

C.3 Estimates on the difference of eigenvalues

Firstly, we are going to estimate the Rayleigh quotient for $\lambda_N(\varepsilon)$. Let $R > 1$. With R_0 as in the previous section, for every $\varepsilon \in (0, \varepsilon_0)$ such that $R\varepsilon < R_0$ we define the function

$$v_{R,\varepsilon} = \begin{cases} v_{R,\varepsilon}^{int}, & \text{in } D_{R\varepsilon}^+, \\ u_N, & \text{in } \Omega \setminus D_{R\varepsilon}^+, \end{cases}$$

where $v_{R,\varepsilon}^{int}$ is the unique solution to

$$\begin{cases} -\Delta v_{R,\varepsilon}^{int} = 0, & \text{in } D_{R\varepsilon}^+, \\ v_{R,\varepsilon}^{int} = u_N, & \text{on } S_{R\varepsilon}^+, \\ v_{R,\varepsilon}^{int} = 0, & \text{on } \Gamma_{R\varepsilon} \setminus \Gamma_\varepsilon, \\ \frac{\partial v_{R,\varepsilon}^{int}}{\partial \nu} = 0, & \text{on } \Gamma_\varepsilon, \end{cases} \tag{63}$$

i.e., by the Dirichlet principle, the unique solution to the minimization problem

$$\int_{D_{R\varepsilon}^+} |\nabla v_{R,\varepsilon}^{int}|^2 dx = \min\left\{ \textstyle\int_{D_{R\varepsilon}^+} |\nabla v|^2 : v \in H^1(D_{R\varepsilon}^+),\ v = u_N \text{ on } S_{R\varepsilon}^+,\ v = 0 \text{ on } \Gamma_{R\varepsilon} \setminus \Gamma_\varepsilon \right\}. \tag{64}$$

In order to handle the denominator of the Rayleigh quotient we proceed with a Gram-Schmidt process. Since we are taking into account $u_1, \dots, u_{N-1}$ as the first $N-1$ test functions for the Rayleigh quotient, which are already orthonormalized in $L^2(\Omega)$, we define

$$\tilde{u}_{R,\varepsilon} = \frac{v_{R,\varepsilon} - \sum_{j=1}^{N-1} \left(\int_\Omega v_{R,\varepsilon}\, u_j \right) u_j}{\left\| v_{R,\varepsilon} - \sum_{j=1}^{N-1} \left(\int_\Omega v_{R,\varepsilon}\, u_j \right) u_j \right\|_{L^2(\Omega)}}.$$

Using the Dirichlet Principle and the asymptotics (10) one can easily prove the following energy estimates for $v_{R,\varepsilon}^{int}$ in small disks.

Lemma C.7. *There exists a constant $C > 0$ (independent of ε and R) such that, for every $R > 1$ and $\varepsilon \in (0, \varepsilon_0)$ such that $R\varepsilon < R_0$, the following estimates hold:*

$$\int_{D_{R\varepsilon}^+} |\nabla v_{R,\varepsilon}^{int}|^2 dx \leq C (R\varepsilon)^{2k}, \tag{65}$$

$$\int_{S_{R\varepsilon}^+} |v_{R,\varepsilon}^{int}|^2 ds \leq C (R\varepsilon)^{2k+1}, \tag{66}$$

$$\int_{D_{R\varepsilon}^+} |v_{R,\varepsilon}^{int}|^2 dx \leq C (R\varepsilon)^{2k+2}. \tag{67}$$

To our aim, for every $R > 1$ we define v_R as the unique solution to the minimization problem

$$\int_{D_R^+} |\nabla v_R|^2 dx = \min \left\{ \int_{D_R^+} |\nabla v|^2 dx : v \in H^1(D_R^+),\ v = \psi_k \text{ on } S_R^+,\ v = 0 \text{ on } \Gamma_R \setminus \Gamma_1 \right\}.$$

The function v_R is the unique weak solution to

$$\begin{cases} -\Delta v_R = 0, & \text{in } D_R^+, \\ v_R = \psi_k, & \text{on } S_R^+, \\ v_R = 0, & \text{on } \Gamma_R \setminus \Gamma_1, \\ \frac{\partial v_R}{\partial \nu} = 0, & \text{on } \Gamma_1. \end{cases} \tag{68}$$

As well, we introduce the following blow-up functions

$$U_\varepsilon(x) := \frac{u_N(\varepsilon x)}{\varepsilon^k}, \quad V_\varepsilon^R(x) := \frac{v_{R,\varepsilon}^{int}(\varepsilon x)}{\varepsilon^k}. \tag{69}$$

Combining (10) with the Dirichlet Principle, we can establish the following convergences

$$U_\varepsilon \to \beta\psi_k \text{ as } \varepsilon \to 0 \text{ in } H^1(D_R^+) \text{ for every } R > 1; \tag{70}$$

$$V_\varepsilon^R \to \beta v_R \text{ for } \varepsilon \to 0 \text{ and for any } R > 1; \tag{71}$$

$$v_R \to \Phi_k \text{ in } H^1(D_r^+) \text{ as } R \to +\infty \text{ for any } r > 1. \tag{72}$$

Proposition C.8. *For any $R > 1$ and $\varepsilon \in (0, \varepsilon_0)$ such that $R\varepsilon < R_0$, we have that*

$$\frac{\lambda_N(\varepsilon) - \lambda_N}{\varepsilon^{2k}} \le f_R(\varepsilon)$$

where

$$\lim_{\varepsilon \to 0} f_R(\varepsilon) = \beta^2 \int_{S_R^+} \psi_k \left(\frac{\partial v_R}{\partial \nu} - \frac{\partial \psi_k}{\partial \nu} \right) ds$$

with ψ_k defined in (11) *and v_R in* (68).

Proof. We note that

$$\begin{aligned} &\left\| v_{R,\varepsilon} - \sum_{j=1}^{N-1} \left(\int_\Omega v_{R,\varepsilon}\, u_j\, dx \right) u_j \right\|_{L^2(\Omega)}^2 = \|v_{R,\varepsilon}\|_{L^2(\Omega)}^2 - \sum_{j=1}^{N-1} \left(\int_\Omega v_{R,\varepsilon}\, u_j\, dx \right)^2 \\ &= 1 - \int_{D_{R\varepsilon}^+} u_N^2\, dx + \int_{D_{R\varepsilon}^+} |v_{R,\varepsilon}^{int}|^2\, dx - \sum_{j=1}^{N-1} \left(\int_\Omega v_{R,\varepsilon}\, u_j\, dx \right)^2 \\ &= 1 + O(\varepsilon^{2k+2}) \quad \text{as } \varepsilon \to 0 \end{aligned} \tag{73}$$

in view of (10) and (67) and since, for all $j < N$,

$$\int_\Omega v_{R,\varepsilon}\, u_j\, dx = -\int_{D_{R\varepsilon}^+} u_N\, u_j\, dx + \int_{D_{R\varepsilon}^+} v_{R,\varepsilon}^{int}\, u_j\, dx = O(\varepsilon^{k+2}) \text{ as } \varepsilon \to 0. \tag{74}$$

