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Elasticity solution for a hollow cylinder under axial end loads: Application to a blister of a stayed bridge

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Starting from an applied problem related to the modeling of an element of a cable-stayed bridge, we compute the elasticity solution for a hollow circular cylinder under axial end loads. We prove results of symmetry for the solution and we expand it in proper Fourier series; computing the Fourier coefficients in adapted power series, we provide the explicit solution. We consider an engineering case of study, applying the corresponding approximate formula and giving some estimates on the error committed with respect to the truncation of the series.

1 | INTRODUCTION

We consider the problem of the stress distribution in a hollow circular cylinder subjected to axial loads on its opposite faces. This theoretical problem has relevance in many engineering applications. Here the applied focus is suggested by the structural civil engineering Studio De Miranda Associati, a company expertized in building long span bridges. We refer to [1–6] for some mathematical models for suspension bridges and plates developed recently.

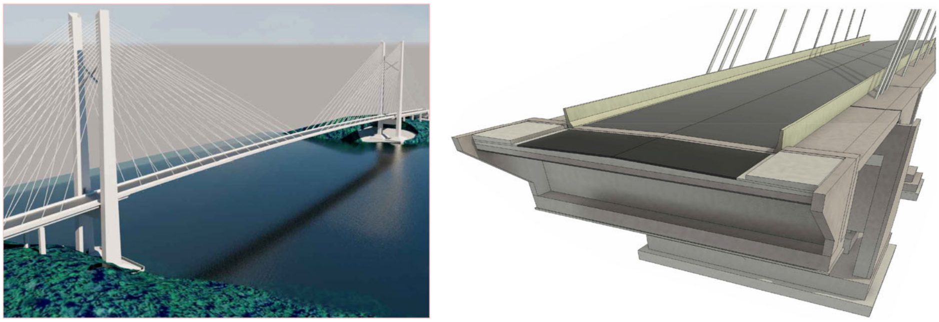


FIGURE 1 From the left a render of a recent cable stayed bridge designed by Studio De Miranda Associati and a detail of its deck.

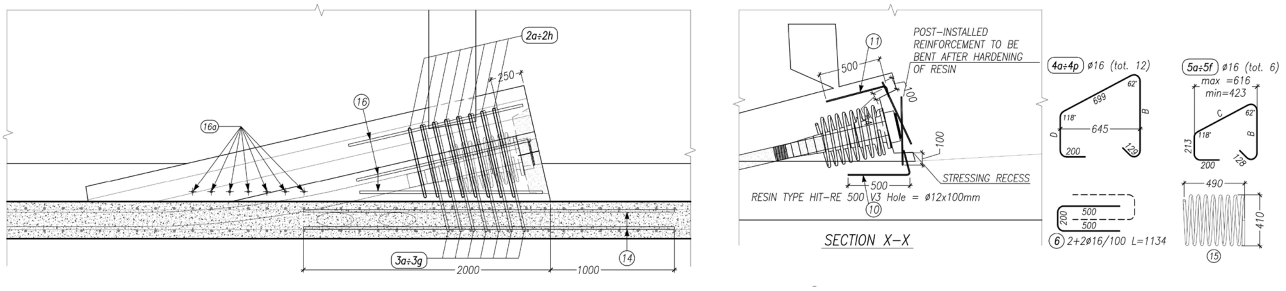


FIGURE 2 Detail of the executive draw of a blister of a recent prestressed concrete bridge.

The aim of this paper is to model the stress diffusion in a constructive detail of a bridge: the blister. In the cable-stayed bridge the blister is the structural element where the steel forestay anchors to the deck. In Figure 1 is shown a render of a future cable-stayed bridge, designed by Studio De Miranda Associati, that will be built in Brazil; a detail of the related blister element is given in Figure 3.

When the deck is built in reinforced concrete, as in this case, high density of the steel reinforcement is present in the blister; this may cause zone with low concrete capacity and possible remarkable cracking. In Figure 2 we show a detail of the executive draw of a blister for another stayed bridge, designed by Studio De Miranda Associati.

Hence, a precise estimate of the stresses acting on the element is fundamental to compute the reinforcing steel without surplus. In engineering literature some of the best known references related to the distribution of the stresses in prisms of concrete are [7, 8]; here the authors consider many combinations of load on the prism and for each one the possible strategies to design the steel bars. These results are obtained from particular solutions of the well known equation of the linear elasticity; we recall it here briefly in the general 3D case, see for example [9–15].

Given $\Omega \subset \mathbb{R}^3$ an elastic homogeneous solid body, we denote by $\mathbf{u} : \Omega \to \mathbb{R}^3$ the displacement vector at any point of the reference configuration of the elastic body itself, see the list of notations at the end of the paper. We denote by $\mathbf{Tu}$ the stress tensor and by λ and μ the classical Lamé constants; it is known that λ and μ may be expressed in terms of the Young modulus E and Poisson ratio $\nu \in \left(-1, \frac{1}{2}\right)$ as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{1.1}$$

The equation of linear elasticity reads

$$\begin{cases} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u}) = \mathbf{f} & \text{in } \Omega, \\ (\mathbf{Tu})\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathbf{f}$ and $\mathbf{g}$ are respectively the forces per unit volume and the boundary forces per unit surface acting on Ω, while $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

In Section 2 we briefly derive (1.2) from variational principles and we recall the existence and uniqueness results in Proposition 2.1; although these are classical topics in linear elasticity, see for example [10, 12], being some questions not trivial and useful for our aims, we provide for completeness some details and the proof in the appendix.

The theoretical solution from which come the applicative cases considered in [7, 8] is given in [16], where Ω is a rectangular prism under end loads. Thanks to this simple geometry and loading condition the authors find explicitly the solution in form of double Fourier series. To find the explicit solution of (1.2) for generic Ω and loading conditions is a very hard task; some results are available simplifying the geometry or the load, see for example [16–18]. For cylindrical domains the solution can be obtained only for particular cases, for instance applying Love representation of the solution [13] and finding a biharmonic function on a cylindrical domain, see for example [14, 19].

In this paper we find the elasticity solution for Ω coincident with a hollow cylinder loaded on the opposite faces, since this geometry fits the modeling of the concrete of the blister, see Figure 3 on the right; indeed, the forestay of the bridge is circular and passes through the cylindrical hole, applying a distributed load on the opposite faces due to its tensioning, see Figure 4 on the right.

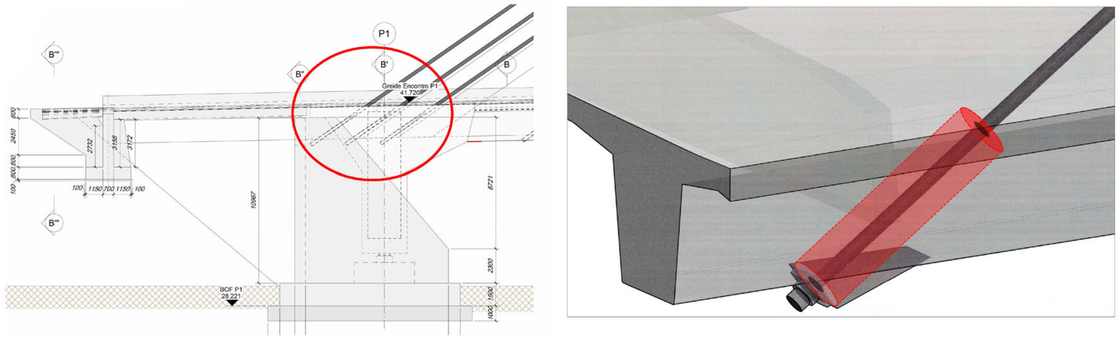


FIGURE 3 From the left a frontal view of the blister and its modelization through the hollow cylinder.

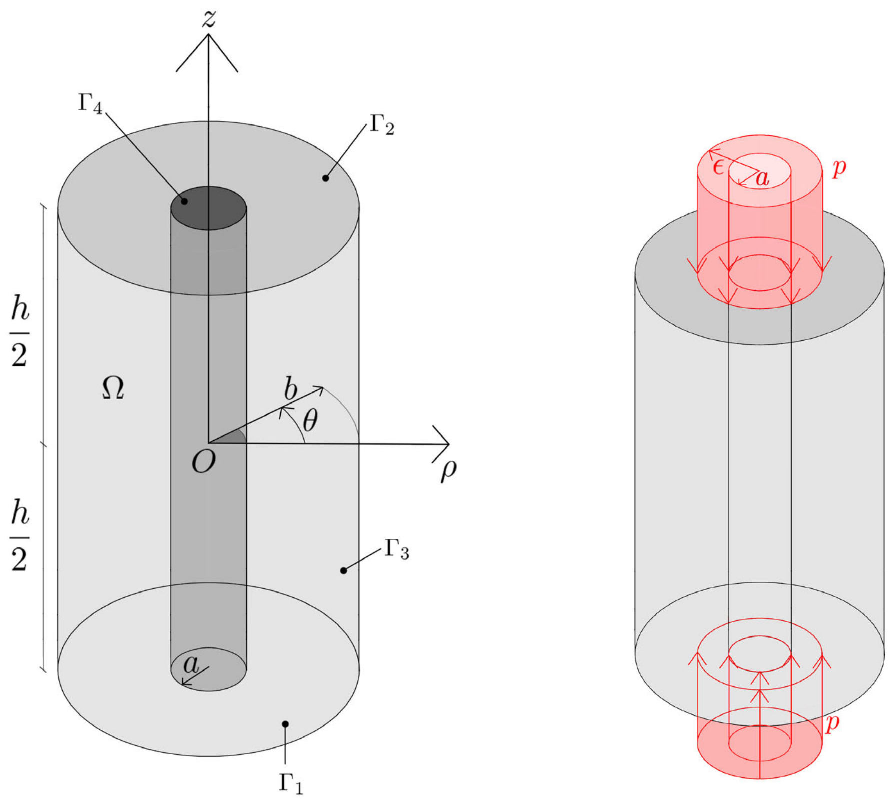


FIGURE 4 The domain Ω on the left and the loading condition in red on the right.

The precise definition of the model is given in Section 4; the application of axial loads leads to a solution having axial symmetric properties, see Proposition 3.1. In the real blister it is also possible to have non radial loadings coming from the deck, but this is a first attempt of modeling that may be implemented in future works; anyway, the solution found here may have general interest beyond this specific application and the technique is not based on the Love representation function.

The definition of the solution is given by steps: in subsection 3.1 we provide a periodic extension of the loads in the variable z corresponding to the symmetry axis of the hollow cylinder, in such a way that it becomes possible to expand the solution in Fourier series with respect to the variable z; then we compute the Fourier coefficients which come to be functions in the other two variables x and y, corresponding to directions orthogonal to the symmetry axis of the hollow cylinder; in subsection 3.2 we pass to the cylindrical coordinates and, exploiting the axial symmetry, we reduce ourself to study a system of ODEs in the radial polar coordinate ρ; we compute the Fourier coefficients as functions of the variable ρ through an adapted expansion in power series so that we are able to state Theorem 3.7, collecting the explicit solution.

In Section 4 we give some hints to truncate the series and we apply the results to an engineering case of study. As it will be explained in details, it will be necessary to compute numerically the first M terms in the Fourier series expansion with

M to be chosen sufficiently large in order to minimize the truncation error. The main question in this procedure is that the computation of those Fourier coefficients, which are solutions of suitable boundary value problems of ODEs, requires the numerical resolution of some algebraic linear systems in four variables which exhibit a condition number higher and higher as M grows; if we need a truncation error smaller than ours, we may consider alternative numerical procedures. We emphasize that the main purpose of this article is to obtain an analytical representation of the unique symmetric solution of (1.2) in the case of the hollow cylinder with the perspective of reproducing such method in more general situations with not necessarily symmetric external loads.

The main analytical and numerical results of the article are stated in Sections 2–4 and their proofs are given in Section 5. The final part of the paper is devoted to the conclusions, see Section 6, the appendix on the details related to existence and uniqueness of solutions, see Section A, and a list of notations which can be helpful for the reader.

2 | THE EQUATION OF THE LINEAR ELASTICITY: EXISTENCE OF A SOLUTION

In this Section we derive the differential equations for the linear elasticity from variational arguments and we state existence results for the solutions. These topics are well known, see for example [9–15], anyway we decided to review them from a mathematical point, basing our approach on the Fredholm alternative. Since many arguments are standard, we recall here only the main results useful to deal with the hollow cylinder problem and we put all the details and the proof in the appendix, see Section A.

We recall that $\Omega \subset \mathbb{R}^3$ is the domain of the elastic body and $\mathbf{u}$ is the displacement function with components $\mathbf{u} = (u_1, u_2, u_3)$. We denote by $\mathbf{Du}$ the linearized strain tensor, which in the sequel will be simply called strain tensor, and by $\mathbf{Tu}$ the stress tensor, recalling its precise definition in the notations at the end of the paper.

The Hooke's Law for isotropic materials reads

$$\mathbf{Tu} = \lambda \mathrm{tr}(\mathbf{Du}) I + 2\mu \mathbf{Du}\,, \tag{2.1}$$

where λ and μ are the Lamé constants. If we assume that on Ω act body forces per unit of volume $\mathbf{f} = (f_1, f_2, f_3)$ and boundary forces per unit of surface $\mathbf{g} = (g_1, g_2, g_3)$ we obtain the total energy of the system

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\int_\Omega \mathbf{Tu} : \mathbf{Du}\, d\mathbf{x} - \int_\Omega \mathbf{f}\cdot\mathbf{u}\, d\mathbf{x} - \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{u}\, dS\,, \tag{2.2}$$

where the first term is the elastic energy related to the internal forces in the configuration corresponding to a generic displacement $\mathbf{u}$.

By looking at the total energy $\mathcal{E}$ in (2.2) as a functional $\mathcal{E} : H^1(\Omega;\mathbb{R}^3) \to \mathbb{R}$ and exploiting the symmetry of the bilinear form, see (A2) in the appendix, we obtain that a critical point $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ of $\mathcal{E}$ solves the variational problem

$$\int_\Omega \mathbf{Tu} : \mathbf{Dv}\, d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\, dS \qquad \text{for any } \mathbf{v} \in H^1(\Omega;\mathbb{R}^3)\,. \tag{2.3}$$

By (A1) in the appendix and a formal integration by parts, we find that (2.3) is the weak formulation of the boundary value problem

$$\begin{cases} -\mathrm{div}(\mathbf{Tu}) = \mathbf{f} & \text{in } \Omega, \\ (\mathbf{Tu})\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Inserting (2.1) into (2.4) we find the well known equations of linear elasticity (1.2).

Let us introduce the space

$$V_0 := \left\{ \mathbf{v}_0 \in H^1(\Omega;\mathbb{R}^3) : \int_\Omega \mathbf{Tv}_0 : \mathbf{Dv}\, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in H^1(\Omega;\mathbb{R}^3) \right\}. \tag{2.5}$$

We observe that V_0 coincides with the eigenspace associated to the first eigenvalue of the following eigenvalue problem: α is an eigenvalue if there exists a nontrivial function $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$, which will be called eigenfunction associated to α, such that

$$\int_\Omega \mathbf{Tu} : \mathbf{Dv}\, d\mathbf{x} = \alpha \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} \qquad \text{for any } \mathbf{v} \in H^1(\Omega;\mathbb{R}^3).$$

In particular if $\alpha = 0$ and $\mathbf{v}_0$ is a corresponding eigenfunction we have

$$\int_\Omega \mathbf{Tv}_0 : \mathbf{Dv}\, d\mathbf{x} = 0 \qquad \text{for any } \mathbf{v} \in H^1(\Omega;\mathbb{R}^3)\,. \tag{2.6}$$

After some computation one can verify that V_0 is the space of functions $\mathbf{v} = (v_1, v_2, v_3)$ admitting the following representation

$$\begin{cases} v_1(x_1,x_2,x_3) = \alpha x_2 + \beta x_3 + \delta_1\,, \\ v_2(x_1,x_2,x_3) = -\alpha x_1 + \gamma x_3 + \delta_2\,, \\ v_3(x_1,x_2,x_3) = -\beta x_1 - \gamma x_2 + \delta_3\,. \end{cases} \tag{2.7}$$

where $\alpha,\beta,\gamma,\delta_1,\delta_2,\delta_3 \in \mathbb{R}$. Roughly speaking, configurations associated with such functions $\mathbf{v} \in V_0$ correspond to translations and rotations of the solid body without deforming it in such a way the elastic energy equals zero.

In the next proposition we recall the existence result for problem (1.2).

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ a bounded domain with Lipschitzian boundary and let $\mathbf{f} \in L^2(\Omega;\mathbb{R}^3)$ and $\mathbf{g} \in L^2(\partial\Omega;\mathbb{R}^3)$. Let us introduce the following compatibility condition*

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\, dS = 0 \qquad \text{for any } \mathbf{v} \in V_0\,. \tag{2.8}$$

Then the following statements hold true:

- *(i) problem (2.3) admits at least one solution $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$, or equivalently the boundary value problem (1.2) admits at least one weak solution, if and only if (2.8) holds true;*
- *(ii) if $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ is a particular solution of (2.3) then for any $\mathbf{v}_0 \in V_0$ the function $\mathbf{u} + \mathbf{v}_0$ is still a solution of (2.3);*
- *(iii) if $\mathbf{u} \in H^1(\Omega;\mathbb{R}^3)$ is a particular solution of (2.3) and if $\mathbf{w} \in H^1(\Omega;\mathbb{R}^3)$ is any other solution of (2.3), then there exists $\mathbf{v}_0 \in V_0$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}_0$;*
- *(iv) letting $V_0^\perp$ be the space orthogonal to V_0 with respect to the scalar product (A4), we have that problem (2.3) admits a unique solution in $V_0^\perp$.*

The proof of Proposition 2.1 is given in subsection A.1 in the appendix. In the next remarks we recall some issues related to the uniqueness of solution for the elasticity equation.

Remark 2.2. The problem (2.3) admits solutions up to translations and rotations. This is why we introduce in Section 2 the space V_0; such space includes functions as (2.6) that does not modify the elastic energy. Assuming for simplicity $\delta_1 = \delta_2 = \delta_3 = 0$ in (2.6), deformations corresponding to displacements $\mathbf{v} \in V_0$ can be considered good approximations of a rotation only for α,β,γ small; when at least one of the constants α,β,γ is not small the corresponding deformation of the solid body is no more negligible. In such a case, one may wonder why the elastic energy remains anyway zero; the answer is that in the linear theory only small deformations are allowed so that large deformations are no more meaningful for our model. Let us recall that we are considering in our model the linearized strain tensor $\mathbf{Dv}$ which is a good approximation of the real strain tensor only for small deformations since the last one also contains quadratic terms in the first order derivatives of v_1, v_2, v_3; these quadratic terms can be neglected when first order derivatives are small.

Remark 2.3. We observe that, as a consequence of Proposition 2.1, if $\mathbf{u}$ and $\mathbf{w}$ are solutions of (2.3), then the two configurations of the elastic body, corresponding to $\mathbf{u}$ and $\mathbf{w}$, generate the same stress state. More precisely we have $\mathbf{Tu} = \mathbf{Tw}$ in Ω as a consequence of the Hooke's law and of the fact that $\mathbf{D}(\mathbf{u} - \mathbf{w})$ vanishes in Ω being $\mathbf{u} - \mathbf{w} \in V_0$. Physically, this is completely reasonable since, given the configuration corresponding to $\mathbf{u}$, the one corresponding to $\mathbf{w}$ can be obtained from the first one by means of rotations and translations of the elastic body, which clearly do not affect the stress state of the solid body itself.

3 | THE HOLLOW CYLINDER UNDER AXIAL END LOADS

We consider a circular, finite, homogeneous, isotropic and elastic cylinder with height h, radius $b > 0$, having a coaxial hole of radius $a > 0$. In this section we use the usual notation x, y, z for the three coordinates in $\mathbb{R}^3$. We maintain the notation $d\mathbf{x}$ to denote the differential volume $dxdydz$.

