

ALEXANDROV THEOREM FOR GENERAL NONLOCAL CURVATURES: THE GEOMETRIC IMPACT OF THE KERNEL

DORIN BUCUR, ILARIA FRAGALÀ

ABSTRACT. For a general radially symmetric, non-increasing, non-negative kernel $h \in L^1_{loc}(\mathbb{R}^d)$, we study the rigidity of measurable sets in \mathbb{R}^d with constant nonlocal h -mean curvature. Under a suitable “improved integrability” assumption on h , we prove that these sets are finite unions of equal balls, as soon as they satisfy a natural nondegeneracy condition. Both the radius of the balls and their mutual distance can be controlled from below in terms of suitable parameters depending explicitly on the measure of the level sets of h . In the simplest, common case, in which h is positive, bounded and decreasing, our result implies that any bounded open set or any bounded measurable set with finite perimeter which has constant nonlocal h -mean curvature has to be a ball.

1. INTRODUCTION

Let $h : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a radially symmetric non-increasing measurable kernel. Given a measurable set $\Omega \subset \mathbb{R}^d$, by seeing the quantity $h(x-y)$ as an interaction density between two points $x \in \Omega$ and $y \in \Omega^c := \mathbb{R}^d \setminus \Omega$, the *nonlocal h -perimeter* of Ω can be defined by

$$P_h(\Omega) := \int_{\Omega} \int_{\Omega^c} h(x-y) dx dy.$$

For typical choices of the kernel, the classical perimeter by Caccioppoli (see [19]) can be recovered from the above nonlocal one by a scaling argument. Moreover, by analogy with the classical case, a natural notion of *nonlocal h -mean curvature* can be associated with the nonlocal h -perimeter, by setting

$$H_h(\Omega) := \int_{\mathbb{R}^d} h(x-y)(\chi_{\Omega^c}(y) - \chi_{\Omega}(y)) dy.$$

Actually, this definition is well-posed as soon as the regularity of Ω ensures the finiteness of the integral, and is justified by the fact that minimizers of P_h under a volume constraint turn out to have constant h -mean curvature.

The concept of nonlocal perimeter has been first considered in [4], and it has been widely developed since then. In particular, based on the seminal papers [8, 9], a wide attention has been devoted to the so-called *fractional* perimeter, which corresponds to the choice of the singular kernel

$$(1) \quad h(x) = \frac{1}{|x|^{d+s}} \quad s \in (0, 1).$$

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Research in the fractional setting has been extended to a broad spectrum of directions, including the study of isoperimetric type inequalities [17], minimal surfaces [15, 16], diffusion processes [2, 5, 3], and mean curvature flows [10, 11, 18] (where references are clearly a sparse sampling).

These topics have been investigated also for another family of kernels, namely the one of bounded integrable kernels, which has been extensively treated in the monograph [20]. A distinguishing feature of such kernels, that we care to mark as a deep difference from the fractional one, is that the notion of nonlocal mean curvature makes sense for general measurable sets.

The present work is focused on a recent trend in nonlocal analysis, namely the rigidity of sets with constant nonlocal mean curvature. The reference milestone result in the local setting is Alexandrov theorem, dating back to 1958: it states that, among connected smooth domains, the only one with constant mean curvature is the ball [1]. Let us also mention that, still in the local framework, a significant extension of Alexandrov theorem has been obtained in the recent paper [14] by Delgadino and Maggi, who have been able to remove any kind of regularity or connectedness assumption: they have proved that, among sets with finite measure and finite perimeter, the only ones with constant mean curvature (now meant in distributional sense) are finite unions of equal balls.

In the nonlocal setting, the problem has been attacked in recent years for two distinct choices of the kernel, that we shortly summarize hereafter.

The first case, which has been solved in 2018, is that of the fractional kernel (1): in two independent papers, Ciruolo-Figalli-Maggi-Novaga [13] and Cabré-Fall-Solà Morales-Weth [7] have proved that, among sets of class $C^{1,\alpha}$, the only one with constant fractional mean curvature is the ball. Notice that the regularity assumption in such result cannot be removed, since it is necessary to give a meaning to the fractional mean curvature; on the other hand, no connectedness hypothesis is needed, because nonlocal interactions automatically rule out the bubbling phenomenon appearing in [14].

The second case, which has been treated in our previous work [6], is that of a completely different kernel, given by

$$(2) \quad h = \chi_{B_r(0)},$$

where r is a *fixed* positive radius, and $B_r(0)$ denotes the ball of radius r centred at the origin. Since such kernel is nonsingular, the main novelty of the corresponding rigidity result is that its validity extends to the broad class of measurable sets. Still as a consequence of the kernel's properties, specifically of the boundedness of its support, some kind of "short-range connectedness" assumption, that we call *r -nondegeneracy*, is needed in order to get rigidity. Under this additional assumption (which holds for free e.g. for open connected sets of diameter larger than r), we proved that the only measurable sets with constant nonlocal mean curvature for the kernel (2) are finite unions of equal balls, of radius $R > r/2$, lying at distance at least r from each other. Thus the initial choice of the positive radius r tunes the rigidity phenomenon, in the sense that r acts as a threshold from below both for the diameter of the balls, and for their mutual distance. Aim of this paper is to go beyond in the study of rigidity in the nonlocal setting, by characterizing measurable sets with constant h -mean curvature when h is an *arbitrary* locally

integrable kernel. These sets, that we call *h-critical*, satisfy the following condition:

$$(3) \quad \exists c > 0 : \int_{\Omega} h(x-y) dy = c \quad \forall x \in \partial^* \Omega,$$

where $\partial^* \Omega$ denotes the essential boundary of Ω , namely the set of points $x \in \mathbb{R}^d$ at which both Ω and its complement Ω^c have a strictly positive d -dimensional upper density.

Apart from the local integrability assumption on h , which is the minimal requirement to make the h -mean curvature well-defined for all measurable sets, our scope is to analyse the problem in a unique framework, including both singular and nonsingular kernels; in particular, the challenge is to throw some light on the delicate interplay between the geometry of the kernel and the corresponding rigidity result. This aspect requires to go into depth in the comprehension of the principles which govern the rigidity phenomenon. A crucial issue is that, in order to get rigidity, the h -criticality condition must be combined with a sort of “short-range connectedness”, which extends in a natural way the above mentioned notion of r -nondegeneracy introduced in [6]. Precisely, we say that a measurable set is *h-nondegenerate* if

$$(4) \quad \inf_{x_1, x_2 \in \partial^* \Omega} \frac{\int_{\Omega} |h(x_1 - y) - h(x_2 - y)| dy}{\|x_1 - x_2\|} > 0.$$

Some sufficient conditions for nondegeneracy will be given in Proposition 12. For instance, if Ω is an open connected set, or an indecomposable set of finite perimeter which is h -critical, condition (4) is fulfilled as soon as the diameter of Ω is larger than the radius of the the level set $\{h = \text{ess sup } h\}$: this means in particular that, if h does not have a plateau of positive measure at its supremum, such a set Ω is automatically h -nondegenerate.

In order to get rigidity for measurable sets which are h -critical and h -nondegenerate, we also need to ask a technical condition on h , which is an *improved integrability assumption*, expressed through the behaviour of its level sets. We ask that

$$(5) \quad \int_1^{+\infty} r^{d-1}(s) ds < +\infty,$$

where $r(s)$ is the distribution function of the map $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$(6) \quad h(x) = \varphi(|x|) \quad \forall x > 0,$$

namely

$$(7) \quad r(s) := \mathcal{L}^1(\{x \in \mathbb{R}_+ : \varphi(x) > s\}) \quad \forall s \geq 0.$$

Condition (5) is satisfied in particular when h is bounded, or when it has a “not too steep” singularity at 0 (see the examples at the end of this Introduction).

Our rigidity result reads:

Theorem 1. *Let h be a radially symmetric, non-increasing, non-negative kernel in $L^1_{\text{loc}}(\mathbb{R}^d)$ satisfying the improved integrability condition (5). Let Ω be a set of finite Lebesgue measure which is h -critical and h -nondegenerate, i.e. satisfies (3) and (4).*

Then Ω is equivalent to a finite union of balls B_i of the same radius R ; moreover,

$$(8) \quad R > \frac{\eta}{2}, \quad \text{with } \eta := \mathcal{L}^1(\{\varphi = \text{ess sup } \varphi\}),$$

$$(9) \quad \text{dist}(B_i, B_j) \geq r(\sigma), \quad \text{with } \sigma := \begin{cases} 0 & \text{if } \text{diam } \Omega \geq r(0) \\ \varphi(\text{diam } \Omega) & \text{if } \text{diam } \Omega < r(0). \end{cases}$$

Remark 2. Let us point out that, in the simplest, common case, in which h is positive, bounded and decreasing, Theorem 1 implies that any bounded open set or any bounded measurable set with finite perimeter which has constant nonlocal h -mean curvature has to be a ball. In more general situations, the interpretation of Theorem 1 (especially concerning the role of the two parameters η and σ) is discussed below and in the next sections.

Remark 3 (on the size of balls). The parameter η depends just on the kernel: in particular, the positivity of η occurs only when the kernel is bounded and reaches its supremum on a plateau of positive measure. Thus we can distinguish two cases:

- Case $\eta = 0$ (no supremal plateau): balls may have arbitrarily small scale.
- Case $\eta > 0$ (supremal plateau): the diameter of balls is bounded from below by the radius of the plateau.

Remark 4 (on the mutual distance of multiple balls). The parameter σ depends on the interplay between the diameter of Ω and the radii of the level sets of the kernel. More precisely, after noticing that $r(0) = \mathcal{L}^1(\text{supp } \varphi)$, we can distinguish two cases:

- Case $\text{diam } \Omega \geq \mathcal{L}^1(\text{supp } \varphi)$ (Ω is “large” compared to the support of the kernel): multiple balls are allowed, at mutual distance bounded from below by $\mathcal{L}^1(\text{supp } \varphi)$ (that is, by the radius of $\text{supp } h$).
- Case $\text{diam } \Omega < \mathcal{L}^1(\text{supp } \varphi)$ (Ω is “small” compared to the support of the kernel): by the properties of the distribution function (see Lemma 9) it holds

$$\mathcal{L}^1(\{\varphi > \varphi(\text{diam } \Omega)\}) = r(\varphi(\text{diam } \Omega)) \leq \text{diam } \Omega \leq r(\varphi(\text{diam } \Omega)^-) = \mathcal{L}^1(\{\varphi \geq \varphi(\text{diam } \Omega)\}).$$

Hence, a necessary condition for Ω to be a *multiple* family of balls is that the kernel has a level set of positive measure, corresponding to a jump in the distribution function r .

Two subcases may occur:

- Case $\mathcal{L}^1(\{\varphi = \varphi(\text{diam } \Omega)\}) = 0$: multiple balls are not allowed, because by inequality (9) they should be at distance equal at least to $\text{diam } \Omega$.
- Case $\mathcal{L}^1(\{\varphi = \varphi(\text{diam } \Omega)\}) > 0$: multiple balls are allowed, at mutual distance bounded from below by $r(\varphi(\text{diam } \Omega))$, and *all of them will be contained into the same level set of h* , given by points $x \in \mathbb{R}^d$ such that $\varphi(|x|) \geq \varphi(\text{diam } \Omega)$.