The functions $u_1, \ldots, u_{N-1}, \tilde{u}_{R,\varepsilon}$ are linearly independent (since they are nontrivial and mutually orthogonal) and belong to $\mathcal{Q}_\varepsilon$; if we plug a linear combination of them into the Rayleigh quotient

(7) we obtain

$$
\begin{aligned}
\lambda_N(\varepsilon)-\lambda_N &\le \left(\max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N\\ \sum_{j=1}^N|\alpha_j|^2=1}} \int_\Omega \left|\nabla\Big(\sum_{j=1}^{N-1}\alpha_j u_j+\alpha_N\tilde u_{R,\varepsilon}\Big)\right|^2\right)-\lambda_N\\
&= \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N\\ \sum_{j=1}^N|\alpha_j|^2=1}} \left[\sum_{j=1}^{N-1}\alpha_j^2\lambda_j+\alpha_N^2\int_\Omega|\nabla\tilde u_{R,\varepsilon}|^2+2\sum_{j=1}^{N-1}\alpha_j\alpha_N\int_\Omega\nabla u_j\cdot\nabla\tilde u_{R,\varepsilon}-\lambda_N\right]\\
&= \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N\\ \sum_{j=1}^N|\alpha_j|^2=1}} \left[\sum_{j=1}^{N-1}\alpha_j^2(\lambda_j-\lambda_N)+\alpha_N^2\left(\int_\Omega|\nabla\tilde u_{R,\varepsilon}|^2-\lambda_N\right)+2\sum_{j=1}^{N-1}\alpha_j\alpha_N\int_\Omega\nabla u_j\cdot\nabla\tilde u_{R,\varepsilon}\right].
\end{aligned}
$$

In view of (65) and (10) we have that

$$
\int_\Omega \nabla v_{R,\varepsilon}\cdot\nabla u_j=-\int_{D^+_{R\varepsilon}}\nabla u_N\cdot\nabla u_j\,dx+\int_{D^+_{R\varepsilon}}\nabla v^{int}_{R,\varepsilon}\cdot\nabla u_j\,dx=O(\varepsilon^{k+1}). \tag{75}
$$

Moreover, from convergences (70)–(72) we have

$$
\begin{aligned}
&\int_\Omega|\nabla v_{R,\varepsilon}|^2dx-\lambda_N=-\int_{D^+_{R\varepsilon}}|\nabla u_N|^2dx+\int_{D^+_{R\varepsilon}}|\nabla v^{int}_{R,\varepsilon}|^2dx\\
&=\varepsilon^{2k}\Big(-\int_{D^+_R}|\nabla U_\varepsilon|^2dx+\int_{D^+_R}|\nabla V^R_\varepsilon|^2dx\Big)=\varepsilon^{2k}\beta^2\Big(-\int_{D^+_R}|\nabla\psi_k|^2dx+\int_{D^+_R}|\nabla v_R|^2dx+o(1)\Big)\\
&=\varepsilon^{2k}\beta^2\left(\int_{S^+_R}\psi_k\Big(\frac{\partial v_R}{\partial\nu}-\frac{\partial\psi_k}{\partial\nu}\Big)ds+o(1)\right)\quad\text{as }\varepsilon\to0.
\end{aligned}\tag{76}
$$

Collecting (73), (74), (75), and (76), we obtain that

$$
\begin{aligned}
&\int_\Omega|\nabla\tilde u_{R,\varepsilon}|^2-\lambda_N\\
&\quad=\frac{\int_\Omega|\nabla v_{R,\varepsilon}|^2+\sum\limits_{j=1}^{N-1}\big(\int_\Omega v_{R,\varepsilon}\,u_j\big)^2\lambda_j-2\sum\limits_{j=1}^{N-1}\big(\int_\Omega v_{R,\varepsilon}\,u_j\big)\int_\Omega\nabla v_{R,\varepsilon}\cdot\nabla u_j}{\left\|v_{R,\varepsilon}-\sum_{j=1}^{N-1}\big(\int_\Omega v_{R,\varepsilon}\,u_j\big)\,u_j\right\|^2_{L^2(\Omega)}}-\lambda_N\\
&\quad=\varepsilon^{2k}\beta^2\left(\int_{S^+_R}\psi_k\Big(\frac{\partial v_R}{\partial\nu}-\frac{\partial\psi_k}{\partial\nu}\Big)ds+o(1)\right)\quad\text{as }\varepsilon\to0.
\end{aligned}
$$

From (73), (74), and (75) it follows that, for every $j<N$,

$$
\int_\Omega\nabla u_j\cdot\nabla\tilde u_{R,\varepsilon}=O(\varepsilon^{k+1})\quad\text{as }\varepsilon\to0.
$$

Hence, the assumptions in [1, Lemma 6.1] are fulfilled by $\mu(\varepsilon)=\int_{S^+_R}\psi_k\big(\frac{\partial v_R}{\partial\nu}-\frac{\partial\psi_k}{\partial\nu}\big)ds+o(1)$, $\alpha=1$, $\sigma(\varepsilon)=\beta^2\varepsilon^{2k}$ and $M=2k-1$ and the conclusion follows. □

In the sequel we denote

$$
\kappa_R:=\int_{S^+_R}\psi_k\Big(\frac{\partial v_R}{\partial\nu}-\frac{\partial\psi_k}{\partial\nu}\Big)\,ds. \tag{77}
$$

Lemma C.9. *Let κ_R be defined in (77). Then $\lim_{R\to+\infty}\kappa_R=2\,\mathfrak{m}_k$, with $\mathfrak{m}_k$ as in (44).*

Proof. From (68) it follows that the function σ_R defined as

$$\sigma_R(r) := \int_0^\pi v_R(r(\cos t, \sin t)) \sin(kt)\, dt, \quad r \in [1, R], \tag{78}$$

satisfies the equation $(r^{1+2k}(r^{-k}\sigma_R)')' = 0$ and hence, for some $c_R \in \mathbb{R}$, $\big(r^{-k}\sigma_R(r)\big)' = \frac{c_R}{r^{1+2k}}$ in $(1, R)$. Integrating the previous equation over $(1, r)$ we obtain

$$r^{-k}\sigma_R(r) - \sigma_R(1) = \frac{c_R}{2k}\bigg(1 - \frac{1}{r^{2k}}\bigg), \quad \text{for all } r \in (1, R]. \tag{79}$$

Since (68) implies that $\sigma_R(R) = \frac{1}{2}\pi R^k$, from (79) we deduce that

$$\frac{c_R}{2k} = \frac{R^{2k}}{R^{2k}-1}\bigg(\frac{\pi}{2} - \sigma_R(1)\bigg)$$

and then

$$\sigma_R(r) = r^k \frac{\frac{\pi}{2}R^{2k} - \sigma_R(1)}{R^{2k}-1} - r^{-k}\frac{R^{2k}}{R^{2k}-1}(\tfrac{\pi}{2} - \sigma_R(1)),$$

for all $r \in (1, R]$. If we differentiate the previous identity and evaluate it in $r = R$, we obtain

$$\sigma_R'(R) = k\,\frac{R^{k-1}}{R^{2k}-1}\Big(\tfrac{\pi}{2}(R^{2k}+1) - 2\sigma_R(1)\Big). \tag{80}$$