Therefore, we introduce the annular domain $C_{a,b} := \{(x, y) \in \mathbb{R}^2 : a^2 < x^2 + y^2 < b^2\}$ in such a way that

$$\Omega = C_{a,b} \times \left(-\frac{h}{2}, \frac{h}{2}\right).$$

In the sequel we want to model a hollow cylinder subject to an external load acting on the upper and lower faces of the cylinder compressing the cylinder itself; we denote this (constant) distributed load by p, see Figure 4 on the right. This loading configuration corresponds to zero volume forces, that is, recalling the notations introduced in (1.2), $\mathbf{f} = \mathbf{0}$ in Ω.

In order to better describe the surface forces represented by the vector function $\mathbf{g}$, we split $\partial\Omega$ in four regular parts

$$\Gamma_1 := C_{a,b} \times \left\{-\frac{h}{2}\right\}, \qquad \Gamma_2 := C_{a,b} \times \left\{\frac{h}{2}\right\},$$

$$\Gamma_3 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = b^2\} \times \left(-\frac{h}{2}, \frac{h}{2}\right), \qquad \Gamma_4 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = a^2\} \times \left(-\frac{h}{2}, \frac{h}{2}\right),$$

having respectively outward unit normal vectors $(0, 0, -1)$, $(0,0,1)$, $(x/b, y/b, 0)$ when $(x, y, z) \in \Gamma_3$ and $(-x/a, -y/a, 0)$ when $(x, y, z) \in \Gamma_4$. In this way, the outward unit normal vector $\mathbf{n}$ is well defined on the whole $\partial\Omega$, see also Figure 4 on the left.

Exploiting the above notations, the vector function $\mathbf{g}$ can be represented in the following way

$$\mathbf{g}(x, y, z) = \begin{cases} (0, 0, \chi_p(x, y)) & \text{for any } (x, y, z) \in \Gamma_1, \\ (0, 0, -\chi_p(x, y)) & \text{for any } (x, y, z) \in \Gamma_2, \\ (0, 0, 0) & \text{for any } (x, y, z) \in \Gamma_3 \cup \Gamma_4, \end{cases} \tag{3.1}$$

where the function $\chi_p : C_{a,b} \to \mathbb{R}$, $p \in \mathbb{R}_+$, is defined by

$$\chi_p(x, y) := \begin{cases} p & \text{if } a^2 \le x^2 + y^2 < \epsilon^2, \\ 0 & \text{if } \epsilon^2 < x^2 + y^2 \le b^2, \end{cases} \tag{3.2}$$

for some $\epsilon \in (a, b)$. The parameter ϵ may be varied until that the distributed load covers entirely the top and bottom faces of the cylinder ($\epsilon = b$); since this study starts from the applied purpose to model a blister, we choose $\epsilon \in (a, b)$, being $\{(x, y) \in C_{a,b} : a^2 \le x^2 + y^2 < \epsilon^2\} \subset C_{a,b}$ the surface where the load of the forestay is transferred to the concrete through anchor plates, see Figure 5 and Section 4 for details.

Therefore, we are led to consider the problem

$$\begin{cases} -\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\mathrm{div}\mathbf{u}) = \mathbf{0} & \text{in } \Omega\,, \\ (\mathbf{Tu})\mathbf{n} = (0, 0, \chi_p) & \text{on } \Gamma_1, \\ (\mathbf{Tu})\mathbf{n} = (0, 0, -\chi_p) & \text{on } \Gamma_2, \\ (\mathbf{Tu})\mathbf{n} = \mathbf{0} & \text{on } \Gamma_3 \cup \Gamma_4\,. \end{cases} \tag{3.3}$$

Among all solutions of (3.3) which can be obtained by a single solution by adding to it a function in the space V_0, we focus our attention on the unique solution $\mathbf{u} = (u_1, u_2, u_3)$ of (3.3) in the space $V_0^\perp$ where orthogonality is meant in the sense of the scalar product defined in (A4), see Proposition 2.1. From a geometric point of view, condition $\mathbf{u} \in V_0^\perp$ avoids translations and rotations of the hollow cylinder, being V_0 the space of displacement functions which generate translations and rotations.

In subsection 5.1 we prove a symmetry result for the unique solution of (3.3) in the space $V_0^\perp$, whose validity is physically evident, but which however needs a rigorous proof:

Proposition 3.1. *Let* $\mathbf{u}$ *be the unique solution of (3.3) in the space* $V_0^\perp$*. Then* $\mathbf{u}$ *satisfies the following symmetry properties:*

(i) for any $(x, y, z) \in \Omega$ *we have*

$$u_1(x, y, -z) = u_1(x, y, z), \quad u_2(x, y, -z) = u_2(x, y, z) \quad u_3(x, y, -z) = -u_3(x, y, z)\,, \tag{3.4}$$

$$u_1(-x, y, z) = -u_1(x, y, z), \quad u_2(-x, y, z) = u_2(x, y, z) \quad u_3(-x, y, z) = u_3(x, y, z)\,, \tag{3.5}$$

$$u_1(x, -y, z) = u_1(x, y, z), \quad u_2(x, -y, z) = -u_2(x, y, z) \quad u_3(x, -y, z) = u_3(x, y, z)\,; \tag{3.6}$$

(ii) the third component u_3 *of the solution* $\mathbf{u}$ *is axially symmetric in the sense that:*

$$u_3(x_1, y_1, z) = u_3(x_2, y_2, z) \quad \forall (x_1, y_1, z), (x_2, y_2, z) \in \Omega \text{ with } x_1^2 + y_1^2 = x_2^2 + y_2^2\,;$$

(iii) the first two components u_1, u_2 *of the solution* $\mathbf{u}$ *form a central vector field in two dimensions in the sense that*

$$|(u_1, u_2)|_{|(x_1,y_1,z)} = |(u_1, u_2)|_{|(x_2,y_2,z)} \quad \forall (x_1, y_1, z), (x_2, y_2, z) \in \Omega \text{ with } x_1^2 + y_1^2 = x_2^2 + y_2^2$$

and

$$(u_1, u_2)_{|(x,y,z)} = |(u_1, u_2)|_{|(x,y,z)} \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \quad \text{for any } (x, y, z) \in \Omega\,.$$

The symmetries described in this proposition are crucial for the method here presented and its possible generalizations. Up to a greater computational burden, it is possible to include non zero body forcing $\mathbf{f}$, having the same symmetries of the domain Ω. Also the load p in (3.2) could be assumed not constant, for example $p = p(x, y)$, but it has to agree the Ω symmetries (i.e., the load has to be axial). Hence, possible changes in the loading conditions are allowed, in principle one could consider also tangential (axially symmetric) loading conditions on $\partial\Omega$. The fundamental point is that Proposition 3.1 has to hold so that the problem can be extended periodically as described in the following section.

3.1 | Periodic extension of the problem

Our next purpose is to look for and construct a solution $\mathbf{u} = (u_1, u_2, u_3)$ of (3.3) admitting a Fourier series expansion in the z variable and, hence, admitting a periodic extension defined on the whole $C_{a,b} \times \mathbb{R}$. A posteriori, we show that it necessarily coincides with the unique solution of (3.3) belonging to $V_0^\perp$, see the end of the proof of Theorem 3.7.

As a first step, since $\mathbf{u} \in H^1(\Omega; \mathbb{R}^3)$, we define a function, still denoted for simplicity by $\mathbf{u}$, on the domain $C_{a,b} \times \left(-\frac{h}{2}, \frac{3h}{2} \right)$ by extending it in suitable way: the new function $\mathbf{u}$ coincides with the original function $\mathbf{u}$ on $C_{a,b} \times \left(-\frac{h}{2}, \frac{h}{2} \right)$ and

$$u_1(x, y, z) = -u_1(x, y, h - z)\,, \quad u_2(x, y, z) = -u_2(x, y, h - z)\,, \quad u_2(x, y, z) = u_3(x, y, h - z)\,, \tag{3.7}$$

for any $(x, y, z) \in C_{a,b} \times \left(\frac{h}{2}, \frac{3h}{2} \right)$. This means that u_1 and u_2 are antisymmetric with respect to $z = \frac{h}{2}$ and u_3 is symmetric with respect to $z = \frac{h}{2}$. The second step is to extend the new function $\mathbf{u} : C_{a,b} \times \left(-\frac{h}{2}, \frac{3h}{2} \right) \to \mathbb{R}$ to the whole $C_{a,b} \times \mathbb{R}$

as a $2h$-periodic function in the variable z, having the obvious symmetries. This construction and the solution properties stated in Proposition 3.1 allow expanding $\mathbf{u} = (u_1, u_2, u_3)$ in Fourier series with respect to the variable z in the following way:

$$u_1(x,y,z) = \sum_{k=0}^{+\infty} \varphi_k^1(x,y)\cos\left(\frac{\pi}{h}kz\right), \qquad u_2(x,y,z) = \sum_{k=0}^{+\infty} \varphi_k^2(x,y)\cos\left(\frac{\pi}{h}kz\right), \tag{3.8}$$

$$u_3(x,y,z) = \sum_{k=0}^{+\infty} \varphi_k^3(x,y)\sin\left(\frac{\pi}{h}kz\right).$$

Remark 3.2. We observe that for $k = 0$, the boundary value problem (3.3), or equivalently (5.17)–(5.18) and (5.20), see the proof in Subsection 5.3, admits an infinite number of solutions. More precisely, these solutions are in form $(\varphi_0^1, \varphi_0^2, \varphi_0^3) = (c_1, c_2, c_3)$ where c_1, c_2, c_3 are three arbitrary constants. We may choose $\varphi_0^3 \equiv 0$ being irrelevant in the Fourier expansion of u_3. Concerning the other two components, we have necessarily $\varphi_0^1 \equiv \varphi_0^2 \equiv 0$ in $C_{a,b}$ due to the odd symmetry of u_1 and u_2 with respect to the variables x and y, as stated in (3.5) and (3.6).

Thanks to the expansion (3.8) and Remark 3.2 we infer that the horizontal displacements u_1 and u_2 vanish on the upper and lower faces of the hollow cylinder Ω:

$$u_1\left(x,y,\frac{h}{2}\right) = u_1\left(x,y,-\frac{h}{2}\right) = 0 \quad \text{and} \quad u_2\left(x,y,\frac{h}{2}\right) = u_2\left(x,y,-\frac{h}{2}\right) = 0\,. \tag{3.9}$$

Therefore, the symmetric extension, for instance with respect to $z = \frac{h}{2}$, produces $\mathbf{u} \in H^1\left(C_{a,b} \times \left(-\frac{h}{2}, \frac{3h}{2}\right); \mathbb{R}^3\right)$. More in general, it is easy to understand that the periodic extension, still denoted for simplicity by $\mathbf{u}$, is a function satisfying $\mathbf{u} \in H^1(C_{a,b} \times I; \mathbb{R}^3)$ for any open bounded interval I.

We state here some lemmas in order to understand the main steps in the construction of the solution of (3.3), given in the final theorem. The periodic extension of the boundary data can be achieved according to the next lemma, proved in subsection 5.2.

Lemma 3.3. *Let* $\mathbf{u}$ *be the periodic extension of the solution of (3.3) defined as above and let* $\mathbf{\Lambda}$ *be the distribution defined by*

$$-\mathrm{div}(\mathbf{Tu}) = \mathbf{\Lambda} \qquad \text{in } \mathcal{D}'(C_{a,b} \times \mathbb{R}; \mathbb{R}^3)\,. \tag{3.10}$$

Then $\mathbf{\Lambda}$ *admits the following Fourier series expansion*

$$\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) = \left(0, 0, \chi_p(x,y) \sum_{m=0}^{+\infty} (-1)^{m+1} \frac{4}{h} \sin\left[\frac{\pi}{h}(2m+1)z\right]\right). \tag{3.11}$$

In subsection 5.3 we prove the following lemma.

Lemma 3.4. *For any* $k \geq 1$ *odd, there exists a unique* $(\varphi_k^1, \varphi_k^2, \varphi_k^3) \in H^1(C_{a,b}; \mathbb{R}^3)$*, satisfying (3.3) and (3.8). For any* $k \geq 2$ *even, there exists a unique trivial* $(\varphi_k^1, \varphi_k^2, \varphi_k^3) \equiv (0,0,0)$ *in* $C_{a,b}$*, satisfying (3.3) and (3.8).*

3.2 | Cylindrical coordinates exchange

The symmetry properties of $\mathbf{u}$ stated in Proposition 3.1 imply that φ_k^3 is a radial function and the vector field $(\varphi_k^1, \varphi_k^2)$ is a central vector field in the plane, in the sense that it is oriented toward the origin and its modulus is a function only of the distance from the origin. This implies that for any $k \geq 1$ odd, there exist two radial functions $Y_k = Y_k(\rho)$ and $Z_k = Z_k(\rho)$ such that in polar coordinates we may write

$$\varphi_k^1(\rho,\theta) = Y_k(\rho)\cos\theta\,, \qquad \varphi_k^2(\rho,\theta) = Y_k(\rho)\sin\theta\,, \qquad \varphi_k^3(\rho,\theta) = Z_k(\rho)\,, \tag{3.12}$$

with $\rho \in [a,b]$ and $\theta \in [0, 2\pi)$.

In Section 5 we show that Y_k and Z_k solve a proper boundary value problem. More precisely, this fact will be shown in subsection 5.4 which is devoted to the proof of the next lemma, where we state existence and uniqueness for solutions of the boundary value problem mentioned above.

Lemma 3.5. *Let $\Psi_k : C_{a,b} \to \mathbb{R}$ be defined as*

$$\Psi_k(x,y) := \begin{cases} (-1)^{\frac{k+1}{2}} \dfrac{4}{h}\,\chi_p(x,y) & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases} \qquad \forall (x,y) \in C_{a,b}. \tag{3.13}$$

For any $k \geq 1$ odd, the boundary value problem

$$\begin{cases} Y_k''(\rho) + \dfrac{Y_k'(\rho)}{\rho} - \dfrac{Y_k(\rho)}{\rho^2} - \dfrac{\mu}{\lambda+2\mu}\dfrac{\pi^2 k^2}{h^2} Y_k(\rho) + \dfrac{\lambda+\mu}{\lambda+2\mu}\dfrac{\pi k}{h} Z_k'(\rho) = 0 & \text{in } (a,b), \\ Z_k''(\rho) + \dfrac{Z_k'(\rho)}{\rho} - \dfrac{\lambda+2\mu}{\mu}\dfrac{\pi^2 k^2}{h^2} Z_k(\rho) - \dfrac{\lambda+\mu}{\mu}\dfrac{\pi k}{h}\left[Y_k'(\rho) + \dfrac{Y_k(\rho)}{\rho}\right] = -\dfrac{1}{\mu}\Psi_k(\rho) & \text{in } (a,b), \\ (\lambda+2\mu) Y_k'(\rho) + \frac{\lambda}{\rho} Y_k(\rho) + \lambda \frac{\pi k}{h} Z_k(\rho) = 0\,, & \rho \in \{a,b\} \\ Z_k'(\rho) - \frac{\pi k}{h} Y_k(\rho) = 0\,, & \rho \in \{a,b\} \end{cases} \tag{3.14}$$

admits a unique solution $(Y_k, Z_k) \in H^1(a,b;\mathbb{R}^2)$.

About existence and uniqueness of solutions of (3.14), in subsection 5.4 we only give an idea of the proof since it can be proved exactly as Lemma 3.4 of which Lemma 3.5 is the radial version.

Now we need a more explicit representation for the unique solution (Y_k, Z_k) of (3.14). This will be done by performing a power series expansion in which the coefficients will be characterized explicitly in terms of a suitable iterative scheme. As a byproduct of this result in Section 4 we also obtain a numerical approximation of the exact solution and we estimate the corresponding error. Being a linear problem, we proceed by applying the superposition principle and we provide the explicit formula in the next lemma.

Lemma 3.6. *For any $k \geq 1$, odd, let $\boldsymbol{\Upsilon}_k = (Y_k, Z_k)$ the unique solution of (3.14). Omitting for brevity the k-index, we have a unique $(C_1, C_2, C_3, C_4) \in \mathbb{R}^4$ such that*

$$\boldsymbol{\Upsilon}(\rho) = C_1 \boldsymbol{\Upsilon}^1(\rho) + C_2 \boldsymbol{\Upsilon}^2(\rho) + C_3 \boldsymbol{\Upsilon}^3(\rho) + C_4 \boldsymbol{\Upsilon}^4(\rho) + \overline{\boldsymbol{\Upsilon}}(\rho), \tag{3.15}$$

where $\boldsymbol{\Upsilon}^j = (Y^j, Z^j)$ with $j = 1, \dots, 4$ are four linear independent solutions of the corresponding homogenous system and $\overline{\boldsymbol{\Upsilon}} = (\overline{Y}, \overline{Z})$ solves

$$\left(\overline{Y}(\rho) \quad \overline{Y}'(\rho) \quad \overline{Z}(\rho) \quad \overline{Z}'(\rho)\right)^T = \mathbf{W}(\rho) \int_a^\rho (\mathbf{W}(r))^{-1} \left(0 \quad 0 \quad 0 \quad -\tfrac{1}{\mu}\Psi_k(r)\right)^T dr\,, \qquad \rho > 0\,, \tag{3.16}$$

being $\mathbf{W}(\rho)$ the wronskian obtained through $\boldsymbol{\Upsilon}^j(\rho)$ ($j = 1, \dots, 4$). Each of the linear independent solutions of the homogeneous system can be written as

$$\begin{cases} Y^j(\rho) = \displaystyle\sum_{n=-1}^{+\infty} a_n^j \rho^n + (\ln \rho) \sum_{n=0}^{+\infty} b_n^j \rho^n\,, \\ Z^j(\rho) = \displaystyle\sum_{n=0}^{+\infty} c_n^j \rho^n + (\ln \rho) \sum_{n=0}^{+\infty} d_n^j \rho^n \end{cases} \qquad (j = 1, \dots, 4), \tag{3.17}$$

where the coefficients are uniquely determined.

In the proof of the Lemma we give all the details related to the computation of the constants C_j in (3.15) and of the coefficients in the series (3.17), see subsection 5.5. As a consequence of Lemmas 3.3-3.4-3.5-3.6 we state the main theorem, whose proof can be found in subsection 5.6.

Theorem 3.7. *Let* $\mathbf{u}$ *be the unique solution of (3.3) satisfying* $\mathbf{u} \in V_0^{\perp}$ *and let* (Y_k, Z_k), $k \geq 1$ *odd, be the unique solution of (3.14). Then, in cylindrical coordinates,* $\mathbf{u} = (u_1, u_2, u_3)$ *admits the following representation:*

$$\begin{cases} u_1(\rho,\theta,z) = \displaystyle\sum_{m=0}^{+\infty} Y_{2m+1}(\rho)\cos\theta\cos\left[\frac{(2m+1)\pi}{h}\, z\right], \\ u_2(\rho,\theta,z) = \displaystyle\sum_{m=0}^{+\infty} Y_{2m+1}(\rho)\sin\theta\cos\left[\frac{(2m+1)\pi}{h}\, z\right], \\ u_3(\rho,\theta,z) = \displaystyle\sum_{m=0}^{+\infty} Z_{2m+1}(\rho)\sin\left[\frac{(2m+1)\pi}{h}\, z\right], \end{cases} \tag{3.18}$$

with $\rho \in (a,b)$, $\theta \in [0, 2\pi)$, $z \in \left(-\frac{h}{2}, \frac{h}{2}\right)$ *where the three series in (3.18) converge weakly in* $H^1(\Omega)$ *and strongly in* $L^2(\Omega)$.