Example 5. Let

$$h(x) = \frac{1}{|x|^\alpha} \quad \text{with } \alpha < d - 1.$$

Due to the choice of α , the improved integrability condition (5) is satisfied. Theorem 1 applies: we have $\eta = 0$, and h does not have any level set of positive measure; hence, the unique h -critical and h -nondegenerate domain of finite measure is a single ball, whose radius can be arbitrarily small.

Example 6. Let

$$h(x) = \sum_{i=1}^N \alpha_i \chi_{B_{r_i}(0) \setminus B_{r_{i-1}}(0)}.$$

where $\alpha_1 > \alpha_2 > \dots > \alpha_N > 0$ and $0 = r_0 < r_1 < r_2 < \dots < r_N$. Since h is bounded, the improved integrability condition (5) is satisfied. By Theorem 1, a domain of finite measure which is h -critical and h -nondegenerate will be a finite union of equal balls, of radius $R > \eta = r_1/2$, with

$$\text{dist}(B_i, B_j) \geq \sigma = \begin{cases} r_N & \text{if } \text{diam } \Omega \geq r_N \\ r_i & \text{if } r_i \leq \text{diam } \Omega < r_{i+i} \text{ for } i = 1, \dots, N. \end{cases}$$

In particular, for $N = 1$, we recover the result proved in [6].

The proof of Theorem 1 is obtained via a new version of the moving planes method valid in the framework of measurable sets, which has been settled in [6]. However, with respect to the case $h = \chi_{B_r(0)}$, dealing with a nonconstant kernel makes the proof considerably more delicate. This is the reason why we decided to omit all the parts of the proof which closely follow [6], and in spite to focus in full detail on all the parts where the kernel plays an important role. Such parts are attacked relying on the basic idea of layering integrals according to Cavalieri's principle: in particular, this allows to set up some key estimates, which require the improved integrability assumption (5).

The paper is organized as follows: the required preliminaries are collected in Section 2, and then the proof is given in Section 3 (which is in turn divided, for the sake of clearness, into four subsections).

2. PRELIMINARIES

Throughout the paper, we assume that h is a radially symmetric, non-increasing, non-negative measurable function in $L^1_{\text{loc}}(\mathbb{R}^d)$. Moreover, for any $x \in \mathbb{R}^d$, we set for brevity

$$h_x(y) := h(x - y) \quad \forall y \in \mathbb{R}^d.$$

2.1. On some plain consequences of criticality.

Lemma 7. *Let Ω be a measurable set of finite Lebesgue measure satisfying the criticality condition (3). Then Ω is bounded.*

Proof. Assume without loss of generality that $\inf_{\mathbb{R}} h = 0$ (otherwise replace h by $h - \inf_{\mathbb{R}} h$). By contradiction, let $\{p_n\}$ be a sequence of points in $\partial^* \Omega$, with $|p_n| \rightarrow +\infty$. Since $|\Omega| < +\infty$, for every $\varepsilon > 0$ there exists R_ε , with $R_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, such that $|\Omega \cap B_{R_\varepsilon}(p_n)| < \omega_d \varepsilon^d$. Thus we have

$$\int_{\Omega} h_{p_n} = \int_{\Omega \setminus B_{R_\varepsilon}(p_n)} h_{p_n} + \int_{\Omega \cap B_{R_\varepsilon}(p_n)} h_{p_n} \leq h(R_\varepsilon) |\Omega| + \int_{B_\varepsilon(0)} h.$$

In the limit as $\varepsilon \rightarrow 0$, since $h \in L^1_{\text{loc}}(\mathbb{R}^d)$, this contradicts assumption (3). \square

Lemma 8. *Let Ω be a measurable set of finite Lebesgue measure.*

- If Ω satisfies the criticality condition (3), the same equality continues to hold at every point $x \in \overline{\partial^* \Omega}$.
- If Ω satisfies the nondegeneracy condition (4), the same strict inequality continues to hold when the infimum is taken over the pairs of points $x_1, x_2 \in \overline{\partial^* \Omega}$.

Proof. Assume that Ω satisfies the criticality condition (3). Let $x_0 = \lim_n x_n$, with $x_n \in \partial^* \Omega$. Let us prove that (3) continues to hold at $x = x_0$. Set $\Omega_n = x_n - \Omega$ and $\Omega_0 = x_0 - \Omega$. We claim that, up to passing to a (not relabeled) subsequence,

$$(10) \quad \chi_{\Omega_n} \rightarrow \chi_{\Omega_0} \quad \text{pointwise a.e. in } \mathbb{R}^d.$$

Indeed, since by assumption Ω has finite Lebesgue measure, the sequence χ_{Ω_n} is bounded in $L^2(\mathbb{R}^d)$ and hence, up to a subsequence, it converges weakly in $L^2(\mathbb{R}^d)$ to some function f . Since, for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \chi_{\Omega_n}(y) \varphi(y) dy = \int_{x_n - \Omega} \varphi(y) dy = \int_{\Omega} \varphi(x_n - y) dy = \int_{\mathbb{R}^d} \chi_{\Omega}(y) \varphi(x_n - y) dy,$$

by dominated convergence we infer that the weak limit f agrees with χ_{Ω_0} . Taking into account that

$$\|\chi_{\Omega_n}\|_{L^2(\mathbb{R}^d)} = |\Omega_n| = |\Omega_0| = \|\chi_{\Omega_0}\|_{L^2(\mathbb{R}^d)},$$

we deduce that the convergence is strong in $L^2(\mathbb{R}^d)$. Up to choosing a further subsequence, we have that (10) is satisfied.

Now consider the sequence $h_n := h \chi_{\Omega_n}$. By (10), up to a subsequence it converges to $h_0 := h \chi_{\Omega_0}$ pointwise a.e. in \mathbb{R}^d . Moreover, we have $|h_n| \leq |h|$. Since by assumption $h \in L^1_{\text{loc}}(\mathbb{R}^d)$, and since by Lemma 7 Ω is bounded, by applying the dominated convergence theorem on a sufficiently large ball, we infer that $h_n \rightarrow h_0$ in $L^1(\mathbb{R}^d)$. Hence,

$$\|h_0\|_{L^1(\mathbb{R}^d)} = \lim_n \|h_n\|_{L^1(\mathbb{R}^d)} = c,$$

as required. The proof of the second claim in the statement is analogous. \square

2.2. On the distribution function and layered integrals. For convenience of the reader, we recall in the next lemma some well-known properties of the function r which maps any $s \in \mathbb{R}_+$ into the radius of the level set $\{h > s\}$ (see for instance [12] and references therein).

Lemma 9. *Let φ be associated with h as in (6), and $r(s)$ be its distribution function defined in (7). Then:*

- (i) *The map $s \mapsto r(s)$ is non-increasing, with $r(s) = 0$ for every $s \geq \text{ess sup } \varphi$.*
- (ii) *The map $s \mapsto r(s)$ is right continuous and, setting $r(s_0^-) := \lim_{s \rightarrow s_0^-} r(s)$, it holds*

$$r(s_0^-) - r(s_0) = \mathcal{L}^1(\{\varphi \geq s_0\}) - \mathcal{L}^1(\{\varphi > s_0\}) = \mathcal{L}^1(\{\varphi = s_0\}).$$

in particular, r is continuous at a given point s_0 if and only if $\mathcal{L}^1(\{\varphi = s_0\}) = 0$;

- (iii) *It holds*

$$\sup_{(0, +\infty)} r(s) = r(0) = \mathcal{L}^1(\{\text{supp } \varphi\}) \quad \text{and} \quad \inf_{(0, \text{ess sup } \varphi)} r(s) = \eta := \mathcal{L}^1(\{\varphi = \text{ess sup } \varphi\}).$$

- (iv) *It holds*

$$\varphi(t) = \sup \{s \geq 0 : r(s) > t\} \quad \forall t \geq 0.$$

Remark 10. We point out for later use that the parameters η and σ introduced in the statement of Theorem 1 enjoy the following properties, which can be easily checked by using the above lemma:

$$(11) \quad \forall \lambda > 0, \quad \mathcal{L}^1(\{s : r(s) \in (0, \lambda)\}) > 0 \Leftrightarrow \lambda > \eta$$

$$(12) \quad r(s) > \text{diam } \Omega \Leftrightarrow s < \sigma.$$

In the following simple lemma, which will be used repeatedly in the sequel, we exploit the layer-cake principle to rewrite the integrals appearing in the criticality and in the nondegeneracy conditions in terms of the function $r(s)$ and of the parameter σ .

Lemma 11. *Let Ω be a measurable set of finite Lebesgue measure satisfying the criticality condition (3), and let σ be defined as in (9).*

(i) *For every $x \in \overline{\partial^* \Omega}$, it holds*

$$(13) \quad \int_{\Omega} h_x(y) dy = \int_0^{+\infty} |\Omega \cap B_{r(s)}(x)| ds = \sigma |\Omega| + \int_{\sigma}^{+\infty} |\Omega \cap B_{r(s)}(x)| ds.$$

(ii) *For every $x_1, x_2 \in \overline{\partial^* \Omega}$, it holds*

$$(14) \quad \int_{\Omega} |h_{x_1} - h_{x_2}| dy = \int_{\sigma}^{+\infty} |\Omega \cap (B_{r(s)}(x_1) \Delta B_{r(s)}(x_2))| ds.$$

Proof. For every $x \in \overline{\partial^* \Omega}$, since by assumption $h_x \in L^1_{\text{loc}}(\mathbb{R}^d)$, and since by Lemma 7 the set Ω is bounded, we have $h_x \in L^1(\Omega)$. Then, by the layer-cake principle and Fubini Theorem, we have

$$\begin{aligned} \int_{\Omega} h_x(y) dy &= \int_{\Omega} \int_0^{+\infty} \chi_{\{h_x > s\}} ds dy = \int_0^{+\infty} \int_{\Omega} \chi_{\{h_x > s\}} dy ds \\ &= \int_0^{+\infty} \int_{\Omega} \chi_{B_{r(s)}(x)}(y) dy ds = \int_0^{+\infty} |\Omega \cap B_{r(s)}(x)| ds. \end{aligned}$$

The equality (13) follows by noticing that, as a consequence of (12), for $s < \sigma$ we have $|\Omega \cap B_{r(s)}(x)| = |\Omega|$.