On the other hand, differentiating (78), we obtain that

$$\sigma_R'(r) = r^{-1-k}\int_{S_r^+} \nabla v_R \cdot \nu\, \psi_k\, ds \tag{81}$$

and then from (80) and (81)

$$\sigma_R'(R) = R^{-1-k}\int_{S_R^+} \psi_k \frac{\partial v_R}{\partial \nu}\, ds = k\frac{R^{k-1}}{R^{2k}-1}\Big(\tfrac{\pi}{2}(R^{2k}+1) - 2\sigma_R(1)\Big). \tag{82}$$

As well, from the definition of ψ_k (11) we have that

$$\int_{S_R^+} \psi_k \frac{\partial \psi_k}{\partial \nu}\, ds = \frac{\pi}{2}k\, R^{2k}. \tag{83}$$

Combining (82) and (83) we obtain that

$$\kappa_R = \frac{2k\,R^{2k}}{R^{2k}-1}\bigg(\frac{\pi}{2} - \sigma_R(1)\bigg) = \frac{2k\,R^{2k}}{R^{2k}-1}\bigg(\frac{\pi}{2} - \int_0^\pi v_R(\cos t, \sin t)\sin(kt)\, dt\bigg)$$

and hence, via (72),

$$\lim_{R\to+\infty} \kappa_R = 2k\bigg(\frac{\pi}{2} - \int_0^\pi \Phi_k(\cos t, \sin t)\sin(kt)\, dt\bigg).$$

By Lemma C.3, the proof is concluded. □

We are now going to estimate the Rayleigh quotient for λ_N. Let $R \geq 1$. Choosing R_0 as in the previous subsection, for every $\varepsilon \in (0, \varepsilon_0)$ such that $R\varepsilon < R_0$ and for any $j = 1, \dots, N$ we define the function

$$w_{j,R,\varepsilon} = \begin{cases} w_{j,R,\varepsilon}^{int}, & \text{in } D_{R\varepsilon}^+, \\ w_{j,R,\varepsilon}^{ext}, & \text{in } \Omega \setminus D_{R\varepsilon}^+, \end{cases} \tag{84}$$

where, letting u_j^ε be as in (50)–(54),

$$w_{j,R,\varepsilon}^{ext} = u_j^\varepsilon \quad \text{in } \Omega \setminus D_{R\varepsilon}^+,$$

and $w_{j,R,\varepsilon}^{int}$ is the unique solution to

$$\begin{cases} -\Delta w_{j,R,\varepsilon}^{int} = 0, & \text{in } D_{R\varepsilon}^+, \\ w_{j,R,\varepsilon}^{int} = u_j^\varepsilon, & \text{on } S_{R\varepsilon}^+, \\ w_{j,R,\varepsilon}^{int} = 0, & \text{on } \Gamma_{R\varepsilon}. \end{cases} \tag{85}$$

By the Dirichlet principle, we have that $w_{j,R,\varepsilon}^{int}$ is the unique solution to the minimization problem

$$\int_{D_{R\varepsilon}^+} |\nabla w_{j,R,\varepsilon}^{int}|^2 dx = \min \left\{ \int_{D_{R\varepsilon}^+} |\nabla v|^2 dx : v \in H^1(D_{R\varepsilon}^+),\ v = u_j^\varepsilon \text{ on } S_{R\varepsilon}^+,\ v = 0 \text{ on } \Gamma_{R\varepsilon} \right\}. \tag{86}$$

In order to handle the denominator we proceed with a Gram-Schmidt process. We then define

$$\hat{u}_{j,R,\varepsilon} := \frac{\tilde{w}_{j,R,\varepsilon}}{\|\tilde{w}_{j,R,\varepsilon}\|_{L^2(\Omega)}}, \quad j = 1, \dots, N, \tag{87}$$

where $\tilde{w}_{N,R,\varepsilon} := w_{N,R,\varepsilon}$ and

$$\tilde{w}_{j,R,\varepsilon} := w_{j,R,\varepsilon} - \sum_{\ell=j+1}^{N} \frac{\int_\Omega w_{j,R,\varepsilon} \tilde{w}_{\ell,R,\varepsilon}\, dx}{\|\tilde{w}_{\ell,R,\varepsilon}\|_{L^2(\Omega)}^2} \tilde{w}_{\ell,R,\varepsilon} \quad \text{for } j = 1, \dots, N-1.$$

We can derive the following estimate of the energy of eigenfunctions u_j^ε in half-disks of radius of order ε.

Lemma C.10. *For $1 \le j \le N$ and $\varepsilon \in (0, \varepsilon_0)$, let u_j^ε be as in (50)–(54). For every $\delta \in (0, 1/2)$, there exists $\mu_\delta > 1$ such that, for all $R \ge \mu_\delta$, $\varepsilon < \frac{R_0}{R}$, and $1 \le j \le N$,*

$$\int_{S_{R\varepsilon}^+} |u_j^\varepsilon|^2\, ds \le C(R\varepsilon)^{3-2\delta}, \tag{88}$$

$$\int_{D_{R\varepsilon}^+} |\nabla u_j^\varepsilon|^2\, dx \le C(R\varepsilon)^{2-2\delta}, \tag{89}$$

$$\int_{D_{R\varepsilon}^+} |u_j^\varepsilon|^2\, dx \le C(R\varepsilon)^{4-2\delta}, \tag{90}$$

$$\int_{S_{R\varepsilon}^+} |w_{j,R,\varepsilon}^{int}|^2\, ds \le C(R\varepsilon)^{3-2\delta}, \tag{91}$$

$$\int_{D_{R\varepsilon}^+} |\nabla w_{j,R,\varepsilon}^{int}|^2\, dx \le C(R\varepsilon)^{2-2\delta}, \tag{92}$$

$$\int_{D_{R\varepsilon}^+} |w_{j,R,\varepsilon}^{int}|^2\, dx \le C(R\varepsilon)^{4-2\delta}, \tag{93}$$

for some constant $C > 0$ depending only on R_0 and λ_N.

Proof. From (52) and (55) we know that $\{u_j^\varepsilon\}_{\varepsilon \in (0,\varepsilon_0)}$ is bounded in H^1; hence, from of property (ii) at page 26 we deduce that, for ε sufficiently small, $N(u_j^\varepsilon, R_0, \lambda_j(\varepsilon))$ is bounded uniformly with respect to ε. Estimates (88)–(90) then follow from Lemma C.6; we refer to [1, Lemma 5.8] for a detailed proof in a similar problem. Estimates (91)–(93) can be proved combining estimates (88)–(90) with the Dirichlet principle (see [1, Lemma 6.2] for details in a similar problem). □

For $\delta \in (0, 1/2)$ fixed, let μ_δ be as in Lemma C.10. For ε sufficiently small in such a way that $\mu_\delta \varepsilon < R_0$, we introduce the following blow-up functions:

$$\hat{U}_\varepsilon(x) := \frac{u_N^\varepsilon(\varepsilon x)}{\sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)}}, \qquad W_\varepsilon^R(x) := \frac{w_{N,R,\varepsilon}^{int}(\varepsilon x)}{\sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)}}. \tag{94}$$