Moreover, letting $\boldsymbol{U}_M = (U_M^1, U_M^2, U_M^3)$ *be the sequence of vector partial sums corresponding to the series expansions in (3.18), we have for any* $M \geq 1$

$$\|U_M^1 - u_1\|_{L^2(\Omega)} \leq \frac{pb^2}{\mu}\sqrt{\frac{h(b-a)}{2a\pi}}\,\frac{1}{\sqrt{M}}, \qquad \|U_M^2 - u_2\|_{L^2(\Omega)} \leq \frac{pb^2}{\mu}\sqrt{\frac{h(b-a)}{2a\pi}}\,\frac{1}{\sqrt{M}}, \tag{3.19}$$

$$\|U_M^3 - u_3\|_{L^2(\Omega)} \leq \frac{p}{\mu a \pi^2}\sqrt{\frac{h^3 b^3 (b-a)}{24}}\,\frac{1}{\sqrt{M^3}}.$$

4 | AN ENGINEERING APPLICATION

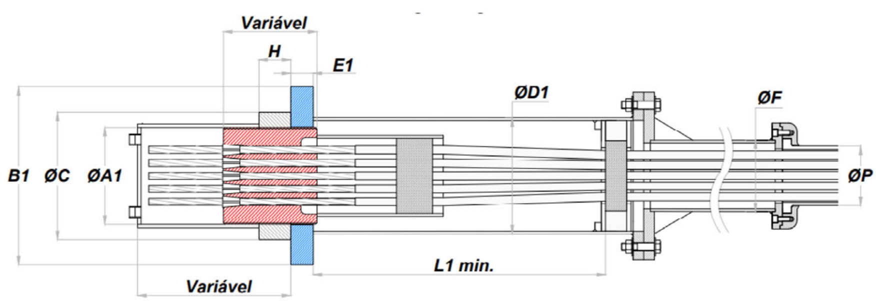
In this section we consider a case of study: a hollow cylinder having the features of a blister for the bridge in Figure 1. In Table 1 we give the mechanical parameters, see also Figure 4. We consider stays composed of 19 strands, see Figure 5, suitable to bear the concentrated load P in Table 1. P is computed from the executive project, while the diameter $2a$ is taken from the catalogue of Protende ABS-2021 [20], a company producing such elements, see in Figure 5 the diameter ϕD_1 for 19 strands anchorage; hence, the distributed load in (3.2) is given by $p = \frac{P}{\pi(\varepsilon^2 - a^2)} = 22.80$ MPa.

Our purpose is to obtain a good approximation of the functions $\boldsymbol{\Upsilon}^j = (Y^j, Z^j)$, $j \in \{1,2,3,4\}$ introduced in Lemma 3.6. For $j = 1, \dots, 4$ we consider the approximate solution ($N \geq 1$)

$$Y_N^j(\rho) = \sum_{n=-1}^{N} a_n\,\rho^n + (\ln\rho)\sum_{n=0}^{N} b_n\,\rho^n \quad \text{and} \quad Z_N^j(\rho) = \sum_{n=0}^{N-1} c_n\,\rho^n + (\ln\rho)\sum_{n=0}^{N-1} d_n\,\rho^n. \tag{4.1}$$

TABLE 1 Mechanical parameters assumed.

h	3.00 m	Height of the cylinder
$2a$	273 mm	Diameter of the cylindrical hollow
$2b$	800 mm	External diameter of the cylinder
2ε	425 mm	External diameter of the load
P	1900 kN	Concentrated load
E	35000 MPa	Young modulus of the concrete
ν	0.2	Poisson ratio of the concrete



NÚMERO DE CORDOALHAS	ØA1	B1	ØC	ØD1	E1	ØF	H	L1 min.	ØP
4	110	270	160	141	20	89	30	430	63
7	150	300	200	168	30	114	40	550	75
13	200	340	256	219	36	168	55	860	110
19	228	380	300	273	53	168	75	960	140
31	280	470	350	323	60	219	90	1280	180
37	280	490	365	323	70	273	100	1410	180
55	323	570	415	355	85	273	125	1720	225
61	349	600	440	355	85	323	130	1870	250
73	410	680	525	457	95	323	145	2030	280
91	450	750	570	508	110	355	160	2280	315
109	470	780	580	508	120	355	175	2400	315
127	500	810	620	559	130	406	190	2700	355
169	570	900	680	609	145	457	225	3120	400

Valores sujeitos a variações de acordo com os requisitos especiais do projeto.

FIGURE 5 Detail of the strands anchorage and in table the geometric features for a 19 strands element, from the commercial catalogue [20].

The reason for in (4.1) we have $n = 0, \dots, N-1$ in the expansion of Z^j_N will be clarified in the proof in subsection 5.7 of the next proposition about an estimate of the truncating error.

Proposition 4.1. *Let $k > 1$, $k \in \mathbb{N}$ odd, and let $N \geq 3$, odd integer, be the truncating index of the series as in (4.1). Then, letting*

$$E_{k,N} := \max_{j\in\{1,2,3,4\}} \left\{ \max \left\{ \max_{\rho\in[a,b]} |Y^j_N(\rho) - Y^j(\rho)|, \max_{\rho\in[a,b]} |Z^j_N(\rho) - Z^j(\rho)| \right\} \right\},$$

we have that

$$E_{k,N} \leq C(a,b,k) \frac{3(2\lambda + 5\mu)(\lambda+\mu)^2}{16\mu^3} \left(\frac{\pi k b}{h}\right)^{N+2} e^{\left(\frac{\pi k b}{h}\right)^2} \frac{(N+3)(3N^3 + 21N^2 + 42N + 32)}{2^N \left[\left(\frac{N+1}{2}\right)!\right]^2}, \tag{4.2}$$

where

$$C(a,b,k) = \max\left\{1, \frac{h}{\pi k b}\right\} \max\left\{1, \left|\ln\left(\frac{\pi k a}{h}\right)\right|, |\ln\left(\frac{\pi k b}{h}\right)|\right\} \max\left\{\frac{\pi k}{h}, \frac{\mu}{\lambda+\mu}\frac{\pi k}{h}\ln\left(\frac{\pi k}{h}\right), \frac{2(\lambda+2\mu)}{\lambda+\mu}\frac{h}{\pi k}\ln\left(\frac{\pi k}{h}\right)\right\}.$$

Once we have (4.2), one may choose N in such a way that

$$\frac{E_{k,N}}{\min_{j\in\{1,2,3,4\}} \left\{ \min \left\{ \max_{\rho\in[a,b]} |Y^j_N(\rho)|, \max_{t\in[a,b]} |Z^j_N(\rho)| \right\} \right\}} < \varepsilon \tag{4.3}$$

with ε small enough. Condition (4.3) means that the truncation error is relatively small compared to the order of magnitude of both functions Y^j_N and Z^j_N for all $j \in \{1,2,3,4\}$.

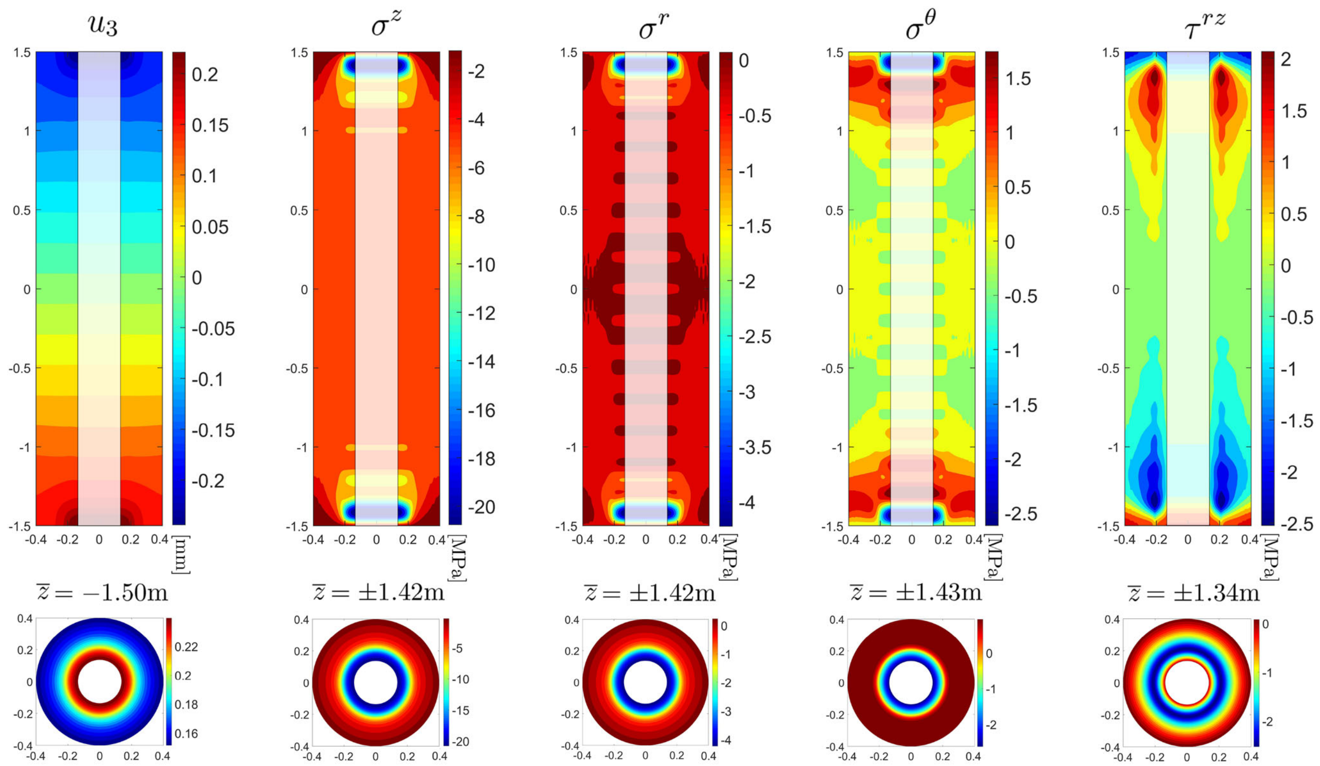


FIGURE 6 From the the left the vertical displacement u_3 in mm, the vertical stress σ^z in MPa, the radial stress σ^r in MPa, the angular stress σ^θ in MPa and the tangential stress τ^{rz} in MPa.

TABLE 2 Maximum absolute values and points of Ω in which they are assumed.

	$\max|\cdot|$	$\bar{\rho}$	$\bar{z}$ [m]
u_3	0.24 mm	a	$\pm$ 1.50
σ^z	20.77 MPa	a	$\pm$ 1.42
σ^r	4.23 MPa	a	$\pm$1.42
σ^θ	2.62 MPa	a	$\pm$ 1.43
τ^{rz}	2.52 MPa	ϵ	$\pm$ 1.34

In our numerical simulation the condition (4.3) is verified by making use of estimate (4.2) on the truncation error $E_{k,N}$, that is, the program verifies at each step the validity of (4.3) in which the numerator of the fraction is replaced by the majorant in (4.2). The program runs until the value of N is sufficiently large to guarantee (4.3).

In Figure 6 we plot a vertical section of the cylinder and the corresponding more stressed horizontal section. We show the vertical displacement u_3 and the following components of the stress tensor in cylindrical coordinates

$$\begin{aligned} \sigma^z &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial u_3}{\partial z} + \nu\left(\frac{u^r}{\rho} + \frac{\partial u^r}{\partial \rho}\right)\right] \qquad \sigma^r = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial u^r}{\partial \rho} + \nu\left(\frac{u^r}{\rho} + \frac{\partial u_3}{\partial z}\right)\right] \\ \sigma^\theta &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{u^r}{\rho} + \nu\left(\frac{\partial u^r}{\partial \rho} + \frac{\partial u_3}{\partial z}\right)\right] \qquad \tau^{rz} = \mu\left[\frac{\partial u^r}{\partial z} + \frac{\partial u_3}{\partial \rho}\right], \end{aligned} \tag{4.4}$$

where $u^r = \sqrt{u_1^2 + u_2^2}$ is the radial displacement. We point out that putting $\mathbf{n} = (\cos\theta, \sin\theta, 0)$, $\mathbf{t} = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{k} = (0, 0, 1)$, the four components introduced in (4.4) are defined by $\sigma^z := (\mathbf{T}\mathbf{u})\mathbf{k}\cdot\mathbf{k}$, $\sigma^r := (\mathbf{T}\mathbf{u})\mathbf{n}\cdot\mathbf{n}$, $\sigma^\theta := (\mathbf{T}\mathbf{u})\mathbf{t}\cdot\mathbf{t}$ and $\tau^{rz} := (\mathbf{T}\mathbf{u})\mathbf{n}\cdot\mathbf{k}$ and the representation (4.4) can be deduced by (A9) and (A10).

We consider an approximate solution $\boldsymbol{U}_M$ as stated in Theorem 3.7 truncating the Fourier series at $M = 29$ with $\varepsilon < 10^{-3}$ in (4.3), implying $N = 123$ in (4.1) and $\|U^1_{29} - u_1\|_{L^2(\Omega)} \leq 4.46\cdot 10^{-5}$ m$^{5/2}$, $\|U^3_{29} - u_3\|_{L^2(\Omega)} \leq 1.02\cdot 10^{-6}$ m$^{5/2}$ in (3.19). In Table 2 we give the maximum absolute values of the variables involved, including the coordinate of the point $(\bar{\rho}, \bar{z})$ where they are assumed (for all $\theta \in [0, 2\pi)$ thanks to the radial symmetry of the problem).

As expected the vertical displacement u_3 achieves its maximum absolute value at $z = \pm\frac{h}{2}$. From the plots we see that there are two (symmetric) critical zones where we observe the loading diffusion; they are close to the upper and bottom faces of the cylinder and involve approximately the 20% of the closest volume, that is, the volume of Ω such that $z \in \left(-\frac{h}{2}, -\frac{2h}{5}\right) \cup \left(\frac{2h}{5}, \frac{h}{2}\right)$.

5 | PROOFS OF THE RESULTS

5.1 | Proof of Proposition 3.1

Concerning part (i) of the Proposition we only give the proof of (3.4) since the proof of (3.5)–(3.6) can be obtained with a similar procedure. For any function $\mathbf{v} \in H^1(\Omega; \mathbb{R}^3)$ we denote by $\bar{\mathbf{v}} = (\bar{v}_1, \bar{v}_2, \bar{v}_2) \in H^1(\Omega; \mathbb{R}^3)$ the function defined by

$$\bar{v}_1(x, y, z) = v_1(x, y, -z), \quad \bar{v}_2(x, y, z) = v_2(x, y, -z), \quad \bar{v}_3(x, y, z) = -v_3(x, y, -z), \quad \forall (x, y, z) \in \Omega. \tag{5.1}$$

Let $\mathbf{u}$ be the unique solution of (3.3) in $V_0^\perp$ and let $\bar{\mathbf{u}}$ be the corresponding function defined by (5.1).

We start by showing that $\bar{\mathbf{u}}$ solves problem (3.3). In doing this we show that it solves the variational problem (2.3) where in the present case $\mathbf{f} = \mathbf{0}$ and $\mathbf{g}$ is the function defined in (3.1).

By direct computation one can see that for any test function $\mathbf{v} \in H^1(\Omega; \mathbb{R}^3)$ we have for any $(x, y, z) \in \Omega$

$$(\mathbf{D}\bar{\mathbf{u}} : \mathbf{D}\mathbf{v})_{|(x,y,z)} = (\mathbf{D}\mathbf{u} : \mathbf{D}\bar{\mathbf{v}})_{|(x,y,-z)}, \qquad [(\operatorname{div}\bar{\mathbf{u}})(\operatorname{div}\mathbf{v}]_{|(x,y,z)} = [(\operatorname{div}\mathbf{u})(\operatorname{div}\bar{\mathbf{v}}]_{|(x,y,-z)}. \tag{5.2}$$

By (2.3), (2.1), (5.2), (3.1) and a change of variables, we obtain

$$\begin{aligned} &2\mu \int_\Omega \mathbf{D}\bar{\mathbf{u}} : \mathbf{D}\mathbf{v}\, d\mathbf{x} + \lambda \int_\Omega (\operatorname{div}\bar{\mathbf{u}})(\operatorname{div}\mathbf{v})\, d\mathbf{x} \\ &\quad = 2\mu \int_\Omega \mathbf{D}\mathbf{u} : \mathbf{D}\bar{\mathbf{v}}\, d\mathbf{x} + \lambda \int_\Omega (\operatorname{div}\mathbf{u})(\operatorname{div}\bar{\mathbf{v}})\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{g} \cdot \bar{\mathbf{v}}\, dS = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\, dS. \end{aligned} \tag{5.3}$$

By (5.3) we deduce that $\bar{\mathbf{u}}$ is a solution of (2.3) and hence a weak solution of (3.3). We now prove that $\bar{\mathbf{u}} \in V_0^\perp$. Indeed, proceeding as in (5.3) one can easily show that $(\bar{\mathbf{u}}, \mathbf{v})_\mathbf{T} = (\mathbf{u}, \bar{\mathbf{v}})_\mathbf{T} = 0$ for any $\mathbf{v} \in V_0$ since $\mathbf{u} \in V_0^\perp$ and $\bar{\mathbf{v}} \in V_0$ whenever $\mathbf{v} \in V_0$, as one can deduce by (2.7). This completes the proof of (3.4).

Let us proceed with the proof of part (ii) and (iii) of the proposition. For any $\theta \in (-2\pi, 2\pi)$ we denote by $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ the anticlockwise rotation of an angle θ and by A_θ the associate matrix. Clearly we have that the inverse map of R_θ is given by $R_{-\theta}$ and $A_\theta^{-1} = A_{-\theta}$.

We use the notation $\mathbf{u} = (u', u_3) \in \mathbb{R}^2 \times \mathbb{R}$ with $u' = (u_1, u_2)$ and we denote by

$$\nabla' u' = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

its Jacobian matrix in the x and y variables, and by $\mathbf{D}'u'$ the corresponding symmetric gradient given by $\frac{1}{2}(\nabla' u' + (\nabla' u')^T)$; more in general, throughout this proof we will use the symbol ∇' for denoting the gradient with respect to the x and y variables.

We now define

$$\mathbf{u}_\theta(x, y, z) = (R_{-\theta}(u_1(R_\theta(x, y), z), u_2(R_\theta(x, y), z)), u_3(R_\theta(x, y), z)) \quad \text{for any } (x, y, z) \in \Omega.$$

Then, the Jacobian matrix $\nabla \mathbf{u}_\theta \in \mathbb{R}^{3\times 3}$ and in turn the matrix $\mathbf{D}\mathbf{u}_\theta$ admit a representation in terms of four blocks of dimensions 2×2, 2×1, 1×2, 1×1 respectively. We proceed directly with the representation of $\mathbf{D}u_\theta$:

$$\mathbf{D}\mathbf{u}_\theta = \begin{pmatrix} A_{-\theta}\,\mathbf{D}'u'(R_\theta(x,y),z)A_\theta & A_{-\theta}\frac{\partial u'}{\partial z}(R_\theta(x,y),z)+\left[\nabla' u_3(R_\theta(x,y),z)A_\theta\right]^T \\ \left[A_{-\theta}\frac{\partial u'}{\partial z}(R_\theta(x,y),z)\right]^T+\nabla' u_3(R_\theta(x,y),z)A_\theta & \frac{\partial u_3}{\partial z}(R_\theta(x,y),z) \end{pmatrix} \tag{5.4}$$

In the same way, for any test function $\mathbf{v} \in H^1(\Omega;\mathbb{R}^3)$ and any $\theta \in (-2\pi, 2\pi)$ we may define the corresponding function $\mathbf{v}_\theta$. Looking at $\mathbf{v}$ as $(\mathbf{v}_{-\theta})_\theta$ and applying (5.4) to $\mathbf{v}_{-\theta}$ we claim that for any $(x,y,z) \in \Omega$

$$\mathbf{D}\mathbf{u}_\theta(x,y,z) : \mathbf{D}\mathbf{v}(x,y,z) = \mathbf{D}\mathbf{u}(R_\theta(x,y),z) : \mathbf{D}\mathbf{v}_{-\theta}(R_\theta(x,y),z)\,. \tag{5.5}$$

This is a consequence of the fact that A_θ is orthogonal and the linear map $\mathcal{L}_\theta : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$, $\mathcal{L}_\theta(X) = A_{-\theta}XA_\theta$ is an isometry in $\mathbb{R}^{2\times2}$ as one can see by verifying the orthogonality of the associated matrix $M_\theta \in \mathbb{R}^{4\times4}$. This implies

$$(A_{-\theta}XA_\theta) : Y = \mathcal{L}_\theta(X) : Y = \mathcal{L}_\theta(X) : \mathcal{L}_\theta(\mathcal{L}_\theta^{-1}(Y)) = X : \mathcal{L}_\theta^{-1}(Y) = X : (A_\theta Y A_{-\theta})$$

for any $X, Y \in \mathbb{R}^{2\times2}$. This arguments allow to treat the scalar products between the 2×2 block appearing in the representation (5.4). Even easier is to treat the scalar products between the 2×1 and 1×2 blocks thanks to the orthogonality of A_θ. This proves the claim (5.5).

