Research article

Global existence for reaction-diffusion evolution equations driven by the p-Laplacian on manifolds[†]

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Abstract: We consider reaction-diffusion equations driven by the *p*-Laplacian on noncompact, infinite volume manifolds assumed to support the Sobolev inequality and, in some cases, to have L^2 spectrum bounded away from zero, the main example we have in mind being the hyperbolic space of any dimension. It is shown that, under appropriate conditions on the parameters involved and smallness conditions on the initial data, global in time solutions exist and suitable smoothing effects, namely explicit bounds on the L^∞ norm of solutions at all positive times, in terms of L^q norms of the data. The geometric setting discussed here requires significant modifications w.r.t. the Euclidean strategies.

Keywords: nonlinear reaction-diffusion equation; Riemannian manifolds; *p*−Laplace equation; smoothing estimates; global existence; Sobolev inequality; Poincaré inequality; porous medium equation

1. Introduction

We investigate existence of nonnegative global in time solutions to the quasilinear parabolic problem

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^\sigma & \text{in } M \times (0, T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{1.1}$$

where M is an N-dimensional, complete, noncompact, Riemannian manifold of infinite volume, whose metric is indicated by g, and where div and ∇ are respectively the divergence and the gradient with

respect to g and $T \in (0, +\infty]$. We shall assume throughout this paper that

$$\frac{2N}{N+1} < p < N, \qquad \sigma > p - 1. \tag{1.2}$$

The problem is posed in the Lebesgue spaces

$$L^q(M) = \left\{ v : M \to \mathbb{R} \text{ measurable }, \ \|v\|_{L^q} := \left(\int_M v^q \, d\mu \right)^{1/q} < +\infty \right\},$$

where μ is the Riemannian measure on M. We also assume the validity of the Sobolev inequality:

$$\text{(Sobolev inequality)} \quad \|v\|_{L^{p^*}(M)} \leq \frac{1}{C_{s,p}} \|\nabla v\|_{L^p(M)} \quad \text{for any } v \in C_c^\infty(M), \tag{1.3}$$

where $C_{s,p} > 0$ is a constant and $p^* := \frac{pN}{N-p}$. In some cases we also assume that the Poincaré inequality is valid, that is

$$\text{(Poincaré inequality)} \quad \|v\|_{L^p(M)} \leq \frac{1}{C_p} \|\nabla v\|_{L^p(M)} \quad \text{for any } v \in C_c^\infty(M), \tag{1.4}$$

for some $C_p > 0$. Observe that, for instance, (1.3) holds if M is a Cartan-Hadamard manifold, i.e., a simply connected Riemannian manifold with nonpositive sectional curvatures, while (1.4) is valid when M is a Cartan-Hadamard manifold satisfying the additional condition of having sectional curvatures bounded above by a constant $-c < 0$ (see, e.g., [15, 16]). Therefore, as it is well known, on $\mathbb{R}^N$ (1.3) holds, but (1.4) fails, whereas on the hyperbolic space both (1.3) and (1.4) are fulfilled.

Global existence and finite time blow-up of solutions for problem (1.1) has been deeply studied when $M = \mathbb{R}^N$, especially in the case $p = 2$ (linear diffusion). The literature for this problem is huge and there is no hope to give a comprehensive review here. We just mention the fundamental result of Fujita, see [10], who shows that blow-up in a finite time occurs for all nontrivial nonnegative data when $\sigma < 1 + \frac{2}{N}$, while global existence holds, for $\sigma > 1 + \frac{2}{N}$, provided the initial datum is small enough in a suitable sense. Furthermore, the critical exponent $\sigma = 1 + \frac{2}{N}$, belongs to the case of finite time blow-up, see e.g., [22] for the one dimensional case, $N = 1$, or [23] for $N > 1$. For further results concerning problem (1.1) with $p = 2$ see e.g., [7, 9, 11, 20, 26, 34–36, 41–43]).

Similarly, the case of problem (1.1) when $M = \mathbb{R}^N$ and $p > 1$ has attracted a lot of attention, see e.g., [12–14, 30–33] and references therein. In particular, in [31], nonexistence of nontrivial weak solutions is proved for problem (1.1) with $M = \mathbb{R}^N$ and

$$p > \frac{2N}{N+1}, \qquad \max\{1, p-1\} < \sigma \leq p - 1 + \frac{p}{N}.$$

Similar *weighted* problems have also been treated. In fact, for any strictly positive measurable function $\rho : \mathbb{R}^N \to \mathbb{R}$, let us consider the weighted L_ρ^q spaces

$$L_\rho^q(\mathbb{R}^N) = \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable }, \ \|v\|_{L_\rho^q} := \left(\int_{\mathbb{R}^N} v^q \rho(x) \, dx \right)^{1/q} < +\infty \right\}.$$

In [27] problem

$$\begin{cases} \rho(x)u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \rho(x)u^\sigma & \text{in } \mathbb{R}^N \times (0, T) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.5}$$

is addressed. In [27, Theorem 1], it is showed that, when $p > 2$, $\rho(x) = (1 + |x|)^{-l}$, $0 \le l < p$, $\sigma > p - 1 + \frac{p}{N}$, $u_0 \in L^1_\rho(\mathbb{R}^N) \cap L^s_\rho(\mathbb{R}^N)$ is sufficiently small, with $s > \frac{(N-l)(\sigma-p+1)}{p-l}$, then problem (1.5) admits a global in time solution. Moreover, the solution satisfies a smoothing estimate $L^1_\rho - L^\infty$, in the sense that for sufficiently small data $u_0 \in L^1_\rho(\mathbb{R}^N)$, the corresponding solution is bounded, and a quantitive bound on the L^∞ norm of the solution holds, in term of the $L^1_\rho(\mathbb{R}^N)$ norm of the initial datum. On the other hand, in [27, Theorem 2], when $p > 2$, $\rho(x) = (1 + |x|)^{-l}$, $l \ge p$, $\sigma > p - 1$, $u_0 \in L^1_\rho(\mathbb{R}^N) \cap L^s_\rho(\mathbb{R}^N)$ is sufficiently small, with $s > \frac{N}{p}(\sigma - p + 1)$, then problem (1.5) admits a global in time solution, which is bounded for positive times.

On the other hand, existence and nonexistence of global in time solutions to problems closely related to problem (1.1) have been investigated also in the Riemannian setting. The situation can be significantly different from the Euclidean situation, especially in the case of negative curvature. Infact, when dealing with the case of the N-dimensional *hyperbolic space*, $M = \mathbb{H}^N$, it is known that when $p = 2$, for all $\sigma > 1$ and sufficiently small nonnegative data there exists a global in time solution, see [3, 34, 39, 40]. A similar result has been also obtain when M is a complete, noncompact, stochastically complete Riemannian manifolds with $\lambda_1(M) > 0$, where $\lambda_1(M) := \inf \operatorname{spec}(-\Delta)$, see [19]. Stochastic completeness amounts to requiring that the linear heat semigroup preserves the identity, and is known to hold e.g., if the sectional curvature satisfies $\sec(x) \ge -cd(x, o)^2$ for all $x \in M$ outside a given compact, and a suitable $c > 0$, where d is the Riemannian distance and o is a given pole. Besides, it is well known that $\lambda_1(M) > 0$ e.g., if $\sec(x) \le -c < 0$ for all $x \in M$. Therefore, the class of manifolds for which the results of [19] hold is large, since it includes e.g., all Cartan-Hadamard manifolds with curvature bounded away from zero and not diverging faster than quadratically at infinity.

Concerning problem (1.1) with $p > 1$, we refer the reader to [28, 29] and references therein. In particular, in [28], nonexistence of global in time solutions on infinite volume Riemannian manifolds M is shown under suitable weighted volume growth conditions. In [29], problem (1.1) with $M = \Omega$ being a bounded domain and u^σ replaced by $V(x, t)u^\sigma$ is addressed, where V is a positive potential. To be specific, nonexistence of nonnegative, global solutions is established under suitable integral conditions involving V, p and σ.

In this paper, we prove the following results. Assume that the bounds (1.2) and the Sobolev inequality (1.3) hold, and besides that $\sigma > p - 1 + \frac{p}{N}$.

(a) If $u_0 \in L^s(M) \cap L^1(M)$ is sufficiently small, with $s > (\sigma - p + 1)\frac{N}{p}$, then a global solution exists. Furthermore, a smoothing estimate of the type $L^1 - L^\infty$ holds (see Theorem 2.2).

(b) If $u_0 \in L^{(\sigma-p+1)\frac{N}{p}}(M)$ is sufficiently small, then a global solution exists. Furthermore, a smoothing estimate of the type $L^{(\sigma-p+1)\frac{N}{p}} - L^\infty$ holds (see Theorem 2.4), this being new even in the Euclidean case.

(c) In addition, in both the latter two cases, we establish a $L^{(\sigma-p+1)\frac{N}{p}} - L^q$ smoothing estimate, for any $(\sigma - p + 1)\frac{N}{p} \le q < +\infty$ and an $L^q - L^q$ estimate for any $1 < q < +\infty$, for suitable initial data u_0.

Now suppose that both the Sobolev inequality (1.3) and the Poincaré inequality (1.4) hold, and

that (1.2) holds. This situation has of course no Euclidean analogue, as it is completely different from the case of a bounded Euclidean domain since M is noncompact and of infinite measure. Then:

(d) If $u_0 \in \mathrm{L}^s(M) \cap \mathrm{L}^{\sigma \frac{N}{p}}(M)$ is sufficiently small, with $s > \max\{(\sigma - p + 1)\frac{N}{p}, 1\}$, then a global solution exists. Furthermore, a smoothing estimate of the type $\mathrm{L}^s - \mathrm{L}^\infty$ holds (see Theorem 2.7).

(e) In addition, we establish and $\mathrm{L}^{\sigma \frac{N}{p}} - \mathrm{L}^q$ estimate, for any $\sigma \frac{N}{p} \le q < +\infty$ and an $\mathrm{L}^q - \mathrm{L}^q$ estimate for any $1 < q < +\infty$, for suitable initial data u_0.

Note that, when we require both (1.3) and (1.4), the assumption on σ can be relaxed.

In order to prove (a), we adapt the methods exploited in [27, Theorem 1]. Moreover, (b), (c) and (e) are obtained by means of an appropriate use of the Moser iteration technique, see also [18] for a similar result in the case of the porous medium equation with reaction. The proof of statement (d) is inspired [27, Theorem 2]; however, significant changes are needed since in [27] the precise form of the weight ρ is used.

As concerns smoothing effects for general nonlinear evolution equations, we refer the reader to the fundamental works of Bénilan [4] and, slightly later but with considerable further generality and methodological simplifications, Véron [38]. Recently, Coulhon and Hauer further generalize such results and give new and abstract ones which even allow to avoid Moser's iteration in a very general functional analytic setting, through an extrapolation argument, see [8]. It should also be remarked that, though we deal with weak solutions to our problems, it is certainly possible to prove existence of solution in stronger senses, e.g., the strong one according to Bénilan and Crandall seminal contribution [5]. In this regard, we also refer to the recent paper [21], in which existence results are proved also for parabolic equations governed by the p-Laplace operator with Lipschitz lower-order terms. We also mention that several important and seminal contributions to regularity results for solutions of general nonlinear parabolic equations and systems can be found in several works by Mingione, see e.g., [1, 6, 24].

The paper is organized as follows. The main results are stated in Section 2. Section 3 is devoted to $\mathrm{L}^{q_0} - \mathrm{L}^q$ and $\mathrm{L}^q - \mathrm{L}^q$ smoothing estimates, mainly instrumental to what follows. Some a priori estimates are obtained in Section 4. In Sections 5–7, Theorems 2.2, 2.4 and 2.7 are proved, respectively. Finally, in Section 8 we state similar results for the porous medium equation with reaction; the proofs are omitted since they are entirely similar to the p-Laplacian case.

2. Statements of the main results

Solutions to (1.1) will be meant in the weak sense, according to the following definition.