For every $x_1, x_2 \in \partial^* \Omega$, we have

$$\int_{\Omega} |h_{x_1} - h_{x_2}| dy = \int_{\Omega \cap \{h_{x_1} > h_{x_2}\}} (h_{x_1} - h_{x_2}) dy + \int_{\Omega \cap \{h_{x_2} > h_{x_1}\}} (h_{x_2} - h_{x_1}) dy =: I' + I''.$$

As above, we use the layer cake principle to rewrite I' as

$$\begin{aligned} I' &= \int_0^{+\infty} \int_{\Omega \cap \{h_{x_1} > h_{x_2}\}} [\chi_{B_{r(s)}(x_1)}(y) - \chi_{B_{r(s)}(x_2)}(y)] ds \\ &= \int_0^{+\infty} |\Omega \cap \{h_{x_1} > h_{x_2}\} \cap (B_{r(s)}(x_1) \setminus B_{r(s)}(x_2))| ds \\ &\quad - \int_0^{+\infty} |\Omega \cap \{h_{x_1} > h_{x_2}\} \cap (B_{r(s)}(x_2) \setminus B_{r(s)}(x_1))| ds. \end{aligned}$$

Then, since h is non-increasing, and since $\Omega \subset B_{r(s)}(x_2)$ for $s > \sigma$ (again by (12)), we obtain

$$\begin{aligned} I' &= \int_0^{+\infty} |\Omega \cap (B_{r(s)}(x_1) \setminus B_{r(s)}(x_2))| ds \\ &= \int_\sigma^{+\infty} |\Omega \cap (B_{r(s)}(x_1) \setminus B_{r(s)}(x_2))| ds. \end{aligned}$$

Likewise, we obtain

$$I'' = \int_\sigma^{+\infty} |\Omega \cap (B_{r(s)}(x_2) \setminus B_{r(s)}(x_1))| ds.$$

By adding the above expressions for I' and I'' , we obtain (14). \square

2.3. On some sufficient conditions for nondegeneracy.

Proposition 12. *Let Ω be a measurable set of finite Lebesgue measure satisfying the criticality condition (3). Assume in addition that Ω is either open or of finite perimeter. Then the nondegeneracy condition (4) is satisfied provided*

$$(15) \quad \inf_i \text{diam}(\Omega_i) > \eta,$$

where $\{\Omega_i\}_i$ the family of the connected or indecomposable components of Ω , and η is defined as in (8).

Proof. Working component by component, we are reduced to show that, if Ω is an indecomposable set of finite perimeter, or an open connected set, it is not h -degenerate provided

$$(16) \quad \text{diam}(\Omega) > \eta.$$

By Lemma 11, we have

$$\inf_{x_1, x_2 \in \partial^* \Omega} \frac{\int_\Omega |h(x_1 - y) - h(x_2 - y)| dy}{\|x_1 - x_2\|} = \inf_{x_1, x_2 \in \partial^* \Omega} \int_\sigma^{+\infty} \frac{|\Omega \cap (B_{r(s)}(x_1) \Delta B_{r(s)}(x_2))|}{\|x_1 - x_2\|} ds.$$

Passing the infimum under the sign of integral, we get

$$\inf_{x_1, x_2 \in \partial^* \Omega} \frac{\int_\Omega |h(x_1 - y) - h(x_2 - y)| dy}{\|x_1 - x_2\|} \geq \int_\sigma^{+\infty} \inf_{x_1, x_2 \in \partial^* \Omega} \frac{|\Omega \cap (B_{r(s)}(x_1) \Delta B_{r(s)}(x_2))|}{\|x_1 - x_2\|} ds.$$

The r.h.s. of the above inequality is strictly positive provided

$$(17) \quad \mathcal{L}^1(\{s : r(s) \in (0, \text{diam}(\Omega))\}) > 0,$$

because, by [6, Proposition 10],

$$\inf_{x_1, x_2 \in \partial^* \Omega} \frac{|\Omega \cap (B_r(x_1) \Delta B_r(x_2))|}{\|x_1 - x_2\|} > 0 \quad \forall r \in (0, \text{diam}(\Omega)).$$

In turn, recalling (11), condition (17) is fulfilled thanks to assumption (16). \square

3. PROOF OF THEOREM 1

Outline. We adopt the moving planes method for measurable sets settled in [6]. As in the classical moving planes method, the idea is to consider, for any fixed direction $\nu \in S^{d-1}$, an initial hyperplane H_0 with unit normal ν , not intersecting $\overline{\partial^*\Omega}$ (this can be done thanks to Lemma 7). Then one starts moving H_0 in the direction of its normal ν to new positions H_t , so that at a certain moment of the process it starts intersecting $\overline{\partial^*\Omega}$. The main novelty of the approach introduced in [6] with respect to the classical case is how to define the stopping time of the movement, and then how to get rigidity (including possibly multiple balls).

Here we follow the same global strategy as in [6], but each part of the proof needs to be significantly changed, due to the much greater generality of the kernel we work with (in the few points where the same arguments apply, the reader is explicitly referred to [6]). Before starting, let us fix some notation and terminology: we denote by H_t^- and H_t^+ the two closed halfspaces determined by H_t (for definiteness, assume that $H_0 \subset H_t^-$); we set

$$\Omega_t := \Omega \cap H_t^-, \quad \mathcal{R}_t := \text{the reflection of } \Omega_t \text{ about } H_t.$$

- We say that *symmetric inclusion* holds at t if

$$(18) \quad \mathcal{R}_t \subset \Omega \quad \text{and} \quad \Omega_t \cup \mathcal{R}_t \text{ is Steiner symmetric about } H_t.$$

(Recall that a measurable set ω is Steiner symmetric about a hyperplane H with unit normal ν if it is equivalent to the set of points $x \in \mathbb{R}^d$ of the form $x = z + t\nu$, with $z \in H$ and $|t| < \frac{1}{2}\mathcal{H}^1(\omega \cap \{z + t\nu : t \in \mathbb{R}\})$).

- We say that symmetric inclusion occurs at t if *with away contact* if (18) holds and there exists an “away contact point”, namely a point

$$(19) \quad p' \in [\overline{\partial^*\mathcal{R}_t} \cap \overline{\partial^*\Omega}] \setminus H_t.$$

when (18) holds but (19) is false, we say that symmetric inclusion at t holds *without away contact*.

- We say that symmetric inclusion occurs at t if *with close contact* if (18) holds and there exists a “close contact point”, namely a point

$$(20) \quad H_t \ni q = \lim_n q_{1,n} = \lim_n q_{2,n}, \quad q_{i,n} \in \overline{\partial^*\Omega} \cap \{q + t\nu : t \in \mathbb{R}\}, \quad q_{1,n} \neq q_{2,n}.$$

Notice that symmetric inclusion can occur at the same t with both away contact and close contact.

We are now ready to start. We proceed in four steps, which are carried over in separate subsections below.

3.1. Step 1 (start). We prove the following claim:

- *Claim 1: There exists $\varepsilon > 0$ such that, for every $t \in [0, \varepsilon)$, symmetric inclusion holds.*

The proof is based on the following

Lemma 13 (no converging pairs). *Under the assumptions of Theorem 1, if Ω is contained into $H_0^+ := \{z + t\nu : z \in H_0, t \geq 0\}$, H_0 being a hyperplane with unit normal ν , there cannot exist two sequences of points $\{p_{1,n}\}, \{p_{2,n}\}$ in $\overline{\partial^*\Omega} \cap H_0^+$ which for every*

fixed n are distinct, with the same projection onto H_0 , and at infinitesimal distance from H_0 as $n \rightarrow +\infty$.

Proof. We argue by contradiction. Setting $t_{i,n} := \text{dist}(p_{i,n}, H_0)$, we can assume up to a subsequence that $t_{1,n} > t_{2,n}$ for every n . We are going to show that

$$(21) \quad \liminf_{n \rightarrow +\infty} \frac{\int_{\Omega} h_{p_{1,n}} - \int_{\Omega} h_{p_{2,n}}}{t_{1,n} - t_{2,n}} > 0 ,$$

against the fact that Ω is h -critical.

By the equality (13) in Lemma 11, we have

$$(22) \quad \begin{aligned} \int_{\Omega} h_{p_{1,n}} - \int_{\Omega} h_{p_{2,n}} &= \int_{\sigma}^{+\infty} |\Omega \cap B_{r(s)}(p_{1,n})| - |\Omega \cap B_{r(s)}(p_{2,n})| ds \\ &= \int_{\sigma}^{+\infty} |\Omega \cap (B_{r(s)}(p_{1,n}) \setminus B_{r(s)}(p_{2,n}))| - |\Omega \cap (B_{r(s)}(p_{2,n}) \setminus B_{r(s)}(p_{1,n}))| ds . \end{aligned}$$

Since Ω is not h -degenerate, and recalling the equality (14) in Lemma 11, there exists a positive constant C such that

$$(23) \quad \frac{\int_{\Omega} |h_{p_{1,n}} - h_{p_{2,n}}| dy}{t_{1,n} - t_{2,n}} = \frac{\int_{\sigma}^{+\infty} |\Omega \cap (B_{r(s)}(p_{1,n}) \Delta B_{r(s)}(p_{2,n}))| ds}{t_{1,n} - t_{2,n}} \geq C .$$

In view of (22) and (23), the inequality (21) holds true provided, for n large enough,

$$(24) \quad \frac{\int_{\sigma}^{+\infty} |\Omega \cap (B_{r(s)}(p_{2,n}) \setminus B_{r(s)}(p_{1,n}))| ds}{t_{1,n} - t_{2,n}} \leq \frac{C}{4} .$$

By the inclusion $\Omega \subset H_0^+$, to prove the inequality (24) it is sufficient to have

$$(25) \quad \int_{\sigma}^{+\infty} |H_0^+ \cap (B_{r(s)}(p_{2,n}) \setminus B_{r(s)}(p_{1,n}))| ds = o(t_{1,n} - t_{2,n}) .$$

It remains to prove (25). To that aim, we are going to provide two distinct estimates valid for n large enough for the integrand in (25), according to the values of the radius $r(s)$. More precisely, we distinguish the two regimes

$$r(s) \leq k_n t_{1,n} \quad \text{and} \quad r(s) > k_n t_{1,n} ,$$

where k_n is a constant larger than 1, which will be suitably chosen at the end of the proof. In the estimates below, we set $\gamma_n := t_{1,n} - t_{2,n}$; moreover, we omit for shortness the index n , by simply writing p_1, p_2, t_1, t_2 , and γ .