We notice that, by scaling,

$$\frac{1}{\mu_\delta} \int_{S_{\mu_\delta}^+} |\hat{U}_\varepsilon|^2 \, ds = 1. \tag{95}$$

Theorem C.11. *Let $\delta \in (0, 1/2)$ be fixed and let $r_\delta > 0$ be as in Lemma C.6. For all $R \geq \mu_\delta$,*

$$\text{the family of functions } \left\{\hat{U}_\varepsilon(x) : \ R\varepsilon < r_\delta\right\} \text{ is bounded in } H^1(D_R^+). \tag{96}$$

In particular, for all $R \geq \mu_\delta$,

$$\int_{D_{R\varepsilon}^+} |\nabla u_N^\varepsilon|^2 \, dx = O(H(u_N^\varepsilon, \mu_\delta \varepsilon)), \quad \text{as } \varepsilon \to 0^+, \tag{97}$$

$$\int_{S_{R\varepsilon}^+} |u_N^\varepsilon|^2 ds = O(\varepsilon H(u_N^\varepsilon, \mu_\delta \varepsilon)), \quad \text{as } \varepsilon \to 0^+, \tag{98}$$

$$\int_{D_{R\varepsilon}^+} |u_N^\varepsilon|^2 dx = O(\varepsilon^2 H(u_N^\varepsilon, \mu_\delta \varepsilon)), \quad \text{as } \varepsilon \to 0^+. \tag{99}$$

Proof. We omit the proof which can be derived from the monotonicity result given in Lemma C.6 following the same argument as Lemma C.10; for details in an analogous problem we refer to [1, Theorem 5.9]. □

By the Dirichlet principle and Theorem C.11 we have also the following estimates.

Lemma C.12. *For all $R > \max\{2, \mu_\delta\}$,*

$$\text{the family of functions } \left\{W_\varepsilon^R : R\varepsilon < r_\delta\right\} \text{ is bounded in } H^1(D_R^+). \tag{100}$$

In particular, for all $R > \max\{2, \mu_\delta\}$,

$$\int_{D_{R\varepsilon}^+} \left|\nabla w_{N,R,\varepsilon}^{int}\right|^2 dx = O(H(u_N^\varepsilon, \mu_\delta \varepsilon)), \quad \text{as } \varepsilon \to 0^+, \tag{101}$$

$$\int_{S_{R\varepsilon}^+} |w_{N,R,\varepsilon}^{int}|^2 dx = O(\varepsilon H(u_N^\varepsilon, \mu_\delta \varepsilon)), \quad \text{as } \varepsilon \to 0^+, \tag{102}$$

$$\int_{D_{R\varepsilon}^+} |w_{N,R,\varepsilon}^{int}|^2 dx = O(\varepsilon^2 H(u_N^\varepsilon, \mu_\delta \varepsilon)), \quad \text{as } \varepsilon \to 0^+. \tag{103}$$

We are now in position to prove a sharp upper bound for the eigenvalue variation $\lambda_N - \lambda_N(\varepsilon)$.

Proposition C.13. *There exists $\tilde{R} > 2$ such that, for all $R > \tilde{R}$ and $\varepsilon > 0$ with $R\varepsilon < R_0$,*

$$\frac{\lambda_N - \lambda_N(\varepsilon)}{H(u_N^\varepsilon, \mu_\delta \varepsilon)} \leq g_R(\varepsilon)$$

where

$$g_R(\varepsilon) = \int_{D_R^+} |\nabla W_\varepsilon^R|^2 \, dx - \int_{D_R^+} |\nabla \hat{U}_\varepsilon|^2 \, dx + o(1) \quad \text{and} \quad g_R(\varepsilon) = O(1) \quad \text{as } \varepsilon \to 0^+, \tag{104}$$

with $\hat{U}_\varepsilon$ and W_ε^R defined in (94).

Proof. As already mentioned, we take into account the Courant–Fisher characterization for λ_N recalled in (6) and consider the N-dimensional space spanned by the functions $\{\hat{u}_{j,R,\varepsilon,}\}_{j=1}^N$ defined in (87). Before proceeding, we note that

$$\|\tilde{w}_{N,R,\varepsilon}\|^2_{L^2(\Omega)} = 1 + O\big(\varepsilon^2 H(u_N^\varepsilon, \mu_\delta\varepsilon)\big),$$
$$d_{N,j}^{R,\varepsilon} := \frac{\int_\Omega w_{j,R,\varepsilon}\tilde{w}_{N,R,\varepsilon}\,dx}{\|\tilde{w}_{N,R,\varepsilon}\|^2_{L^2(\Omega)}} = O\Big(\varepsilon^{3-\delta}\sqrt{H(u_N^\varepsilon, \mu_\delta\varepsilon)}\Big), \quad \text{for all } j < N, \tag{105}$$

as $\varepsilon \to 0$, thanks to (99), (103), (90) and (93). On the other hand,

$$\|\tilde{w}_{j,R,\varepsilon}\|^2_{L^2(\Omega)} = 1 + O(\varepsilon^{4-2\delta}),$$
$$d_{\ell,j}^{R,\varepsilon} := \frac{\int_\Omega w_{j,R,\varepsilon}\tilde{w}_{\ell,R,\varepsilon}\,dx}{\|\tilde{w}_{\ell,R,\varepsilon}\|^2_{L^2(\Omega)}} = O(\varepsilon^{4-2\delta}), \quad \text{for all } j \neq \ell, \tag{106}$$

as $\varepsilon \to 0$ and for any $j = 1, \dots, N-1$ thanks to (93) and (90).

If we plug a linear combination of $\{\hat{u}_{j,R,\varepsilon,}\}_{j=1}^N$ into the Rayleigh quotient we obtain that

$$\lambda_N \le \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N \\ \sum_{j=1}^N |\alpha_j|^2=1}} \int_\Omega \left|\nabla\left(\sum_{j=1}^N \alpha_j \hat{u}_{j,R,\varepsilon}\right)\right|^2 dx,$$

and then

$$\lambda_N - \lambda_N(\varepsilon) \le \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{R}^N \\ \sum_{j=1}^N |\alpha_j|^2=1}} \sum_{j,n=1}^N m_{j,n}^{\varepsilon,R}\alpha_j\alpha_n, \tag{107}$$

where

$$m_{j,n}^{\varepsilon,R} = \int_\Omega \nabla\hat{u}_{j,R,\varepsilon}\cdot\nabla\hat{u}_{n,R,\varepsilon}\,dx - \lambda_N(\varepsilon)\delta_{jn},$$

with $\delta_{jn} = 1$ if $j = n$ and $\delta_{jn} = 0$ if $j \neq n$.