The invariance of the trace of a matrix X under maps of the form $X \mapsto A^{-1}XA$ combined with (5.4) shows that $\operatorname{div}\mathbf{u}_\theta(x,y,z) = \operatorname{div}\mathbf{u}(R_\theta(x,y),z)$ and in particular for any $(x,y,z)\in\Omega$ we have

$$(\operatorname{div}\mathbf{u}_\theta(x,y,z))(\operatorname{div}\mathbf{v}(x,y,z)) = (\operatorname{div}\mathbf{u}(R_\theta(x,y),z))(\operatorname{div}\mathbf{v}_{-\theta}(R_\theta(x,y),z))\,. \tag{5.6}$$

By (2.3), (3.1), (3.2), (5.5), (5.6), two changes of variables and the definitions of $\mathbf{v}_{-\theta}$ and $\mathbf{g}$, we obtain

$$2\mu\int_\Omega \mathbf{D}\mathbf{u}_\theta : \mathbf{D}\mathbf{v}\,d\mathbf{x} + \lambda\int_\Omega(\operatorname{div}\mathbf{u}_\theta)(\operatorname{div}\mathbf{v})\,d\mathbf{x} \tag{5.7}$$

$$= 2\mu\int_\Omega \mathbf{D}\mathbf{u} : \mathbf{D}\mathbf{v}_{-\theta}\,d\mathbf{x} + \lambda\int_\Omega(\operatorname{div}\mathbf{u})(\operatorname{div}\mathbf{v}_{-\theta})\,d\mathbf{x} = \int_{\partial\Omega}\mathbf{g}\cdot\mathbf{v}_{-\theta}\,dS = \int_{\partial\Omega}\mathbf{g}\cdot\mathbf{v}\,dS\,.$$

We have just proved that $\mathbf{u}_\theta$ is still a weak solution of (3.3). We now show that $\mathbf{u}_\theta \in V_0^\perp$ as a consequence of the fact that $\mathbf{u} \in V_0^\perp$. Proceeding as in (5.7), we infer

$$(\mathbf{u}_\theta, \mathbf{v})_{\mathbf{T}} = (\mathbf{u}, \mathbf{v}_{-\theta})_{\mathbf{T}} \qquad \text{for any } \mathbf{v} \in H^1(\Omega;\mathbb{R}^3)\,. \tag{5.8}$$

We need to prove that if $\mathbf{v} \in V_0$ then $\mathbf{v}_{-\theta} \in V_0$. For any $\theta \in (-2\pi, 2\pi)$, let B_θ be the 3×3 matrix corresponding to an anticlockwise rotation of an angle θ around the z axis. Clearly B_θ is orthogonal and $B_\theta^{-1} = B_{-\theta}$. With this notation we may write

$$\mathbf{v}_{-\theta}(\mathbf{x}) = B_\theta\,\mathbf{v}(B_{-\theta}\,\mathbf{x}) \qquad \text{for any } \mathbf{x} \in \mathbb{R}^3 \tag{5.9}$$

where both $\mathbf{x}$ and $\mathbf{v}$ have to be considered vector columns in the right hand side of the identity.

If $\mathbf{v} \in V_0$, then by (2.7) we have that $\mathbf{v}$ admits the following matrix representation

$$\mathbf{v}(\mathbf{x}) = M\mathbf{x} + \boldsymbol{\delta} \qquad \text{for any } \mathbf{x} \in \mathbb{R}^3 \tag{5.10}$$

where M is an antisymmetric matrix and $\boldsymbol{\delta} = (\delta_1\ \ \delta_2\ \ \delta_3)^T$.

Combining (5.9) and (5.10) we obtain $\mathbf{v}_{-\theta}(\mathbf{x}) = B_\theta M B_{-\theta}\,\mathbf{x} + B_\theta\boldsymbol{\delta}$ where the matrix $B_\theta M B_{-\theta}$ is antisymmetric since

$$(B_\theta M B_{-\theta})^T = B_{-\theta}^T M^T B_\theta^T = B_{-\theta}^{-1}(-M)B_\theta^{-1} = -B_\theta M B_{-\theta}\,.$$

This proves that also $\mathbf{v}_{-\theta} \in V_0$ since it admits a representation like in (2.7).

Now, if we choose $\mathbf{v} \in V_0$ in (5.8), we readily see that $(\mathbf{u}_\theta, \mathbf{v})_{\mathbf{T}} = 0$ being $\mathbf{u} \in V_0^\perp$ and $\mathbf{v}_{-\theta} \in V_0$. This proves that $\mathbf{u}_\theta \in V_0^\perp$.

By the uniqueness result stated in Proposition 2.1 (iv) we infer that $\mathbf{u}_\theta = \mathbf{u}$ for any $\theta \in (-2\pi, 2\pi)$.

Now the validity of (ii) and of the first part of (iii) follows immediately from the definition of $\mathbf{u}_\theta$.

It remains to observe that the vector field u' is oriented radially in the xy-plane. To do this, it is sufficient to combine the identity $\mathbf{u} = \mathbf{u}_\theta$ with the identity $u_2(x, 0, z) = 0$, valid for any $a < x < b$ and $z \in \left(-\frac{h}{2}, \frac{h}{2}\right)$, as a consequence of (3.6). □

5.2 | Proof of Lemma 3.3

Let us introduce the sequence of intervals $I_k := \left(-\frac{h}{2} + kh, \frac{h}{2} + kh\right)$, the corresponding sequence of domains $\Omega_k := C_{a,b} \times I_k$ and the sequence of functions $\mathbf{g}_k : \partial\Omega_k \to \mathbb{R}^3$

$$\mathbf{g}_k(x, y, z) := \begin{cases} (0, 0, (-1)^k \chi_p(x, y)) & \text{if } (x, y, z) \in C_{a,b} \times \left\{-\frac{h}{2} + kh\right\}, \\ (0, 0, (-1)^{k+1} \chi_p(x, y)) & \text{if } (x, y, z) \in C_{a,b} \times \left\{\frac{h}{2} + kh\right\}, \\ (0, 0, 0) & \text{if } (x, y, z) \in \partial C_{a,b} \times I_k . \end{cases} \tag{5.11}$$

We know that the original function $\mathbf{u}$ is a weak solution of problem (3.3) in the sense that

$$\int_\Omega \mathbf{T}\mathbf{u} : \mathbf{D}\mathbf{v}\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\, dS \qquad \text{for any } \mathbf{v} \in H^1(\Omega; \mathbb{R}^3). \tag{5.12}$$

We need to find, starting from (5.12), the equation solved, in the sense of distributions, by the periodic extension. First of all, we observe that by (3.7), (5.11), (5.12) and some computations, we have

$$\int_{\Omega_k} \mathbf{T}\mathbf{u} : \mathbf{D}\mathbf{v}\, d\mathbf{x} = \int_{\partial\Omega_k} \mathbf{g}_k \cdot \mathbf{v}\, dS \qquad \text{for any } \mathbf{v} \in H^1(\Omega_k; \mathbb{R}^3). \tag{5.13}$$

Now, letting $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) \in \mathcal{D}(C_{a,b} \times \mathbb{R}; \mathbb{R}^3)$, by (5.13) we infer

$$\int_{C_{a,b}\times\mathbb{R}} \mathbf{T}\mathbf{u} : \mathbf{D}\boldsymbol{\phi}\, d\mathbf{x} = \sum_{k\in\mathbb{Z}} \int_{\Omega_k} \mathbf{T}\mathbf{u} : \mathbf{D}\boldsymbol{\phi}\, d\mathbf{x} = \sum_{k\in\mathbb{Z}} \int_{\partial\Omega_k} \mathbf{g}_k \cdot \boldsymbol{\phi}\, dS$$

$$= \sum_{k\in\mathbb{Z}} 2(-1)^{k+1} \int_{C_{a,b}} \chi_p(x, y)\, \phi_3\left(x, y, \frac{h}{2} + kh\right) dx dy .$$

This proves (3.10), where $\boldsymbol{\Lambda}$ is the distribution defined by

$$\langle \boldsymbol{\Lambda}, \boldsymbol{\phi} \rangle := \int_{C_{a,b}} \chi_p(x, y) \sum_{k\in\mathbb{Z}} 2(-1)^{k+1} \phi_3\left(x, y, \frac{h}{2} + kh\right) dx dy \tag{5.14}$$

for any $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) \in \mathcal{D}(C_{a,b} \times \mathbb{R}; \mathbb{R}^3)$.

The distribution $\boldsymbol{\Lambda}$ admits a sort of factorization as a product of a function in the variables x and y and of a distribution acting on functions of the variable z:

$$\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3) = \left(0, 0, 2\, \chi_p \sum_{k\in\mathbb{Z}} (-1)^{k+1}\, \delta_{\frac{h}{2}+kh}\right)$$

where $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{D}'(C_{a,b} \times \mathbb{R}; \mathbb{R})$ are the *scalar distributions* defined by

$$\langle \Lambda_i, \phi \rangle := \langle \boldsymbol{\Lambda}, \phi\, \mathbf{e}_i \rangle \qquad \text{for any } \phi \in \mathcal{D}(C_{a,b} \times \mathbb{R}; \mathbb{R}),$$

with $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, $\mathbf{e}_3 = \mathbf{k}$, and $\delta_{\frac{h}{2}+kh}$ are Dirac delta distributions concentrated at $z = \frac{h}{2} + kh$.

Expanding in Fourier series the periodic distribution $\sum_{k\in\mathbb{Z}} 2(-1)^{k+1}\,\delta_{\frac{h}{2}+kh}$ we obtain (3.11), where the Fourier series converges in the sense of distributions. For more details on this convergence see the arguments introduced in subsection 5.6. □

5.3 | Proof of Lemma 3.4

First of all we insert (3.8) into (3.10); recalling the Hooke's law (2.1) and exploiting (3.11), we obtain

$$\begin{cases} -\mu\Delta\varphi_k^1 + \mu\dfrac{\pi^2k^2}{h^2}\,\varphi_k^1 - (\lambda+\mu)\left[\dfrac{\partial^2\varphi_k^1}{\partial x^2} + \dfrac{\partial^2\varphi_k^2}{\partial x\partial y} + \dfrac{\pi k}{h}\dfrac{\partial\varphi_k^3}{\partial x}\right] = 0 & \text{in } C_{a,b}\,, \\ -\mu\Delta\varphi_k^2 + \mu\dfrac{\pi^2k^2}{h^2}\,\varphi_k^2 - (\lambda+\mu)\left[\dfrac{\partial^2\varphi_k^1}{\partial x\partial y} + \dfrac{\partial^2\varphi_k^2}{\partial y^2} + \dfrac{\pi k}{h}\dfrac{\partial\varphi_k^3}{\partial y}\right] = 0 & \text{in } C_{a,b}\,, \\ -\mu\Delta\varphi_k^3 + \mu\dfrac{\pi^2k^2}{h^2}\,\varphi_k^3 + \dfrac{\pi k}{h}(\lambda+\mu)\left[\dfrac{\partial\varphi_k^1}{\partial x} + \dfrac{\partial\varphi_k^2}{\partial y} + \dfrac{\pi k}{h}\varphi_k^3\right] = \Psi_k & \text{in } C_{a,b}\,, \end{cases} \tag{5.15}$$

where the forcing term is defined in (3.13). We observe that in (5.15), the operator Δ stands for the Laplace operator in the variables x and y, that is, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Putting $\Phi_k := (\varphi_k^1, \varphi_k^2)$ and $\bar{\mathbf{n}} \in \mathbb{R}^2$ the outward unit normal to $\partial C_{a,b}$, system (5.15) may be rewritten in the following form

$$\begin{cases} -\mu\Delta\Phi_k + \mu\frac{\pi^2k^2}{h^2}\Phi_k - (\lambda+\mu)\nabla(\operatorname{div}\Phi_k) - (\lambda+\mu)\frac{\pi k}{h}\nabla\varphi_k^3 = \mathbf{0} & \text{in } C_{a,b}\,, \\ -\mu\Delta\varphi_k^3 + \mu\frac{\pi^2k^2}{h^2}\varphi_k^3 + \frac{\pi k}{h}(\lambda+\mu)\left(\operatorname{div}\Phi_k + \frac{\pi k}{h}\varphi_k^3\right) = \Psi_k & \text{in } C_{a,b}\,, \end{cases} \tag{5.16}$$

or equivalently in the following form

$$\begin{cases} -\operatorname{div}(\lambda(\operatorname{div}\Phi_k)I + 2\mu\mathbf{D}\Phi_k) + \mu\frac{\pi^2k^2}{h^2}\Phi_k - (\lambda+\mu)\frac{\pi k}{h}\nabla\varphi_k^3 = \mathbf{0} & \text{in } C_{a,b}\,, \\ -\mu\Delta\varphi_k^3 + (\lambda+2\mu)\frac{\pi^2k^2}{h^2}\varphi_k^3 + (\lambda+\mu)\frac{\pi k}{h}\operatorname{div}\Phi_k = \Psi_k & \text{in } C_{a,b}\,, \end{cases} \tag{5.17}$$

where $\mathbf{D}$ represents here the symmetric gradient in the two-dimensional case and I is the 2×2 identity matrix.

We also recall that by (3.3), $(\mathbf{Tu})\mathbf{n} = \mathbf{0}$ on $\partial C_{a,b}\times\mathbb{R}$ so that by the Hooke's law (2.1) we obtain

$$\begin{cases} (\lambda+2\mu)x\dfrac{\partial u_1}{\partial x} + \lambda x\dfrac{\partial u_2}{\partial y} + \lambda x\dfrac{\partial u_3}{\partial z} + \mu y\left(\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x}\right) = 0 & \text{on } \partial C_{a,b}\times\mathbb{R}\,, \\ \lambda y\dfrac{\partial u_1}{\partial x} + (\lambda+2\mu)y\dfrac{\partial u_2}{\partial y} + \lambda y\dfrac{\partial u_3}{\partial z} + \mu x\left(\dfrac{\partial u_1}{\partial y} + \dfrac{\partial u_2}{\partial x}\right) = 0 & \text{on } \partial C_{a,b}\times\mathbb{R}\,, \\ \mu x\left(\dfrac{\partial u_1}{\partial z} + \dfrac{\partial u_3}{\partial x}\right) + \mu y\left(\dfrac{\partial u_2}{\partial z} + \dfrac{\partial u_3}{\partial y}\right) = 0 & \text{on } \partial C_{a,b}\times\mathbb{R}\,, \end{cases}$$

and by (3.8) we obtain

$$\begin{cases} \lambda(\operatorname{div}\Phi_k)\bar{\mathbf{n}} + 2\mu(\mathbf{D}\Phi_k)\bar{\mathbf{n}} + \lambda\frac{\pi k}{h}\varphi_k^3\,\bar{\mathbf{n}} = \mathbf{0} & \text{on } \partial C_{a,b}\,, \\ \mu\nabla\varphi_k^3\cdot\bar{\mathbf{n}} - \mu\frac{\pi k}{h}\,\Phi_k\cdot\bar{\mathbf{n}} = 0 & \text{on } \partial C_{a,b}\,. \end{cases} \tag{5.18}$$

Let us derive the weak formulation of (5.16)–(5.18). Testing (5.17) with (w^1, w^2, w^3), putting $W = (w^1, w^2)$ and integrating by parts we obtain

$$-\int_{\partial C_{a,b}} [(\lambda(\operatorname{div}\Phi_k)I + 2\mu\mathbf{D}\Phi_k)\bar{\mathbf{n}}]\cdot W\,ds + \int_{C_{a,b}} (\lambda(\operatorname{div}\Phi_k)I + 2\mu\mathbf{D}\Phi_k) : \nabla W\,dxdy \tag{5.19}$$

$$+\mu\frac{\pi^2k^2}{h^2}\int_{C_{a,b}}\Phi_k\cdot W\,dxdy-\mu\frac{\pi k}{h}\int_{C_{a,b}}\nabla\varphi_k^3\cdot W\,dxdy-\lambda\frac{\pi k}{h}\int_{\partial C_{a,b}}\varphi_k^3\,\bar{\mathbf{n}}\cdot W\,ds+\lambda\frac{\pi k}{h}\int_{C_{a,b}}\varphi_k^3\operatorname{div}W\,dxdy$$

$$-\int_{\partial C_{a,b}}\mu\frac{\partial\varphi_k^3}{\partial\bar{\mathbf{n}}}\,w^3ds+\mu\int_{C_{a,b}}\nabla\varphi_k^3\cdot\nabla w^3\,dxdy+(\lambda+2\mu)\frac{\pi^2k^2}{h^2}\int_{C_{a,b}}\varphi_k^3\,w^3\,dxdy$$

$$+\mu\frac{\pi k}{h}\int_{\partial C_{a,b}}w^3\Phi_k\cdot\bar{\mathbf{n}}\,ds-\mu\frac{\pi k}{h}\int_{C_{a,b}}\Phi_k\cdot\nabla w^3\,dxdy+\lambda\frac{\pi k}{h}\int_{C_{a,b}}\operatorname{div}\Phi_k\,w^3\,dxdy=\int_{C_{a,b}}\Psi_k\,w^3\,dxdy\,.$$

We observe that by (5.18) the boundary integrals in (5.19) disappear; on the other hand collecting the double integrals and recalling that $\mathbf{D}\Phi_k : \nabla W = \mathbf{D}\Phi_k : \mathbf{D}W$, we may write (5.19) in the form

$$2\mu\int_{C_{a,b}}\mathbf{D}\Phi_k:\mathbf{D}W\,dxdy+\lambda\int_{C_{a,b}}\left(\operatorname{div}\Phi_k+\frac{\pi k}{h}\varphi_k^3\right)\left(\operatorname{div}W+\frac{\pi k}{h}w^3\right)dxdy \tag{5.20}$$

$$+\mu\int_{C_{a,b}}(\nabla\varphi_k^3-\frac{\pi k}{h}\Phi_k)\cdot(\nabla w^3-\frac{\pi k}{h}W)dxdy+2\mu\frac{\pi^2k^2}{h^2}\int_{C_{a,b}}\varphi_k^3w^3dxdy=\int_{C_{a,b}}\Psi_kw^3dxdy$$

for any $\mathbf{w}\in H^1(C_{a,b};\mathbb{R}^3)$, where $W=(w^1,w^2)$. This represents the weak form of (5.16)–(5.18).

For any $k\geq 2$ even we observe that, by (3.13), $\varphi_k^1\equiv\varphi_k^2\equiv\varphi_k^3\equiv 0$ in $C_{a,b}$, as one can deduce by testing (5.20) with $(w^1,w^2,w^2)=(\varphi_k^1,\varphi_k^2,\varphi_k^3)$.

For any $k\geq 1$ odd, we define the following bilinear form

$$\bar{a}_k(\boldsymbol{\varphi},\mathbf{w}):=2\mu\int_{C_{a,b}}\mathbf{D}\Phi:\mathbf{D}W\,dxdy+\lambda\int_{C_{a,b}}\left(\operatorname{div}\Phi+\frac{\pi k}{h}\,\varphi_k^3\right)\left(\operatorname{div}W+\frac{\pi k}{h}\,w^3\right)dxdy$$

$$+\mu\int_{C_{a,b}}(\nabla\varphi^3-\frac{\pi k}{h}\,\Phi)\cdot(\nabla w^3-\frac{\pi k}{h}\,W)\,dxdy+2\mu\frac{\pi^2k^2}{h^2}\int_{C_{a,b}}\varphi^3\,w^3\,dxdy\quad\text{for any }\boldsymbol{\varphi},\mathbf{w}\in H^1(C_{a,b};\mathbb{R}^3) \tag{5.21}$$

where $\boldsymbol{\varphi}=(\varphi^1,\varphi^2,\varphi^3)$, $\Phi:=(\varphi^1,\varphi^2)$, $\mathbf{w}=(w^1,w^2,w^3)$ and $W=(w^1,w^2)$.