- For $r(s) \leq kt_1$, we have

$$|H_0^+ \cap (B_{r(s)}(p_2) \setminus B_{r(s)}(p_1))| \leq |B_{r(s)}(p_2) \setminus B_{r(s)}(p_1)| \leq \omega_{d-1} r(s)^{d-1} \gamma .$$

- For $r(s) > kt_1$, since $k > 1$ both the balls $B_{r(s)}(p_1)$ and $B_{r(s)}(p_2)$ intersect H_0 . Let us denote by $z (= z_n)$ the common projection of p_1 and p_2 onto H_0 . The measure of $H_0^+ \cap (B_{r(s)}(p_2) \setminus B_{r(s)}(p_1))$ is not larger than the measure of the region $D(s)$ obtained as the difference between two right cylinders having as axis the perpendicular to H_0 through z , as bases the $(d-1)$ -dimensional ball contained

into H_0 centred at z with radii $r_2 := (r(s)^2 - t_2^2)^{1/2}$ and $r_1 := (r(s)^2 - t_1^2)^{1/2}$, and as height $t_2 + \frac{\gamma}{2}$. We have

$$|D(s)| = \omega_{d-1}(r_2^{d-1} - r_1^{d-1}) \left(t_2 + \frac{\gamma}{2}\right).$$

By the convexity of the map $t \mapsto t^{d-1}$ (for $d \geq 2$) we infer

$$\begin{aligned} |D(s)| &\leq (d-1)\omega_{d-1}r_2^{d-2}(r_2 - r_1) \left(t_2 + \frac{\gamma}{2}\right) \\ &= (d-1)\omega_{d-1}r_2^{d-2}(\sqrt{r(s)^2 - t_2^2} - \sqrt{r(s)^2 - t_1^2}) \left(t_2 + \frac{\gamma}{2}\right) \\ &= (d-1)\omega_{d-1}r_2^{d-2} \frac{t_1^2 - t_2^2}{r_1 + r_2} \left(t_2 + \frac{\gamma}{2}\right) \\ &\leq 2(d-1)\omega_{d-1}r_2^{d-3}t_1 \left(t_2 + \frac{\gamma}{2}\right) \gamma, \\ &\leq 2(d-1)\omega_{d-1}r(s)^{d-3}t_1^2 \gamma \\ &\leq \frac{2(d-1)\omega_{d-1}}{k^2} r(s)^{d-1} \gamma. \end{aligned}$$

Now, if we set

$$s(\lambda) := \sup\{s : r(s) > \lambda\}, \quad \forall \lambda > 0,$$

for n sufficiently large it holds

$$\sigma = s(\text{diam}\Omega) < s(kt_1).$$

Hence,

$$\begin{aligned} &\int_{\sigma}^{+\infty} |H_0^+ \cap (B_{r(s)}(p_2) \setminus B_{r(s)}(p_1))| ds = \\ &\int_{s(kt_1)}^{+\infty} |H_0^+ \cap (B_{r(s)}(p_2) \setminus B_{r(s)}(p_1))| ds + \int_{\sigma}^{s(kt_1)} |H_0^+ \cap (B_{r(s)}(p_2) \setminus B_{r(s)}(p_1))| ds \leq \\ &\left[\omega_{d-1} \int_{s(kt_1)}^{+\infty} r(s)^{d-1} ds + \frac{2(d-1)\omega_{d-1}}{k^2} \int_{\sigma}^{s(kt_1)} r(s)^{d-1} ds \right] \gamma. \end{aligned}$$

Finally we claim that, by choosing $k = k_n \rightarrow +\infty$ in such way that $kt_1 \rightarrow 0$, the two addenda in square bracket are infinitesimal. To prove such claim it is enough to have

$$(26) \quad \int_{\sigma}^{+\infty} r(s)^{d-1} ds < +\infty.$$

Indeed in this case the first addendum will be infinitesimal because $s(kt_1)$ tends to $+\infty$, while the second one will be infinitesimal because it is bounded from above by a finite integral times a ratio which tends to zero. Eventually, condition (26) holds true since the convergence of the integral near $+\infty$ is guaranteed by assumption (5), while the convergence near σ is guaranteed by the fact that, if $\sigma = 0$, we have $r(0) \leq \text{diam}\Omega$. \square

Assume now that Claim 1 is false. Then, either there exists $\{t_n\} \rightarrow 0$ such that $\forall n$ $\Omega_{t_n} \cup \mathcal{R}_{t_n}$ is *not* Steiner symmetric about H_{t_n} , or there exists $\{t_n\} \rightarrow 0$ such that $\forall n$

$|\mathcal{R}_{t_n} \setminus \Omega| > 0$. Lemma 13 ensures that none of these two cases is possible (the detailed contradiction argument works as in the proof of Step 1 in [6, Theorem 1]).

3.2. Step 2 (away contact at the stopping time). We set

$$T := \sup \left\{ t > 0 : \text{for all } s \in [0, t), \text{ symmetric inclusion occurs without away contact} \right\}.$$

Since Ω is bounded, we have $T < +\infty$. Then we prove the following claims:

- *Claim 2a. Symmetric inclusion holds at T with away contact or with close contact.*
- *Claim 2b. Symmetric inclusion cannot hold with close contact and no away contact.*

In order to prove these claims, we need to establish preliminarily that symmetric inclusion without away contact implies the “away inclusion properties” stated in the next lemma (precisely, in (27) and (28)). Below, for any $\delta > 0$ and $s \geq 0$, we set

$$U_{T-\delta}^s := \left\{ x + (2\delta + 2s)\nu : x \in \mathcal{R}_{T-\delta} \right\}.$$

Moreover, we denote by $E \oplus B_R$ the collection of points of \mathbb{R}^d with distance less than R from a set E .

Lemma 14. *Assume that symmetric inclusion occurs without away contact at T , and let $\delta > 0$ be fixed. Then:*

- *There exists $s_\delta > 0$ such that, for every $s \in [0, s_\delta]$,*

$$(27) \quad U_{T-\delta}^s \subset \Omega.$$

- *There exists $\eta = \eta_\delta$ such that*

$$(28) \quad (U_{T-\delta}^0 \oplus B_\eta) \cap H_{T+\delta}^+ \subseteq \Omega.$$

Proof. We argue by contradiction. If the inclusion (27) was false, we could find an infinitesimal sequence $\{s_n\}$ of positive numbers, and a sequence of points $\{x'_n\}$ of density 1 for $U_{T-\delta}^{s_n}$ but of density 0 for Ω . Up to a subsequence, there exists $x' := \lim_n x'_n$. By construction, we have $x' \in \{x + 2\delta\nu : x \in \overline{\partial^*\mathcal{R}_{T-\delta}}\} \subset \overline{\partial^*\mathcal{R}_T}$. But, since we are assuming that symmetric inclusion occurs without away contact at T , it is readily checked that $\overline{\partial^*\mathcal{R}_T} \subseteq \text{int}(\Omega^{(1)})$. Then $x' \in \text{int}(\Omega^{(1)})$, against the fact that x'_n are points of density 0 for Ω .

In a similar way, if the inclusion (28) was false, we could find a sequence $\{x_n\} \in (\Omega^e)^{(1)}$ such that $x_n \in (U_{T-\delta}^0 \oplus B_{\frac{1}{n}}) \cap H_{T+\delta}^+$. Since $U_{T-\delta}^0 \subseteq \Omega$ is open (as a consequence of Proposition 13 in [6]), we could also find $y_n \in (U_{T-\delta}^0 \oplus B_{\frac{3}{n}}) \cap H_{T+\delta_0}^+$ such that $y_n \in \partial^*\Omega$ (otherwise by Federer’s Theorem, the perimeter of Ω inside the set $B_{\frac{3}{n}}(y_n)$ would be zero). By compactness, we would obtain a limit point of the sequence $\{y_n\}$ lying both in $\overline{\partial^*\Omega}$ and in $\overline{U_{T-\delta}^0} \cap \overline{H_{T+\delta_0}^+}$, in contradiction with our assumption of symmetric inclusion without away contact at T . \square

Proof of Claim 2a. The same arguments used to obtain the homonym claim in the proof of [6, Theorem 1] apply.

Proof of Claim 2b.

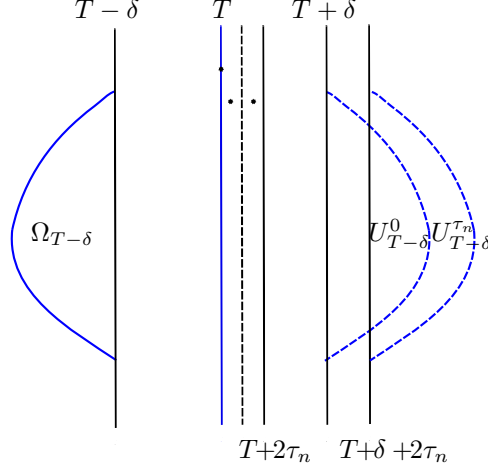


FIGURE 1. The geometry in the proof of Claim 2b: the vertical lines represent, starting from the left, the hyperplanes $H_{T-\delta}$, H_T (in blue), $H_{T+2\tau_n}$, $H_{T+\delta}$, and $H_{T+\delta+2\tau_n}$.

Assume that symmetric inclusion holds at T with close contact. We are going to contradict (3) by showing that, if $\{q_{1,n}\}$ and $\{q_{2,n}\}$ are sequences converging to a point $q \in H_T$ as in (20), it holds

$$(29) \quad \liminf_{n \rightarrow +\infty} \frac{\int_{\Omega} h_{q_{1,n}} - \int_{\Omega} h_{q_{2,n}}}{\|q_{1,n} - q_{2,n}\|} > 0.$$

Below, we set for brevity

$$\gamma_n := \|q_{1,n} - q_{2,n}\|,$$

We fix $\delta > 0$ small enough (to be chosen later), and we let $\tau_n > 0$ be such that $H_{T+\tau_n}$ contains the midpoint of the segment $(q_{1,n}, q_{2,n})$ (see Figure 1). Up to working with n sufficiently large, since $q_{1,n}$ and $q_{2,n}$ converge to a point of H_T , thanks to the away inclusion property (27) we can assume that

$$(30) \quad U_{T-\delta}^{\tau_n} \subset \Omega.$$

We can also assume that

$$(31) \quad \gamma_n + \text{dist}(q_{1,n}, H_T) < \frac{\delta}{4}.$$

We decompose

$$\int_{\Omega} h_{q_{1,n}} - h_{q_{2,n}} = X_n(\delta) + Y_n(\delta) + Z_n(\delta)$$

where

$$X_n(\delta) = \int_{\Omega \cap (H_{T+\tau_n} \oplus B_{\delta+\tau_n})} (h_{q_{1,n}} - h_{q_{2,n}})$$

$$Y_n(\delta) = \int_{\Omega_{T-\delta} \cup U_{T-\delta}^{\tau_n}} (h_{q_{1,n}} - h_{q_{2,n}})$$

$$Z_n(\delta) = \int_{\Omega \cap H_{T+\delta+2\tau_n}^+ \setminus U_{T-\delta}^{\tau_n}} (h_{q_{1,n}} - h_{q_{2,n}}).$$

By symmetry, it holds

$$Y_n(\delta) = 0.$$

From the non-degeneracy hypothesis we know that

$$(32) \quad C := \inf_n \frac{\int_{\Omega} |h_{q_{1,n}} - h_{q_{2,n}}|}{\gamma_n} > 0.$$

In order to prove (29) we are going to show first that

$$(33) \quad |X_n(\delta)| \leq C(\delta)\gamma_n, \quad \text{with } C(\delta) = o(1) \text{ as } \delta \rightarrow 0$$

so that in particular we can choose δ so small that $C(\delta) < \frac{C}{16}$, for C as in (32), and second that

$$(34) \quad Z_n(\delta) \geq C'\gamma_n \quad \text{for some } C' > \frac{C}{16}.$$

Proof of (33). We decompose

$$\int_{\Omega} |h_{q_{1,n}} - h_{q_{2,n}}| = X'_n(\delta) + Y'_n(\delta) + Z'_n(\delta),$$

where

$$\begin{aligned} X'_n(\delta) &= \int_{\Omega \cap (H_{T+\tau_n} \oplus B_{\delta+\tau_n})} |h_{q_{1,n}} - h_{q_{2,n}}| \\ Y'_n(\delta) &= \int_{\Omega_{T-\delta} \cup U_{T-\delta}^{\tau_n}} |h_{q_{1,n}} - h_{q_{2,n}}| \\ Z'_n(\delta) &= \int_{\Omega \cap H_{T+\delta+2\tau_n}^+ \setminus U_{T-\delta}^{\tau_n}} |h_{q_{1,n}} - h_{q_{2,n}}|. \end{aligned}$$

Assume by a moment to know that there exists a positive constant $C = C(\delta)$ such that

$$(35) \quad X'_n(\delta) \leq C(\delta)\gamma_n, \quad \text{with } C(\delta) = o(1) \text{ as } \delta \rightarrow 0.$$

If this is the case, since $|X_n| \leq X'_n$, (33) is satisfied. In particular, we can assume that δ is fixed such that $X'_n(3\delta) \leq \frac{C}{16}\gamma_n$ for n large enough. Note that X'_n is non decreasing in δ , so that $X'_n(\delta) \leq \frac{C}{16}\gamma_n$.