From (56) and Lemma C.6, if $R \ge \mu_\delta$ and $R\varepsilon < r_\delta$ we have

$$\frac{1}{H(u_N^\varepsilon, r)}\frac{d}{dr}H(u_N^\varepsilon, r) = \frac{2}{r}\mathcal{N}(u_N^\varepsilon, r, \lambda_N(\varepsilon)) \le \frac{2}{r}(k+\delta) \quad \text{for all } \mu_\delta\varepsilon \le r \le r_\delta, \tag{108}$$

Integration of (108) over the interval $(\mu_\delta\varepsilon, r_\delta)$ and property (ii) at page 26 yield

$$H(u_N^\varepsilon, \mu_\delta\varepsilon) \ge C_\delta\varepsilon^{2k+2\delta}, \quad \text{if } \mu_\delta\varepsilon < r_\delta, \tag{109}$$

for some $C_\delta > 0$ independent of ε. Estimate (88) implies that

$$H(u_N^\varepsilon, \mu_\delta\varepsilon) = O(\varepsilon^{2-2\delta}) \quad \text{as } \varepsilon \to 0. \tag{110}$$

From (105), (94), Theorem C.11, and Lemma C.12 we deduce that

$$m_{N,N}^{\varepsilon,R} = \frac{\lambda_N(\varepsilon)(1 - \|w_{N,R,\varepsilon}\|^2_{L^2(\Omega)})}{\|w_{N,R,\varepsilon}\|^2_{L^2(\Omega)}} + \frac{\Big(\int_{D^+_{R\varepsilon}} \big|\nabla w^{int}_{N,R,\varepsilon}\big|^2 dx - \int_{D^+_{R\varepsilon}} \big|\nabla u_N^\varepsilon\big|^2 dx\Big)}{\|w_{N,R,\varepsilon}\|^2_{L^2(\Omega)}}$$
$$= H(u_N^\varepsilon, \mu_\delta\varepsilon)\Big(\int_{D_R^+} |\nabla W_\varepsilon^R|^2\,dx - \int_{D_R^+} |\nabla\hat{U}_\varepsilon|^2\,dx + o(1)\Big), \tag{111}$$

as $\varepsilon \to 0^+$. On the other hand, if $j < N$, by the convergence of the perturbed eigenvalue, (106), (92), (89) we have that

$$\begin{aligned}
m_{j,j}^{\varepsilon,R} &= -\lambda_N(\varepsilon) + \frac{1}{\|\tilde{w}_{j,R,\varepsilon}\|^2_{L^2(\Omega)}} \left(\lambda_j(\varepsilon) - \int_{D^+_{R\varepsilon}} |\nabla u_j^\varepsilon|^2 \, dx + \int_{D^+_{R\varepsilon}} |\nabla w^{int}_{j,R,\varepsilon}|^2 \, dx \right) \\
&\quad + \frac{1}{\|\tilde{w}_{j,R,\varepsilon}\|^2_{L^2(\Omega)}} \int_\Omega \left| \nabla \Big(\sum_{\ell > j} d_{\ell,j}^{R,\varepsilon} \tilde{w}_{\ell,R,\varepsilon} \Big) \right|^2 dx \\
&\quad - \frac{2}{\|\tilde{w}_{j,R,\varepsilon}\|^2_{L^2(\Omega)}} \left(\int_\Omega \nabla w_{j,R,\varepsilon} \cdot \nabla \Big(\sum_{\ell > j} d_{\ell,j}^{R,\varepsilon} \tilde{w}_{\ell,R,\varepsilon} \Big) \, dx \right) \\
&= (\lambda_j - \lambda_N) + o(1) \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

From (105), (106), (89), (92), (97), and (101), it follows that, for all $j < N$,

$$\begin{aligned}
\|\tilde{w}_{j,R,\varepsilon}\|_{L^2(\Omega)} \|\tilde{w}_{N,R,\varepsilon}\|_{L^2(\Omega)} m_{j,N}^{\varepsilon,R} &= \int_{D^+_{R\varepsilon}} \Big(\nabla w^{int}_{j,R,\varepsilon} \cdot \nabla w^{int}_{N,R,\varepsilon} - \nabla u_j^\varepsilon \cdot \nabla u_N^\varepsilon \Big) \, dx \\
&\quad - \int_\Omega \nabla \Big(\sum_{\ell > j} d_{\ell,j}^{R,\varepsilon} \tilde{w}_{\ell,R,\varepsilon} \Big) \cdot \nabla w_{N,R,\varepsilon} \, dx = O\Big(\varepsilon^{1-\delta} \sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)} \Big).
\end{aligned}$$

Hence, by (105) and (106), we have that

$$m_{j,N}^{\varepsilon,R} = O\Big(\varepsilon^{1-\delta} \sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)} \Big) \quad \text{and} \quad m_{N,j}^{\varepsilon,R} = m_{j,N}^{\varepsilon,R} = O\Big(\varepsilon^{1-\delta} \sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)} \Big)$$

as $\varepsilon \to 0^+$. From (106), (89), (92), we deduce that, for all $j, n < N$ with $j \neq n$,

$$\begin{aligned}
&\|\tilde{w}_{j,R,\varepsilon}\|_{L^2(\Omega)} \|\tilde{w}_{n,R,\varepsilon}\|_{L^2(\Omega)} m_{j,n}^{\varepsilon,R} \\
&\quad = \int_{D^+_{R\varepsilon}} \Big(\nabla w^{int}_{j,R,\varepsilon} \cdot \nabla w^{int}_{n,R,\varepsilon} - \nabla u_j^\varepsilon \cdot \nabla u_n^\varepsilon \Big) \, dx \\
&\quad\quad + \int_\Omega \nabla \big(\sum_{\ell > j} d_{\ell,j}^{R,\varepsilon} \tilde{w}_{\ell,R,\varepsilon} \big) \cdot \nabla \big(\sum_{h > n} d_{h,n}^{R,\varepsilon} \tilde{w}_{h,R,\varepsilon} \big) \, dx \\
&\quad\quad - \int_\Omega \nabla \big(\sum_{\ell > j} d_{\ell,j}^{R,\varepsilon} \tilde{w}_{\ell,R,\varepsilon} \big) \cdot \nabla w_{n,R,\varepsilon} \, dx \\
&\quad\quad - \int_\Omega \nabla w_{j,R,\varepsilon} \cdot \nabla \big(\sum_{h > n} d_{h,n}^{R,\varepsilon} \tilde{w}_{h,R,\varepsilon} \big) \, dx = O(\varepsilon^{2-2\delta}) \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Hence, in view of (106),

$$m_{j,n}^{\varepsilon,R} = O(\varepsilon^{2-2\delta}) \quad \text{as } \varepsilon \to 0.$$

Taking into account (109), we can then apply [1, Lemma 6.1] with $\sigma(\varepsilon) = H(u_N^\varepsilon, \mu_\delta \varepsilon)$, $\mu(\varepsilon) = g_R(\varepsilon)$, $\alpha = 1 - \delta$ and $M = 4k$ in order to deduce

$$\max_{\substack{(\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N \\ \sum_{j=1}^N |\alpha_j|^2 = 1}} \sum_{j,n=1}^N m_{j,n}^{\varepsilon,R} \alpha_j \alpha_n = H(u_N^\varepsilon, \mu_\delta \varepsilon) \Big(\int_{D_R^+} |\nabla W_\varepsilon^R|^2 \, dx - \int_{D_R^+} |\nabla \hat{U}_\varepsilon|^2 \, dx + o(1) \Big)$$

as $\varepsilon \to 0^+$, which, in view of (107), yields $\frac{\lambda_N - \lambda_N(\varepsilon)}{H(u_N^\varepsilon, \mu_\delta \varepsilon)} \leq g_R(\varepsilon)$ with g_R as in (104). We notice that, from Theorem C.11 and Lemma C.12, for all $R > \max\{2, \mu_\delta\}$, $g_R(\varepsilon) = O(1)$ as $\varepsilon \to 0^+$. The proof is now complete. □

Combining Proposition C.8, Lemma C.9 and Proposition C.13 we obtain the following upper/lower estimates for $\lambda_N - \lambda_N(\varepsilon)$.