Definition 2.1. *Let M be a complete noncompact Riemannian manifold of infinite volume. Let $p > 1$, $\sigma > p - 1$ and $u_0 \in L^1_{loc}(M)$, $u_0 \ge 0$. We say that the function u is a weak solution to problem* (1.1) *in the time interval* $[0, T)$ *if*

$$u \in L^2((0, T); W^{1,p}_{loc}(M)) \cap L^\sigma_{loc}(M \times (0, T))$$

and for any $\varphi \in C^\infty_c(M \times [0, T])$ such that $\varphi(x, T) = 0$ for any $x \in M$, u satisfies the equality:

$$-\int_0^T \int_M u\,\varphi_t\,d\mu\,dt = -\int_0^T \int_M |\nabla u|^{p-2}\,\langle \nabla u, \nabla \varphi \rangle\,d\mu\,dt + \int_0^T \int_M u^\sigma\,\varphi\,d\mu\,dt$$
$$+ \int_M u_0(x)\,\varphi(x, 0)\,d\mu.$$

First we consider the case that $\sigma > p - 1 + \frac{p}{N}$ and that the Sobolev inequality holds on M. In order to state our results, we define

$$\sigma_0 := (\sigma - p + 1)\frac{N}{p}. \tag{2.1}$$

Observe that $\sigma_0 > 1$ whenever $\sigma > p - 1 + \frac{p}{N}$. Our first result is a generalization of [27] to the geometric setting considered here.

Theorem 2.2. *Let M be a complete, noncompact, Riemannian manifold of infinite volume such that the Sobolev inequality (1.3) holds. Assume (1.2) holds and, besides, that $\sigma > p - 1 + \frac{p}{N}$, $s > \sigma_0$ and $u_0 \in L^s(M) \cap L^1(M)$, $u_0 \geq 0$ where σ_0 has been defined in (2.1).*

(i) Assume that

$$\|u_0\|_{L^s(M)} < \varepsilon_0, \quad \|u_0\|_{L^1(M)} < \varepsilon_0, \tag{2.2}$$

with $\varepsilon_0 = \varepsilon_0(\sigma, p, N, C_{s,p}) > 0$ sufficiently small. Then problem (1.1) admits a solution for any $T > 0$, in the sense of Definition 2.1. Moreover, for any $\tau > 0$, one has $u \in L^\infty(M \times (\tau, +\infty))$ and there exists a constant $\Gamma > 0$ such that, one has

$$\|u(t)\|_{L^\infty(M)} \leq \Gamma \, t^{-\alpha} \|u_0\|_{L^1(M)}^{\frac{p}{N(p-2)+p}} \quad \text{for all } t > 0, \tag{2.3}$$

where

$$\alpha := \frac{N}{N(p-2)+p}.$$

(ii) Let $\sigma_0 \leq q < \infty$. If

$$\|u_0\|_{L^{\sigma_0}(M)} < \hat{\varepsilon}_0 \tag{2.4}$$

for $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(\sigma, p, N, C_{s,p}, q) > 0$ small enough, then there exists a constant $C = C(\sigma, p, N, \hat{\varepsilon}_0, C_{s,p}, q) > 0$ such that

$$\|u(t)\|_{L^q(M)} \leq C \, t^{-\gamma_q} \|u_0\|_{L^{\sigma_0}(M)}^{\delta_q} \quad \text{for all } t > 0, \tag{2.5}$$

where

$$\gamma_q = \frac{1}{\sigma - 1}\left[1 - \frac{N(\sigma - p + 1)}{pq}\right], \quad \delta_q = \frac{\sigma - p + 1}{\sigma - 1}\left[1 + \frac{N(p-2)}{pq}\right].$$

(iii) Finally, for any $1 < q < \infty$, if $u_0 \in L^q(M) \cap L^{\sigma_0}(M)$ and

$$\|u_0\|_{L^{\sigma_0}(M)} < \varepsilon \tag{2.6}$$

with $\varepsilon = \varepsilon(\sigma, p, N, C_{s,p}, q) > 0$ sufficiently small, then

$$\|u(t)\|_{L^q(M)} \leq \|u_0\|_{L^q(M)} \quad \text{for all } t > 0. \tag{2.7}$$

Remark 2.3. Observe that the choice of ε_0 in (2.2) is made in Lemma 5.1. Moreover, the proof of the above theorem will show that one can take an explicit value of $\hat{\varepsilon}_0$ in (2.4) and ε in (2.6). In fact, let $q_0 > 1$ be fixed and $\{q_n\}_{n \in \mathbb{N}}$ be the sequence defined by:

$$q_n = \frac{N}{N - p}(p + q_{n-1} - 2), \quad \text{for all } n \in \mathbb{N},$$

so that

$$q_n = \left(\frac{N}{N-p}\right)^n q_0 + \frac{N}{N-p}(p-2)\sum_{i=0}^{n-1}\left(\frac{N}{N-p}\right)^i. \qquad (2.8)$$

Clearly, $\{q_n\}$ is increasing and $q_n \longrightarrow +\infty$ as $n \to +\infty$. Fix $q \in [q_0, +\infty)$ and let $\bar{n}$ be the first index such that $q_{\bar{n}} \geq q$. Define

$$\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\sigma, p, N, C_{s,p}, q, q_0) := \left[\min\left\{\min_{n=0,\dots,\bar{n}}\left(\frac{p(q_n-1)^{1/p}}{p+q_n-2}\right)^p ; \left(\frac{p(\sigma_0-1)^{1/p}}{p-\sigma_0-2}\right)^p\right\}\frac{C_{s,p}^p}{2}\right]^{\frac{1}{\sigma-p+1}}. \qquad (2.9)$$

Observe that $\tilde{\varepsilon}_0$ in (2.9) depends on the value q through the sequence $\{q_n\}$. More precisely, $\bar{n}$ is increasing with respect to q, while the quantity $\min_{n=0,\dots,\bar{n}}(q_n-1)\left(\frac{p}{p+q_n-2}\right)^p\frac{C_{s,p}^p}{2}$ decreases w.r.t. q. Then, in (2.4) we can take

$$\hat{\varepsilon}_0 = \hat{\varepsilon}_0(\sigma, p, N, C_{s,p}, q) = \tilde{\varepsilon}_0(\sigma, p, N, C_{s,p}, q, \sigma_0).$$

Similarly, in (2.6), we can take

$$\varepsilon = \bar{\varepsilon}_0 \wedge \hat{\varepsilon}_0,$$

where

$$\bar{\varepsilon}_0 = \bar{\varepsilon}_0(\sigma, p, C_{s,p}, q) := \left[\min\left\{\left(\frac{p(q-1)^{1/p}}{p+q-2}\right)^p C_{s,p}^p ; \left(\frac{p(\sigma_0-1)^{1/p}}{p-\sigma_0-2}\right)^p C_{s,p}^p\right\}\right]^{\frac{1}{\sigma-p+1}}.$$

The next result involves a similar smoothing effect for a different class of data. Such result seems to be new also in the Euclidean setting.

Theorem 2.4. *Let M be a complete, noncompact, Riemannian manifold of infinite volume such that the Sobolev inequality* (1.3) *holds. Assume* (1.2) *and, besides, that $\sigma > p - 1 + \frac{p}{N}$ and $u_0 \in L^{\sigma_0}(M)$, $u_0 \geq 0$, with σ_0 as in* (2.1). *Assume that*

$$\|u_0\|_{L^{\sigma_0}(M)} < \varepsilon_2, \qquad (2.10)$$

with $\varepsilon_2 = \varepsilon_2(\sigma, p, N, C_{s,p}, q) > 0$ sufficiently small. Then problem (1.1) *admits a solution for any $T > 0$, in the sense of Definition 2.1. Moreover, for any $\tau > 0$, one has $u \in L^\infty(M \times (\tau, +\infty))$ and for any $\sigma > \sigma_0$, there exists a constant $\Gamma > 0$ such that, one has*

$$\|u(t)\|_{L^\infty(M)} \leq \Gamma\, t^{-\frac{1}{\sigma-1}} \|u_0\|_{L^{\sigma_0}(M)}^{\frac{\sigma-p+1}{\sigma-1}} \quad \text{for all } t > 0. \qquad (2.11)$$

Moreover, (ii) and (iii) of Theorem 2.2 hold.

Remark 2.5. *We comment that, as in Remark 2.3, one can choose an explicit value for ε_2 in* (2.10). *In fact, let $q_0 = \sigma_0$ in* (2.9). *It can be shown that one can take, with this choice of q_0:*

$$\varepsilon_2 = \varepsilon_2(\sigma, p, N, C_{s,p}, \sigma_0) := \min\left\{\tilde{\varepsilon}_0(\sigma, p, N, C_{s,p}, q, \sigma_0) ; \left(\frac{1}{C\tilde{C}}\right)^{\frac{1}{\sigma-p+1}}\right\},$$

where $C > 0$ and $\tilde{C} > 0$ are defined in Proposition 3.3 and Lemma 4.3, respectively.

Remark 2.6. *Observe that, due to the assumption $\sigma > p - 1 + \frac{p}{N}$, one has*

$$\frac{1}{\sigma - 1} < \frac{N}{N(p-2) + p}.$$

Hence, for large times, the decay given by Theorem 2.4 is worse than the one of Theorem 2.2; however, in this regards, note that the assumptions on the initial datum u_0 are different in the two theorems. On the other hand, estimates (2.11) and (2.3), are not sharp in general for small times. For example, when $u_0 \in L^\infty(M)$, $u(t)$ remains bounded for any $t \in [0, T)$, where T is the maximal existence time.

In the next theorem, we address the case $\sigma > p - 1$, assuming that both the inequalities (1.3) and (1.4) hold on M, hence with stronger assumptions on the manifold considered. This has of course no Euclidean analogue, as the noncompactness of the manifold considered, as well as the fact that it has infinite volume, makes the situation not comparable to the case of a bounded Euclidean domain.

Theorem 2.7. *Let M be a complete, noncompact manifold of infinite volume such that the Sobolev inequality (1.3) and the Poincaré inequality (1.4) hold. Assume that (1.2) holds, and besides that $p > 2$. Let $u_0 \geq 0$ be such that $u_0 \in L^s(M) \cap L^{\sigma \frac{N}{p}}(M)$, for some $s > \max\{\sigma_0, 1\}$ and $q_0 > 1$. Assume also that*

$$\|u_0\|_{L^s(M)} < \varepsilon_1, \quad \|u_0\|_{L^{\sigma \frac{N}{p}}(M)} < \varepsilon_1,$$

with $\varepsilon_1 = \varepsilon_1(\sigma, p, N, C_{s,p}, C_p, s)$ sufficiently small. Then problem (1.1) admits a solution for any $T > 0$, in the sense of Definition 2.1. Moreover, for any $\tau > 0$, one has $u \in L^\infty(M \times (\tau, +\infty))$ and, for any $q > s$, there exists a constant $\Gamma > 0$ such that, one has

$$\|u(t)\|_{L^\infty(B_R)} \leq \Gamma \, t^{-\beta_{q,s}} \|u_0\|_{L^s(B_R)}^{\frac{ps}{N(p-2)+pq}} \quad \textit{for all } t > 0, \tag{2.12}$$

where

$$\beta_{q,s} := \frac{1}{p-2}\left(1 - \frac{ps}{N(p-2)+pq}\right) > 0.$$

Moreover, let $s \leq q < \infty$ and

$$\|u_0\|_{L^s(M)} < \hat{\varepsilon}_1$$

for $\hat{\varepsilon}_1 = \hat{\varepsilon}_1(\sigma, p, N, C_{s,p}, C_p, q, s)$ small enough. Then there exists a constant $C = C(\sigma, p, N, \varepsilon_1, C_{s,p}, C_p, q, s) > 0$ such that

$$\|u(t)\|_{L^q(M)} \leq C \, t^{-\gamma_q} \|u_0\|_{L^s(M)}^{\delta_q} \quad \textit{for all } t > 0, \tag{2.13}$$

where

$$\gamma_q = \frac{s}{p-2}\left[\frac{1}{s} - \frac{1}{q}\right], \quad \delta_q = \frac{s}{q}.$$

Finally, for any $1 < q < \infty$, if $u_0 \in L^q(M) \cap L^s(M)$ and

$$\|u_0\|_{L^s(M)} < \varepsilon$$

with $\varepsilon = \varepsilon(\sigma, p, N, C_{s,p}, C_p, q)$ sufficiently small, then

$$\|u(t)\|_{L^q(M)} \leq \|u_0\|_{L^q(M)} \quad \textit{for all } t > 0. \tag{2.14}$$