Since $|X_n| \leq X'_n$, we have $|X_n(3\delta)| \leq \frac{C}{16}\gamma_n$ and $|X_n(\delta)| \leq \frac{C}{16}\gamma_n$.

We now prove (35). To that aim, we are going to proceed in a similar way as done in the proof of Lemma 13. We set

$$\begin{aligned} t_{1,n} &:= \text{dist}(q_{1,n}, H_{T-\delta}) = \delta + \tau_n - \frac{\gamma_n}{2} \\ t_{2,n} &:= \text{dist}(q_{2,n}, H_{T-\delta}) = \delta + \tau_n + \frac{\gamma_n}{2}, \end{aligned}$$

We have

$$X'_n(\delta) = \int_{\sigma}^{+\infty} |\Omega \cap (H_{T+\tau_n} \oplus B_{\delta+\tau_n}) \cap (B_{r(s)}(q_{2,n}) \setminus B_{r(s)}(q_{1,n}))| ds.$$

Let us provide two distinct estimates valid for n large enough for the above integrand in the two regimes

$$r(s) \leq \gamma_n + \sqrt{\delta} \quad \text{and} \quad r(s) > \gamma_n + \sqrt{\delta}.$$

In the estimates below, we omit for shortness the index n , by simply writing q_1, q_2, t_1, t_2, τ , and γ .

- For $r(s) \leq \gamma + \sqrt{\delta}$, we have:

$$\begin{aligned} |\Omega \cap (H_{T+\tau} \oplus B_{\delta+\tau}) \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))| &\leq |B_{r(s)}(q_2) \setminus B_{r(s)}(q_1)| \\ &\leq \omega_{d-1} r(s)^{d-1} \gamma. \end{aligned}$$

- For $r(s) > \gamma + \sqrt{\delta}$, since for n large enough $t_1 < \gamma + \sqrt{\delta}$, both the balls $B_{r(s)}(q_1)$ and $B_{r(s)}(q_2)$ intersect $H_{T-\delta}^+$. We have

$$\begin{aligned} |\Omega \cap (H_{T+\tau} \oplus B_{\delta+\tau}) \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))| &\leq \\ |H_{T-\delta}^+ \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))|. \end{aligned}$$

Let us denote by $z(= z_n)$ the common projection of q_1 and q_2 onto $H_{T-\delta}$. The measure of $H_0^+ \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))$ is not larger than the measure of the region $D(s)$ obtained as the difference between two right cylinders having as axis the perpendicular to H_0 through z , as bases the $(d-1)$ -dimensional ball contained into H_0 with centre z and radii $r_2 := (r(s)^2 - t_2^2)^{1/2}$ and $r_1 := (r(s)^2 - t_1^2)^{1/2}$, and as height $t_2 + \frac{\gamma}{2}$. As in the proof of Lemma 13, we have

$$|D(s)| \leq 2(d-1)\omega_{d-1}r(s)^{d-3}t_1^2\gamma,$$

Since, for n large enough, we have $t_1 \leq \delta \leq r(s)\sqrt{\delta}$, we infer that

$$|D(s)| \leq 2(d-1)\omega_{d-1}r(s)^{d-1}\delta\gamma.$$

If $s(\lambda) := \sup\{s : r(s) > \lambda\}$, for large n it holds $\sigma = s(\text{diam}\Omega) < s(\gamma + \sqrt{\delta})$, and hence

$$\begin{aligned} &\int_{\sigma}^{+\infty} |H_0^+ \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))| ds = \\ &\int_{s(\gamma+\sqrt{\delta})}^{+\infty} |H_0^+ \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))| ds + \int_{\sigma}^{s(\gamma+\sqrt{\delta})} |H_0^+ \cap (B_{r(s)}(q_2) \setminus B_{r(s)}(q_1))| ds \leq \\ &\left[\omega_{d-1} \int_{s(\gamma+\sqrt{\delta})}^{+\infty} r(s)^{d-1} ds + 2(d-1)\omega_{d-1}\delta \int_{\sigma}^{s(\gamma+\sqrt{\delta})} r(s)^{d-1} ds \right] \gamma. \end{aligned}$$

Then the proof of (35) is achieved by taking $C(\delta)$ equal to the expression in square bracket. Indeed, recalling (26), we see that such expression is infinitesimal as $\delta \rightarrow 0$: the first addendum is infinitesimal because $s(\gamma + \sqrt{\delta})$ tends to $+\infty$, while the second one is bounded from above by a multiple of δ .

Proof of (34). Since on $H_{T+\delta+2\tau_n}^+$ the difference $h_{q_1,n} - h_{q_2,n}$ is non negative, we get that $Z_n(\delta) = Z'_n(\delta)$. There are two situations:

$$\text{either } Y'_n(\delta) \geq Z'_n(\delta), \quad \text{or } Z'_n(\delta) > Y'_n(\delta).$$

If $Z'_n(\delta) > Y'_n(\delta)$, then from the non degeneracy hypothesis together with the estimate of $X'_n(\delta)$ (see respectively (32) and (35)), we get

$$Z'_n(\delta) \geq \frac{C}{4}\gamma_n$$

hence we get (34).

Assume now that $Y'_n(\delta) > Z'_n(\delta)$. By arguing as above, we see that $Y'_n(\delta) \geq \frac{C}{4}\gamma_n$. We shall prove that there exists a constant $K(\delta) > 0$ such that $Z'_n(\delta) \geq K(\delta)\gamma_n$. Although we are not able to evaluate $K(\delta)$, its strict positivity will be sufficient. In fact, following from our geometric construction and in particular from the definition of $Z'_n(\delta)$, the map $\delta \mapsto K(\delta)$ is non increasing. Consequently, if one replaces δ with a smaller value $\tilde{\delta} < \delta$, it holds $K(\tilde{\delta}) \geq K(\delta)$ so that $Z'_n(\tilde{\delta}) \geq K(\tilde{\delta})\gamma_n \geq K(\delta)\gamma_n$. In order to conclude the proof, it is then enough to fix $\tilde{\delta}$ small enough such that $X_n(3\tilde{\delta}) \leq \frac{K(\tilde{\delta})}{4}\gamma_n$ and reproduce the same reasoning with $\tilde{\delta}$ instead of δ .

It remains to show that there exists a constant $K(\delta) > 0$ such that $Z'_n(\delta) \geq K(\delta)\gamma_n$. We know that $Y'_n(\delta) \geq \frac{C}{4}\gamma_n$ or, equivalently,

$$\int_{\sigma}^{+\infty} |(\Omega_{T-\delta} \cup U_{T-\delta}^{\tau_n}) \cap (B_{r(s)}(q_{1,n}) \Delta B_{r(s)}(q_{2,n}))| ds \geq \frac{C}{4}\gamma_n.$$

By symmetry,

$$\int_{\sigma}^{+\infty} |U_{T-\delta}^{\tau_n} \cap (B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n}))| ds \geq \frac{C}{8}\gamma_n.$$

Since $X'_n(3\delta) \leq \frac{C}{16}\gamma_n$, then

$$(36) \quad \int_{[\sigma, +\infty]} 1_{r(s) \in [2\delta, \text{diam}\Omega]} |U_{T-\delta}^{\tau_n} \cap (B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n}))| ds > 0.$$

By (36), setting

$$A(n, \delta, s) = U_{T-\delta}^{\tau_n} \cap (B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})),$$

the one dimensional measure of the set

$$\mathcal{S} := \left\{ s : r(s) \in [2\delta, \text{diam}\Omega] \text{ and } |A(n, \delta, s)| > 0 \right\}$$

is strictly positive.

We have

$$Z'_n(\delta) \geq \int_{\sigma}^{+\infty} 1_{|A(n, \delta, s)| > 0} |\Omega \cap (H_{T+\delta+2\tau_n}^+ \setminus U_{T-\delta}^{\tau_n}) \cap (B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n}))| ds.$$

Now, for any $s \in \mathcal{S}$, we want to estimate from below the measure of the set

$$\Omega \cap (H_{T+\delta+2\tau_n}^+ \setminus U_{T-\delta}^{\tau_n}) \cap (B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})).$$

To that end, on the straight line through $q_{1,n}$ and $q_{2,n}$, we look at the segment

$$\{P_{H_T}(q_{1,n}) + t\nu : t \in [\delta, \delta + \text{diam}\Omega]\},$$

where $P_{H_T}(q_{1,n})$ is the orthogonal projection of $q_{1,n}$ onto H_T . This segment can contain neither interior points of $U_{T-\delta}^{\tau_n}$ nor points of its essential boundary, otherwise we would have an away contact point. Consequently, its distance to the closure of the set $U_{T-\delta}^{\tau_n}$ is strictly positive, say $\eta' > 0$.

There are two possibilities: either all the set $(B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})) \cap H_{T+\delta+2\tau_n}^+$ is contained in Ω , or not.