For the uniqueness issue we claim that for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\bar{a}_k(\boldsymbol{\varphi},\boldsymbol{\varphi})+\varepsilon\|\boldsymbol{\varphi}\|_{L^2}^2\geq C_\varepsilon\|\boldsymbol{\varphi}\|_{H^1}^2\qquad\text{for any }\boldsymbol{\varphi}\in H^1(C_{a,b};\mathbb{R}^3)\,. \tag{5.22}$$

Suppose by contradiction that there exists $\varepsilon>0$ such that for any $m\geq 1$ there exists $\boldsymbol{\varphi}_m\in H^1(C_{a,b};\mathbb{R}^3)$ such that

$$\bar{a}_k(\boldsymbol{\varphi}_m,\boldsymbol{\varphi}_m)+\varepsilon\|\boldsymbol{\varphi}_m\|_{L^2}^2\leq\frac{1}{m}\|\boldsymbol{\varphi}_m\|_{H^1}^2\,. \tag{5.23}$$

Up to normalization, it is not restrictive to assume that the sequence $\{\boldsymbol{\varphi}_m\}$ satisfies $\|\boldsymbol{\varphi}_m\|_{H^1}=1$ for any $m\geq 1$, so that by (5.21) and (5.23) we infer

$$\boldsymbol{\varphi}_m\to\mathbf{0}\quad\text{in }L^2(C_{a,b};\mathbb{R}^3)\,,\qquad\int_{C_{a,b}}|\mathbf{D}\Phi_m|^2dxdy\to 0\,,\qquad\nabla\varphi_m^3-k\Phi_m\to\mathbf{0}\quad\text{in }L^2(C_{a,b};\mathbb{R}^2) \tag{5.24}$$

as $m\to+\infty$. Applying (A3) in the two-dimensional case we obtain

$$\int_{C_{a,b}}|\nabla\Phi_m|^2dxdy\leq C\left(\int_{C_{a,b}}|\mathbf{D}\Phi_m|^2dxdy+\int_{C_{a,b}}|\Phi_m|^2\,dxdy\right)$$

for some constant $C>0$. This, combined with (5.24), proves that

$$\nabla\Phi_m\to\mathbf{0}\quad\text{in }L^2(C_{a,b};\mathbb{R}^{2\times 2})\,,\qquad\nabla\varphi_m^3\to\mathbf{0}\qquad\text{in }L^2(C_{a,b};\mathbb{R}^2)$$

and, in turn, that $\boldsymbol{\varphi}_m\to\mathbf{0}$ in $H^1(C_{a,b};\mathbb{R}^3)$. This contradicts the assumption $\|\boldsymbol{\varphi}_m\|_{H^1}=1$. We have completed the proof of the claim (5.22).

Thanks to (5.22), we may proceed as in the proof of Proposition 2.1 and apply the Fredholm alternative to show that (5.17)–(5.18) admits a solution if and only if

$$\int_{C_{a,b}} \Psi_k\, w^3\, dxdy = 0 \qquad \text{for any } \mathbf{w} = (w^1, w^2, w^3) \in \bar{V}_k \tag{5.25}$$

where $\bar{V}_k := \{\mathbf{w} \in H^1(C_{a,b};\mathbb{R}^3) : \bar{a}_k(\mathbf{w},\mathbf{v}) = 0 \text{ for any } \mathbf{v} \in H^1(C_{a,b};\mathbb{R}^3)\}$. Testing the variational identity in the definition of $\bar{V}_k$ with $\mathbf{v} = \mathbf{w}$, we readily see that for any $k \geq 1$ we $\bar{V}_k = \{\mathbf{0}\}$ and hence, condition (5.25) is always satisfied. This completes the proof of the lemma. □

5.4 | Proof of Lemma 3.5

Before proceeding with the proof of the lemma, we devote the first part of this subsection to show that the functions Y_k and Z_k introduced in (3.12) really satisfy (3.14).

In order to simplify the notations we denote by Y and Z the unknown functions, omitting the index k. Testing (5.20) with a test function (w^1, w^2, w^3) admitting in polar coordinates the following representation

$$w^1(\rho,\theta) = H(\rho)\cos\theta\,, \quad w^2(\rho,\theta) = H(\rho)\sin\theta\,, \quad w^3(\rho,\theta) = K(\rho)\,,$$

by (3.12) we obtain

$$2\mu\int_a^b \left[\rho Y'(\rho)H'(\rho) + \frac{Y(\rho)H(\rho)}{\rho}\right]d\rho + \lambda\int_a^b \rho\left[Y'(\rho) + \frac{Y(\rho)}{\rho} + \frac{\pi k}{h}Z(\rho)\right]\left[H'(\rho) + \frac{H(\rho)}{\rho} + \frac{\pi k}{h}K(\rho)\right]d\rho \tag{5.26}$$

$$+ \mu\int_a^b \rho\left[Z'(\rho) - \frac{\pi k}{h}Y(\rho)\right]\left[K'(\rho) - \frac{\pi k}{h}H(\rho)\right]d\rho + 2\mu\frac{\pi^2k^2}{h^2}\int_a^b \rho Z(\rho)K(\rho)\,d\rho = \int_a^b \rho\Psi_k(\rho)K(\rho)\,d\rho$$

with obvious meaning of the notation $\Psi_k(\rho)$ being it a radial function.

Collecting in a proper way the terms of (5.26), we may rewrite it in the form

$$\int_a^b \left[(\lambda+2\mu)\rho Y'(\rho) + \lambda Y(\rho) + \lambda\frac{\pi k}{h}\rho Z(\rho)\right]H'(\rho)\,d\rho \tag{5.27}$$

$$+ \int_a^b \left[\lambda Y'(\rho) + (\lambda+2\mu)\frac{Y(\rho)}{\rho} + \mu\frac{\pi^2k^2}{h^2}\rho Y(\rho) - \mu\frac{\pi k}{h}\rho Z'(\rho) + \lambda\frac{\pi k}{h}Z(\rho)\right]H(\rho)\,d\rho$$

$$+ \int_a^b \left[\mu\rho Z'(\rho) - \mu\frac{\pi k}{h}\rho Y(\rho)\right]K'(\rho)\,d\rho + \int_a^b \left[(\lambda+2\mu)\frac{\pi^2k^2}{h^2}\rho Z(\rho) + \lambda\frac{\pi k}{h}\rho Y'(\rho) + \lambda\frac{\pi k}{h}Y(\rho)\right]K(\rho)\,d\rho$$

$$= \int_a^b \rho\Psi_k(\rho)K(\rho)\,d\rho$$

Integrating by parts the terms in (5.27) containing $H'(\rho)$ and $K'(\rho)$, we see that (5.27) is the variational formulation of (3.14).

Let us proceed now with the proof of the lemma which is the main point of this section. Actually, we give here only a sketch of the proof since it essentially follows the ideas already introduced in the proof of Lemma 3.4.

About the uniqueness issue, on the space $H^1(a,b;\mathbb{R}^2)$ it sufficient to define the bilinear form

$$b_k : H^1(a,b;\mathbb{R}^2) \times H^1(a,b;\mathbb{R}^2) \to \mathbb{R}$$

corresponding to the left hand side of (5.26) and prove for it an estimate of the type (5.22).

Then, following again the proof of Lemma 3.4, one finds that the compatibility condition for Ψ_k is given by

$$\int_a^b \rho \Psi_k(\rho) K(\rho)\, d\rho = 0 \tag{5.28}$$

for any $(H, K) \in H^1(a, b; \mathbb{R}^2)$ satisfying $b_k((H, K), (H, K)) = 0$. A simple check shows that $(H, K) \equiv (0, 0)$ so that (5.28) is trivially satisfied.

The Fredholm alternative then implies the existence of a solution. □

5.5 | Proof of Lemma 3.6

We omit for simplicity the dependence from the index k in the unknowns Y_k and Z_k. For more clarity we divide the construction of this representation of Y and Z into different steps each of them is contained in the next subsections.

5.5.1 | The solution of the homogeneous system

We consider the homogeneous version of the system in (3.14)

$$\begin{cases} Y''(\rho) + \dfrac{Y'(\rho)}{\rho} - \dfrac{Y(\rho)}{\rho^2} - \alpha\, k^2\, Y(\rho) + \beta\, k Z'(\rho) = 0 \qquad \rho > 0\,, \\ Z''(\rho) + \dfrac{Z'(\rho)}{\rho} - \gamma\, k^2 Z(\rho) - \delta\, k \left[Y'(\rho) + \dfrac{Y(\rho)}{\rho} \right] = 0 \quad \rho > 0\,, \end{cases} \tag{5.29}$$

where we put for simplicity

$$\alpha = \frac{\pi^2 \mu}{h^2(\lambda + 2\mu)}\,, \quad \beta = \frac{\pi(\lambda + \mu)}{h(\lambda + 2\mu)}\,, \quad \gamma = \frac{\pi^2(\lambda + 2\mu)}{h^2 \mu}\,, \quad \delta = \frac{\pi(\lambda + \mu)}{h\mu}\,.$$

We look for a solution admitting the following expansion

$$\begin{cases} Y(\rho) = \displaystyle\sum_{n=-1}^{+\infty} a_n\, \rho^n + (\ln \rho) \sum_{n=0}^{+\infty} b_n\, \rho^n\,, \\ Z(\rho) = \displaystyle\sum_{n=0}^{+\infty} c_n\, \rho^n + (\ln \rho) \sum_{n=0}^{+\infty} d_n\, \rho^n\,. \end{cases} \tag{5.30}$$

Inserting the representation (5.30) in the system (5.29), we obtain for each of the two equations the following identities:

$$\begin{aligned} &\sum_{n=-1}^{+\infty} n(n-1) a_n\, \rho^n + \sum_{n=0}^{+\infty} (n-1) b_n\, \rho^n + \sum_{n=0}^{+\infty} n b_n\, \rho^n + (\ln \rho) \sum_{n=0}^{+\infty} n(n-1) b_n\, \rho^n \\ &\quad + \sum_{n=-1}^{+\infty} n a_n\, \rho^n + \sum_{n=0}^{+\infty} b_n\, \rho^n + (\ln \rho) \sum_{n=0}^{+\infty} n b_n\, \rho^n \\ &\quad -\alpha k^2 \sum_{n=1}^{+\infty} a_{n-2}\, \rho^n - \alpha k^2 (\ln \rho) \sum_{n=2}^{+\infty} b_{n-2}\, \rho^n - \sum_{n=-1}^{+\infty} a_n\, \rho^n - (\ln \rho) \sum_{n=0}^{+\infty} b_n\, \rho^n \\ &+\beta k \sum_{n=1}^{+\infty} (n-1) c_{n-1}\, \rho^n + \beta k \sum_{n=1}^{+\infty} d_{n-1}\, \rho^n + \beta k (\ln \rho) \sum_{n=1}^{+\infty} (n-1) d_{n-1}\, \rho^n = 0, \end{aligned} \tag{5.31}$$

$$
\begin{aligned}
&\sum_{n=0}^{+\infty} n(n-1)c_n\,\rho^n + \sum_{n=0}^{+\infty}(n-1)d_n\,\rho^n + \sum_{n=0}^{+\infty} nd_n\,\rho^n + (\ln\rho)\sum_{n=0}^{+\infty} n(n-1)d_n\,\rho^n \\
&\quad + \sum_{n=0}^{+\infty} nc_n\,\rho^n + \sum_{n=0}^{+\infty} d_n\,\rho^n + (\ln\rho)\sum_{n=0}^{+\infty} nd_n\,\rho^n - \gamma k^2\sum_{n=2}^{+\infty} c_{n-2}\,\rho^n - \gamma k^2(\ln\rho)\sum_{n=2}^{+\infty} d_{n-2}\,\rho^n \\
&\quad -\delta k\sum_{n=0}^{+\infty}(n-1)a_{n-1}\,\rho^n - \delta k\sum_{n=1}^{+\infty} b_{n-1}\,\rho^n - \delta k(\ln\rho)\sum_{n=1}^{+\infty}(n-1)b_{n-1}\,\rho^n \\
&-\delta k\sum_{n=0}^{+\infty} a_{n-1}\,\rho^n - \delta k(\ln\rho)\sum_{n=1}^{+\infty} b_{n-1}\,\rho^n = 0.
\end{aligned}
\tag{5.32}
$$

To determine the values of the coefficients a_n, b_n, c_n, d_n we need an iterative scheme starting from the values of the coefficients $a_{-1}, a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$. The values of these nine parameters have to be determined collecting the coefficients of the terms $\rho^{-1}, \rho^0, \rho^0\ln\rho, \rho, \rho\ln\rho$ appearing in (5.31)–(5.32) and equating them to zero.

As a result of this procedure we obtain the following constraint:

$$
\begin{cases}
a_0 = b_0 = c_1 = d_1 = 0\,, \\
2b_1 + \beta k d_0 = \alpha k^2 a_{-1}\,.
\end{cases}
\tag{5.33}
$$

Among the left five parameters $a_{-1}, a_1, b_1, c_0, d_0$ that may be possibly different from zero, a_1, c_0 and two among a_{-1}, b_1, d_0 can be chosen arbitrarily, while the remaining one is determined by the equation in the second line of (5.33); for example, we may choose arbitrarily a_{-1}, a_1, b_1, c_0 and put $d_0 = \frac{\alpha k}{\beta}\,a_{-1} - \frac{2}{\beta k}\,b_1$.

In particular, we are interested in finding the general solution of (5.29) as a linear combination of four linearly independent special solutions, denoted by $\boldsymbol{\Upsilon}^j = (Y^j, Z^j)$ with $j = 1, \dots, 4$. A possible choice for the independent solutions is given respectively by the assumption on the following combinations of coefficients:

$$
\begin{aligned}
&\boldsymbol{\Upsilon}^1 :\ (a_{-1}, a_1, b_1, c_0) = (1,0,0,0)\,, \qquad &&\boldsymbol{\Upsilon}^2 :\ (a_{-1}, a_1, b_1, c_0) = (0,1,0,0)\,, \\
&\boldsymbol{\Upsilon}^3 :\ (a_{-1}, a_1, b_1, c_0) = (0,0,1,0)\,, \qquad &&\boldsymbol{\Upsilon}^4 :\ (a_{-1}, a_1, b_1, c_0) = (0,0,0,1).
\end{aligned}
\tag{5.34}
$$

By (5.31)–(5.32) we deduce the following linear system in the unknowns $a_n, b_n, c_{n-1}, d_{n-1}$ with data expressed in terms of $a_{n-2}, b_{n-2}, c_{n-3}, d_{n-3}$:

$$
\begin{cases}
(n^2-1)a_n + 2nb_n + \beta k(n-1)c_{n-1} + \beta k d_{n-1} = \alpha k^2 a_{n-2} \\
(n^2-1)b_n + \beta k(n-1)d_{n-1} = \alpha k^2 b_{n-2} \\
(n-1)^2 c_{n-1} + 2(n-1)d_{n-1} = \delta k(n-1)a_{n-2} + \delta k b_{n-2} + \gamma k^2 c_{n-3} \\
(n-1)^2 d_{n-1} = \delta k(n-1)b_{n-2} + \gamma k^2 d_{n-3}
\end{cases}
\qquad (n \geq 3).
\tag{5.35}
$$

We observe that the matrix of coefficients associated to system (5.35) is given by

$$
\begin{pmatrix}
n^2-1 & 2n & \beta(n-1)k & \beta k \\
0 & n^2-1 & 0 & \beta(n-1)k \\
0 & 0 & (n-1)^2 & 2(n-1) \\
0 & 0 & 0 & (n-1)^2
\end{pmatrix}
$$

whose determinant is given by $(n-1)^6(n+1)^2 \neq 0$, thus showing that the system is not singular for $n \geq 2$ and hence admits a unique solution.

With the restriction $n \geq 3$ the coefficients a_2, b_2 remained excluded, but their calculation can be obtained from the first two equations of (5.35) by choosing $n = 2$; this gives $a_2 = b_2 = 0$.

The linear independence of $\boldsymbol{\Upsilon}^1, \boldsymbol{\Upsilon}^2, \boldsymbol{\Upsilon}^3, \boldsymbol{\Upsilon}^4$ can be verified by looking at the asymptotic behavior of $Y^j(\rho)$, $j = 1, 2, 3, 4$ as $\rho \to 0^+$ in the four cases (5.34):

$$\textbf{case 1:} \quad Y^1(\rho) \sim \rho^{-1} \quad \text{as } \rho \to 0^+\,; \qquad \textbf{case 2:} \quad Y^2(\rho) \sim \rho \quad \text{as } \rho \to 0^+\,;$$

$$\textbf{case 3:} \quad Y^3(\rho) \sim \rho \ln \rho \quad \text{as } \rho \to 0^+\,; \qquad \textbf{case 4:} \quad Y^4(\rho) = O(\rho^2 \ln \rho) \quad \text{as } \rho \to 0^+\,.$$

Remark 5.1. We observe that, after a suitable scaling, the dependence of system (5.29) from the parameter k can be dropped: given a solution (Y, Z) of (5.29), we may define the functions $\widetilde{Y}(t) = Y\left(\frac{h}{\pi k}\, t\right)$ and $\widetilde{Z}(t) = Z\left(\frac{h}{\pi k}\, t\right)$ in such a way that the couple $(\widetilde{Y}, \widetilde{Z})$ solves system

$$\begin{cases} \widetilde{Y}''(t) + \dfrac{\widetilde{Y}'(t)}{t} - \dfrac{\widetilde{Y}(t)}{t^2} - \widetilde{\alpha}\, \widetilde{Y}(t) + \widetilde{\beta}\, \widetilde{Z}'(t) = 0 & t > 0\,, \\ \widetilde{Z}''(t) + \dfrac{\widetilde{Z}'(t)}{t} - \widetilde{\gamma}\, \widetilde{Z}(t) - \widetilde{\delta} \left[\widetilde{Y}'(t) + \dfrac{\widetilde{Y}(t)}{t} \right] = 0 & t > 0\,, \end{cases} \tag{5.36}$$

where $\widetilde{\alpha} = \mu/(\lambda + 2\mu)$, $\widetilde{\beta} = (\lambda + \mu)/(\lambda + 2\mu)$, $\widetilde{\gamma} = (\lambda + 2\mu)/\mu$ and $\widetilde{\delta} = (\lambda + \mu)/\mu$.

5.5.2 | The particular solution

We write the nonhomogeneous system in the matrix form

$$\begin{pmatrix} Y(\rho) \\ Y'(\rho) \\ Z(\rho) \\ Z'(\rho) \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\rho^2} + \alpha k^2 & -\frac{1}{\rho} & 0 & -\beta k \\ 0 & 0 & 0 & 1 \\ \frac{\delta k}{\rho} & \delta k & \gamma k^2 & -\frac{1}{\rho} \end{pmatrix} \begin{pmatrix} Y(\rho) \\ Y'(\rho) \\ Z(\rho) \\ Z'(\rho) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\mu} \Psi_k(\rho) \end{pmatrix}, \qquad \rho > 0\,, \tag{5.37}$$

where the function $\Psi_k = \Psi_k(\rho)$ is extended trivially outside the interval (a, b).

Maintaining the order of the components, we may write the Wronskian matrix associated with $\boldsymbol{\Upsilon}^1$, $\boldsymbol{\Upsilon}^2$, $\boldsymbol{\Upsilon}^3$, $\boldsymbol{\Upsilon}^4$ in the form

$$\mathbf{W}(\rho) = \begin{pmatrix} Y^1(\rho) & Y^2(\rho) & Y^3(\rho) & Y^4(\rho) \\ (Y^1(\rho))' & (Y^2(\rho))' & (Y^3(\rho))' & (Y^4(\rho))' \\ Z^1(\rho) & Z^2(\rho) & Z^3(\rho) & Z^4(\rho) \\ (Z^1(\rho))' & (Z^2(\rho))' & (Z^3(\rho))' & (Z^4(\rho))' \end{pmatrix},$$

so that a particular solution $\overline{\boldsymbol{\Upsilon}} = (\overline{Y}, \overline{Z})$ of (5.37) is given by (3.16).