Remark 2.8. *It is again possible to give an explicit estimate on the smallness parameter ε_1 above. In fact, let $q_0 > 1$ be fixed and $\{q_m\}_{m\in\mathbb{N}}$ be the sequence defined by:*

$$q_m = p + q_{m-1} - 2, \quad \text{for all } m \in \mathbb{N},$$

so that

$$q_m = q_0 + m(p - 2). \tag{2.15}$$

Clearly, $\{q_m\}$ is increasing and $q_m \longrightarrow +\infty$ as $m \to +\infty$. Fix $q \in [q_0, +\infty)$ and let $\bar{m}$ be the first index such that $q_{\bar{m}} \geq q$. Define $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(\sigma, p, N, C_{s,p}, C_p, q, q_0)$ such that

$$\tilde{\varepsilon}_1 := \min\left\{ \left[\min_{m=0,\ldots,\bar{m}} \left(\frac{p(q_m - 1)^{1/p}}{p + q_m - 2} \right)^p C \right]^{\frac{\sigma+p+q_m-2}{\sigma(\sigma+q_m-1)-p(p+q_m-2)}} ; \right.$$

$$\left. \left[\left(\frac{p(\sigma\frac{N}{p} - 1)^{1/p}}{(p + \sigma\frac{N}{p} - 2)} \right)^p C \right]^{\frac{\sigma+p+\sigma\frac{N}{p}-2}{\sigma(\sigma+\sigma\frac{N}{p}-1)-p(p+\sigma\frac{N}{p}-2)}} \right\}$$

where $C = \tilde{C} C_p^{p(\frac{p-1}{\sigma})}$ and $\tilde{C} = \tilde{C}(C_s, p, \sigma, q) > 0$ is defined in (3.37). Observe that $\tilde{\varepsilon}_1$ depends on q through the sequence $\{q_m\}$. More precisely, $\bar{m}$ is increasing with respect to q, while the quantity $\min_{m=0,\ldots,\bar{m}} \left(\frac{p(q_m-1)^{1/p}}{p+q_m-2} \right)^p C$ decreases w.r.t. q_m. Furthermore, let $\delta_1 > 0$ be such that

$$\tilde{C} \delta_1^{\frac{ps(\sigma-1)}{N(p-2)+ps}} + \frac{C\tilde{C}}{4} \delta_1^{\frac{ps(\sigma-1)}{N(p-2)+pq}} < 1,$$

where $C > 0$ and $\tilde{C} > 0$ are defined in Proposition 3.3 and Lemma 4.3, respectively. Then, let $q_0 = s$ with s as in Theorem 2.7 and define

$$\varepsilon_1 = \varepsilon_1(\sigma, p, N, C_{s,p}, C_p, q, s) = \min\left\{ \tilde{\varepsilon}_1(\sigma, p, N, C_{s,p}, C_p, q, s) ; \delta_1 \right\}.$$

3. $L^q - L^q$ and $L^{q_0} - L^q$ estimates

Let $x_0, x \in M$. We denote by $r(x) = \text{dist}(x_0, x)$ the Riemannian distance between x_0 and x. Moreover, we let $B_R(x_0) := \{x \in M : \text{dist}(x_0, x) < R\}$ be the geodesic ball with centre $x_0 \in M$ and radius $R > 0$. If a reference point $x_0 \in M$ is fixed, we shall simply denote by B_R the ball with centre x_0 and radius R. We also recall that μ denotes the Riemannian measure on M.

For any given function v, we define for any $k \in \mathbb{R}^+$

$$T_k(v) := \begin{cases} k & \text{if } v \geq k, \\ v & \text{if } |v| < k, \\ -k & \text{if } v \leq -k; \end{cases} \tag{3.1}$$

For every $R > 0$, $k > 0$, consider the problem

$$\begin{cases} u_t = \text{div}\left(|\nabla u|^{p-2}\nabla u\right) + T_k(u^\sigma) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, +\infty) \\ u = u_0 & \text{in } B_R \times \{0\}, \end{cases} \tag{3.2}$$

where $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Solutions to problem (3.2) are meant in the weak sense as follows.

Definition 3.1. *Let $p > 1$ and $\sigma > p - 1$. Let $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. We say that a nonnegative function u is a solution to problem* (3.2) *if*

$$u \in L^\infty(B_R \times (0, +\infty)), \quad u \in L^2((0, T); W_0^{1,p}(B_R)) \qquad \text{for any } T > 0,$$

and for any $T > 0$, $\varphi \in C_c^\infty(B_R \times [0, T])$ such that $\varphi(x, T) = 0$ for every $x \in B_R$, u satisfies the equality:

$$- \int_0^T \int_{B_R} u \, \varphi_t \, d\mu \, dt = - \int_0^T \int_{B_R} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, d\mu \, dt + \int_0^T \int_{B_R} T_k(u^\sigma) \, \varphi \, d\mu \, dt$$

$$+ \int_{B_R} u_0(x) \, \varphi(x, 0) \, d\mu.$$

3.1. $L^q - L^q$ estimate for $\sigma > \sigma_0$

First we consider the case $\sigma > \sigma_0$ where σ_0 has been defined in (2.1). Moreover, we assume that the Sobolev inequality (1.3) holds on M.

Lemma 3.2. *Assume* (1.2) *and, besides, that $\sigma > p - 1 + \frac{p}{N}$. Assume that inequality* (1.3) *holds. Suppose that $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Let $1 < q < \infty$ and assume that*

$$\|u_0\|_{L^{\sigma_0}(B_R)} < \bar{\varepsilon} \tag{3.3}$$

with $\bar{\varepsilon} = \bar{\varepsilon}(\sigma, p, q, C_{s,p}) > 0$ sufficiently small. Let u be the solution of problem (3.2) *in the sense of Definition 3.1, and assume that $u \in C([0, T], L^q(B_R))$ for any $q \in (1, +\infty)$, for any $T > 0$. Then*

$$\|u(t)\|_{L^q(B_R)} \leq \|u_0\|_{L^q(B_R)} \quad \text{for all } t > 0. \tag{3.4}$$

Note that the request $u \in C([0, T], L^q(B_R))$ for any $q \in (1, \infty)$, for any $T > 0$ is not restrictive, since we will construct solutions belonging to that class. This remark also applies to several other intermediate results below.

Proof. Since u_0 is bounded and $T_k(u^\sigma)$ is a bounded and Lipschitz function, by standard results, there exists a unique solution of problem (3.2) in the sense of Definition 3.1. We now multiply both sides of the differential equation in problem (3.2) by u^{q-1},

$$\int_{B_R} u_t \, u^{q-1} \, dx = \int_{B_R} \text{div} \left(|\nabla u|^{p-2} \nabla u \right) u^{q-1} \, dx + \int_{B_R} T_k(u^\sigma) \, u^{q-1} \, dx.$$

Now, we formally integrate by parts in B_R. This can be justified by standard tools, by an approximation procedure. We get

$$\frac{1}{q} \frac{d}{dt} \int_{B_R} u^q \, d\mu = -(q - 1) \int_{B_R} u^{q-2} \, |\nabla u|^p \, d\mu + \int_{B_R} T_k(u^\sigma) \, u^{q-1} \, d\mu. \tag{3.5}$$

Observe that, thanks to Sobolev inequality (1.3), we have

$$\int_{B_R} u^{q-2} \, |\nabla u|^p \, d\mu = \left(\frac{p}{p + q - 2} \right)^p \int_{B_R} \left| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right|^p \, d\mu$$

$$\geq \left(\frac{p}{p + q - 2} \right)^p C_{s,p}^p \left(\int_{B_R} u^{\frac{p+q-2}{p} \frac{pN}{N-p}} \, d\mu \right)^{\frac{N-p}{N}}. \tag{3.6}$$

Moreover, the last term in the right hand side of (3.5), by using the Hölder inequality with exponents $\frac{N}{N-p}$ and $\frac{N}{p}$, becomes

$$\int_{B_R} T_k(u^\sigma) \, u^{q-1} \, dx \le \int_{B_R} u^\sigma \, u^{q-1} \, dx = \int_{B_R} u^{\sigma-p+1} \, u^{p+q-2} \, dx$$

$$\le \|u(t)\|_{L^{(\sigma-p+1)\frac{N}{p}}(B_R)}^{\sigma-p+1} \, \|u(t)\|_{L^{(p+q-2)\frac{N}{N-p}}(B_R)}^{p+q-2} \,. \tag{3.7}$$

Combining (3.6) and (3.7) we get

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|_{L^q(B_R)}^q \le -\left[(q-1)\left(\frac{p}{p+q-2}\right)^p C_{s,p}^p - \|u(t)\|_{L^{\sigma_0}(B_R)}^{\sigma-p+1}\right] \|u(t)\|_{L^{(p+q-2)\frac{N}{N-p}}(B_R)}^{p+q-2} \tag{3.8}$$

Take $T > 0$. Observe that, due to hypotheses (3.3) and the known continuity in L^{σ_0} of the map $t \mapsto u(t)$ in $[0, T]$, there exists $t_0 > 0$ such that

$$\|u(t)\|_{L^{\sigma_0}(B_R)} \le 2\,\bar{\varepsilon} \quad \text{for any} \quad t \in [0, t_0].$$

Hence (3.8) becomes, for any $t \in (0, t_0]$,

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|_{L^q(B_R)}^q \le -\left[\left(\frac{p}{p+q-2}\right)^p (q-1)C_{s,p}^p - (2\,\bar{\varepsilon})^{\sigma-p+1}\right] \|u(t)\|_{L^{(p+q-2)\frac{N}{N-p}}(B_R)}^{p+q-2} \le 0,$$

where the last inequality is obtained by using (3.3). We have proved that $t \mapsto \|u(t)\|_{L^q(B_R)}$ is decreasing in time for any $t \in (0, t_0]$, thus

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_0]. \tag{3.9}$$

In particular, inequality (3.9) follows for the choice $q = \sigma_0$ in view of hypothesis (3.3). Hence we have

$$\|u(t)\|_{L^{\sigma_0}(B_R)} \le \|u_0\|_{L^{\sigma_0}(B_R)} < \bar{\varepsilon} \quad \text{for any} \quad t \in (0, t_0].$$

Now, we can repeat the same argument in the time interval $(t_0, t_1]$, with $t_1 = 2t_0$. This can be done due to the uniform continuity of the map $t \mapsto u(t)$ in $[0, T]$. Hence, we can write that

$$\|u(t)\|_{L^{\sigma_0}(B_R)}^{\sigma-p+1} \le 2\,\bar{\varepsilon} \quad \text{for any} \quad t \in (t_0, t_1].$$

Thus we get

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_1].$$

Iterating this procedure we obtain that $t \mapsto \|u(t)\|_{L^q(B_R)}$ is decreasing in $[0, T]$. Since $T > 0$ was arbitrary, the thesis follows.

$\square$

3.2. $L^{q_0} - L^q$ estimate for $\sigma > \sigma_0$

Using a Moser type iteration procedure we prove the following result:

Proposition 3.3. *Assume* (1.2) *and, besides, that* $\sigma > p - 1 + \frac{p}{N}$. *Assume that inequality* (1.3) *holds. Suppose that* $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. *Let* u *be the solution of problem* (3.2), *so that* $u \in C([0, T], L^q(B_R))$ *for any* $q \in (1, +\infty)$, *for any* $T > 0$. *Let* $1 < q_0 \leq q < +\infty$ *and assume that*

$$\|u_0\|_{L^{\sigma_0}(B_R)} \leq \tilde{\varepsilon}_0 \tag{3.10}$$

for $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\sigma, p, N, C_{s,p}, q, q_0)$ *sufficiently small. Then there exists* $C(p, q_0, C_{s,p}, \tilde{\varepsilon}_0, N, q) > 0$ *such that*

$$\|u(t)\|_{L^q(B_R)} \leq C\, t^{-\gamma_q} \|u_0\|_{L^{q_0}(B_R)}^{\delta_q} \quad \textit{for all } t > 0, \tag{3.11}$$

where

$$\gamma_q = \left(\frac{1}{q_0} - \frac{1}{q}\right) \frac{N q_0}{p\, q_0 + N(p - 2)}, \quad \delta_q = \frac{q_0}{q}\left(\frac{q + \frac{N}{p}(p - 2)}{q_0 + \frac{N}{p}(p - 2)}\right). \tag{3.12}$$

Proof. Let $\{q_n\}$ be the sequence defined in (2.8). Let $\bar{n}$ be the first index such that $q_{\bar{n}} \geq q$. Observe that $\bar{n}$ is well defined in view of the mentioned properties of $\{q_n\}$, see (2.8). We start by proving a smoothing estimate from q_0 to $q_{\bar{n}}$ using a Moser iteration technique (see also [2]). Afterwards, if $q_{\bar{n}} \equiv q$ then the proof is complete. Otherwise, if $q_{\bar{n}} > q$ then, by interpolation, we get the thesis.