In the first situation, the set

$$(B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})) \cap B_{\eta'}(P_{H_T}(q_{1,n}) + (r(s) + \tau_n)\nu)$$

is contained in $\Omega \cap H_{T+\delta+2\tau_n}^+$ but does not intersect $U_{T-\delta}^{\tau_n}$. Moreover, thanks to (31), there exists a constant $C(\delta, d)$ depending on δ and the dimension of the space such that

$$|(B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})) \cap B_{\eta'}(P_{H_T}(q_{1,n}) + (r(s) + \tau_n)\nu)| \geq \gamma_n C(\delta, d)(\eta')^{d-1}.$$

In the second situation, since $(B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})) \cap H_{T+\delta+2\tau_n}^+$ contains already points from Ω (precisely from $U_{T-\delta}^{\tau_n}$), it also contains points from $\partial^*\Omega$. Since any such boundary point lies at distance at least η from $U_{T-\delta}^{\tau_n}$ (recall (28)), we can find a point x_n such that $B_{\frac{\eta}{2}}(x_n) \cap ((B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})) \cap H_{T+\delta+2\tau_n}^+) \subseteq \Omega \setminus U_{T-\delta}^{\tau_n}$. As before, thanks to (31), there exists a constant $C(\delta, d, \text{diam}\Omega)$, now depending on the diameter as well, such that

$$|B_{\frac{\eta}{2}}(x_n) \cap ((B_{r(s)}(q_{1,n}) \setminus B_{r(s)}(q_{2,n})) \cap H_{T+\delta+2\tau_n}^+)| \geq \gamma_n C(\delta, d, \text{diam}\Omega)\eta^{d-1}.$$

Finally,

$$Z'_n(\delta) \geq \int_{\sigma}^{+\infty} 1_{|A(n,\delta,s)|>0} \gamma_n (C(\delta, d) \wedge C(\delta, d, \text{diam}\Omega)) (\eta \wedge \eta')^{d-1} ds \geq K(\delta)\gamma_n,$$

for some positive constant $K(\delta)$.

If $K(\delta) > \frac{C}{8}$, the proof is achieved. Otherwise, we choose $\tilde{\delta} < \delta$ such that $X_n(3\tilde{\delta}) \leq \frac{K(\tilde{\delta})}{4}\gamma_n$. Reproducing the same arguments and using the fact that $K(\tilde{\delta}) \geq K(\delta)$, we conclude the proof. \square

3.3. Step 3 (decomposition of Ω into symmetric and non-symmetric part). We show that Ω can be decomposed as

$$\Omega = \Omega^s \sqcup \Omega^{ns},$$

where Ω^s is an *open set* representing the Steiner symmetric part of Ω , given by

$$\Omega^s := \bigcup \left\{ (p, p') : p' \text{ is an away contact point, } p \text{ is its symmetric about } H_T \right\},$$

(p, p') being the open segment with endpoints p and p' , and $\Omega^{ns} := \Omega \setminus \Omega^s$ represents the non-symmetric part. More precisely, we prove the following two claims:

- *Claim 3a.* If p' is an away contact point and p is its symmetric about H_T ,

$$(37) \quad |(B_{r(s)}(p') \setminus B_{r(s)}(p)) \cap \Omega^{ns}| = 0 \text{ for a.e. } s \in (\sigma, +\infty);$$

$$(38) \quad \exists \varepsilon > 0 : |B_\varepsilon(p') \cap (\Omega \setminus \mathcal{R}_T)| = 0, \text{ and hence } \Omega^s \text{ is open.}$$

- *Claim 3b.* Denoting by Ω_i^s the open connected components of Ω^s , it holds

$$(39) \quad \overline{\partial^*\Omega_i^s} \cap (H_T^\pm \setminus H_T) \text{ are connected sets;}$$

$$(40) \quad \overline{\partial^*\Omega^s} \cap \overline{\partial^*\Omega^{ns}} \subset H_T.$$

Remark 15. We point out that (37) implies that every connected component Ω_i^s of Ω^s satisfies

$$(41) \quad \text{dist}(\Omega_i^s, \Omega^{ns}) \geq \sup_{s \in (\sigma, +\infty)} r(s) = r(\sigma).$$

Proof of Claim 3a. Starting from the assumption of h -criticality, we obtain

$$\begin{aligned} 0 &= \int_{\Omega} h_{p'} - \int_{\Omega_T \cup \mathcal{R}_T} h_{p'} - \int_{\Omega} h_p + \int_{\Omega_T \cup \mathcal{R}_T} h_p \\ &= \int_{\Omega \setminus (\Omega_T \cup \mathcal{R}_T)} h_{p'} - \int_{\Omega \setminus (\Omega_T \cup \mathcal{R}_T)} h_p \\ &= \int_0^{+\infty} \int_{\Omega \setminus (\Omega_T \cup \mathcal{R}_T)} [\chi_{B_{r(s)}(p')}(y) - \chi_{B_{r(s)}(p)}(y)] dy ds \\ &= \int_0^{+\infty} \int_{\Omega \setminus (\Omega_T \cup \mathcal{R}_T)} \chi_{B_{r(s)}(p')}(y) [1 - \chi_{B_{r(s)}(p)}(y)] dy ds \\ &= \int_0^{+\infty} |(\Omega \setminus (\Omega_T \cup \mathcal{R}_T)) \cap (B_{r(s)}(p') \setminus B_{r(s)}(p))| ds \\ &= \int_0^{+\infty} |(\Omega \setminus \mathcal{R}_T) \cap (B_{r(s)}(p') \setminus B_{r(s)}(p))| ds \\ &= \int_{\sigma}^{+\infty} |(\Omega \setminus \mathcal{R}_T) \cap (B_{r(s)}(p') \setminus B_{r(s)}(p))| ds, \end{aligned}$$

which proves (37).

Let us prove (38). We claim that there exists $s_0 \in (\sigma, +\infty)$ such that

$$(42) \quad 0 < |[B_{r(s_0)}(p') \setminus B_{r(s_0)}(p)] \cap \Omega| < |B_{r(s_0)}(p') \setminus B_{r(s_0)}(p)|.$$

Indeed, since Ω is not h -degenerate, by equality (14) in Lemma 11 we have

$$\mathcal{L}^1(\{s > \sigma : |\Omega \cap (B_{r(s)}(p) \Delta B_{r(s)}(p'))| > 0\}) > 0.$$

Hence, using also (37), we can pick $s_0 \in (\sigma, +\infty)$ such that the left inequality in (42) is satisfied and

$$(43) \quad |(B_{r(s_0)}(p') \setminus B_{r(s_0)}(p)) \cap \Omega^{ns}| = 0.$$

We observe that, for such s_0 , also the right inequality in (42) is necessarily satisfied. Indeed, if this is not the case, we have

$$|[B_{r(s_0)}(p') \setminus B_{r(s_0)}(p)] \cap \Omega| = |B_{r(s_0)}(p') \setminus B_{r(s_0)}(p)|.$$

In view of (43), this implies that $B_{r(s_0)}(p') \setminus B_{r(s_0)}(p)$ is contained into \mathcal{R}_T , and hence $B_{r(s_0)}(p) \setminus B_{r(s_0)}(p')$ is contained into Ω_T . Since $\Omega_T \cup \mathcal{R}_T$ is Steiner-symmetric about H_T , we obtain (via Fubini Theorem) that p and p' belong to $\text{int}(\Omega^{(1)})$, contradicting the fact that they belong to $\overline{\partial^* \Omega}$.

Now we observe that

$$(44) \quad \exists y' \in [B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}] \cap \overline{\partial^* \Omega}.$$

Indeed, if (44) was false, $B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}$ would be contained either into $\text{int}(\Omega^{(1)})$ or into $\text{int}(\Omega^{(0)})$, against (42). In view of (43), the two sets Ω and \mathcal{R}_T have the same density at every point of $B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}$, and hence

$$[B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}] \cap \partial^* \Omega = [B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}] \cap \partial^* \mathcal{R}_T;$$

consequently, since the set $B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}$ is open, we have

$$(45) \quad [B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}] \cap \overline{\partial^* \Omega} = [B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}] \cap \overline{\partial^* \mathcal{R}_T}.$$

By (44) and (45), it turns out that the point y' is itself an away contact point. Therefore, denoting by y its symmetric about H_T , in the same way as we obtained (37), replacing the pair p, p' by the pair y, y' , we obtain

$$(46) \quad |[B_{r(\tau)}(y') \setminus B_{r(\tau)}(y)] \cap (\Omega \setminus \mathcal{R}_T)| = 0 \quad \text{for a.e. } \tau \in (\sigma, +\infty).$$

Since the set $B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}$ is open, for every $\varepsilon > 0$ sufficiently small the ball $B_\varepsilon(y')$ is contained into $B_{r(s_0)}(p') \setminus \overline{B_{r(s_0)}(p)}$, and hence

$$(47) \quad \exists \varepsilon(s_0) > 0 : B_{\varepsilon(s_0)}(p') \subset [B_{r(s_0)}(y') \setminus \overline{B_{r(s_0)}(y)}].$$

This achieves the proof of (38) in case the equality in (46) is satisfied at $\tau = s_0$. But, since (46) holds merely for a.e. $\tau \in (0, +\infty)$, we have to refine the argument as follows. By (47), for λ sufficiently close to $r(s_0)$, we have

$$(48) \quad p' \in [B_\lambda(y') \setminus \overline{B_\lambda(y)}].$$

Then, by the continuity from the right of the map $r \mapsto r(s)$, there exists $\delta > 0$ such that

$$(49) \quad \forall s \in [s_0, s_0 + \delta), \quad p' \in [B_{r(s)}(y') \setminus \overline{B_{r(s)}(y)}]$$

and hence

$$(50) \quad \forall s \in [s_0, s_0 + \delta), \quad \exists \varepsilon(s) > 0 : B_{\varepsilon(s)}(p') \subset [B_{r(s)}(y') \setminus \overline{B_{r(s)}(y)}].$$

Eventually, the proof of (38) is achieved by choosing $s \in [s_0, s_0 + \delta)$ such that the equality in (46) is satisfied at $\tau = s$. \square

Proof of Claim 3b. The same arguments used to obtain the homonym claim in the proof of [6, Theorem 1] apply.

3.4. Step 4 (conclusion). We show that the open connected components of Ω^s are balls of the same radius $R > \eta/2$, lying at distance larger than or equal to $r(\sigma)$, while the set Ω^{ns} is Lebesgue negligible.

In order to formulate more precisely the claims which conclude our proof, we need to set up some additional definitions and notation.

Given two different open connected components Ω_i^s, Ω_j^s of Ω^s , we say that Ω_i^s is in *h-contact with* Ω_j^s if there exists an away contact point $p' \in \overline{\partial^* \Omega_i^s} \setminus H_T$ such that, denoting by p its symmetric about H_T , it holds

$$\int_{\Omega_j^s} |h_p - h_{p'}| > 0.$$

It is not difficult to check that, if Ω_i^s is in *h-contact with* Ω_j^s , Ω_j^s is in *h-contact with* Ω_i^s . If Ω_i^s is not in contact with any other component of Ω^s , we say that Ω_i^s is *h-isolated*.

Remark 16. We observe that, if Ω_i^s is h -isolated, for every $p, p' \in \overline{\partial^* \Omega_i^s} \setminus H_T$ symmetric about H_T , it holds

$$\int_{\Omega^s} |h_p - h_{p'}| = \int_{\Omega_i^s} |h_p - h_{p'}|.$$

Hence, if Ω_j^s is any other component of Ω^s , we have

$$\int_{\sigma}^{\infty} |(B_{r(s)}(p') \setminus B_{r(s)}(p)) \cap \Omega_j^s| ds = 0,$$

which implies

$$(51) \quad \text{dist}(\Omega_j^s, \Omega_i^s) > \sup_{s \in (\sigma, +\infty)} r(s) = r(\sigma).$$

Since our strategy will require to let the initial hyperplane vary, we will write

$$\Omega = \Omega^{\nu, s} \sqcup \Omega^{\nu, ns},$$

where the additional superscript ν indicates the direction of the parallel movement, namely the normal to the initial hyperplane H_0 (and the decomposition is always meant with respect to the parallel hyperplane H_T at the stopping time T defined in Step 2).