5.5.3 | The unique solution of (3.14)

Applying the superposition principle we get (3.15). In order to obtain the unique solution (Y, Z) of the boundary value problem (3.14), it remains to determine the constants C_1, C_2, C_3, C_4 so that the boundary conditions at $\rho = a$ and $\rho = b$ are satisfied.

We check that the constants C_1, C_2, C_3, C_4 are uniquely determined. They solve the system

$$A\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} -\left[(\lambda+2\mu)\overline{Y}'(a) + \frac{\lambda}{a}\,\overline{Y}(a) + \lambda\frac{\pi k}{h}\,\overline{Z}(a)\right] \\ -\left[(\lambda+2\mu)\overline{Y}'(b) + \frac{\lambda}{b}\,\overline{Y}(b) + \lambda\frac{\pi k}{h}\,\overline{Z}(b)\right] \\ -\left[\overline{Z}'(a) - \frac{\pi k}{h}\,\overline{Y}(a)\right] \\ -\left[\overline{Z}'(b) - \frac{\pi k}{h}\,\overline{Y}(b)\right] \end{pmatrix}$$

where the matrix $A = (a_{ij})$, $i, j \in \{1, 2, 3, 4\}$, is given by

$$a_{1j} = (\lambda+2\mu)(Y^j)'(a) + \frac{\lambda}{a}\,Y^j(a) + \lambda\frac{\pi k}{h}\,Z^j(a)\,, \quad a_{2j} = (\lambda+2\mu)(Y^j)'(b) + \frac{\lambda}{b}\,Y^j(b) + \lambda\frac{\pi k}{h}\,Z^j(b)\,,$$

$$a_{3j} = (Z^j)'(a) - \frac{\pi k}{h}\,Y^j(a)\,, \qquad a_{4j} = (Z^j)'(b) - \frac{\pi k}{h}\,Y^j(b)\,.$$

We claim that the matrix A is not singular. Consider the homogeneous linear system $A\mathbf{d} = \mathbf{0}$ with $\mathbf{d} = (D_1, D_2, D_3, D_4)^T$. Then the function $\boldsymbol{\Gamma} = (G, H)$ given by

$$\boldsymbol{\Gamma}(\rho) = D_1\boldsymbol{\Upsilon}^1(\rho) + D_2\boldsymbol{\Upsilon}^2(\rho) + D_3\boldsymbol{\Upsilon}^3(\rho) + D_4\boldsymbol{\Upsilon}^4(\rho)$$

solves system (5.29) coupled with the boundary conditions

$$\begin{cases} (\lambda+2\mu)G'(a) + \frac{\lambda}{a}\,G(a) + \lambda\frac{\pi k}{h}H(a) = 0\,, \\ (\lambda+2\mu)G'(b) + \frac{\lambda}{b}\,G(b) + \lambda\frac{\pi k}{h}H(b) = 0\,, \\ H'(a) - \frac{\pi k}{h}G(a) = 0\,, \\ H'(b) - \frac{\pi k}{h}G(b) = 0\,. \end{cases}$$

By Lemma 3.5 we then have that $\boldsymbol{\Gamma} \equiv (0,0)$ in (a,b) but being $\boldsymbol{\Gamma}$ also a solution of system (5.29) for $\rho \in (0,+\infty)$, by local uniqueness for Cauchy problems, $\boldsymbol{\Gamma} \equiv (0,0)$ in $(0,+\infty)$. The linear independence of the functions $\boldsymbol{\Upsilon}^1$, $\boldsymbol{\Upsilon}^2$, $\boldsymbol{\Upsilon}^3$, $\boldsymbol{\Upsilon}^4$, then implies $D_1 = D_2 = D_3 = D_4 = 0$.

We just proved that the linear system $A\mathbf{d} = \mathbf{0}$ admits only the trivial solution, thus completing the proof. □

5.6 | Proof of Theorem 3.7

The formal series contained in (3.18) are a consequence of (3.8), Lemma 3.4 and Lemma 3.5.

It remains to show how those series converge. We start by proving the weak convergence in $H^1(\Omega)$. Let $\boldsymbol{\mathcal{F}}$ be the linear functional defined by $\boldsymbol{\mathcal{F}}(\mathbf{v}) := \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\,dS$ for any $\mathbf{v} \in H^1(\Omega;\mathbb{R}^3)$ with $\mathbf{g}$ as in (3.1). We observe that thanks to the Hölder inequality and the trace inequality $\boldsymbol{\mathcal{F}} \in (H^1(\Omega;\mathbb{R}^3))'$:

$$\left|{}_{(H^1(\Omega;\mathbb{R}^3))'}\langle \boldsymbol{\mathcal{F}}, \mathbf{v}\rangle_{H^1(\Omega;\mathbb{R}^3)}\right| \le 2p\sqrt{\pi(b^2-a^2)}\,C(\Omega)\,\|\mathbf{v}\|_{H^1(\Omega;\mathbb{R}^3)} \qquad \text{for any } \mathbf{v} \in H^1(\Omega;\mathbb{R}^3)\,,$$

where $C(\Omega)$ is such that $\|\mathrm{trace}(v)\|_{L^2(\partial\Omega)} \le C(\Omega)\|v\|_{H^1(\Omega)}$ for any $v \in H^1(\Omega)$.

Writing $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ we have that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in (H^1(\Omega))'$ with $\mathcal{F}_1 = \mathcal{F}_2$ are the null functionals and

$${}_{(H^1(\Omega))'}\langle \mathcal{F}_3, v\rangle_{H^1(\Omega)} = -\int_{C_{a,b}} \chi_p(x,y)\left[v\left(x,y,\frac{h}{2}\right) - v\left(x,y,-\frac{h}{2}\right)\right]dxdy \quad \text{for any } v \in H^1(\Omega)\,.$$

Let us define the sequence of partial sums corresponding to the Fourier expansion in (3.11):

$$S_M(x,y,z) := \chi_p(x,y)\sum_{m=0}^{M}(-1)^{m+1}\frac{4}{h}\sin\left[\frac{\pi}{h}(2m+1)z\right].$$

We claim that $S_M \rightharpoonup \mathcal{F}_3$ weakly in $(H^1(\Omega))'$ as $M \to +\infty$. We first prove that the sequence $\{S_M\}$ is bounded in $(H^1(\Omega))'$. In the next estimate we use the following notations: we put $\widetilde{\Omega} := C_{a,b} \times \left(-\frac{h}{2}, \frac{3h}{2}\right)$, we still denote by v the symmetric and $2h$-periodic extension of a function $v \in H^1(\Omega)$ (see Section 3.1) and by $v\left(x, y, \frac{3h}{2}\right)$ and $v\left(x, y, -\frac{h}{2}\right)$, the traces of a function $v \in H^1(\widetilde{\Omega})$ on the upper and lower faces of the hollow cylinder $\widetilde{\Omega}$, respectively.

$$\left|{}_{(H^1(\Omega))'}\langle S_M, v\rangle_{H^1(\Omega)}\right| = \left|\sum_{m=0}^{M}(-1)^{m+1}\frac{4}{h}\int_{\Omega}\chi_p(x,y)\sin\left[\frac{\pi}{h}(2m+1)z\right]v(x,y,z)\,dxdydz\right|$$

$$= \left|\sum_{m=0}^{M}(-1)^{m+1}\frac{2}{h}\int_{\widetilde{\Omega}}\chi_p(x,y)\sin\left[\frac{\pi}{h}(2m+1)z\right]v(x,y,z)\,dxdydz\right|$$

$$\text{(integration by parts)} = \left|\sum_{m=0}^{M}(-1)^{m+1}\frac{2}{h}\left\{\int_{C_{a,b}} -\frac{h\cos\left[\frac{3\pi}{2}(2m+1)\right]\chi_p(x,y)}{\pi(2m+1)}\, v\left(x,y,\frac{3h}{2}\right)dxdy\right.\right.$$

$$+ \int_{C_{a,b}}\frac{h\cos\left[-\frac{\pi}{2}(2m+1)\right]\chi_p(x,y)}{\pi(2m+1)}\, v\left(x,y,-\frac{h}{2}\right)dxdy$$

$$\left.\left. + \int_{\widetilde{\Omega}}\frac{h\chi_p(x,y)}{\pi(2m+1)}\cos\left[\frac{\pi}{h}(2m+1)z\right]\frac{\partial v}{\partial z}(x,y,z)\,dxdydz\right\}\right|$$

$$\text{(}2h\text{-periodicity of } v\text{)} = \left|\sum_{m=0}^{M}(-1)^{m+1}\frac{2}{h}\left\{\int_{\widetilde{\Omega}}\frac{h\chi_p(x,y)}{\pi(2m+1)}\cos\left[\frac{\pi}{h}(2m+1)z\right]\frac{\partial v}{\partial z}(x,y,z)\,dxdydz\right\}\right|$$

$$= \left|\frac{2}{\pi}\int_{C_{a,b}}\left\{\sum_{m=0}^{M}\frac{(-1)^{m+1}\chi_p(x,y)}{2m+1}\int_{-\frac{h}{2}}^{\frac{3h}{2}}\cos\left[\frac{\pi}{h}(2m+1)z\right]\frac{\partial v}{\partial z}(x,y,z)\,dz\right\}dxdy\right|$$

(Cauchy-Schwarz inequality in $\mathbb{R}^{n+1}$)

$$\le \frac{2p}{\pi}\left[\sum_{m=0}^{M}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}}\int_{C_{a,b}}\left[\sum_{m=0}^{M}\left(\int_{-\frac{h}{2}}^{\frac{3h}{2}}\cos\left[\frac{\pi}{h}(2m+1)z\right]\frac{\partial v}{\partial z}(x,y,z)\,dz\right)^2\right]^{\frac{1}{2}}dxdy$$

$$\text{(Bessel inequality)} \le \frac{2p}{\pi}\left[\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}}\int_{C_{a,b}}\left[\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{3h}{2}}\left(\frac{\partial v}{\partial z}(x,y,z)\right)^2dz\right]^{\frac{1}{2}}dxdy$$

$$\text{(Hölder inequality)} \le \frac{2p}{\pi}\left[\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}}\sqrt{\pi(b^2-a^2)}\left\{\int_{C_{a,b}}\left[\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{3h}{2}}\left(\frac{\partial v}{\partial z}(x,y,z)\right)^2dz\right]dxdy\right\}^{\frac{1}{2}}$$

$$= 2p\sqrt{\frac{b^2-a^2}{\pi h}}\left[\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}}\left\|\frac{\partial v}{\partial z}\right\|_{L^2(\widetilde{\Omega})}$$

$$\le 4p\sqrt{\frac{b^2-a^2}{\pi h}}\left[\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}}\|v\|_{H^1(\Omega)}.$$

This readily implies

$$\|S_M\|_{(H^1(\Omega))'} \leq 4p\sqrt{\frac{b^2-a^2}{\pi h}}\left[\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}} \tag{5.38}$$

and boundedness of $\{S_M\}$ in $(H^1(\Omega))'$ is proved.

Now we claim that

$$\int_\Omega S_M\phi\, d\mathbf{x} \to -\int_{C_{a,b}} \chi_p(x,y)\left[\phi\left(x,y,\frac{h}{2}\right)-\phi\left(x,y,-\frac{h}{2}\right)\right]dxdy =_{(H^1(\Omega))'} \langle \mathcal{F}_3,\phi\rangle_{H^1(\Omega)} \tag{5.39}$$

as $M\to+\infty$, for any $\phi\in C^\infty(\overline{\Omega})$. First of all, by using the classical results about pointwise convergence of the Fourier Series applied to suitable $2h$-periodic extensions of the functions

$$z\mapsto\phi(x,y,z)\,,\; z\in\left[-\frac{h}{2},0\right] \qquad \text{and} \qquad z\mapsto\phi(x,y,z)\,,\; z\in\left[0,\frac{h}{2}\right]$$

one can show that for any $(x,y)\in C_{a,b}$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} S_M(x,y,z)\phi(x,y,z)dz \to -\chi_p(x,y)\left[\phi\left(x,y,\frac{h}{2}\right)-\phi\left(x,y,-\frac{h}{2}\right)\right]. \tag{5.40}$$

Then applying to the test function ϕ the estimates used for proving boundedness of $\{S_M\}$ in $(H^1(\Omega))'$, one can show that for any M

$$\left|\int_{-\frac{h}{2}}^{\frac{h}{2}} S_M(x,y,z)\phi(x,y,z)dz\right| \leq \frac{2\sqrt{2}p}{\pi}\left[\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2}\right]^{\frac{1}{2}}\left\|\frac{\partial\phi}{\partial z}\right\|_{L^\infty(\Omega)} \qquad \text{for any } (x,y)\in C_{a,b}\,. \tag{5.41}$$

By (5.40), (5.41) and the Dominated Convergence Theorem the proof of (5.39) follows. With an essentially similar procedure one can prove that S_M converges in the sense of distributions to Λ_3 where Λ_3 is the third component of the vector distribution $\boldsymbol{\Lambda}$ defined in (5.14).

Since $(H^1(\Omega))'$ is a reflexive Banach space, by (5.38) we infer that along suitable subsequences, the partial sums are weakly convergent in $(H^1(\Omega))'$. Thanks to (5.39), we deduce that the weak limits of this subsequences coincide on the space $C^\infty(\overline{\Omega})$ and they equal $\mathcal{F}_3$ on it. By density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$, they actually coincide on the whole $H^1(\Omega)$. This proves that all weakly convergent subsequences weakly converge to $\mathcal{F}_3$ and hence the sequence S_M is itself weakly convergent to $\mathcal{F}_3$ in $(H^1(\Omega))'$. We can now denote by $\boldsymbol{S}_M=(0,0,S_M)$ the sequence of vector partial sums in such a way that $\boldsymbol{S}_M \rightharpoonup \boldsymbol{\mathcal{F}}$ weakly in $(H^1(\Omega;\mathbb{R}^3))'$ as $M\to+\infty$.

Now, let us consider the linear continuous operator L introduced in the proof of Proposition 2.1 and its restriction to $V_0^\perp$, where we recall that orthogonality is with respect to the scalar product (A4). Then, by Proposition 2.1 (iv) we deduce that $L_{|V_0^\perp}: V_0^\perp\to(H^1(\Omega;\mathbb{R}^3))'$ is invertible and by the Open Mapping Theorem it follows that its inverse is continuous.

If we define $\boldsymbol{U}_M := L^{-1}_{|V_0^\perp}\boldsymbol{S}_M$, then $\boldsymbol{U}_M$ is the vector partial sum corresponding to the Fourier expansion (3.18). Since $\boldsymbol{S}_M$ is weakly convergent in $(H^1(\Omega;\mathbb{R}^3))'$ to $\boldsymbol{\mathcal{F}}$, then the continuity of $L^{-1}_{|V_0^\perp}$ implies that $\boldsymbol{U}_M$ is weakly convergent in $H^1(\Omega;\mathbb{R}^3)$ to the unique solution $\mathbf{u}$ of (3.3) as $M\to+\infty$.

The strong convergence $\boldsymbol{U}_M\to\mathbf{u}$ in $L^2(\Omega;\mathbb{R}^3)$ as $M\to+\infty$ is a consequence of the compactness of the embedding $H^1(\Omega;\mathbb{R}^3)\subset L^2(\Omega;\mathbb{R}^3)$.

It remains to prove (3.19). In order to emphasize the dependence on k we reintroduce it for denoting the functions Y_k and Z_k appearing in the proof of Lemma 3.5. Testing (5.26) with $(H,K)=(Y_k,Z_k)$ we have

$$\frac{2\mu\pi^2}{h^2}k^2\int_a^b \rho(Z_k(\rho))^2d\rho \leq \int_a^b \rho\Psi_k(\rho)Z_k(\rho)\,d\rho$$

from which we obtain

$$\left(\int_a^b (Z_k(\rho))^2 d\rho\right)^{\frac{1}{2}} \leq \frac{2pbh\sqrt{b-a}}{\mu a\pi^2}\frac{1}{k^2}. \tag{5.42}$$

Testing again (5.26) with $(H, K) = (Y_k, Z_k)$ we also have

$$2\mu\int_a^b \frac{(Y_k(\rho))^2}{\rho}\,d\rho \leq \int_a^b \rho\Psi_k(\rho)Z_k(\rho)\,d\rho$$

from which we obtain

$$\int_a^b (Y_k(\rho))^2\,d\rho \leq \frac{2pb^2\sqrt{b-a}}{\mu h}\left(\int_a^b (Z_k(\rho))^2 d\rho\right)^{\frac{1}{2}} \leq \frac{4p^2b^3(b-a)}{\mu^2 a\pi^2}\frac{1}{k^2} \tag{5.43}$$

where in the last inequality we used (5.42).

Let us proceed by considering the difference between the partial sum for u_1 and u_1 itself:

$$\begin{aligned}
\|U_M^1 - u_1\|^2_{L^2(\Omega)} &= \int_{C_{a,b}} \left\| \sum_{m=M+1}^{+\infty} Y_{2m+1}(\rho)\cos\theta\cos\left[\frac{(2m+1)\pi}{h}z\right]\right\|^2_{L^2\left(-\frac{h}{2},\frac{h}{2}\right)} dxdy \\
&= \frac{h}{2}\sum_{m=M+1}^{+\infty}\int_{C_{a,b}} (\cos^2\theta)(Y_{2m+1}(\rho))^2 dxdy = \frac{\pi h}{2}\sum_{m=M+1}^{+\infty}\int_a^b (Y_{2m+1}(\rho))^2\rho\,d\rho \\
&\leq \frac{2p^2b^4(b-a)h}{\mu^2 a\pi}\sum_{m=M+1}^{+\infty}\frac{1}{(2m+1)^2} \leq \frac{p^2b^4(b-a)h}{2\mu^2 a\pi}\sum_{m=M+1}^{+\infty}\frac{1}{m^2} \leq \frac{p^2b^4(b-a)h}{2\mu^2 a\pi}\frac{1}{M},
\end{aligned}$$

where we also used (5.43). The estimate for $\|U_n^2 - u_2\|_{L^2(\Omega)}$ gives the same result for obvious reasons.

With a completely similar procedure by exploiting this time (5.42), we obtain

$$\|U_n^3 - u_3\|^2_{L^2(\Omega)} \leq \frac{2p^2b^3h^3(b-a)}{\mu^2a^2\pi^4}\sum_{m=M+1}^{+\infty}\frac{1}{(2m+1)^4} \leq \frac{p^2b^3h^3(b-a)}{8\mu^2a^2\pi^4}\sum_{m=M+1}^{+\infty}\frac{1}{m^4} \leq \frac{p^2b^3h^3(b-a)}{24\mu^2a^2\pi^4}\frac{1}{M^3}.$$

Let us denote by $\mathbf{u}$ the solution we found by means of the Fourier series expansion. Then, $\mathbf{u}$ satisfies (3.9) in the sense of traces of H^1-functions. Moreover, by construction, see subsections 3.1–3.2 for more details, the function $\mathbf{u}$ is a solution of the variational problem (2.3) with $\mathbf{f} \equiv 0$ and $\mathbf{g}$ as in (3.1). In particular, as it follows from a classical argument based on integration by parts, $\mathbf{u}$ is a solution of (3.3), in the sense that it satisfies the equations of the linear elasticity coupled with the Neumann-type boundary conditions on Γ_1, Γ_2 and Γ_3.