Let $t > 0$, we define

$$r = \frac{t}{2^{\bar{n}} - 1}, \quad t_n = (2^n - 1)r. \tag{3.13}$$

Observe that $t_0 = 0$, $t_{\bar{n}} = t$, $\{t_n\}$ is an increasing sequence w.r.t. n. Now, for any $1 \leq n \leq \bar{n}$, we multiply Eq (3.2) by $u^{q_{n-1}-1}$ and integrate in $B_R \times [t_{n-1}, t_n]$. Thus we get

$$\int_{t_{n-1}}^{t_n} \int_{B_R} u_t\, u^{q_{n-1}-1}\, d\mu\, dt - \int_{t_{n-1}}^{t_n} \int_{B_R} \text{div}\left(|\nabla u|^{p-2}\nabla u\right) u^{q_{n-1}-1}\, d\mu\, dt$$
$$= \int_{t_{n-1}}^{t_n} \int_{B_R} T_k(u^\sigma)\, u^{q_{n-1}-1}\, d\mu\, dt.$$

Then we integrate by parts in $B_R \times [t_{n-1}, t_n]$. Due to Sobolev inequality (1.3) and assumption (3.10), we get

$$\frac{1}{q_{n-1}}\left[\|u(\cdot, t_n)\|_{L^{q_{n-1}}(B_R)}^{q_{n-1}} - \|u(\cdot, t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{q_{n-1}}\right]$$
$$\leq -\left[\left(\frac{p}{p + q_{n-1} - 2}\right)^p (q_{n-1} - 1)C_{s,p}^p - 2\tilde{\varepsilon}_0\right] \int_{t_{n-1}}^{t_n} \|u(\tau)\|_{L^{(p+q_{n-1}-2)\frac{N}{N-p}}(B_R)}^{p+q_{n-1}-2}\, d\tau, \tag{3.14}$$

where we have made use of inequality $T_k(u^\sigma) \leq u^\sigma$. We define q_n as in (2.8), so that $(p + q_{n-1} - 2)\frac{N}{N-p} = q_n$. Hence, in view of hypotheses (3.10) we can apply Lemma 3.2 to the integral on the right hand side of (3.14), hence we get

$$\frac{1}{q_{n-1}}\left[\|u(\cdot, t_n)\|_{L^{q_{n-1}}(B_R)}^{q_{n-1}} - \|u(\cdot, t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{q_{n-1}}\right]$$
$$\leq -\left[\left(\frac{p}{p + q_{n-1} - 2}\right)^p (q_{n-1} - 1)C_{s,p}^p - 2\tilde{\varepsilon}_0\right] \|u(\cdot, t_n)\|_{L^{(p+q_{n-1}-2)\frac{N}{N-p}}(B_R)}^{p+q_{n-1}-2}\, |t_n - t_{n-1}|. \tag{3.15}$$

Observe that

$$\|u(\cdot, t_n)\|_{L^{q_{n-1}}(B_R)}^{q_{n-1}} \geq 0,$$

$$|t_n - t_{n-1}| = \frac{2^{n-1} t}{2^{\bar{n}} - 1}.$$

(3.16)

We define

$$d_{n-1} := \left[\left(\frac{p}{p + q_{n-1} - 2}\right)^p (q_{n-1} - 1)C_{s,p}^p - 2\tilde{\varepsilon}_0\right]^{-1} \frac{1}{q_{n-1}}.$$

(3.17)

By plugging (3.16) and (3.17) into (3.15) we get

$$\|u(\cdot, t_n)\|_{L^{(p+q_{n-1}-2)\frac{N}{N-p}}(B_R)}^{p+q_{n-1}-2} \leq \frac{(2^{\bar{n}} - 1)d_n}{2^{n-1} t} \|u(\cdot, t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{q_{n-1}}.$$

The latter can be rewritten as

$$\|u(\cdot, t_n)\|_{L^{(p+q_{n-1}-2)\frac{N}{N-p}}(B_R)} \leq \left(\frac{(2^{\bar{n}} - 1)d_n}{2^{n-1}}\right)^{\frac{1}{p+q_{n-1}-2}} t^{-\frac{1}{p+q_{n-1}-2}} \|u(\cdot, t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{\frac{q_{n-1}}{p+q_{n-1}-2}}.$$

Due to to the definition of the sequence $\{q_n\}$ in (2.8) we write

$$\|u(\cdot, t_n)\|_{L^{q_n}(B_R)} \leq \left(\frac{(2^{\bar{n}} - 1)d_{n-1}}{2^{n-1}}\right)^{\frac{N}{N-p}\frac{1}{q_n}} t^{-\frac{N}{N-p}\frac{1}{q_n}} \|u(\cdot, t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{\frac{q_{n-1}}{q_n}\frac{N}{N-p}}.$$

(3.18)

We define

$$s := \frac{N}{N - p}.$$

(3.19)

Observe that, for any $1 \leq n \leq \bar{n}$, we have

$$\left(\frac{(2^{\bar{n}} - 1)d_{n-1}}{2^{n-1}}\right)^s = \left\{\frac{2^{\bar{n}} - 1}{2^{n-1}} \left[\left(\frac{p}{p + q_{n-1} - 2}\right)^p (q_{n-1} - 1)C_{s,p}^p - 2\varepsilon\right]^{-1} \frac{1}{q_{n-1}}\right\}^s$$

$$= \left[\frac{2^{\bar{n}} - 1}{2^{n-1}} \frac{1}{q_{n-1}(q_{n-1} - 1)\left(\frac{p}{p + q_{n-1} - 2}\right)^p C_{s,p}^p - 2\varepsilon q_{n-1}}\right]^s,$$

(3.20)

and

$$\frac{2^{\bar{n}} - 1}{2^{n-1}} \leq 2^{\bar{n}+1} \qquad \text{for all } 1 \leq n \leq \bar{n}.$$

(3.21)

Consider the function

$$g(x) := \left[(x - 1)\left(\frac{p}{p + x - 2}\right)^p C_{s,p}^p - 2\varepsilon\right] x \qquad \text{for } q_0 \leq x \leq q_{\bar{n}}, \ x \in \mathbb{R}.$$

Observe that, due to (2.9), $g(x) > 0$ for any $q_0 \leq x \leq q_{\bar{n}}$. Moreover, g has a minimum in the interval $q_0 \leq x \leq q_{\bar{n}}$; call $\tilde{x}$ the point at which the minimum is attained. Then we have

$$\frac{1}{g(x)} \leq \frac{1}{g(\tilde{x})} \qquad \text{for any } q_0 \leq x \leq q_{\bar{n}}.$$

(3.22)

Thanks to (3.20)–(3.22), there exist a positive constant C, where $C = C(N, C_{s,p}, \tilde{\varepsilon}_0, \bar{n}, p, q_0)$ such that

$$\left(\frac{(2^{\bar{n}} - 1)d_{n-1}}{2^{n-1}}\right)^s \leq C, \quad \text{for all } 1 \leq n \leq \bar{n}. \tag{3.23}$$

By plugging (3.19) and (3.23) into (3.18) we get, for any $1 \leq n \leq \bar{n}$

$$\|u(\cdot, t_n)\|_{L^{q_n}(B_R)} \leq C^{\frac{1}{q_n}} t^{-\frac{s}{q_n}} \|u(\cdot, t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{\frac{s\,q_{n-1}}{q_n}}. \tag{3.24}$$

Let us set

$$U_n := \|u(\cdot, t_n)\|_{L^{q_n}(B_R)}.$$

Then (3.24) becomes

$$\begin{aligned}
U_n &\leq C^{\frac{1}{q_n}} t^{-\frac{s}{q_n}} U_{n-1}^{\frac{q_{n-1}\,s}{q_n}} \\
&\leq C^{\frac{1}{q_n}} t^{-\frac{s}{q_n}} \left[C^{\frac{s}{q_n}} t^{-\frac{s^2}{q_n}} U_{k-2}^{s^2 \frac{q_{n-2}}{q_n}} \right] \\
&\leq \dots \\
&\leq C^{\frac{1}{q_n} \sum_{i=0}^{n-1} s^i} t^{-\frac{s}{q_n} \sum_{i=0}^{n-1} s^i} U_0^{s^n \frac{q_0}{q_n}}.
\end{aligned}$$

We define

$$\begin{aligned}
\alpha_n &:= \frac{1}{q_n} \sum_{i=0}^{n-1} s^i, \\
\beta_n &:= \frac{s}{q_n} \sum_{i=0}^{n-1} s^i = s\,\alpha_n, \\
\delta_n &:= s^n \frac{q_0}{q_n}.
\end{aligned} \tag{3.25}$$

By substituting n with $\bar{n}$ into (3.25) we get

$$\begin{aligned}
\alpha_{\bar{n}} &:= \frac{N - p}{p} \frac{A}{q_{\bar{n}}}, \\
\beta_{\bar{n}} &:= \frac{N}{p} \frac{A}{q_{\bar{n}}}, \\
\delta_{\bar{n}} &:= (A + 1) \frac{q_0}{q_{\bar{n}}}.
\end{aligned} \tag{3.26}$$

where $A := \left(\frac{N}{N-p}\right)^{\bar{n}} - 1$. Hence, in view of (3.13) and (3.26), (3.24) with $n = \bar{n}$ yields

$$\|u(\cdot, t)\|_{L^{q_{\bar{n}}}(B_R)} \leq C^{\frac{N-p}{p} \frac{A}{q_{\bar{n}}}} t^{-\frac{N}{p} \frac{A}{q_{\bar{n}}}} \|u_0\|_{L^{q_0}(B_R)}^{q_0 \frac{A+1}{q_{\bar{n}}}}. \tag{3.27}$$

We have proved a smoothing estimate from q_0 to $q_{\bar{n}}$. Observe that if $q_{\bar{n}} = q$ then the thesis is proved. Now suppose that $q_{\bar{n}} > q$. Observe that $q_0 \leq q < q_{\bar{n}}$ and define

$$B := N(p - 2)A + p\,q_0(A + 1).$$

From (3.27) and Lemma 3.2, we get, by interpolation,

$$
\begin{aligned}
\|u(\cdot, t)\|_{L^q(B_R)} &\leq \|u(\cdot, t)\|_{L^{q_0}(B_R)}^{\theta} \|u(\cdot, t)\|_{L^{q_{\bar{n}}}(B_R)}^{1-\theta} \\
&\leq \|u_0(\cdot)\|_{L^{q_0}(B_R)}^{\theta} C\, t^{-\frac{NA}{B}(1-\theta)} \|u_0\|_{L^{q_0}(B_R)}^{pq_0 \frac{A+1}{B}(1-\theta)} \\
&= C\, t^{-\frac{NA}{B}(1-\theta)} \|u_0\|_{L^{q_0}(B_R)}^{pq_0 \frac{A+1}{B}(1-\theta)+\theta},
\end{aligned}
\tag{3.28}
$$

where

$$
\theta = \frac{q_0}{q}\left(\frac{q_{\bar{n}} - q}{q_{\bar{n}} - q_0}\right).
\tag{3.29}
$$

Observe that

(i) $\quad \dfrac{NA}{B}(1-\theta) = \dfrac{N}{p}\left(\dfrac{q - q_0}{q}\right)\dfrac{1}{q_0 + \frac{N}{p}(p-2)};$

(ii) $\quad p\, q_0 \dfrac{A+1}{B}(1-\theta) + \theta = \dfrac{q_0}{q}\dfrac{q + \frac{N}{p}(p-2)}{q_0 + \frac{N}{p}(p-2)}.$

Combining (3.28), (3.12) and (3.29) we get the claim, noticing that q was arbitrarily in $[q_0, +\infty)$. $\qquad\square$

Remark 3.4. *One can not let $q \to +\infty$ is the above bound. In fact, one can show that $\varepsilon \longrightarrow 0$ as $q \to \infty$. So in such limit the hypothesis on the norm of the initial datum (2.9) is satisfied only when $u_0 \equiv 0$.*

3.3. $L^q - L^q$ estimates for $\sigma > p - 1$

We now consider the case $\sigma > p - 1$ and that the Sobolev and Poincaré inequalities (1.3), (1.4) hold on M.