The fourth and final step of our proof consists in showing the following claims:

- *Claim 4a.* Given $\nu \in \mathbb{S}^{d-1}$, let Ω_b be a h -isolated open connected component of $\Omega^{\nu, s}$. Then Ω_b is a ball of radius at least $\eta/2$, and $\Omega \setminus \Omega_b$ is h -critical and not h -degenerate, unless it has measure zero.
- *Claim 4b.* The following family is empty:

$$\mathcal{F} := \bigcup_{\nu \in \mathbb{S}^{d-1}} \left\{ \text{open connected components not } h\text{-isolated of } \Omega^{\nu, s} \right\}.$$

- *Claim 4c (conclusion).* Ω is equivalent to a finite union of balls of radius $R > \eta/2$, at mutual distance larger than or equal to $r(\sigma)$.

Proof of claim 4a. Given $\nu \in \mathbb{S}^{d-1}$, let Ω_b be a h -isolated open connected component of $\Omega^{\nu, s}$. Assume by a moment to know that

$$(52) \quad \Omega_b \text{ is } h\text{-critical and not } h\text{-degenerate.}$$

In this case, we can restart our proof, with Ω_b in place of Ω . Given an arbitrary direction $\tilde{\nu} \in \mathbb{S}^{d-1}$, we make the decomposition

$$\Omega_b = \Omega_b^{\tilde{\nu}, s} \sqcup \Omega_b^{\tilde{\nu}, ns}.$$

It is not difficult to show that, unless $\Omega_b^{\tilde{\nu}, ns}$ is empty, this decomposition splits Ω_b into two open sets, contradicting the connectedness of Ω_b .

(The detailed argument can be found in the proof of [6, Theorem 1], see Claim 4a. therein). Hence Ω_b is Steiner symmetric about a hyperplane with unit normal $\tilde{\nu}$. By the arbitrariness of $\tilde{\nu}$, we deduce that Ω_b is a ball. Denote by R the radius of this ball. We observe that a ball of radius R is not h -degenerate if and only if $\mathcal{L}^1(\{s : r(s) \in (0, 2R)\}) > 0$. Recalling (11), the nondegeneracy of Ω_b yields the lower bound $R > \eta/2$.

To conclude the proof of Claim 4a., it remains to show that (52) holds true and that the same property is valid for $\Omega \setminus \Omega_b$, unless it has measure zero.

For the sake of clearness, we split the proof into three consecutive lemmas.

Lemma 17. *Under the assumptions of Theorem 1, given $\nu \in \mathbb{S}^{d-1}$, let Ω_b be a h -isolated open connected component of $\Omega^{\nu,s}$. Then*

$$\inf_{x_1, x_2 \in \partial^* \Omega_b} \frac{\int_{\Omega^{\nu,s}} |h_{x_1} - h_{x_2}|}{\|x_1 - x_2\|} > 0.$$

Proof. Assume by contradiction that

$$(53) \quad \inf_{x_1, x_2 \in \partial^* \Omega_b} \frac{\int_{\Omega^{\nu,s}} |h_{x_1} - h_{x_2}|}{\|x_1 - x_2\|} = 0.$$

Then there exist sequences of distinct points $\{x_{1,n}\}, \{x_{2,n}\} \subset \partial^* \Omega_b$, with $\|x_{1,n} - x_{2,n}\| \rightarrow 0$, such that

$$\frac{\int_{\Omega^{\nu,s}} |h_{x_{1,n}} - h_{x_{2,n}}|}{\|x_{1,n} - x_{2,n}\|} \rightarrow 0.$$

Up to subsequences, we may assume that $\|x_{1,n} - x_{2,n}\|$ converges to 0 decreasingly, and that $\{x_{1,n}\}$ and $\{x_{2,n}\}$ converge to some point $\bar{x} \in \partial^* \Omega_b$, which may belong or not to H_T , being as usual T the stopping time defined as in Step 2 for the parallel movement with normal ν . Let us examine the two cases separately.

In case $\bar{x} \notin H_T$, we may assume without loss of generality that $\{x_{1,n}\}, \{x_{2,n}\} \subset H_T^+ \setminus H_T$. Since Ω is not h -degenerate, (53) implies

$$\inf_{x_1, x_2 \in \partial^* \Omega_b} \frac{\int_{\Omega^{\nu,ns}} |h_{x_1} - h_{x_2}|}{\|x_1 - x_2\|} > 0.$$

In particular, for $n = 1$, we have

$$\int_{\sigma}^{+\infty} |\Omega^{\nu,ns} \cap (B_{r(s)}(x_{1,1}) \Delta B_{r(s)}(x_{2,1}))| ds = \int_{\Omega^{\nu,ns}} |h_{x_{1,1}} - h_{x_{2,1}}| > 0.$$

We infer that there exists $s_0 \in (\sigma, +\infty)$ such that

$$|\Omega^{\nu,ns} \cap (B_{r(s_0)}(x_{1,1}) \Delta B_{r(s_0)}(x_{2,1}))| > 0.$$

Hence we can pick a point $p \in \text{int}(B_{r(s_0)}(x_{1,1}) \Delta B_{r(s_0)}(x_{2,1}))$ of density 1 for $\Omega^{\nu,ns}$, and a radius $\varepsilon > 0$ sufficiently small so that

$$(54) \quad |B_{\varepsilon}(p) \cap \Omega^{\nu,ns}| \geq \frac{1}{2} |B_{\varepsilon}(p)|.$$

Possibly reducing ε we can also assume that $B_{\varepsilon}(p) \subseteq (B_{r(s_0)}(x_{1,1}) \Delta B_{r(s_0)}(x_{2,1}))$.

Now we recall that, by (39), the set $\partial^* \Omega_b \cap (H_T^+ \setminus H_T)$ is connected. Hence for every $n \geq 1$ we can join $x_{1,n}$ to $x_{1,n+1}$ by a continuous arc $\gamma_{1,n}$ contained into $\partial^* \Omega_b \cap (H_T^+ \setminus H_T)$. We can repeat the same procedure for the second sequence, constructing a family of continuous arcs $\gamma_{2,n}$ joining $x_{2,n}$ to $x_{2,n+1}$ for every $n \geq 1$.

We look at the boundaries of the balls of radius $r(s_0)$ whose centre moves along $\gamma_{1,n}$ and $\gamma_{2,n}$. Clearly these balls tends to superpose in the limit as $n \rightarrow +\infty$, since $\|x_{1,n} - x_{2,n}\|$ decreases to 0. Moreover, we know from (41) that, during the continuous movement of

their centre along along $\gamma_{1,n}$ and $\gamma_{2,n}$, the boundary of these balls cannot cross points of density 1 for $\Omega^{\nu,ns}$. Hence, for n large,

$$B_\varepsilon(p) \cap \Omega^{\nu,ns} \subseteq B_{r(s_0)}(x_{1,n}) \Delta B_{r(s_0)}(x_{2,n});$$

hence, still for n sufficiently large,

$$|B_\varepsilon(p) \cap \Omega^{\nu,ns}| \leq |B_{r(s_0)}(x_{1,n}) \Delta B_{r(s_0)}(x_{2,n})| < \frac{1}{4}|B_\varepsilon(p)|,$$

against (54).

In case $\bar{x} \in H_T$, we proceed in the same way, except that we cannot ensure any more that both sequences $\{x_{1,n}\}$ and $\{x_{2,n}\}$ belong to the same halfspace H_T^+ or H_T^- . Thus, when we construct the continuous arcs $\gamma_{1,n}$ and $\gamma_{2,n}$, they may belong indistinctly to $\partial^*\Omega_b \cap (H_T^- \setminus H_T)$ or to $\partial^*\Omega_b \cap (H_T^+ \setminus H_T)$, but this does not affect the validity of the proof since the contradiction follows as soon as $x_{1,n}$ and $x_{2,n}$ are close enough. \square

Lemma 18. *Under the assumptions of Theorem 1, given $\nu \in \mathbb{S}^{d-1}$, let Ω_b be a h -isolated open connected component of $\Omega^{\nu,s}$. There exists a constant $c_b > 0$ such that*

$$(55) \quad \int_{\Omega^{\nu,s}} h_x = c_b \quad \forall x \in \overline{\partial^*\Omega_b}.$$

Moreover, the constant is the same for any other open connected component of $\Omega^{\nu,s}$ such that the closure of its essential boundary intersects $\overline{\partial^*\Omega_b}$.

Proof. We argue in a similar way as in the proof of the previous lemma. Given $x_1, x_2 \in \overline{\partial^*\Omega_b} \cap (H_T^+ \setminus H_T)$, by (39), they can be joined by a continuous arc γ contained into $\partial^*\Omega_b \cap (H_T^+ \setminus H_T)$. By (41), for \mathcal{L}^1 -a.e. $s \in (\sigma, +\infty)$, the boundary of the ball of radius $r(s)$ centred at any point along γ cannot cross points of density 1 for $\Omega^{\nu,ns}$. We deduce that, still for \mathcal{L}^1 -a.e. $s \in (\sigma, +\infty)$, $B_{r(s)}(x_1) \Delta B_{r(s)}(x_2)$ cannot contain points of density 1 for $\Omega^{\nu,ns}$. Therefore, by the equality (14) in Lemma 11, we get

$$\int_{\Omega^{\nu,ns}} |h_{x_1} - h_{x_2}| = 0,$$

and hence, using also the fact that Ω is h -critical,

$$\int_{\Omega^{\nu,s}} h_{x_1} = \int_{\Omega} h_{x_1} - \int_{\Omega^{\nu,ns}} h_{x_1} = \int_{\Omega} h_{x_2} - \int_{\Omega^{\nu,ns}} h_{x_2} = \int_{\Omega^{\nu,s}} h_{x_2}.$$

By the arbitrariness of x_1, x_2 , we infer that there exists a constant $c_b^+ > 0$ such that $\int_{\Omega^{\nu,s}} h_x = c_b^+$ for every $x \in \overline{\partial^*\Omega_b} \cap (H_T^+ \setminus H_T)$. In the same way, we obtain that there exists a constant $c_b^- > 0$ such that $\int_{\Omega^{\nu,s}} h_x = c_b^-$ for every $x \in \overline{\partial^*\Omega_b} \cap (H_T^- \setminus H_T)$. Since the two sets $\overline{\partial^*\Omega_b} \cap H_T^\pm$ have common points on H_T , we conclude that $c_b^+ = c_b^-$. The same argument proves also the last assertion of the lemma. \square

Lemma 19. *Under the assumptions of Theorem 1, given $\nu \in \mathbb{S}^{d-1}$, let Ω_b be a h -isolated open connected component of $\Omega^{\nu,s}$. Then Ω_b is h -critical and not h -degenerate. The same assertions hold true for its complement $\Omega \setminus \Omega_b$, unless it is of measure zero.*

Proof. We know from Lemma 17 that

$$\inf_{x_1, x_2 \in \partial^* \Omega_b} \frac{\int_{\Omega^{\nu, s}} |h_{x_1} - h_{x_2}|}{\|x_1 - x_2\|} > 0.$$

But, since Ω_b is h -isolated, we have

$$\int_{\Omega^{\nu, s}} |h_{x_1} - h_{x_2}| = \int_{\Omega_b} |h_{x_1} - h_{x_2}| \quad \forall x_1, x_2 \in \partial^* \Omega_b$$

and hence Ω_b is not h -degenerate.