We conclude the proof of the theorem by observing that $\mathbf{u}$ coincides with the unique solution of (3.3) belonging to $V_0^\perp$. To see this, denote by $\mathbf{w}$ the solution in $V_0^\perp$. Both $\mathbf{u}$ and $\mathbf{w}$ possesses the symmetry properties stated in Proposition 3.1 as it occurs to their difference $\mathbf{u} - \mathbf{w}$. But from Proposition 2.1 we have that $\mathbf{u} - \mathbf{w} \in V_0$ and it is readily seen from (2.7) that functions in V_0 satisfying those symmetry properties are necessarily the null function. This proves that $\mathbf{u} = \mathbf{w}$ and completes the proof of the theorem. □

5.7 | Proof of Proposition 4.1

We rewrite the homogeneous system (5.29) as in (5.36) so that the corresponding series expansion can be written in the form

$$\begin{cases} \widetilde{Y}(t) = \sum_{n=-1}^{+\infty} \widetilde{a}_n t^n + (\ln t) \sum_{n=0}^{+\infty} \widetilde{b}_n t^n , \\ \widetilde{Z}(t) = \sum_{n=0}^{+\infty} \widetilde{c}_n t^n + (\ln t) \sum_{n=0}^{+\infty} \widetilde{d}_n t^n . \end{cases} \tag{5.44}$$

The coefficients $\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n, \widetilde{d}_n$ are related to the corresponding coefficients a_n, b_n, c_n, d_n appearing in (5.30), by the formulas

$$\widetilde{a}_{-1} = \frac{\pi k}{h} a_{-1}, \qquad \widetilde{a}_n = \left(\frac{h}{\pi k}\right)^n \left[a_n - \ln\left(\frac{\pi k}{h}\right) b_n\right], \qquad \widetilde{b}_n = \left(\frac{h}{\pi k}\right)^n b_n, \tag{5.45}$$

$$\widetilde{c}_n = \left(\frac{h}{\pi k}\right)^n \left[c_n - \ln\left(\frac{\pi k}{h}\right) d_n\right], \qquad \widetilde{d}_n = \left(\frac{h}{\pi k}\right)^n d_n .$$

Inserting (5.44) into (5.36) or alternatively combining (5.45) and (5.35), we see that $\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_n, \widetilde{d}_n$ solve the system

$$\begin{cases} (n^2-1)\widetilde{a}_n + 2n\widetilde{b}_n + \beta(n-1)\widetilde{c}_{n-1} + \widetilde{\beta}\widetilde{d}_{n-1} = \widetilde{\alpha}\widetilde{a}_{n-2} \\ (n^2-1)\widetilde{b}_n + \widetilde{\beta}(n-1)\widetilde{d}_{n-1} = \widetilde{\alpha}\widetilde{b}_{n-2} \\ (n-1)^2\widetilde{c}_{n-1} + 2(n-1)\widetilde{d}_{n-1} = \widetilde{\delta}(n-1)\widetilde{a}_{n-2} + \widetilde{\delta}\widetilde{b}_{n-2} + \widetilde{\gamma}\widetilde{c}_{n-3} \\ (n-1)^2\widetilde{d}_{n-1} = \widetilde{\delta}(n-1)\widetilde{b}_{n-2} + \widetilde{\gamma}\widetilde{d}_{n-3} \end{cases} \tag{5.46}$$

for $n \geq 3$; moreover $\widetilde{a}_0 = \widetilde{b}_0 = \widetilde{c}_1 = \widetilde{d}_1 = \widetilde{a}_2 = \widetilde{b}_2 = 0$, the coefficients $\widetilde{a}_{-1}, \widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_0$ may be chosen arbitrarily and $\widetilde{d}_0 = \frac{\widetilde{\alpha}}{\widetilde{\beta}} \widetilde{a}_{-1} - \frac{2}{\widetilde{\beta}} \widetilde{b}_1$.

By direct computation one can verify that the unique solution of system (5.46) can be written in form

$$\begin{pmatrix} \widetilde{a}_n \\ \widetilde{b}_n \\ \widetilde{c}_{n-1} \\ \widetilde{d}_{n-1} \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{\mu}\frac{1}{(n+1)(n-1)} & \frac{2\lambda}{\mu}\frac{n}{(n+1)^2(n-1)^2} & -\frac{\lambda+\mu}{\mu}\frac{1}{(n+1)(n-1)^2} & \frac{\lambda+\mu}{\mu}\frac{3n+1}{(n+1)^2(n-1)^3} \\ 0 & -\frac{\lambda}{\mu}\frac{1}{(n+1)(n-1)} & 0 & -\frac{\lambda+\mu}{\mu}\frac{1}{(n+1)(n-1)^2} \\ \frac{\lambda+\mu}{\mu}\frac{1}{n-1} & -\frac{\lambda+\mu}{\mu}\frac{1}{(n-1)^2} & \frac{\lambda+2\mu}{\mu}\frac{1}{(n-1)^2} & -\frac{2(\lambda+2\mu)}{\mu}\frac{1}{(n-1)^3} \\ 0 & \frac{\lambda+\mu}{\mu}\frac{1}{n-1} & 0 & \frac{\lambda+2\mu}{\mu}\frac{1}{(n-1)^2} \end{pmatrix} \begin{pmatrix} \widetilde{a}_{n-2} \\ \widetilde{b}_{n-2} \\ \widetilde{c}_{n-3} \\ \widetilde{d}_{n-3} \end{pmatrix} \tag{5.47}$$

for any $n \geq 3$.

We are interested in the case n odd since when n is even, thanks to (5.46), we know that $\widetilde{a}_n = \widetilde{b}_n = \widetilde{c}_{n-1} = \widetilde{d}_{n-1} = 0$. Looking at (5.47), for any $n \geq 3$ odd, we introduce the matrices

$$S_n := \begin{pmatrix} -\frac{\lambda}{\mu}\frac{1}{(n+1)(n-1)} & -\frac{\lambda+\mu}{\mu}\frac{1}{(n+1)(n-1)^2} \\ \frac{\lambda+\mu}{\mu}\frac{1}{n-1} & \frac{\lambda+2\mu}{\mu}\frac{1}{(n-1)^2} \end{pmatrix}, \qquad T_n := \begin{pmatrix} \frac{2\lambda}{\mu}\frac{n}{(n+1)^2(n-1)^2} & \frac{\lambda+\mu}{\mu}\frac{3n+1}{(n+1)^2(n-1)^3} \\ -\frac{\lambda+\mu}{\mu}\frac{1}{(n-1)^2} & -\frac{2(\lambda+2\mu)}{\mu}\frac{1}{(n-1)^3} \end{pmatrix}.$$

In this way, system (5.47) may written in the form

$$\begin{pmatrix} \widetilde{b}_n \\ \widetilde{d}_{n-1} \end{pmatrix} = S_n \begin{pmatrix} \widetilde{b}_{n-2} \\ \widetilde{d}_{n-3} \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{a}_n \\ \widetilde{c}_{n-1} \end{pmatrix} = S_n \begin{pmatrix} \widetilde{a}_{n-2} \\ \widetilde{c}_{n-3} \end{pmatrix} - T_n \begin{pmatrix} \widetilde{b}_{n-2} \\ \widetilde{d}_{n-3} \end{pmatrix}.$$

After an iterative procedure we may write

$$\begin{cases} \begin{pmatrix} \widetilde{b}_n \\ \widetilde{d}_{n-1} \end{pmatrix} = \left(\prod_{m=0}^{(n-3)/2} S_{n-2m} \right) \begin{pmatrix} \widetilde{b}_1 \\ \widetilde{d}_0 \end{pmatrix}, \\ \begin{pmatrix} \widetilde{a}_n \\ \widetilde{c}_{n-1} \end{pmatrix} = \left(\prod_{m=0}^{(n-3)/2} S_{n-2m} \right) \begin{pmatrix} \widetilde{a}_1 \\ \widetilde{c}_0 \end{pmatrix} - \sum_{j=0}^{(n-3)/2} \left[\left(\prod_{m=1}^{j} S_{n-2m+2} \right) T_{n-2j} \left(\prod_{m=j+1}^{(n-3)/2} S_{n-2m} \right) \begin{pmatrix} \widetilde{b}_1 \\ \widetilde{d}_0 \end{pmatrix} \right], \end{cases} \tag{5.48}$$

for any $n \geq 3$ odd, with the convention that for any sequence of matrices $A_m \in \mathbb{R}^{2\times 2}$

$$\prod_{m=m_1}^{m_2} A_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \sum_{m=m_1}^{m_2} A_m = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

whenever $m_1 > m_2$.

By induction one can verify that for any $j \leq \frac{n-3}{2}$

$$\prod_{m=0}^{j} S_{n-2m} = \begin{pmatrix} -\frac{(j+1)\lambda + j\mu}{\mu(n+1)[n+1-2(j+1)]\prod_{m=1}^{j}(n+1-2m)^2} & -\frac{(j+1)(\lambda+\mu)}{\mu(n+1)\prod_{m=1}^{j+1}(n+1-2m)^2} \\ \frac{(j+1)(\lambda+\mu)}{\mu[n+1-2(j+1)]\prod_{m=1}^{j}(n+1-2m)^2} & \frac{(j+1)\lambda+(j+2)\mu}{\mu\prod_{m=1}^{j+1}(n+1-2m)^2} \end{pmatrix}$$

and, in turn, by (A12) we infer

$$\left\| \prod_{m=0}^{j} S_{n-2m} \right\|_\infty \leq \frac{(\lambda+\mu)(n-2j)(j+2)}{\mu\prod_{m=1}^{j+1}(n+1-2m)^2}. \tag{5.49}$$

In particular, with appropriate choices of the minimum and the maximum values of the index in the product (5.49) and with appropriate changes of index, for any $n \geq 3$ odd, we obtain the estimates

$$\left\| \prod_{m=0}^{(n-3)/2} S_{n-2m} \right\|_\infty \leq \frac{3(\lambda+\mu)(n+1)}{\mu 2^n \left[\left(\frac{n-1}{2}\right)!\right]^2}, \qquad \left\| \prod_{m=1}^{j} S_{n-2m+2} \right\|_\infty \leq \frac{(\lambda+\mu)(n-2j+2)(j+1)}{\mu\prod_{m=1}^{j}(n+1-2m)^2}, \tag{5.50}$$

$$\left\| \prod_{m=j+1}^{(n-3)/2} S_{n-2m} \right\|_\infty \leq \frac{3(\lambda+\mu)(n-2j-1)}{2\mu\prod_{m=j+2}^{(n-1)/2}(n+1-2m)^2}.$$

On the other hand, we observe that for the components of the matrices S_n and T_n the following inequalities hold true:

$$|(T_n)_{ij}| \leq \frac{3}{n-1} |(S_n)_{ij}| \qquad \text{for any } i, j \in \{1,2\} \text{ and } n \geq 3,$$

which, in turn, implies $\|T_n\|_\infty \leq \frac{3}{n-1}\|S_n\|_\infty = 3(\lambda+\mu)\frac{2n}{\mu(n-1)^3}$; the last inequality is obtained by (5.49) with $j = 0$.

Therefore, combining (5.49) and (5.50), for any $n \geq 3$ odd, we obtain

$$\left\| \sum_{j=0}^{(n-3)/2} \left[\left(\prod_{m=1}^{j} S_{n-2m+2} \right) T_{n-2j} \left(\prod_{m=j+1}^{(n-3)/2} S_{n-2m} \right) \right] \right\|_\infty \tag{5.51}$$

$$\leq \sum_{j=0}^{(n-3)/2} \left\| \prod_{m=1}^{j} S_{n-2m+2} \right\|_\infty \left\| T_{n-2j} \right\|_\infty \left\| \prod_{m=j+1}^{(n-3)/2} S_{n-2m} \right\|_\infty$$

$$\le \sum_{j=0}^{(n-3)/2} \frac{18(\lambda+\mu)^3(n-2j+2)(n-2j)(j+1)}{\mu^3\,2^n\left[\left(\frac{n-1}{2}\right)!\right]^2} \le \frac{9(\lambda+\mu)^3\,n(n+2)(n^2-1)}{4\mu^3\,2^n\left[\left(\frac{n-1}{2}\right)!\right]^2},$$

where in the last inequality we used the estimate $(n-2j+2)(n-2j) \le n(n+2)$ and the identity $\sum_{j=0}^{(n-3)/2}(j+1) = \frac{n^2-1}{8}$.

Combining (A11) with (5.48), (5.50) and (5.51), for any $n \ge 3$ odd, we obtain

$$\left|\begin{pmatrix}\widetilde{a}_n\\ \widetilde{c}_{n-1}\end{pmatrix}\right|_\infty \le \frac{3(\lambda+\mu)(n+1)}{\mu 2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\left|\begin{pmatrix}\widetilde{a}_1\\ \widetilde{c}_0\end{pmatrix}\right|_\infty + \frac{9(\lambda+\mu)^3\,n(n+2)(n^2-1)}{4\mu^3\,2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\left|\begin{pmatrix}\widetilde{b}_1\\ \widetilde{d}_0\end{pmatrix}\right|_\infty \tag{5.52}$$

$$\le \frac{3(2\lambda+5\mu)(\lambda+\mu)^2\,(n+1)(3n^3+3n^2-6n+4)}{4\mu^3\,2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}$$

and

$$\left|\begin{pmatrix}\widetilde{b}_n\\ \widetilde{d}_{n-1}\end{pmatrix}\right|_\infty \le \frac{3(\lambda+\mu)(n+1)}{\mu 2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\left|\begin{pmatrix}\widetilde{b}_1\\ \widetilde{d}_0\end{pmatrix}\right|_\infty \le \frac{3(2\lambda+5\mu)(n+1)}{\mu 2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\} \tag{5.53}$$

where we exploited the fact that $\widetilde{d}_0 = \frac{\widetilde{\alpha}}{\widetilde{\beta}}\widetilde{a}_{-1} - \frac{2}{\widetilde{\beta}}\widetilde{b}_1$, accordingly with what already explained in the lines below (5.46), so that

$$|\widetilde{d}_0| \le \frac{\widetilde{\alpha}+2}{\widetilde{\beta}}\max\{\widetilde{a}_{-1},\widetilde{b}_1\} = \frac{2\lambda+5\mu}{\lambda+\mu}\max\{\widetilde{a}_{-1},\widetilde{b}_1\},$$

from which it follows that

$$\max\left\{\left|\begin{pmatrix}\widetilde{a}_1\\ \widetilde{c}_0\end{pmatrix}\right|_\infty, \left|\begin{pmatrix}\widetilde{b}_1\\ \widetilde{d}_0\end{pmatrix}\right|_\infty\right\} \le \frac{2\lambda+5\mu}{\lambda+\mu}\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}.$$

Since we are interested to the restrictions of the functions Y and Z to the interval $[a,b]$, we have to evaluate the series expansion (5.44) of the functions $\widetilde{Y}$ and $\widetilde{Z}$ for $t \in [\frac{\pi k}{h}a, \frac{\pi k}{h}b]$.

Let N odd be the number at which we want to truncate the series expansions in (5.44). Recalling that the coefficients $\widetilde{a}_n, \widetilde{b}_n, \widetilde{c}_{n-1}, \widetilde{d}_{n-1}$ vanish for n even, we may write

$$\begin{cases} \widetilde{Y}(t) = \left(\displaystyle\sum_{n=-1}^{N}\widetilde{a}_n\,t^n + (\ln t)\sum_{n=0}^{N}\widetilde{b}_n\,t^n\right) + \left(\displaystyle\sum_{n=N+2}^{+\infty}\widetilde{a}_n\,t^n + (\ln t)\sum_{n=N+2}^{+\infty}\widetilde{b}_n\,t^n\right), \\ \widetilde{Z}(t) = \left(\displaystyle\sum_{n=0}^{N-1}\widetilde{c}_n\,t^n + (\ln t)\sum_{n=0}^{N-1}\widetilde{d}_n\,t^n\right) + \left(\displaystyle\sum_{n=N+1}^{+\infty}\widetilde{c}_n\,t^n + (\ln t)\sum_{n=N+1}^{+\infty}\widetilde{d}_n\,t^n\right), \end{cases}$$

and define the truncation error as

$$E_{k,N} = \max\left\{\max_{t\in\left[\frac{\pi k}{h}a,\frac{\pi k}{h}b\right]}\left|\sum_{n=N+2}^{+\infty}\widetilde{a}_n\,t^n + (\ln t)\sum_{n=N+2}^{+\infty}\widetilde{b}_n\,t^n\right|, \max_{t\in\left[\frac{\pi k}{h}a,\frac{\pi k}{h}b\right]}\left|\sum_{n=N+1}^{+\infty}\widetilde{c}_n\,t^n + (\ln t)\sum_{n=N+1}^{+\infty}\widetilde{d}_n\,t^n\right|\right\}$$

By (5.52) and (5.53), we see that for any $t \in [\frac{\pi k}{h}a, \frac{\pi k}{h}b]$ we have

$$0 \le E_{k,N} \le \widetilde{C}(a,b,k)\left[\sum_{n=N+2}^{+\infty}\left(\frac{\pi k b}{h}\right)^n\left|\begin{pmatrix}\widetilde{a}_n\\ \widetilde{c}_{n-1}\end{pmatrix}\right|_\infty + \sum_{n=N+2}^{+\infty}\left(\frac{\pi k b}{h}\right)^n\left|\begin{pmatrix}\widetilde{b}_n\\ \widetilde{d}_{n-1}\end{pmatrix}\right|_\infty\right]$$

$$\leq \widetilde{C}(a,b,k)\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}\sum_{\substack{n=N+2\\ n\text{ odd}}}^{+\infty}\left(\frac{\pi kb}{h}\right)^n\frac{3(2\lambda+5\mu)(\lambda+\mu)^2\,(n+1)(3n^3+3n^2-6n+8)}{4\mu^3\,2^n\left[\left(\frac{n-1}{2}\right)!\right]^2}$$

where we put $\widetilde{C}(a,b,k)=\max\{1,\frac{h}{\pi kb}\}\max\{1,|\ln(\frac{\pi ka}{h})|,|\ln(\frac{\pi kb}{h})|\}$.

Since we are interested to truncation of the series expansion with a sufficiently large number of terms, letting $P(n):=(n+1)(3n^3+3n^2-6n+8)$, it is not restrictive to assume $N\geq 3$ in such a way that the sequence $n\mapsto 2^{-n}P(n)$ becomes decreasing for $n\geq N+2\geq 5$.

In this way, for $N\geq 3$ odd, we obtain for all $t\in[\frac{\pi k}{h}a,\frac{\pi k}{h}b]$

$$0\leq E_{k,N}\leq \widetilde{C}(a,b,k)\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}\frac{3(2\lambda+5\mu)(\lambda+\mu)^2\,P(N+2)}{4\mu^3\,2^{N+2}\left(\frac{N+1}{2}\right)!}\sum_{m=\frac{N+1}{2}}^{+\infty}\frac{\left(\frac{\pi kb}{h}\right)^{2m+1}}{m!} \tag{5.54}$$

$$\leq \widetilde{C}(a,b,k)\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}\left(\frac{\pi kb}{h}\right)^{N+2}e^{\left(\frac{\pi kb}{h}\right)^2}\frac{3(2\lambda+5\mu)(\lambda+\mu)^2\,P(N+2)}{16\mu^3\,2^N\left[\left(\frac{N+1}{2}\right)!\right]^2},$$

where in the last estimate we used the Lagrange form of the reminder in the Taylor formula for the exponential function and $P(N+2)=(N+3)(3N^3+21N^2+42N+32)$.

According to the rescaling introduced in (5.36) one may define the functions $\widetilde{\boldsymbol{\Upsilon}}^j$, whose series expansions are given by (5.44) with coefficients in (5.45) and with a_{-1},a_1,b_1,c_0 given by (5.34) in the cases corresponding to $j\in\{1,2,3,4\}$.