Proposition C.14. *There exists a positive constant $C^* > 0$ such that*

$$-2\beta^2 \mathfrak{m}_k\, \varepsilon^{2k}(1+o(1)) \le \lambda_N - \lambda_N(\varepsilon) \le C^*\, H(u_N^\varepsilon, \mu_\delta \varepsilon), \quad \text{as } \varepsilon \to 0,$$

with $\mathfrak{m}_k < 0$ as in (44) *and* (42).

C.4 Sharp blow-up analysis and asymptotics

Let us consider the function

$$\begin{aligned} F : \mathbb{R} \times H_0^1(\Omega) &\longrightarrow \mathbb{R} \times H^{-1}(\Omega) \\ (\lambda, \varphi) &\longmapsto \Big(q(\varphi) - \lambda_N,\ -\Delta\varphi - \lambda\varphi\Big), \end{aligned} \tag{112}$$

where q is defined in (5) and $-\Delta\varphi - \lambda\varphi \in H^{-1}(\Omega)$ acts as

$$ {}_{H^{-1}(\Omega)}\Big\langle -\Delta\varphi - \lambda\varphi, u \Big\rangle_{H_0^1(\Omega)} = \int_\Omega \nabla\varphi \cdot \nabla u \, dx - \lambda \int_\Omega \varphi u \, dx$$

for all $\varphi \in H_0^1(\Omega)$. We have that $F(\lambda_N, u_N) = (0,0)$, F is Fréchet-differentiable at (λ_N, u_N) and its Fréchet-differential $dF(\lambda_N, u_N) \in \mathcal{L}\big(\mathbb{R} \times H_0^1(\Omega), \mathbb{R} \times H^{-1}(\Omega)\big)$ is invertible. Therefore we can control $|\lambda_N(\varepsilon) - \lambda_N|$ and $\|w_{N,R,\varepsilon} - u_N\|_{H_0^1(\Omega)}\|$ with $\|F(\lambda_N(\varepsilon) - w_{N,R,\varepsilon}\|_{\mathbb{R}\times H^{-1}(\Omega)}$. Then the norm $\|F(\lambda_N(\varepsilon) - w_{N,R,\varepsilon}\|_{\mathbb{R}\times H^{-1}(\Omega)}$ can be estimated taking advantage of the computations performed in Section C.3, thus yielding

$$\|w_{N,R,\varepsilon} - u_N\|_{H_0^1(\Omega)} = O\Big(\sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)}\Big)$$

as $\varepsilon \to 0^+$ for every $R > 2$, μ_δ being as in Lemma C.10 for some $\delta \in (0, 1/2)$ fixed.

As a consequence, for every $R > 2$

$$\int_{\left(\frac{1}{\varepsilon}\Omega\right) \setminus D_R^+} \left| \nabla \Big(\hat{U}_\varepsilon - \frac{\varepsilon^k}{\sqrt{H(u_N^\varepsilon, \mu_\delta\varepsilon)}} U_\varepsilon \Big) \right|^2 dx = O(1), \quad \text{as } \varepsilon \to 0^+. \tag{113}$$

Using (113) and the uniqueness part of Lemma C.2, we can identify univocally the limit of the blow-up family $\{\hat{U}_\varepsilon\}_\varepsilon$ introduced in (94) and prove that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^k}{\sqrt{H(u_N^\varepsilon, \mu_\delta \varepsilon)}} = \frac{1}{|\beta|} \sqrt{\frac{\mu_\delta}{\int_{S^+_{\mu_\delta}} |\Phi_k|^2 ds}}$$

and

$$\hat{U}_\varepsilon \to \frac{\beta}{|\beta|} \sqrt{\frac{\mu_\delta}{\int_{S^+_{\mu_\delta}} |\Phi_k|^2 ds}}\, \Phi_k \quad \text{as } \varepsilon \to 0^+ \tag{114}$$

in $H^1(D_R^+)$ for every $R > 1$ and in $C^2_{\rm loc}(\overline{\mathbb{R}^2_+} \setminus \{\mathbf{e}, -\mathbf{e}\})$, see [1, Theorem 8.1] for details.

Combining (114) with the Dirichlet principle, we can prove convergence of the blow-up family W_ε^R introduced in (94): for all $R > 2$,

$$W_\varepsilon^R \to \frac{\beta}{|\beta|} \sqrt{\frac{\mu_\delta}{\int_{S^+_{\mu_\delta}} |\Phi_k|^2 ds}}\, w_R \qquad \text{as } \varepsilon \to 0^+ \text{ in } H^1(D_R^+), \tag{115}$$

where w_R is the unique solution to the minimization problem

$$\int_{D_R^+} |\nabla w_R(x)|^2 \, dx = \min \left\{ \int_{D_R^+} |\nabla u|^2 \, dx : \ u \in H^1(D_R^+),\ u = \Phi_k \text{ on } S_R^+,\ u = 0 \text{ on } \Gamma_R \right\},$$

which then solves

$$\begin{cases} -\Delta w_R = 0, & \text{in } D_R^+, \\ w_R = \Phi_k, & \text{on } S_R^+, \\ w_R = 0, & \text{on } \Gamma_R. \end{cases}$$

To obtain the exact asymptotics for $\lambda_N - \lambda_N(\varepsilon)$ it remains to determine the limit of the function $g_R(\varepsilon)$ defined in (104) of Proposition C.13 as $\varepsilon \to 0$ and $R \to +\infty$.

Lemma C.15. *For all $R > \tilde{R}$ and $\varepsilon > 0$ with $R\varepsilon < R_0$, let $g_R(\varepsilon)$ be as in Proposition C.13. Then*

$$\lim_{\varepsilon \to 0^+} g_R(\varepsilon) = \frac{\mu_\delta}{\int_{S^+_{\mu_\delta}} |\Phi_k|^2 ds} \tilde{\kappa}_R \tag{116}$$

where

$$\tilde{\kappa}_R = \int_{S_R^+} \Big(\nabla w_R \cdot \nu - \nabla \Phi_k \cdot \nu \Big) \Phi_k \, ds, \tag{117}$$

with $\nu = \frac{x}{|x|}$. Furthermore $\lim_{R \to +\infty} \tilde{\kappa}_R = -2\mathfrak{m}_k$, where $\mathfrak{m}_k$ is defined in (42) *and* (44)*.*