Lemma 3.5. *Assume (1.2) and, besides, that $p > 2$. Assume that inequalities (1.3) and (1.4) hold. Suppose that $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Let $1 < q < \infty$ and assume that*

$$
\|u_0\|_{L^{\sigma\frac{N}{p}}(B_R)} < \bar{\varepsilon}_1
\tag{3.30}
$$

for a suitable $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(\sigma, p, N, C_p, C_{s,p}, q)$ sufficiently small. Let u be the solution of problem (3.2) in the sense of Definition 3.1, such that in addition $u \in C([0, T); L^q(B_R))$. Then

$$
\|u(t)\|_{L^q(B_R)} \leq \|u_0\|_{L^q(B_R)} \quad \text{for all } t > 0.
\tag{3.31}
$$

Proof. Since u_0 is bounded and $T_k(u^\sigma)$ is a bounded and Lipschitz function, by standard results, there exists a unique solution of problem (3.2) in the sense of Definition 3.1. We now multiply both sides of the differential equation in problem (3.2) by u^{q-1}, therefore

$$
\int_{B_R} u_t\, u^{q-1}\, d\mu = \int_{B_R} \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) u^{q-1}\, d\mu + \int_{B_R} T_k(u^\sigma)\, u^{q-1}\, d\mu.
$$

We integrate by parts. This can again be justified by a standard approximation procedure. By using the fact that $T(u^\sigma) \leq u^\sigma$, we can write

$$
\frac{1}{q}\frac{d}{dt}\int_{B_R} u^q\, d\mu \leq -(q-1)\left(\frac{p}{p+q-2}\right)^p \int_{B_R}\left|\nabla\left(u^{\frac{p+q-2}{p}}\right)\right|^p d\mu + \int_{B_R} u^{\sigma+q-1}\, d\mu.
\tag{3.32}
$$

Now we take $c_1 > 0$, $c_2 > 0$ such that $c_1 + c_2 = 1$ so that

$$\int_{B_R} \left| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right|^p d\mu = c_1 \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)}^p + c_2 \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)}^p. \tag{3.33}$$

Take $\alpha \in (0, 1)$. Thanks to (1.4), (3.33) we get

$$\begin{aligned}
\int_{B_R} \left| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right|^2 d\mu &\geq c_1 C_p^p \|u\|_{L^{p+q-2}(B_R)}^{p+q-2} + c_2 \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)}^p \\
&\geq c_1 C_p^p \|u\|_{L^{p+q-2}(B_R)}^{p+q-2} + c_2 \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)}^{p+p\alpha-p\alpha} \\
&\geq c_1 C_p^p \|u\|_{L^{p+q-2}(B_R)}^{p+q-2} + c_2 C_p^{p\alpha} \|u\|_{L^{p+q-2}(B_R)}^{\alpha(p+q-2)} \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)}^{p-p\alpha}
\end{aligned} \tag{3.34}$$

Moreover, using the interpolation inequality, Hölder inequality and (1.3), we have

$$\begin{aligned}
\int_{B_R} u^{\sigma+q-1} d\mu, &= \|u\|_{L^{\sigma+q-1}}^{\sigma+q-1} \\
&\leq \|u\|_{L^{p+q-2}(B_R)}^{\theta(\sigma+q-1)} \|u\|_{L^{\sigma+p+q-2}(B_R)}^{(1-\theta)(\sigma+q-1)} \\
&\leq \|u\|_{L^{p+q-2}(B_R)}^{\theta(\sigma+q-1)} \left[\|u\|_{L^{\sigma\frac{N}{p}}(B_R)}^{\sigma} \|u\|_{L^{(p+q-2)\frac{N}{N-p}}(B_R)}^{p+q-2} \right]^{\frac{(1-\theta)(\sigma+q-1)}{\sigma+p+q-2}} \\
&\leq \|u\|_{L^{p+q-2}(B_R)}^{\theta(\sigma+q-1)} \|u\|_{L^{\sigma\frac{N}{p}}(B_R)}^{(1-\theta)\frac{\sigma(\sigma+q-1)}{\sigma+p+q-2}} \left(\frac{1}{C_{s,p}} \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)} \right)^{p(1-\theta)\frac{\sigma+q-1}{\sigma+p+q-2}}
\end{aligned} \tag{3.35}$$

where $\theta := \frac{(p-1)(p+q-2)}{\sigma(\sigma+q-1)}$. By plugging (3.34) and (3.35) into (3.32) we obtain

$$\begin{aligned}
\frac{1}{q} \frac{d}{dt} \|u(t)\|_{L^q(B_R)}^q &\leq -(q-1) \left(\frac{p}{p+q-2} \right)^p c_1 C_p^p \|u\|_{L^{p+q-2}(B_R)}^{p+q-2} \\
&\quad -(q-1) \left(\frac{p}{p+q-2} \right)^p c_2 C_p^{p\alpha} \|u\|_{L^{p+q-2}(B_R)}^{\alpha(p+q-2)} \left\| \nabla \left(u^{\frac{p+q-2}{p}} \right) \right\|_{L^p(B_R)}^{p-p\alpha} \\
&\quad + \tilde{C} \|u\|_{L^{p+q-2}(B_R)}^{\theta(\sigma+q-1)} \|u\|_{L^{\sigma\frac{N}{p}}(B_R)}^{(1-\theta)\frac{\sigma(\sigma+q-1)}{\sigma+p+q-2}} \| \nabla \left(u^{\frac{p+q-2}{p}} \right) \|_{L^p(B_R)}^{p(1-\theta)\frac{\sigma+q-1}{\sigma+p+q-2}},
\end{aligned} \tag{3.36}$$

where

$$\tilde{C} = \left(\frac{1}{C_{s,p}} \right)^{p(1-\theta)\frac{\sigma+q-1}{\sigma+p+q-2}}. \tag{3.37}$$

Let us now fix $\alpha \in (0, 1)$ such that

$$p - p\alpha = p(1-\theta) \frac{\sigma+q-1}{\sigma+p+q-2}.$$

Hence, we have

$$\alpha = \frac{p-1}{\sigma}. \tag{3.38}$$

By substituting (3.38) into (3.36) we obtain

$$
\begin{aligned}
\frac{1}{q}\frac{d}{dt}\|u(t)\|_{L^q(B_R)}^q \leq &-(q-1)\left(\frac{p}{p+q-2}\right)^p c_1\, C_p^p\, \|u\|_{L^{p+q-2}(B_R)}^{p+q-2}\\
&-\frac{1}{\tilde{C}}\left\{(q-1)\left(\frac{p}{p+q-2}\right)^p C - \|u\|_{L^{\sigma\frac{N}{p}}(B_R)}^{\frac{\sigma(\sigma+q-1)-(p-1)(p+q-2)}{\sigma+p+q-2}}\right\}\\
&\times \|u\|_{L^{p+q-2}(B_R)}^{\alpha(p+q-2)}\left\|\nabla\left(u^{\frac{p+q-2}{p}}\right)\right\|_{L^p(B_R)}^{p-p\alpha},
\end{aligned}
\tag{3.39}
$$

where C has been defined in Remark 2.8. Observe that, due to hypotheses (3.30) and by the continuity of the solution $u(t)$, there exists $t_0 > 0$ such that

$$
\|u(t)\|_{L^{\sigma\frac{N}{p}}(B_R)} \leq 2\,\tilde{\varepsilon}_1 \quad \text{for any} \quad t \in (0, t_0]\,.
$$

Hence, (3.39) becomes, for any $t \in (0, t_0]$

$$
\begin{aligned}
\frac{1}{q}\frac{d}{dt}\|u(t)\|_{L^q(B_R)}^q \leq &-(q-1)\left(\frac{p}{p+q-2}\right)^p c_1 C_p^p\, \|u\|_{L^{p+q-2}(B_R)}^{p+q-2}\\
&-\frac{1}{\tilde{C}}\left\{(q-1)\left(\frac{p}{p+q-2}\right)^p C - 2\tilde{\varepsilon}_1^{\frac{\sigma(\sigma+q-1)-(p-1)(p+q-2)}{\sigma+p+q-2}}\right\} \|u\|_{L^{p+q-2}(B_R)}^{\alpha(p+q-2)}\left\|\nabla\left(u^{\frac{p+q-2}{p}}\right)\right\|_{L^p(B_R)}^{p-p\alpha}\\
\leq &\ 0\,,
\end{aligned}
$$

provided $\tilde{\varepsilon}_1$ is small enough. Hence we have proved that $\|u(t)\|_{L^q(B_R)}$ is decreasing in time for any $t \in (0, t_0]$, thus

$$
\|u(t)\|_{L^q(B_R)} \leq \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_0]\,.
\tag{3.40}
$$

In particular, inequality (3.40) holds $q = \sigma\frac{N}{p}$. Hence we have

$$
\|u(t)\|_{L^{\sigma\frac{N}{p}}(B_R)} \leq \|u_0\|_{L^{\sigma\frac{N}{p}}(B_R)} < \tilde{\varepsilon}_1 \quad \text{for any} \quad t \in (0, t_0]\,.
$$

Now, we can repeat the same argument in the time interval $(t_0, t_1]$ with $t_1 = 2t_0$. This can be done due to the uniform continuity of the map $t \mapsto u(t)$ in $[0, T]$. Hence, we can write that

$$
\|u(t)\|_{L^{\sigma\frac{N}{p}}(B_R)} \leq 2\,\tilde{\varepsilon}_1 \quad \text{for any} \quad t \in (t_0, t_1]\,.
$$

Thus we get

$$
\|u(t)\|_{L^q(B_R)} \leq \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_1]\,.
$$

Iterating this procedure we obtain the thesis.

$\square$

3.4. $L^{q_0} - L^q$ estimate for $\sigma > p - 1$

Using a Moser type iteration procedure we prove the following result:

Proposition 3.6. *Assume* (1.2) *and, besides, that* $p > 2$. *Let M be such that* (1.3) *and* (1.4) *hold. Suppose that* $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. *Let u be the solution of problem* (3.2) *in the sense of Definition 3.1 such that in addition* $u \in C([0, T], L^q(B_R))$ *for any* $q \in (1, +\infty)$, *for any* $T > 0$. *Let* $1 < q_0 \leq q < +\infty$ *and assume that*

$$\|u_0\|_{L^{\sigma \frac{N}{p}}(B_R)} < \tilde{\varepsilon}_1 \tag{3.41}$$

for $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(\sigma, p, N, C_{s,p}, C_p, q, q_0)$ *sufficiently small. Then there exists* $C(p, q_0, C_{s,p}, \tilde{\varepsilon}_1, N, q) > 0$ *such that*

$$\|u(t)\|_{L^q(B_R)} \leq C \, t^{-\gamma_q} \|u_0\|_{L^{q_0}(B_R)}^{\delta_q} \quad \text{for all } t > 0, \tag{3.42}$$

where

$$\gamma_q = \frac{q_0}{p-2}\left(\frac{1}{q_0} - \frac{1}{q}\right), \quad \delta_q = \frac{q_0}{q}. \tag{3.43}$$

Proof. Arguing as in the proof of Proposition 3.3, let $\{q_m\}$ be the sequence defined in (2.15). Let $\overline{m}$ be the first index such that $q_{\overline{m}} \geq q$. Observe that $\bar{m}$ is well defined in view of the mentioned properties of $\{q_m\}$, see (2.15). We start by proving a smoothing estimate from q_0 to $q_{\overline{m}}$ using again a Moser iteration technique. Afterwards, if $q_{\overline{m}} \equiv q$ then the proof is complete. Otherwise, if $q_{\overline{m}} > q$ then, by interpolation, we get the thesis.