In these four cases, the quantity $\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}$, appearing in the right hand side of (5.54), admits the following estimates:

$$\begin{cases}\max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}=\frac{\pi k}{h}\,\max\left\{1,\frac{\mu}{\lambda+\mu}\,\ln\left(\frac{\pi k}{h}\right)\right\} & \text{if } j=1,\\ \max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}=\frac{h}{\pi k} & \text{if } j=2,\\ \max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}=\frac{h}{\pi k}\,\max\left\{1,\frac{2(\lambda+2\mu)}{\lambda+\mu}\,\ln\left(\frac{\pi k}{h}\right)\right\} & \text{if } j=3,\\ \max\{\widetilde{a}_{-1},\widetilde{a}_1,\widetilde{b}_1,\widetilde{c}_0\}=1 & \text{if } j=4.\end{cases} \tag{5.55}$$

For $k>1$ is easy to see that all the maximum in (5.55) are less or equal than

$$\max\left\{\frac{\pi k}{h},\frac{\mu}{\lambda+\mu}\,\frac{\pi k}{h}\ln\left(\frac{\pi k}{h}\right),\frac{2(\lambda+2\mu)}{\lambda+\mu}\,\frac{h}{\pi k}\ln\left(\frac{\pi k}{h}\right)\right\},$$

so that (4.2) follows. □

6 | CONCLUSIONS

In this work we started from an applied problem, suggested by Studio De Miranda Associati. They proposed to study the blister, a structural element in bridges where the steel forestay anchors to the deck. The aim is to obtain an explicit formula to estimate the tensions in the blister, useful for the practical design of bridges.

The problem can be solved through the resolution of the elasticity equation with a specific geometry and load configuration. Hence, the first step was to define the geometry of the element. Through some simplifications we end up with a hollow circular cylinder axially loaded at the end faces; the volume of the cylinder represents the portion of the deck concrete where the stresses diffusion happens, while the applied load is given by the force that the stay has to transfer to the deck. Clearly this geometry and load configuration can be refined in order to model a real blister, but this is a first step in this way and we leave more sophisticated models to future works.

As matter of fact, from literature we learn that the elasticity equation was explicitly solved only for very particular domains and loading conditions. In this paper we provide the explicit solution for the hollow cylinder axially loaded, proceeding by steps: first of all we provide a periodic extension of the load in z direction, so that we expand the solution in Fourier series with respect to the variable z. Then we compute the Fourier coefficients in x and y passing to cylindrical coordinates and expanding such functions in power series. In Theorem 3.7 we write the explicit solution for the problem, written in series expansion. We point out that this solution may have an own interest in the construction science field, beyond the application to the blister.

To employ directly the formula in real situations, such as the blister design, it is necessary to consider approximated solutions, giving some estimates on the errors due to the truncating of the series. In Section 4 we proposed a case of study, where, fixing the parameters involved in the problem, we are able to find the distribution of the stresses in the cylinder. These plots can be obtained through a simple code, written in MATLAB or GNU Octave, running in brief time, for example 1–3 min, depending on the number where we truncate the series.

From these results it is possible to find the maximum and the minimum of the different stresses acting on the cylinder, their position on the element and an estimate on the error due to the truncation of the series. Knowing these values, the engineering designer can choice for instance the most appropriate strand anchorage from the commercial catalogue, see Figure 5, in order to not exceed specific limit stresses in the reinforced concrete. Since the map of the tensions is given, see for example Figure 6, the engineer can design the steel reinforcements in the concrete, at least on a pre-dimensioning level, and can check the concrete cracking stresses.

As we explained, to get more precise results on realistic blisters we should modify the geometry of the element and the configuration of the loads; this may be a future work, but we point out that, more the geometry and the distribution of the loads are complex more the expectations to find explicit solutions are few, so that the finite element analysis may be preferred.

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APPENDIX A

We collect in this appendix all the details related to Section 2. First of all we recall some properties useful to write the weak form of the elasticity problem (2.3) and to infer (2.4).

Thanks to the symmetry of the stress tensor $\mathbf{Tu} = (\mathbf{Tu})^T$ we infer that for any $\mathbf{u}, \mathbf{v} \in H^1(\Omega; \mathbb{R}^3)$

$$\mathbf{Tu} : \nabla\mathbf{v} = (\mathbf{Tu})^T : (\nabla\mathbf{v})^T = \mathbf{Tu} : (\nabla\mathbf{v})^T$$

so that

$$2(\mathbf{Tu} : \nabla\mathbf{v}) = \mathbf{Tu} : \nabla\mathbf{v} + \mathbf{Tu} : (\nabla\mathbf{v})^T \quad \Rightarrow \quad \mathbf{Tu} : \nabla\mathbf{v} = \mathbf{Tu} : \mathbf{Dv}\,. \tag{A1}$$

Recalling the Hooke's law (2.1), we observe that the bilinear form

$$(\mathbf{u}, \mathbf{v}) \mapsto \int_\Omega \mathbf{Tu} : \mathbf{Dv}\, dx\,, \qquad (\mathbf{u}, \mathbf{v}) \in H^1(\Omega; \mathbb{R}^3) \times H^1(\Omega; \mathbb{R}^3)$$

is symmetric, since

$$\mathbf{Tu} : \mathbf{Dv} = \lambda\,(\mathrm{div}\mathbf{u})\,(\mathrm{div}\mathbf{v}) + 2\mu\,\mathbf{Du} : \mathbf{Dv}\,. \tag{A2}$$

We recall here the Korn inequality which is known to be fundamental in the study of the equations of linear elasticity. This inequality admits a general validity for vector valued functions in $\mathbb{R}^N$ for any $N \geq 1$. Clearly, in the present paper we will be mainly interested to the case $N = 3$, being $\mathbb{R}^3$ the natural space where a solid elastic body can be modeled. For completeness, we state the inequality in the general N-dimensional case. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, that is, an open connected bounded set of $\mathbb{R}^N$, with Lipschitz boundary. Let us denote by $\mathbf{x} = (x_1, \dots, x_N)$ the generic variable of a function defined in a domain of $\mathbb{R}^N$ and $d\mathbf{x} = dx_1 \dots dx_N$ the N-dimensional volume integral in $\mathbb{R}^N$. Then there exists $C > 0$ such that

$$\int_\Omega |\nabla\mathbf{u}|^2 d\mathbf{x} \leq C\left(\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \int_\Omega |\mathbf{Du}|^2 d\mathbf{x}\right) \qquad \text{for any } \mathbf{u} \in H^1(\Omega; \mathbb{R}^N)\,. \tag{A3}$$

Among the others, for a clear and elegant proof of (A3), we address the reader to [21] by V. A. Kondrat'ev & O. A. Oleinik.

As a consequence of the Korn inequality we have that the following symmetric continuous bilinear form

$$(\mathbf{u}, \mathbf{v})_\mathbf{T} = \int_\Omega \mathbf{Tu} : \mathbf{Dv}\, dx + \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx \qquad \text{for any } \mathbf{u}, \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) \tag{A4}$$

is coercive in $H^1(\Omega;\mathbb{R}^N)$ and in particular $(\cdot,\cdot)_{\mathbf{T}}$ is a scalar product in $H^1(\Omega;\mathbb{R}^N)$ which is equivalent to its natural scalar product

$$(\mathbf{u},\mathbf{v})_{H^1} = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{u}\cdot\mathbf{v}\, dx \qquad \text{for any } \mathbf{u},\mathbf{v}\in H^1(\Omega;\mathbb{R}^N)\,.$$

Therefore, $H^1(\Omega;\mathbb{R}^N)$ still remains a Hilbert space if endowed with the equivalent scalar product $(\cdot,\cdot)_{\mathbf{T}}$.

In the next subsection we provide a proof of the existence results stated in Proposition 2.1.

A.1 | Proof of Proposition 2.1

Let us introduce the following continuous and symmetric bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{Tu} : \mathbf{Dv}\, d\mathbf{x}\,, \qquad \mathbf{u},\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)$$

and the linear continuous functional $\Lambda \in (H^1(\Omega;\mathbb{R}^3))'$ defined by

$$\langle\Lambda,\mathbf{v}\rangle = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}\, d\mathbf{x}\,, \qquad \mathbf{v}\in H^1(\Omega;\mathbb{R}^3)\,.$$

With these notations, the variational problem (2.3) may be written in the form

$$a(\mathbf{u},\mathbf{v}) = \langle\Lambda,\mathbf{v}\rangle \qquad \text{for any } \mathbf{v}\in H^1(\Omega;\mathbb{R}^3)\,.$$

Introducing the linear continuous operator $L : H^1(\Omega;\mathbb{R}^3)\to(H^1(\Omega;\mathbb{R}^3))'$ defined by

$$\langle L\mathbf{u},\mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) \qquad \text{for any } \mathbf{u},\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)\,,$$

we may write (2.3) in the form

$$L\mathbf{u} = \Lambda\,, \tag{A5}$$

as an identity between elements of the dual space $(H^1(\Omega;\mathbb{R}^3))'$.

The next step is to introduce the following operator $R : (H^1(\Omega;\mathbb{R}^3))' \to H^1(\Omega;\mathbb{R}^3)$ which maps each element $\mathbf{h}\in(H^1(\Omega;\mathbb{R}^3))'$ into the unique solution $\mathbf{w}$ of the variational problem

$$(\mathbf{w},\mathbf{v})_{\mathbf{T}} = \langle\mathbf{h},\mathbf{v}\rangle \qquad \text{for any } \mathbf{v}\in H^1(\Omega;\mathbb{R}^3)\,.$$

This problem admits a unique solution by the Lax-Milgram Theorem, being $(\cdot,\cdot)_{\mathbf{T}}$ a scalar product in $H^1(\Omega;\mathbb{R}^3)$ equivalent to the original one, as already explained above. In particular R is well defined and continuous. Moreover, R is invertible and by the Open Mapping Theorem its inverse is also continuous.

In the rest of the proof we denote by $J : H^1(\Omega;\mathbb{R}^3)\to(H^1(\Omega;\mathbb{R}^3))'$ the linear operator defined by

$$\langle J\mathbf{u},\mathbf{v}\rangle = \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} \qquad \text{for any } \mathbf{u},\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)\,,$$

which is compact as a consequence of the compact embedding $H^1(\Omega;\mathbb{R}^3)\subset L^2(\Omega;\mathbb{R}^3)$.

We now introduce the linear compact operator $T : H^1(\Omega;\mathbb{R}^3)\to H^1(\Omega;\mathbb{R}^3)$ defined by $R\circ J$. Then, T is self-adjoint with respect to the scalar product $(\cdot,\cdot)_{\mathbf{T}}$, that is, $(T\mathbf{u},\mathbf{v})_{\mathbf{T}} = (\mathbf{u},T\mathbf{v})_{\mathbf{T}}$ for any $\mathbf{u},\mathbf{v}\in H^1(\Omega;\mathbb{R}^3)$, as one can verify by direct computation.

By definition of R and J we observe that $L = R^{-1} - J$ and hence $\mathbf{u}$ is a solution of (A5) if and only if $\mathbf{u} - T\mathbf{u} = R\Lambda$. Then, applying the Fredholm alternative to the self-adjoint compact operator T, we deduce that (A5), or equivalently (2.3), admits a solution $\mathbf{u}\in H^1(\Omega;\mathbb{R}^3)$ if and only if

$$R\Lambda \in (\mathrm{Ker}(T^* - I_{H^1}))^\perp = (\mathrm{Ker}(T - I_{H^1}))^\perp \tag{A6}$$

where T^* denotes the adjoint operator of T, I_{H^1} denotes the identity map in $H^1(\Omega;\mathbb{R}^3)$ and the orthogonal spaces are defined in the sense of the scalar product $(\cdot,\cdot)_{\mathbf{T}}$. It can be verified that $\mathrm{Ker}(T - I_{H^1}) = V_0$ as we deduce by (2.5).

We now have all the tools to proceed with the proofs of (i)–(iv).

The proof of (i) is complete once we show that (A6) is equivalent to condition (2.8). Condition (A6) is equivalent to

$$(R\Lambda, \mathbf{v})_{\mathbf{T}} = 0 \qquad \text{for any } \mathbf{v} \in V_0, \tag{A7}$$

being $\mathrm{Ker}(T - I_{H^1}) = V_0$. On the other hand, by definition of R, we have that

$$(R\Lambda, \mathbf{v})_{\mathbf{T}} = \langle \Lambda, \mathbf{v} \rangle \qquad \text{for any } \mathbf{v} \in H^1(\Omega; \mathbb{R}^3)\,. \tag{A8}$$

Combining (A7) and (A8) we finally obtain $\langle \Lambda, \mathbf{v} \rangle = 0$ for any $\mathbf{v} \in V_0$, which is exactly (2.8) in view of the definition of the functional Λ.

For the proof of (ii) we observe that by (2.1), (2.3), (2.5) and (2.6) we have for any $\mathbf{v} \in H^1(\Omega; \mathbb{R}^3)$

$$a(\mathbf{u} + \mathbf{v}_0, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + a(\mathbf{v}_0, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) = \langle \Lambda, \mathbf{v} \rangle$$

which shows that $\mathbf{u} + \mathbf{v}_0$ is a solution of (2.3).

For the proof of (iii) we consider two solutions $\mathbf{u}$ and $\mathbf{w}$ of (2.3) and let $\mathbf{v}_0 = \mathbf{w} - \mathbf{u}$. By (2.1) and (2.3) we obtain

$$a(\mathbf{v}_0, \mathbf{v}) = a(\mathbf{w}, \mathbf{v}) - a(\mathbf{u}, \mathbf{v}) = \langle \Lambda, \mathbf{v} \rangle - \langle \Lambda, \mathbf{v} \rangle = 0 \qquad \text{for any } \mathbf{v} \in H^1(\Omega; \mathbb{R}^3)$$

which immediately gives $\mathbf{v}_0 \in V_0$ thanks to (2.5).

Finally, let us proceed with the proof of (iv). First we prove the existence of a solution of (2.3) in $V_0^\perp$.

Let $\mathbf{u}$ be a generic solution of (2.3) and consider its orthogonal decomposition $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \in V_0 \oplus V_0^\perp$ with respect to the scalar product (A4). Then, $\mathbf{u}_1 = \mathbf{u} - \mathbf{u}_0 \in V_0^\perp$ and by part (ii) we deduce that $\mathbf{u}_1$ is still a solution of (2.3).

Once we have proved existence, let us prove uniqueness. Let $\mathbf{u}, \mathbf{w} \in V_0^\perp$ be two solutions of (2.3). Then, on one hand we have that $\mathbf{u} - \mathbf{w} \in V_0^\perp$ and on the other hand $\mathbf{u} - \mathbf{v} \in V_0$ thanks to part (iii). Therefore, $\mathbf{u} - \mathbf{w} \in V_0 \cap V_0^\perp = \{0\}$ and this readily implies $\mathbf{u} = \mathbf{w}$ thus completing the proof of (iv). □

Notations. We give some notations that will be used throughout this paper about functional spaces and differential operators acting on scalar functions, vector valued functions, matrix valued functions. We denote by Ω a general domain in $\mathbb{R}^N$, $N \geq 1$ where by domain we mean a connected open set in $\mathbb{R}^N$.

- For $N = 3$ we have

$$\mathbf{T}\mathbf{u} = \begin{bmatrix} \sigma^1 & \tau^{12} & \tau^{13} \\ \tau^{12} & \sigma^2 & \tau^{23} \\ \tau^{13} & \tau^{23} & \sigma^3 \end{bmatrix}, \tag{A9}$$

where, combining (1.1) and (2.1), we infer

$$\begin{aligned}
\sigma^1 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u_1}{\partial x} + \nu\left(\frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right)\right] & \tau^{12} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\
\sigma^2 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u_2}{\partial y} + \nu\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}\right)\right] & \tau^{13} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) \\
\sigma^3 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u_3}{\partial z} + \nu\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)\right] & \tau^{23} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right).
\end{aligned} \tag{A10}$$

- Given two vectors $\mathbf{x} = (x_1, \dots, x_N), \mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}^N$ we denote by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^N x_i y_i$ their Euclidean scalar product and by $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ the Euclidean modulus of $\mathbf{x}$;
- the ∞-norm of vectors is $|\mathbf{x}|_\infty := \max\limits_{1 \leq i \leq N} |x_i|$;
- $\mathbb{R}^{M \times N}$: space of $M \times N$ matrices;
- if $A \in \mathbb{R}^{M \times N}$ and $\mathbf{x} \in \mathbb{R}^N$ is a vector, $A\mathbf{x}$ denotes the usual product of matrices where $\mathbf{x}$ has to be seen as a vector column;
- letting $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{N \times N}$ we denote by $A : B = \sum_{i,j=1}^N a_{ij} b_{ij}$ their Euclidean scalar product and by $|A| = \sqrt{A : A}$ its Euclidean modulus;
- given $A \in \mathbb{R}^{M \times N}$ we denote by $A^T \in \mathbb{R}^{N \times M}$ its transpose;

- given $A \in \mathbb{R}^{N\times N}$ we introduce the operator ∞-norm of matrices by $\|A\|_\infty := \sup_{\mathbf{x}\in\mathbb{R}^N\setminus\{\mathbf{0}\}} \frac{|A\mathbf{x}|_\infty}{|x|_\infty}$ so that we have in particular

$$|A\mathbf{x}|_\infty \leq \|A\|_\infty\, |\mathbf{x}|_\infty \qquad \text{for any } \mathbf{x}\in\mathbb{R}^N. \tag{A11}$$

  Letting $A = (a_{ij}), \in \mathbb{R}^N$, the following characterization of $\|\cdot\|_\infty$ holds:

$$\|A\|_\infty = \max_{1\leq i\leq N} \sum_{j=1}^{N} |a_{ij}|; \tag{A12}$$

  being $\|\cdot\|_\infty$ an operator norm, it is *sub-multiplicative* in the sense that $\|AB\|_\infty \leq \|A\|_\infty\, \|B\|_\infty$ for any $A, B \in \mathbb{R}^{N\times N}$.
- some well known functional spaces of functions defined from on an open set $\Omega \subset \mathbb{R}^N$ to a vector space V which could be $\mathbb{R}^M$ or a space of matrices: $C^k(\Omega; V)$, $L^p(\Omega; V)$, $H^k(\Omega; V)$ with $0 \leq k \leq \infty$ integer and $1 \leq p \leq \infty$;
- for $0 \leq k \leq \infty$ integer, $C^k(\overline{\Omega}; V)$ denotes the space of restrictions to $\overline{\Omega}$ of functions in $C^k(\mathbb{R}^N; V)$;
- $\mathcal{D}(\Omega; V)$: space of $C^\infty(\Omega; V)$ with compact support in Ω;
- $\mathcal{D}'(\Omega; V)$: space of vector distributions, that is, the dual space of $\mathcal{D}(\Omega; V)$;
- given a scalar function $g \in C^1(\Omega; \mathbb{R})$, we denote by $\nabla g \in C^0(\Omega; \mathbb{R}^n)$ its gradient;
- given a vector valued function $\mathbf{u} \in C^1(\Omega; \mathbb{R}^M)$, we denote by $\nabla\mathbf{u} \in C^0(\Omega; \mathbb{R}^{M\times N})$ its Jacobian matrix;
- given a vector valued function $\mathbf{u} \in C^1(\Omega; \mathbb{R}^N)$, $\Omega \subseteq \mathbb{R}^N$, we denote by $\mathbf{Du} \in C^0(\Omega; \mathbb{R}^{N\times N})$ its symmetric gradient defined by $\mathbf{Du} = \dfrac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2}$ (linearized strain tensor when $N = 3$);
- given $U \in C^1(\Omega; \mathbb{R}^{M\times N})$, $\Omega \subseteq \mathbb{R}^N$, we denote by $\operatorname{div} U \in C^0(\Omega; \mathbb{R}^M)$ the vector field $\mathbf{v} = (v_1, \dots, v_M)$ such that $v_i = \sum_{j=1}^N \frac{\partial U_{ij}}{\partial x_j}$, $i = 1, \dots, M$;
- given $\mathbf{u} = (u_1, \dots, u_M) \in C^2(\Omega; \mathbb{R}^M)$, we denote by $\Delta\mathbf{u} \in C^0(\Omega; \mathbb{R}^M)$ the Laplacian of $\mathbf{u}$ defined component by component, that is $\Delta\mathbf{u} = (\Delta u_1, \dots, \Delta u_M)$ where in the last identity Δ denotes the usual Laplacian of a real valued function.