Proof. We first observe that, by (104) and convergences (114)–(115),

$$\lim_{\varepsilon \to 0^+} g_R(\varepsilon) = \frac{\mu_\delta}{\int_{S^+_{\mu_\delta}} |\Phi_k|^2 ds} \bigg(\int_{D_R^+} |\nabla w_R|^2 \, dx - \int_{D_R^+} |\nabla \Phi_k|^2 \, dx \bigg) = \frac{\mu_\delta}{\int_{S^+_{\mu_\delta}} |\Phi_k|^2 ds} \tilde{\kappa}_R$$

with $\tilde{\kappa}_R$ as in (117). We observe that

$$\tilde{\kappa}_R = \int_{S_R^+} \Big(\nabla w_R \cdot \nu - \nabla \Phi_k \cdot \nu \Big) \psi_k \, ds + I_1(R) + I_2(R) \tag{118}$$

where

$$I_1(R) = \int_{S_R^+} \Big(\Phi_k - \psi_k \Big) \nabla \Big(\psi_k - \Phi_k \Big) \cdot \nu \, ds, \quad I_2(R) = \int_{S_R^+} \Big(\Phi_k - \psi_k \Big) \nabla \Big(w_R - \psi_k \Big) \cdot \nu \, ds.$$

Testing the equation $-\Delta\big(\psi_k - \Phi_k\big) = 0$ with the function $\psi_k - \Phi_k$, recalling that $\psi_k - \Phi_k = 0$ on s, and integrating it over $\mathbb{R}^2_+ \setminus D_R^+$, thanks to Lemma C.2 we obtain that

$$I_1(R) = \int_{\mathbb{R}^2_+ \setminus D_R^+} |\nabla(\Phi_k - \psi_k)|^2 = \int_{\mathbb{R}^2_+ \setminus D_R^+} |\nabla w_k|^2 \to 0 \qquad \text{as } R \to +\infty.$$

Let $\eta_R : \mathbb{R}^2_+ \to \mathbb{R}$ be a smooth cut-off function such that $\eta_R \equiv 0$ in $D^+_{R/2}$, $\eta_R \equiv 1$ in $\mathbb{R}^2_+ \setminus D_R^+$, $0 \le \eta_R \le 1$, and $|\nabla \eta_R| \le \frac{4}{R}$. Testing the equation $-\Delta(\psi_k - w_R) = 0$ in D_R^+ with the function $\eta_R(\Phi_k - \psi_k)$ we obtain

$$I_2(R) = \int_{D_R^+} \nabla(w_R - \psi_k) \cdot \nabla((\Phi_k - \psi_k)\eta_R)$$

so that, in view of the Dirichlet Principle, Lemma C.2 and the fact that $w_k \in \mathcal{Q}$,

$$\begin{aligned} |I_2(R)| &\le \int_{D_R^+} |\nabla((\Phi_k - \psi_k)\eta_R)|^2 dx \le 2 \int_{D_R^+ \setminus D^+_{R/2}} |\nabla(\Phi_k - \psi_k)|^2 \, dx + \frac{32}{R^2} \int_{D_R^+ \setminus D^+_{R/2}} |\Phi_k - \psi_k|^2 \, dx \\ &\le 2 \int_{D_R^+ \setminus D^+_{R/2}} |\nabla w_k|^2 \, dx + 128 \int_{D_R^+ \setminus D^+_{R/2}} \frac{w_k^2}{|x - \mathbf{e}|^2} \, dx \to 0 \end{aligned}$$

as $R \to +\infty$, where in the last line of the above estimate we have used that $\frac{1}{R} \le \frac{2}{|x - \mathbf{e}|}$ for all $x \in D_R^+$.

Therefore we need just to study the limit of the quantity $\int_{S_R^+} \frac{\partial}{\partial\nu}(w_R - \Phi_k)\psi_k$ as $R \to +\infty$. To this aim, we consider the function

$$\xi(r) := \int_0^\pi \Phi_k(r\cos t, r\sin t)\sin(kt)\,dt, \qquad r \geq 1, \tag{119}$$

and notice that it satisfies the differential equation $\xi'' + \frac{1}{r}\xi' - \frac{k^2}{r^2}\xi = 0$ which can be rewritten as $(r^{1+2k}(r^{-k}\xi)')' = 0$ in $[1, +\infty)$. Therefore there exists some $C_\xi \in \mathbb{R}$ such that

$$\big(r^{-k}\xi(r)\big)' = \frac{C_\xi}{r^{1+2k}} \quad \text{in } [1, +\infty).$$

Integrating the previous equation over $[1, r]$ we obtain that

$$r^{-k}\xi(r) - \xi(1) = \frac{C_\xi}{2k}\left(1 - \frac{1}{r^{2k}}\right). \tag{120}$$

From (11), Lemma C.1, and Lemma C.2 it follows that

$$\begin{aligned}\xi(r) &= \int_0^\pi \psi_k(r\cos t, r\sin t)\sin(kt)\,dt + \int_0^\pi \Big(\Phi_k(r\cos t, r\sin t) - \psi_k(r\cos t, r\sin t)\Big)\sin(kt)\,dt \\ &= \frac{\pi}{2}r^k + O(r^{-1}), \quad \text{as } r \to +\infty,\end{aligned}$$

and hence $r^{-k}\xi(r) \to \frac{\pi}{2}$ as $r \to +\infty$. Letting $r \to +\infty$ in (120), this implies that $\frac{C_\xi}{2k} = \frac{\pi}{2} - \xi(1)$, so that

$$\xi(r) = \tfrac{\pi}{2}\,r^k + \big(\xi(1) - \tfrac{\pi}{2}\big)r^{-k}, \quad \xi'(r) = k\tfrac{\pi}{2}r^{k-1} + k\big(\tfrac{\pi}{2} - \xi(1)\big)r^{-k-1} \tag{121}$$

for all $r \geq 1$. In particular, from (121) we have that

$$\tfrac{\pi}{2} - \xi(1) = \tfrac{\pi}{2}r^{2k} - r^k\xi(r), \quad \text{for all } r \geq 1,$$

whose substitution into (121) yields

$$\xi'(r) = k\pi r^{k-1} - k\,\frac{\xi(r)}{r}, \quad \text{for all } r \geq 1.$$

On the other hand, differentiating (119) we obtain also

$$\xi'(r) = \frac{1}{r^{1+k}}\int_{S_r^+} \frac{\partial \Phi_k}{\partial \nu}\,\psi_k\,ds \tag{122}$$

so that

$$\int_{S_r^+} \frac{\partial \Phi_k}{\partial \nu}\,\psi_k\,ds = k(\pi r^{2k} - r^k\xi(r)) \quad \text{for all } r \geq 1. \tag{123}$$

Now we turn to

$$\zeta_R(r) := \int_0^\pi w_R(r\cos t, r\sin t)\,\sin(kt)\,dt$$

which is the k-th Fourier coefficient of the harmonic function w_R and hence satisfies, for some $C_R \in \mathbb{R}$, $\big(r^{-k}\zeta_R(r)\big)' = \frac{C_R}{r^{1+2k}}$ in $(0, R]$. Integrating the previous equation over $[r, R]$ we obtain that

$$R^{-k}\zeta_R(R) - r^{-k}\zeta_R(r) = \frac{C_R}{2k}\left(\frac{1}{r^{2k}} - \frac{1}{R^{2k}}\right), \quad \text{for all } r \in (0, R).$$

By regularity of w_R we necessarily have that $C_R = 0$. Hence

$$\zeta_R(r) = \tfrac{\zeta_R(R)}{R^k}r^k \quad \text{and} \quad \zeta_R'(r) = k\tfrac{\zeta_R(R)}{R^k}r^{k-1}, \quad \text{for all } r \in (0, R]. \tag{124}$$