Let $t > 0$, we define

$$r = \frac{t}{2^{\overline{m}} - 1}, \quad t_m = (2^m - 1)r. \tag{3.44}$$

Observe that

$$t_0 = 0, \quad t_{\overline{m}} = t, \quad \{t_m\} \text{ is an increasing sequence w.r.t. } m.$$

Now, for any $1 \leq m \leq \overline{m}$, we multiply Eq (3.2) by $u^{q_{m-1}-1}$ and integrate in $B_R \times [t_{m-1}, t_m]$. Thus we get

$$\int_{t_{m-1}}^{t_m} \int_{B_R} u_t \, u^{q_{m-1}-1} \, d\mu \, d\tau - \int_{t_{m-1}}^{t_m} \int_{B_R} \text{div}\left(|\nabla u^{p-2}|\nabla u\right) u^{q_{m-1}-1} \, d\mu \, d\tau$$
$$= \int_{t_{m-1}}^{t_m} \int_{B_R} T_k(u^\sigma) \, u^{q_{m-1}-1} \, d\mu \, d\tau.$$

Then we integrate by parts in $B_R \times [t_{m-1}, t_m]$, hence we get

$$\frac{1}{q_{m-1}} \left[\|u(\cdot, t_m)\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} - \|u(\cdot, t_{m-1})\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}}\right]$$
$$\leq -(q_{m-1} - 1)\left(\frac{p}{p + q_{m-1} - 2}\right)^p \int_{t_{m-1}}^{t_m} \int_{B_R} \left|\nabla\left(u^{\frac{p+q_{m-1}-2}{p}}\right)\right|^p \, d\mu \, d\tau$$
$$+ \int_{t_{m-1}}^{t_m} \int_{B_R} u^\sigma \, u^{q_{m-1}-1} \, d\mu \, d\tau.$$

where we have made use of inequality

$$T_k(u^\sigma) \leq u^\sigma.$$

Now, by arguing as in the proof of Lemma 3.5, by using (3.33) and (3.34) with $q = q_{m-1}$, we get

$$\int_{B_R} \left| \nabla \left(u^{\frac{p+q_{m-1}-2}{p}} \right) \right|^p d\mu$$

$$\geq c_1 C_p^p \|u\|_{L^{p+q_{m-1}-2}(B_R)}^{p+q_{m-1}-2} + c_2 C_p^{p\alpha} \|u\|_{L^{p+q_{m-1}-2}(B_R)}^{\alpha(p+q_{m-1}-2)} \left\| \nabla \left(u^{\frac{p+q_{m-1}-2}{p}} \right) \right\|_{L^p(B_R)}^{p-p\alpha}$$

where $\alpha \in (0, 1)$ and $c_1 > 0$, $c_2 > 0$ with $c_1 + c_2 = 1$. Similarly, from (3.35) with $q = q_{m-1}$ we can write

$$\int_{B_R} u^\sigma u^{q_{m-1}-1} d\mu = \|u\|_{L^{\sigma+q_{m-1}-1}(B_R)}^{\sigma+q_{m-1}-1}$$

$$\leq \|u\|_{L^{p+q_{m-1}-2}(B_R)}^{\theta(\sigma+q_{m-1}-1)} \|u\|_{L^{\sigma \frac{N}{p}}(B_R)}^{(1-\theta)\frac{\sigma(\sigma+q_{m-1}-1)}{\sigma+p+q_{m-1}-2}}$$

$$\times \left(\frac{1}{C_{s,p}} \left\| \nabla (u^{\frac{p+q_{m-1}-2}{p}}) \right\|_{L^p(B_R)} \right)^{p(1-\theta)\frac{\sigma+q_{m-1}-1}{\sigma+p+q_{m-1}-2}}$$

where $\theta := \frac{(p-1)(p+q_{m-1}-2)}{\sigma(\sigma+q_{m-1}-1)}$. Now, due to assumption (3.30), the continuity of u, by choosing $\tilde{C}$ and α as in (3.37) and (3.38) respectively, we can argue as in the proof of Lemma 3.5 (see (3.39)), hence we obtain

$$\frac{1}{q_{m-1}} \left[\|u(\cdot, t_m)\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} - \|u(\cdot, t_{m-1})\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} \right]$$

$$\leq -(q_{m-1} - 1) \left(\frac{p}{p + q_{m-1} - 2} \right)^p c_1 C_p^p \int_{t_{m-1}}^{t_m} \|u(\cdot, \tau)\|_{L^{p+q_{m-1}-2}(B_R)}^{p+q_{m-1}-2} d\tau$$

$$- \frac{1}{\tilde{C}} \left\{ (q_{m-1} - 1) \left(\frac{p}{p + q_{m-1} - 2} \right)^p C - 2\tilde{\varepsilon}_1^{\frac{\sigma(\sigma+q_{m-1}-1)-(p-1)(p+q_{m-1}-2)}{\sigma+p+q_{m-1}-2}} \right\}$$ (3.45)

$$\times \int_{t_{m-1}}^{t_m} \|u(\cdot, \tau)\|_{L^{p+q_{m-1}-2}(B_R)}^{\alpha(p+q_{m-1}-2)} \left\| \nabla \left(u^{\frac{p+q_{m-1}-2}{p}} \right)(\cdot, \tau) \right\|_{L^p(B_R)}^{p-p\alpha} d\tau,$$

where C has been defined in Remark 2.8. Finally, provided $\tilde{\varepsilon}_1$ is small enough, (3.45) can be rewritten as

$$\frac{1}{q_{m-1}} \left[\|u(\cdot, t_m)\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} - \|u(\cdot, t_{m-1})\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} \right]$$

$$\leq -(q_{m-1} - 1) \left(\frac{p}{p + q_{m-1} - 2} \right)^p c_1 C_p^p \int_{t_{m-1}}^{t_m} \|u(\cdot, \tau)\|_{L^{p+q_{m-1}-2}(B_R)}^{p+q_{m-1}-2} d\tau.$$

We define q_m as in (2.15), so that $q_m = p + q_{m-1} - 2$. Then, in view of hypothesis (3.41), we can apply Lemma 3.5 to the integral in the right-hand side of the latter, hence we get

$$\frac{1}{q_{m-1}} \left[\|u(\cdot, t_m)\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} - \|u(\cdot, t_{m-1})\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} \right]$$ (3.46)

$$\leq -(q_{m-1} - 1) \left(\frac{p}{p + q_{m-1} - 2} \right)^p c_1 C_p^p \|u(\cdot, t_m)\|_{L^{q_m}(B_R)}^{q_m} |t_m - t_{m-1}|.$$

Observe that

$$\|u(\cdot, t_m)\|_{L^{q_{m-1}}(B_R)}^{q_{m-1}} \geq 0,$$

$$|t_m - t_{m-1}| = \frac{2^{m-1} t}{2^m - 1}.$$ (3.47)

We define

$$d_{m-1} := \left(\frac{p}{p + q_{m-1} - 2}\right)^{-p} \frac{1}{c_1 C_p^p} \frac{1}{q_{m-1}(q_{m-1} - 1)}. \tag{3.48}$$

By plugging (3.47) and (3.48) into (3.46), we get

$$\|u(\cdot, t_m)\|_{L_\rho^{q_m}(B_R)}^{q_m} \leq \frac{2^{\bar{m}} - 1}{2^{m-1}t} d_{m-1} \|u(\cdot, t_{m-1})\|_{L_\rho^{q_{m-1}}(B_R)}^{q_{m-1}}.$$

The latter can be rewritten as

$$\|u(\cdot, t_m)\|_{L^{q_m}(B_R)} \leq \left(\frac{2^{\bar{m}} - 1}{2^{m-1}} d_{m-1}\right)^{\frac{1}{q_m}} t^{-\frac{1}{q_m}} \|u(\cdot, t_{m-1})\|_{L^{q_{m-1}}(B_R)}^{\frac{q_{m-1}}{q_m}} \tag{3.49}$$

Observe that, for any $1 \leq m \leq \bar{m}$, we have

$$\begin{aligned}
\frac{2^{\bar{m}} - 1}{2^{m-1}} d_{m-1} &= \frac{2^{\bar{m}} - 1}{2^{m-1}} \left(\frac{p}{p + q_{m-1} - 2}\right)^{-p} \frac{1}{c_1 C_p^p} \frac{1}{q_{m-1}(q_{m-1} - 1)} \\
&\leq 2^{\bar{m}+1} \frac{1}{c_1 C_p^p} \left(\frac{p + q_{m-1} - 2}{p}\right)^p \frac{1}{q_{m-1}(q_{m-1} - 1)}.
\end{aligned} \tag{3.50}$$

Consider the function

$$h(x) := \frac{(p + x - 2)^p}{x(x - 1)}, \quad \text{for } q_0 \leq x \leq q_{\bar{m}}, \quad x \in \mathbb{R}.$$

Observe that $h(x) \geq 0$ for any $q_0 \leq x \leq q_{\bar{m}}$. Moreover, h has a maximum in the interval $q_0 \leq x \leq q_{\bar{m}}$, call $\tilde{x}$ the point at which it is attained. Hence

$$h(x) \leq h(\tilde{x}) \quad \text{for any } q_0 \leq x \leq q_{\bar{m}}, \quad x \in \mathbb{R}. \tag{3.51}$$

Due to (3.50) and (3.51), we can say that there exists a positive constant C, where $C = C(C_p, \bar{m}, p, q_0)$, such that

$$\frac{2^{\bar{m}} - 1}{2^{m-1}} d_{m-1} \leq C \quad \text{for all } 1 \leq m \leq \bar{m}. \tag{3.52}$$

By using (3.52) and (3.49), we get, for any $1 \leq m \leq \bar{m}$

$$\|u(\cdot, t_m)\|_{L^{q_m}(B_R)} \leq C^{\frac{1}{q_m}} t^{-\frac{1}{q_m}} \|u(\cdot, t_{m-1})\|_{L^{q_{m-1}}(B_R)}^{\frac{q_{m-1}}{q_m}}. \tag{3.53}$$

Let us set

$$U_m := \|u(\cdot, t_m)\|_{L^{q_m}(B_R)}$$

Then (3.53) becomes

$$\begin{aligned}
U_m &\leq C^{\frac{1}{q_m}} t^{-\frac{1}{q_m}} U_{n-1}^{\frac{q_{m-1}}{q_m}} \\
&\leq C^{\frac{1}{q_m}} t^{-\frac{1}{q_m}} \left[C^{\frac{1}{q_{m-1}}} t^{-\frac{1}{q_{m-1}}} U_{m-2}^{\frac{q_{m-2}}{q_{m-1}}}\right] \\
&\leq \dots \\
&\leq C^{\frac{m}{q_m}} t^{-\frac{m}{q_m}} U_0^{\frac{q_0}{q_m}}.
\end{aligned}$$

We define

$$\alpha_m := \frac{m}{q_m}, \quad \delta_m := \frac{q_0}{q_m}. \tag{3.54}$$

Substituting m with $\bar{m}$ into (3.54) and in view of (3.44), (3.53) with $m = \bar{m}$, we have

$$\|u(\cdot, t)\|_{L^{q_{\bar{m}}}(B_R)} \leq C^{\alpha_{\bar{m}}} t^{-\alpha_{\bar{m}}} \|u_0\|_{L^{q_0}(B_R)}^{\delta_{\bar{m}}}.$$

Observe that if $q_{\bar{m}} = q$ then the thesis is proved and one has

$$\alpha_{\bar{m}} = \frac{1}{p-2}\left(1 - \frac{q_0}{q}\right), \quad \delta_{\bar{m}} = \frac{q_0}{q}.$$

Now suppose that $q < q_{\bar{m}}$, then in particular $q_0 \leq q \leq q_{\bar{m}}$. By interpolation and Lemma 3.5 we get

$$
\begin{aligned}
\|u(\cdot, t)\|_{L^q(B_R)} &\leq \|u(\cdot, t)\|_{L^{q_0}(B_R)}^{\theta} \|u(\cdot, t)\|_{L^{q_{\bar{m}}}(B_R)}^{1-\theta} \\
&\quad \|u(\cdot, t)\|_{L^{q_0}(B_R)}^{\theta} C^{\alpha_{\bar{m}}(1-\theta)} t^{-\alpha_{\bar{m}}(1-\theta)} \|u_0\|_{L^{q_0}(B_R)}^{\delta_{\bar{m}}(1-\theta)} \\
&\leq C^{\alpha_{\bar{m}}(1-\theta)} t^{-\alpha_{\bar{m}}(1-\theta)} \|u_0\|_{L^{q_0}(B_R)}^{\delta_{\bar{m}}(1-\theta)+\theta},
\end{aligned}
\tag{3.55}
$$

where

$$\theta = \frac{q_0}{q}\left(\frac{q_{\bar{m}} - q}{q_{\bar{m}} - q_0}\right). \tag{3.56}$$

Combining (3.43), (3.55) and (3.56), we get the claim by noticing that q was arbitrary fixed in $[q_0, +\infty)$.
□

4. Auxiliary results

In what follows, we will deal with solutions u_R to problem (3.2) for arbitrary fixed $R > 0$. For notational convenience, we will simply write u instead of u_R since no confusion will occur in the present section. We define

$$G_k(v) := v - T_k(v). \tag{4.1}$$

where $T_k(v)$ has been defined in (3.1). Let $a_1 > 0$, $a_2 > 0$ and $t > \tau_1 > \tau_2 > 0$. We consider, for any $i \in \mathbb{N} \cup \{0\}$, the sequences

$$
\begin{aligned}
k_i &:= a_2 + (a_1 - a_2)2^{-i}; \\
\theta_i &:= \tau_2 + (\tau_1 - \tau_2)2^{-i};
\end{aligned}
\tag{4.2}
$$

and the cylinders

$$U_i := B_R \times (\theta_i, t). \tag{4.3}$$

Observe that the sequence $\{\theta_i\}_{i \in \mathbb{N}}$ is monotone decreasing w.r.t. i. Furthermore, we define, for any $i \in \mathbb{N}$, the cut-off functions $\xi_i(\tau)$ such that

$$\xi_i(\tau) := \begin{cases} 1 & \theta_{i-1} < \tau < t \\ 0 & 0 < \tau < \theta_i \end{cases} \quad \text{and} \quad |(\xi_i)_\tau| \leq \frac{2^i}{\tau_1 - \tau_2}. \tag{4.4}$$