Let us prove that Ω_b is h -critical. For every $x \in \overline{\partial^* \Omega_b}$, recalling equality (55) in Lemma 18, we have

$$\int_{\Omega^{\nu, s}} h_x = \sigma |\Omega^{\nu, s}| + \int_{\sigma}^{+\infty} |\Omega^{\nu, s} \cap B_{r(s)}(x)| ds = c_b.$$

We infer that

$$\begin{aligned} \int_{\Omega_b} h_x &= \sigma |\Omega_b| + \int_{\sigma}^{+\infty} |\Omega_b \cap B_{r(s)}(x)| ds \\ &= \sigma |\Omega_b| + \int_{\sigma}^{+\infty} |\Omega^{\nu, s} \cap B_{r(s)}(x)| ds = \sigma |\Omega_b| + c_b - \sigma |\Omega^{\nu, s}|. \end{aligned}$$

where the second equality follows from (51).

Let us now consider the complement $\Omega \setminus \Omega_b$. Assume it is of positive measure, and hence that $\partial^*(\Omega \setminus \Omega_b)$ is not empty.

For every $x_1, x_2 \in \partial^*(\Omega \setminus \Omega_b)$, by (41) and (51), it holds

$$\begin{aligned} \int_{\Omega \setminus \Omega_b} |h_{x_1} - h_{x_2}| &= \int_{\sigma}^{+\infty} |(\Omega \setminus \Omega_b) \cap (B_{r(s)}(x_1) \Delta B_{r(s)}(x_2))| ds \\ &= \int_{\sigma}^{+\infty} |\Omega \cap (B_{r(s)}(x_1) \Delta B_{r(s)}(x_2))| ds = \int_{\Omega} |h_{x_1} - h_{x_2}|, \end{aligned}$$

Thus $\Omega \setminus \Omega_b$ is not h -degenerate since by assumption Ω is not h -degenerate.

In a similar way, for every $x \in \partial^*(\Omega \setminus \Omega_b)$, since Ω is h critical, we have

$$\int_{\Omega} h_x = \sigma |\Omega| + \int_{\sigma}^{+\infty} |\Omega \cap B_{r(s)}(x)| ds = c.$$

We infer that

$$\begin{aligned} \int_{\Omega \setminus \Omega_b} h_x &= \sigma |\Omega \setminus \Omega_b| + \int_{\sigma}^{+\infty} |(\Omega \setminus \Omega_b) \cap B_{r(s)}(x)| ds \\ &= \sigma |\Omega \setminus \Omega_b| + \int_{\sigma}^{+\infty} |\Omega \cap B_{r(s)}(x)| ds = \sigma |\Omega \setminus \Omega_b| + c - \sigma |\Omega|, \end{aligned}$$

where the second equality follows from (41) and (51). \square

Proof of Claim 4b. First let us observe that the family \mathcal{F} is at most countable. This follows from the facts that any open set of \mathbb{R}^d has at most countable connected components, and that, for two different directions ν_1 and ν_2 , a connected component of $\Omega^{\nu_1, s}$ cannot intersect another connected component of $\Omega^{\nu_2, s}$ without being equal.

We now prove Claim 4b. by contradiction. If the family \mathcal{F} is not empty, it turns out to contain an element $\Omega_{\#}$ which is Steiner symmetric about d hyperplanes with linearly independent normals ν_1, \dots, ν_d (for the detailed justification, see Claim 4b. in the proof of [6, Theorem 1]).

Next we consider any other element $\Omega_{\#\#}$ of \mathcal{F} which is in h -contact with $\Omega_{\#}$ in the decomposition with respect to one among the directions ν_1, \dots, ν_d , say ν_1 . If T_1 is the stopping time for the parallel movement with normal ν_1 , there exist $p, p' \in \overline{\partial^* \Omega_{\#}} \setminus H_{T_1}$, symmetric about H_{T_1} , such that

$$\int_{\Omega_{\#\#}} |h_p - h_{p'}| = \int_{\sigma}^{+\infty} |(B_{r(s)}(p) \Delta B_{r(s)}(p')) \cap \Omega_{\#\#}| ds > 0.$$

We infer that

$$\mathcal{L}^1(\{s \in (\sigma, +\infty) : |(B_{r(s)}(p) \Delta B_{r(s)}(p')) \cap \Omega_{\#\#}| > 0\}) > 0,$$

and hence, since we are assuming that $\Omega_{\#\#}$ is Steiner symmetric with respect to H_{T_1} ,

$$\mathcal{L}^1(\{s \in (\sigma, +\infty) : \partial B_{r(s)}(p) \cap \Omega_{\#\#}^{(1)} \neq \emptyset\}) > 0.$$

Recalling (37), this implies that $\Omega_{\#\#}$ is itself Steiner symmetric about the same hyperplanes as $\Omega_{\#}$ is. Then, Lemma 18 in [6] implies that the set $\Omega_{\#} \cup \Omega_{\#\#}$ is connected, yielding a contradiction.

Proof of Claim 4c. We start the procedure by choosing a direction $\nu \in \mathbb{S}^{d-1}$. By Claim 4b., we can pick a h -isolated open connected component of $\Omega^{\nu, s}$, which by Claim 4a. turns out to be a ball B_1 of radius $R_1 > \eta/2$. We remove this ball from Ω . By Claim 4a., we are left with a set Ω' which is still h -critical and not h -degenerate (unless it has measure zero). So we can restart the process with Ω' in place of Ω . Again, by Claim 4b., we can pick a h -isolated open connected component of $(\Omega')^{\nu, s}$, which by Claim 4a. turns out to be a ball B_2 of radius $R_2 > \eta/2$. We remove this ball from Ω' . We observe that, by (51), we have

$$(56) \quad \text{dist}(B_1, B_2) \geq \sup_{s \in (\sigma, +\infty)} r(s) = r(\sigma).$$

Then, for every $p_1 \in \partial B_1$ and $p_2 \in \partial B_2$, we have

$$\begin{aligned} \int_{\sigma}^{+\infty} |B_1 \cap B_{r(s)}(p_1)| ds &= \int_{\sigma}^{+\infty} |(B_1 \cup B_2) \cap B_{r(s)}(p_1)| ds = \\ &= \int_{\sigma}^{+\infty} |(B_1 \cup B_2) \cap B_{r(s)}(p_2)| ds = \int_{\sigma}^{+\infty} |B_2 \cap B_{r(s)}(p_2)| ds. \end{aligned}$$

where the first and third equalities follow from (56), while the second one is consequence of the h -criticality of Ω and of $\Omega \setminus (B_1 \cup B_2)$. Taking into account that, for every fixed s , the map $R \mapsto |B_R \cap B_{r(s)}(p)|$, with $p \in \partial B_R$, is strictly increasing, we deduce that $R_1 = R_2$.

Since Ω has finite measure, we can repeat this process a finite number of times, until when we are left with a set of measure zero. \square

REFERENCES

- [1] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large. V, Amer. Math. Soc. Transl. (2) **21** (1962), 412–416.
- [2] B. Barrios, I. Peral, F. Soria, and E. Valdinoci, A Widder’s type theorem for the heat equation with nonlocal diffusion, Arch. Ration. Mech. Anal. **213** (2014), no. 2, 629–650.
- [3] M. Bonforte, Y. Sire, and J.L. Vázquez, Optimal existence and uniqueness theory for the fractional heat equation, Nonlinear Anal. **153** (2017), 142–168.
- [4] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.
- [5] C. Bucur and E. Valdinoci, Nonlocal diffusion and applications, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.
- [6] D. Bucur and I. Fragalà, Rigidity for measurable sets, Arxiv arXiv:2102.12389, 2021.
- [7] X. Cabré, M.M. Fall, J. Solà-Morales, and T. Weth, Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay, J. Reine Angew. Math. **745** (2018), 253–280.
- [8] L. A. Caffarelli and P.E. Souganidis, A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs, Comm. Pure Appl. Math. **61** (2008), no. 1, 1–17.
- [9] L.A. Caffarelli, J.M. Roquejoffre, and O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144.
- [10] A. Chambolle, M. Morini, and M. Ponsiglione, Nonlocal curvature flows, Arch. Ration. Mech. Anal. **218** (2015), no. 3, 1263–1329. MR 3401008
- [11] A. Chambolle, M. Novaga, and B. Ruffini, Some results on anisotropic fractional mean curvature flows, Interfaces Free Bound. **19** (2017), no. 3, 393–415.
- [12] A. Cianchi and N. Fusco, Functions of bounded variation and rearrangements, Arch. Ration. Mech. Anal. **165** (2002), no. 1, 1–40.
- [13] G. Ciraolo, A. Figalli, F. Maggi, and M. Novaga, Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature, J. Reine Angew. Math. **741** (2018), 275–294. MR 3836150
- [14] M.G. Delgadino and F. Maggi, Alexandrov’s theorem revisited, Anal. PDE **12** (2019), no. 6, 1613–1642.
- [15] S. Dipierro, O. Savin, and E. Valdinoci, Boundary behavior of nonlocal minimal surfaces, J. Funct. Anal. **272** (2017), no. 5, 1791–1851.
- [16] A. Figalli and E. Valdinoci, Regularity and Bernstein-type results for nonlocal minimal surfaces, J. Reine Angew. Math. **729** (2017), 263–273.
- [17] R. L. Frank and R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, J. Funct. Anal. **255** (2008), no. 12, 3407–3430.
- [18] C. Imbert, Level set approach for fractional mean curvature flows, Interfaces Free Bound. **11** (2009), no. 1, 153–176.
- [19] F. Maggi, Sets of finite perimeter and geometric variational problems, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012, An introduction to geometric measure theory. MR 2976521
- [20] J.M. Mazón, J.D. Rossi, and J.J. Toledo, Nonlocal perimeter, curvature and minimal surfaces for measurable sets, Frontiers in Mathematics, Birkhäuser/Springer, Cham, 2019.

(Dorin Bucur) UNIVERSITÉ SAVOIE MONT BLANC, LABORATOIRE DE MATHÉMATIQUES CNRS UMR 5127, CAMPUS SCIENTIFIQUE, 73376 LE-BOURGET-DU-LAC (FRANCE)

Email address: dorin.bucur@univ-savoie.fr

(Ilaria Fragalà) DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI, 32, 20133 MILANO (ITALY)

Email address: ilaria.fragala@polimi.it