From the definition of ζ_R we have that $\zeta_R'(r) = \frac{1}{r^{1+k}} \int_{S_r^+} \nabla w_R \cdot \nu \, \psi_k \, ds$. Hence

$$\int_{S_r^+} \nabla w_R \cdot \nu \, \psi_k \, ds = k \tfrac{\zeta_R(R)}{R^k} \, r^{2k}$$

from which, taking into account the boundary conditions for w_R, it follows that

$$\int_{S_R^+} \nabla w_R \cdot \nu \, \psi_k \, ds = kR^k \xi(R). \tag{125}$$

Combining (123), (125), and (121) we conclude that

$$\int_{S_R^+} \left(\frac{\partial w_R}{\partial \nu} \psi_k - \frac{\partial \Phi}{\partial \nu} \psi_k \right) ds = 2kR^k \xi(R) - k\pi R^{2k} = 2k \left(\xi(1) - \frac{\pi}{2} \right) = -2\mathfrak{m}_k$$

by virtue of Lemma C.3. □

By combining the previous results we obtain the following asymptotics for the eigenvalue variation.

Theorem C.16. *Let Ω be a bounded open set in $\mathbb{R}^2$ satisfying* (1) *and* (2)*. Let $N \geq 1$ be such that the N-th eigenvalue λ_N of q_0 on Ω is simple with associated eigenfunctions having in* 0 *a zero of order k with k as in* (10)*. For $\varepsilon \in (0, \varepsilon_0)$ let $\lambda_N(\varepsilon)$ be the N-th eigenvalue of q_ε on Ω. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\lambda_N - \lambda_N(\varepsilon)}{\varepsilon^{2k}} = -2\beta^2 \, \mathfrak{m}_k$$

with β being as in (10) *and $\mathfrak{m}_k$ being as in* (42) *and* (44)*.*

In particular, Theorem C.16 and (114) above provide a proof of Theorem 3.1 that is alternative to the one given in [12].

Acknowledgements The authors thank the anonymous reviewers for their suggestions and comments. They are also indebted to André Froehly for bringing relevant references to their attention. L. Abatangelo, V. Felli and C. Léna are partially supported by the project ERC Advanced Grant 2013 n. 339958: "Complex Patterns for Strongly Interacting Dynamical Systems – COMPAT". L. Abatangelo and V. Felli are partially supported by the INDAM-GNAMPA 2018 grant "Formula di monotonia e applicazioni: problemi frazionari e stabilità spettrale rispetto a perturbazioni del dominio". V. Felli is partially supported by the PRIN 2015 grant "Variational methods, with applications to problems in mathematical physics and geometry". C. Léna is partially supported by the Portuguese FCT (Project OPTFORMA, IF/00177/2013) and the Swedish Research Council (Grant D0497301).

References

[1] L. Abatangelo and V. Felli. Sharp asymptotic estimates for eigenvalues of Aharonov-Bohm operators with varying poles. *Calc. Var. Partial Differential Equations*, 54(4):3857–3903, 2015.

[2] L. Abatangelo and V. Felli. On the leading term of the eigenvalue variation for Aharonov-Bohm operators with a moving pole. *SIAM J. Math. Anal.*, 48(4):2843–2868, 2016.

[3] L. Abatangelo, V. Felli, L. Hillairet, and C. Lena. Spectral stability under removal of small capacity sets and applications to Aharonov-Bohm operators. *J. Spectr. Theory*, online first 2018, doi: 10.4171/JST/251.

[4] L. Abatangelo, V. Felli, and C. Léna. On Aharonov-Bohm operators with two colliding poles. *Adv. Nonlinear Stud.*, 17(2):283–296, 2017.

[5] L. Abatangelo, V. Felli, B. Noris, and M. Nys. Sharp boundary behavior of eigenvalues for Aharonov-Bohm operators with varying poles. *J. Funct. Anal.*, 273(7):2428-2487, 2017.

[6] V. Bonnaillie–Noël, B. Helffer, and T. Hoffmann-Ostenhof. Aharonov–Bohm Hamiltonians, isospectrality and minimal partitions. *J. Phys. A*, 42, 185203, 2009.

[7] V. Bonnaillie-Noël, B. Noris, M. Nys, S. Terracini. Aharonov-Bohm operators with varying poles. *Anal. PDE*, 7 (2014), 1365–1395, doi:10.2140/apde.2014.7.1365.

[8] E. Colorado and I. Peral. Semilinear elliptic problems with mixed Dirichlet–Neumann boundary conditions. *Journal of Functional Analysis*, 199(2):468–507, 2003.

[9] V. Felli and A. Ferrero. Almgren-type monotonicity methods for the classification of behaviour at corners of solutions to semilinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 143(5):957–1019, 2013.

[10] M. M. Fall, V. Felli, A. Ferrero, and A. Niang. Asymptotic expansions and unique continuation at Dirichlet-Neumann boundary junctions for planar elliptic equations. *Mathematics in Engineering*, 1(1):84–117, 2018.

[11] V. Felli, A. Ferrero, and S. Terracini. Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. *J. Eur. Math. Soc. (JEMS)*, 13(1):119–174, 2011.

[12] R.R. Gadyl'shin. Splitting of a multiple eigenvalue of the Dirichlet problem for the Laplace operator under singular perturbation of the boundary condition. (Russian) *Mat. Zametki*, 52(4): 42–55, 1992; translation in *Math. Notes* 52(3–4): 1020–1029 (1993).

[13] M. Kassmann and W. R. Madych. Difference quotients and elliptic mixed boundary value problems of second order. *Indiana Univ. Math. J.*, 56(3):1047–1082, 2007.

[14] A. Laptev and T. Weidl. Hardy inequalities for magnetic Dirichlet forms. In *Mathematical results in quantum mechanics (Prague, 1998)*, volume 108 of *Oper. Theory Adv. Appl.*, pages 299–305. Birkhäuser, Basel, 1999.

[15] C. Léna. Eigenvalues variations for Aharonov-Bohm operators. *J. Math. Phys.*, 56(1):011502, 18, 2015.

[16] B. Noris and S. Terracini. Nodal sets of magnetic Schrödinger operators of Aharonov-Bohm type and energy minimizing partitions. *Indiana Univ. Math. J.*, 59(4):1361–1403, 2010.

[17] B. Noris, M. Nys, and S. Terracini. On the Aharonov–Bohm operators with varying poles: the boundary behavior of eigenvalues. *Communications in Mathematical Physics*, 339(3):1101–1146, 2015.

[18] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I. Functional Analysis.* Academic Press, New York, second edition, 1980.

[19] J. D. Sally and P. J. Sally. *Roots to research.* American Mathematical Society, Providence, RI, 2007.

[20] G. Savaré. Regularity and perturbation results for mixed second order elliptic problems. *Comm. Partial Differential Equations*, 22(5-6):869–899, 1997.

[21] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana.* Springer, Berlin; UMI, Bologna, 2007.

[22] G.N. Watson. A treatise on the theory of the Bessel functions. *Cambridge at the University press*, 1944.