Finally, we define

$$S(t) := \sup_{0 < \tau < t} \left(\tau \|u(\tau)\|_{L^\infty(B_R)}^{\sigma-1}\right). \tag{4.5}$$

We can now state the following

Lemma 4.1. *Let $i \in \mathbb{N}$, k_i, θ_i, U_i be defined in (4.2), (4.3) and $R > 0$. Let u be a solution to problem (3.2). Then, for any $q > 1$, we have that*

$$\sup_{\tau_1 < \tau < t} \int_{B_R} [G_{k_0}(u)]^q \, d\mu + \iint_{U_{i-1}} \left| \nabla [G_{k_i}(u)]^{\frac{p+q-2}{p}} \right|^p \, d\mu d\tau \le 2^i \gamma \, C_1 \iint_{U_i} [G_{k_{i+1}}(u)]^q \, d\mu d\tau.$$

where $\gamma = \gamma(p, q)$ and

$$C_1 := \frac{1}{\tau_1 - \tau_2} + \frac{S(t)}{\tau_1} \frac{2a_1}{a_1 - a_2}. \tag{4.6}$$

Proof. For any $i \in \mathbb{N}$, we multiply both sides of the differential equation in problem (3.2) by $[G_{k_i}(u)]^{q-1} \xi_i$, $q > 1$, and we integrate on the cylinder U_i, yielding:

$$\iint_{U_i} u_\tau \, [G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau$$
$$= \iint_{U_i} \operatorname{div}(|\nabla u|^{p-2} \nabla u)[G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau + \iint_{U_i} T_k(u^\sigma) \, [G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau. \tag{4.7}$$

We integrate by parts. Thus we write, due to (4.4),

$$\iint_{U_i} u_\tau \, [G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau = \frac{1}{q} \iint_{U_i} \frac{d}{d\tau}[(G_{k_i}(u))^q] \xi_i \, d\mu d\tau$$
$$= -\frac{1}{q} \iint_{U_i} [G_{k_i}(u)]^q (\xi_i)_\tau \, d\mu d\tau + \frac{1}{q} \int_{B_R} [G_{k_i}(u(x,t))]^q \, d\mu \tag{4.8}$$

Moreover,

$$- \iint_{U_i} \operatorname{div}(|\nabla u|^{p-2} \nabla u)[G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau = \iint_{U_i} |\nabla u|^{p-2} \nabla u \cdot \nabla [G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau$$
$$\ge (q-1) \iint_{U_i} [G_{k_i}(u)]^{q-2} |\nabla [G_{k_i}(u)]|^p \, \xi_i \, d\mu d\tau. \tag{4.9}$$

Now, combining (4.7), (4.8) and (4.9), using the fact that $T(u^\sigma) \le u^\sigma$ and (4.4), we can write

$$\frac{1}{q} \int_{B_R} [G_{k_i}(u(x,t))]^q \, d\mu + (q-1) \iint_{U_i} [G_{k_i}(u)]^{q-2} |\nabla [G_{k_i}(u)]|^p \, \xi_i \, d\mu d\tau$$
$$\le \frac{1}{q} \iint_{U_i} [G_{k_i}(u)]^q (\xi_i)_\tau \, d\mu d\tau + \iint_{U_i} u^\sigma \, [G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau \tag{4.10}$$
$$\le \frac{2^i}{\tau_1 - \tau_2} \iint_{U_i} [G_{k_i}(u)]^q \, d\mu d\tau + \iint_{U_i} u^\sigma \, [G_{k_i}(u)]^{q-1} \xi_i \, d\mu d\tau.$$

Let us define

$$\tilde{\gamma} := \left[\min \left\{ \frac{1}{q}, q-1 \right\} \right]^{-1},$$

thus (4.10) reads

$$\int_{B_R} [G_{k_i}(u(x,t))]^q \, d\mu + \iint_{U_i} [G_{k_i}(u)]^{q-2} |\nabla [G_{k_i}(u)]|^p \xi_i \, d\mu d\tau$$
$$\le \tilde{\gamma} \frac{2^i}{\tau_1 - \tau_2} \iint_{U_i} [G_{k_i}(u)]^q \, d\mu d\tau + \tilde{\gamma} \iint_{U_i} u^\sigma \, [G_{k_i}(u)]^{q-1} \xi_i d\mu d\tau. \tag{4.11}$$

Observe that the sequence $\{k_i\}_{i\in\mathbb{N}}$ is monotone decreasing, hence

$$G_{k_0}(u) \le G_{k_i}(u) \le G_{k_{i+1}}(u) \le u \qquad \text{for all } i \in \mathbb{N}.$$

Thus (4.11) can be rewritten as

$$\int_{B_R} [G_{k_0}(u(x,t))]^q \, d\mu + \iint_{U_{i-1}} [G_{k_i}(u)]^{q-2} |\nabla[G_{k_i}(u)]|^p \, d\mu d\tau$$

$$\le \frac{2^i \tilde{\gamma}}{\tau_1 - \tau_2} \iint_{U_i} [G_{k_{i+1}}(u)]^q \, d\mu d\tau + \tilde{\gamma} \iint_{U_i} u^\sigma \, [G_{k_{i+1}}(u)]^{q-1} d\mu d\tau. \tag{4.12}$$

Let us now define

$$I := \tilde{\gamma} \iint_{U_i} u^{\sigma-1} \, u \, [G_{k_{i+1}}(u)]^{q-1} d\mu d\tau$$

Observe that, for any $i \in \mathbb{N}$,

$$\frac{u}{k_i}\chi_i \le \frac{u - k_{i+1}}{k_i - k_{i+1}}\chi_i$$

where χ_i is the characteristic function of $D_i := \{(x,t) \in U_i : u(x,t) \ge k_i\}$. Then, by using (4.5), we get:

$$I \le \tilde{\gamma} \int_{\theta_i}^t \frac{1}{\tau}\tau\|u(\tau)\|_{L^\infty(B_R)}^{\sigma-1} \int_{B_R} u\,[G_{k_{i+1}}(u)]^{q-1} \, d\mu d\tau$$

$$= \tilde{\gamma} \int_{\theta_i}^t \frac{1}{\tau}\tau\|u(\tau)\|_{L^\infty(B_R)}^{\sigma-1} \int_{B_R} k_i\frac{u}{k_i}\,[G_{k_{i+1}}(u)]^{q-1} \, d\mu d\tau \tag{4.13}$$

$$\le \tilde{\gamma}\frac{k_i}{k_i - k_{i+1}}S(t) \int_{\theta_i}^t \frac{1}{\tau}\int_{B_R} [G_{k_{i+1}}(u)]^q \, d\mu d\tau.$$

By substituting (4.13) into (4.12) we obtain

$$\sup_{\tau_1 < \tau < t} \int_{B_R} [G_{k_0}(u(x,t))]^q \, d\mu + \left(\frac{p}{p+q-2}\right)^p \iint_{U_{i-1}} \left|\nabla[G_{k_i}(u)]^{\frac{p+q-2}{p}}\right|^p \, d\mu d\tau$$

$$\le \frac{2^i \tilde{\gamma}}{\tau_1 - \tau_2} \iint_{U_i} [G_{k_{i+1}}(u)]^q \, d\mu d\tau + \frac{k_i \tilde{\gamma}}{k_i - k_{i+1}}\frac{S(t)}{\theta_0} \iint_{U_i} [G_{k_{i+1}}(u)]^q d\mu d\tau.$$

To proceed further, observe that

$$\frac{k_i}{k_i - k_{i+1}} = \frac{2^{i+1}a_2}{a_1 - a_2} + 2, \quad \text{and} \quad \theta_0 \equiv \tau_1.$$

Consequently, by choosing C_1 as in (4.6), we get

$$\sup_{\tau_1 < \tau < t} \int_{B_R} [G_{k_0}(u(x,t))]^q \, d\mu + \left(\frac{p}{p+q-2}\right)^p \iint_{U_{i-1}} |\nabla[G_{k_i}(u)]^{\frac{p+q-2}{p}}|^p d\mu d\tau$$

$$\le 2^i \tilde{\gamma} C_1 \int \int_{U_i} [G_{k_{i+1}}(u)]^q \, d\mu d\tau.$$

The thesis follows, letting

$$\gamma := \left[\min\left\{1; \left(\frac{p}{p+q-2}\right)^p\right\}\right]^{-1} \tilde{\gamma}. \tag{4.14}$$

$\square$

Lemma 4.2. *Assume* (1.2)*, let* $1 < r < q$ *and assume that* (1.3) *holds. Let* k_i*,* θ_i*,* U_i *be defined in* (4.2)*,* (4.3) *and* $R > 0$*. Let* u *be a solution to problem* (3.2)*. Then, for every* $i \in \mathbb{N}$ *and* $\varepsilon > 0$*, we have*

$$
\sup_{\tau_1 < \tau < t} \int_{B_R} [G_{k_0}(u)]^q \, d\mu + \iint_{U_{i-1}} \left| \nabla [G_{k_i}(u)]^{\frac{p+q-2}{p}} \right|^p \, d\mu d\tau
$$

$$
\leq \varepsilon \iint_{U_i} \left| \nabla [G_{k_{i+1}}(u)]^{\frac{p+q-2}{p}} \right|^p \, d\mu d\tau
$$

$$
+ C(\varepsilon)(2^i \gamma C_1)^{\frac{N(p+q-2-r)+pr}{N(p-2)+pr}} (t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{B_R} [G_{k_\infty}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}},
$$

with C_1 *and* γ *defined as in* (4.6) *and* (4.14) *respectively and for some* $C(\varepsilon) > 0$*.*

Proof. Let us fix $q > 1$ and $1 < r < q$. We define

$$
\alpha := r \, \frac{N(p-2) + pq}{N(p+q-2-r) + pr}. \tag{4.15}
$$

Observe that, since $1 < r < q$, one has $0 < \alpha < q$. By Hölder inequality with exponents $\frac{pN}{N-p} \left(\frac{p+q-2}{p(q-\alpha)} \right)$ and $\frac{N(p+q-2)}{N(p+\alpha-2)+p(q-\alpha)}$, we thus have:

$$
\int_{B_R} [G_{k_{i+1}}(u)]^q \, d\mu = \int_{B_R} [G_{k_{i+1}}(u)]^{q-\alpha} [G_{k_{i+1}}(u)]^{\alpha} \, d\mu
$$

$$
\leq \left(\int_{B_R} [G_{k_{i+1}}(u)]^{\left(\frac{p+q-2}{p} \right) \frac{pN}{N-p}} \, d\mu \right)^{\left(\frac{p(q-\alpha)}{p+q-2} \right) \frac{N-p}{pN}}
$$

$$
\times \left(\int_{B_R} [G_{k_{i+1}}(u)]^{\frac{\alpha N(p+q-2)}{N(p+\alpha-2)+p(q-\alpha)}} \, d\mu \right)^{\frac{N(p+\alpha-2)+p(q-\alpha)}{N(p+q-2)}} \tag{4.16}
$$

$$
\leq \left(\left\| [G_{k_{i+1}}(u)]^{\frac{p+q-2}{p}} \right\|_{L^{p^*}(B_R)} \right)^{\frac{p(q-\alpha)}{p+q-2}}
$$

$$
\times \left(\int_{B_R} [G_{k_{i+1}}(u)]^{\frac{\alpha N(p+q-2)}{N(p+\alpha-2)+p(q-\alpha)}} \, d\mu \right)^{\frac{N(p+\alpha-2)+p(q-\alpha)}{N(p+q-2)}}.
$$

By the definition of α in (4.15) and inequality (1.3), (4.16) becomes

$$
\int_{B_R} [G_{k_{i+1}}(u)]^q \, d\mu \leq \left(\frac{1}{C_{s,p}} \left\| \nabla [G_{k_{i+1}}(u)]^{\frac{p+q-2}{p}} \right\|_{L^p(B_R)} \right)^{\frac{p(q-\alpha)}{p+q-2}} \left(\int_{B_R} [G_{k_{i+1}}(u)]^r \, d\mu \right)^{\frac{\alpha}{r}}. \tag{4.17}
$$

We multiply both sides of (4.17) by $2^i \gamma C_1$ with C_1 and γ as in (4.6) and (4.14), respectively. Then, we apply Young's inequality with exponents $\frac{p+q-2}{q-\alpha}$ and $\frac{p+q-2}{p+\alpha-2}$ to get:

$$
2^i \gamma C_1 \int_{B_R} [G_{k_{i+1}}(u)]^q \, d\mu
$$

$$
\leq \varepsilon \int_{B_R} \left| \nabla [G_{k_{i+1}}(u)]^{\frac{p+q-2}{p}} \right|^p \, d\mu + C(\varepsilon)(2^i \gamma C_1)^{\frac{p+q-2}{p+\alpha-2}} \left(\int_{B_R} [G_{k_{i+1}}(u)]^r \, d\mu \right)^{\frac{\alpha}{r} \frac{p+q-2}{p+\alpha-2}} \tag{4.18}
$$

Define

$$\lambda := \frac{\alpha}{r}\left(\frac{p+q-2}{p+\alpha-2}\right) = \frac{N(p-2)+pq}{N(p-2)+pr}.$$

Observe that $\lambda > 1$ since $r < q$. By Lemma 4.1,

$$\sup_{\tau_1<\tau<t}\int_{B_R}[G_{k_0}(u)]^q\,d\mu + \iint_{U_{i-1}}\left|\nabla[G_{k_i}(u)]^{\frac{p+q-2}{p}}\right|^p\,d\mu d\tau \le 2^i\gamma C_1\int_{\theta_i}^t\int_{B_R}[G_{k_{i+1}}(u)]^q\,d\mu d\tau \qquad (4.19)$$

Moreover, let us integrate inequality (4.18) in the time interval $\tau \in (\theta_i, t)$. Then, we observe that

$$\begin{aligned}
C(\varepsilon)(2^i\gamma C_1)^{\frac{p+q-2}{p+\alpha-2}}&\int_{\theta_i}^t\left(\int_{B_R}[G_{k_{i+1}}(u)]^r\,d\mu\right)^\lambda\,d\tau \\
&\le C(\varepsilon)(2^i\gamma C_1)^{\frac{p+q-2}{p+\alpha-2}}(t-\tau_2)\left(\sup_{\tau_2<\tau<t}\int_{B_R}[G_{k_{i+1}}(u)]^r\,d\mu\right)^\lambda
\end{aligned} \qquad (4.20)$$

where we have used that $\tau_2 < \theta_i$ for every $i \in \mathbb{N}$. Finally, we substitute (4.19) and (4.20) into (4.18), thus we get

$$\begin{aligned}
\sup_{\tau_1<\tau<t}&\int_{B_R}[G_{k_0}(u)]^q\,d\mu + \iint_{U_{i-1}}\left|\nabla[G_{k_i}(u)]^{\frac{p+q-2}{p}}\right|^p\,d\mu d\tau \\
&\le \varepsilon\iint_{U_i}\left|\nabla[G_{k_{i+1}}(u)]^{\frac{p+q-2}{p}}\right|^p\,d\mu d\tau + C(\varepsilon)(2^i\gamma C_1)^{\frac{p+q-2}{p+\alpha-2}}(t-\tau_2)\left(\sup_{\tau_2<\tau<t}\int_{B_R}[G_{k_{i+1}}(u)]^r\,d\mu\right)^\lambda
\end{aligned}$$

The thesis follows by noticing that, for any $i \in \mathbb{N}$

$$G_{k_i}(u) \le G_{k_{i+1}}(u) \le \ldots \le G_{k_\infty}(u),$$

and that

$$\frac{p+q-2}{p+\alpha-2} = \frac{N(p+q-2-r)+pr}{N(p-2)+pr}.$$

$\square$

Proposition 4.3. *Assume that* (1.2) *and* (1.3) *holds. Let* $S(t)$ *be defined as in* (4.5). *Let* u *be a solution to problem* (3.2). *Suppose that, for all* $t \in (0, T)$,

$$S(t) \le 1.$$

Let $r \ge 1$, *then there exists* $k = k(p, r)$ *such that*

$$\|u(x,\tau)\|_{L^\infty\left(B_R\times\left(\frac{t}{2},t\right)\right)} \le k\,t^{-\frac{N}{N(p-2)+pr}}\left[\sup_{\frac{t}{4}<\tau<t}\int_{B_R}u^r\,d\mu\right]^{\frac{p}{N(p-2)+pr}},$$

for all $t \in (0, T)$.

Proof. Let us define, for any $j \in \mathbb{N}$,

$$J_i := \iint_{U_i} \left| \nabla \left[G_{k_{i+1}}(u) \right]^{\frac{p+q-2}{p}} \right|^p d\mu \, dt, \tag{4.21}$$

where G_k, $\{k_i\}_{i \in \mathbb{N}}$ and U_i have been defined in (4.1), (4.2) and (4.3) respectively. Let us fix $1 \le r < q$ and define

$$\beta := \frac{N(p + q - 2 - r) + pr}{N(p - 2) + pr}.$$

By means of Lemma 4.2 and (4.21), we can write, for any $i \in \mathbb{N} \cup \{0\}$

$$\sup_{\tau_1 < \tau < t} \int_{B_R} [G_{k_0}(u)]^q \, d\mu + J_0$$

$$\le \varepsilon J_1 + C(\varepsilon)(2\gamma C_1)^\beta (t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{B_R} [G_{k_\infty}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}}$$

$$\le \varepsilon \left\{ \varepsilon J_2 + C(\varepsilon)(2^2\gamma C_1)^\beta (t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{B_R} [G_{k_\infty}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}} \right\} \tag{4.22}$$

$$+ C(\varepsilon)(2\gamma C_1)^\beta (t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{B_R} [G_{k_\infty}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}}$$

$$\le \ldots$$

$$\le \varepsilon^i J_i + \sum_{j=0}^{i-1} (2^\beta \varepsilon)^j (2\gamma C_1)^\beta C(\varepsilon)(t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{B_R} [G_{k_\infty}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}}.$$

Fix now $\varepsilon > 0$ such that $\varepsilon 2^\beta < \frac{1}{2}$. Taking the limit as $i \longrightarrow +\infty$ in (4.22) we have:

$$\sup_{\tau_1 < \tau < t} \int_{B_R} [G_{k_0}(u)]^q \, d\mu \le \tilde{C}(2\gamma C_1)^\beta (t - \tau_2) \left(\sup_{\tau_2 < \tau < t} \int_{B_R} [G_{k_\infty}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}}. \tag{4.23}$$

Observe that, due to the definition of the sequence $\{k_i\}_{i \in \mathbb{N}}$ in (4.2), one has

$$k_0 = a_1, \qquad\qquad k_\infty = a_2;$$
$$G_{k_0}(u) = G_{a_1}(u), \qquad G_{k_\infty}(u) = G_{a_2}(u).$$

For $n \in \mathbb{N} \cup \{0\}$, consider, for some $C_0 > 0$ to be fixed later, the following sequences

$$t_n = \frac{1}{2} t(1 - 2^{-n-1});$$
$$h_n = C_0(1 - 2^{-n-1}); \tag{4.24}$$
$$\overline{h}_n = \frac{1}{2}(h_n + h_{n+1}).$$

Let us now set in (4.23):

$$\tau_1 = t_{n+1}; \quad \tau_2 = t_n; \quad a_1 = \overline{h}_n; \quad a_2 = h_n. \tag{4.25}$$

Then the coefficient C_1 defined in (4.6), by (4.24) and (4.25), satisfies, since for any $t \in (0, T)$ one has $S(t) \leq 1$,

$$2C_1 \leq \frac{C_2^n}{t} \quad \text{for some } C_2 > 1.$$

Due to the latter bound and to (4.25), (4.23) reads

$$\sup_{t_{n+1} < \tau < t} \int_{B_R} [G_{\bar{h}_n}(u)]^q \, d\mu \leq \tilde{C} \gamma \, C_2^{n\beta} t^{-\beta+1} \left(\sup_{t_n < \tau < t} \int_{B_R} [G_{h_n}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}}. \tag{4.26}$$

Furthermore, observe that

$$\int_{B_R} [G_{h_{n+1}}(u)]^r \, d\mu \leq (h_{n+1} - \bar{h}_n)^{r-q} \int_{B_R} \left[G_{\bar{h}_n}(u) \right]^q \, d\mu. \tag{4.27}$$

By combining together (4.26) and (4.27), we derive the following inequalities:

$$\sup_{t_{n+1} < \tau < t} \int_{B_R} [G_{h_{n+1}}(u)]^r \, d\mu \leq (h_{n+1} - \bar{h}_n)^{r-q} \sup_{t_{n+1} < \tau < t} \int_{B_R} \left[G_{\bar{h}_n}(u) \right]^q \, d\mu$$

$$\leq \tilde{C} \gamma \, C_2^{n\beta} \left(\frac{h_{n+1} - h_n}{2} \right)^{r-q} t^{-\beta+1} \left(\sup_{t_n < \tau < t} \int_{B_R} [G_{h_n}(u)]^r \, d\mu \right)^{\frac{N(p-2)+pq}{N(p-2)+pr}}. \tag{4.28}$$

Let us finally define

$$Y_n := \sup_{t_n < \tau < t} \int_{B_R} [G_{h_n}(u)]^r \, d\mu.$$

Hence, by using (4.24), (4.28) reads,

$$Y_{n+1} \leq \tilde{C} \gamma \, C_2^{n\beta} \left(\frac{h_{n+1} - h_n}{2} \right)^{r-q} t^{-\beta+1} Y_n^{\frac{N(p-2)+pq}{N(p-2)+pr}}$$

$$\leq \tilde{C} \gamma \, C_2^{n\beta} 2^{(n+3)(q-r)} C_0^{r-q} t^{-\beta+1} Y_n^{\frac{N(p-2)+pq}{N(p-2)+pr}}$$

$$\leq k^{n(q-r)} C_0^{r-q} t^{-\beta+1} Y_n^{\frac{N(p-2)+pq}{N(p-2)+pr}},$$

for some $k = k(p, r) > 1$. From [25, Chapter 2, Lemma 5.6] it follows that

$$Y_n \longrightarrow 0 \quad \text{as } n \to +\infty, \tag{4.29}$$

provided

$$C_0^{r-q} t^{-\beta+1} Y_0^{\frac{N(p-2)+pq}{N(p-2)+pr} - 1} \leq k^{r-q}. \tag{4.30}$$

Now, (4.29), in turn, reads

$$\|u\|_{L^\infty \left(B_R \times \left(\frac{t}{2}, t \right) \right)} \leq C_0.$$

Moreover, (4.30) is fulfilled since

$$C_0 = k \, t^{\frac{-\beta+1}{q-r}} Y_0^{\left(\frac{N(p-2)+pq}{N(p-2)+pr} - 1 \right) \left(\frac{1}{q-r} \right)} \leq k \, t^{-\frac{N}{N(p-2)+pr}} \left[\sup_{\frac{t}{4} < \tau < t} \int_{B_R} u^r \, d\mu \right]^{\frac{p}{N(p-2)+pr}}.$$

This concludes the proof. □

5. Proof of Theorem 2.2

By Lemma 4.3, using the same arguments as in the proof of [27, Lemmata 4 and 5, and subsequent remarks], we get the following result.

Lemma 5.1. *Assume* (1.2) *and* $\sigma > p - 1 + \frac{p}{N}$. *Suppose that* (1.3) *and* (2.2) *hold. Let* $S(t)$ *be defined as in* (4.5). *Define*

$$T := \sup\{t > 0 : S(t) \le 1\}. \tag{5.1}$$

Then

$$T = +\infty.$$

Proof of Theorem 2.2. Let $\{u_{0,h}\}_{h \ge 0}$ be a sequence of functions such that

> (a) $u_{0,h} \in L^\infty(M) \cap C_c^\infty(M)$ for all $h \ge 0$,
>
> (b) $u_{0,h} \ge 0$ for all $h \ge 0$,
>
> (c) $u_{0,h_1} \le u_{0,h_2}$ for any $h_1 < h_2$,
>
> (d) $u_{0,h} \longrightarrow u_0$ in $L^s(M) \cap L^1(M)$ as $h \to +\infty$,

Observe that, due to assumptions (c) and (d), $u_{0,h}$ satisfies (2.2). For any $R > 0$, $k > 0$, $h > 0$, consider the problem

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + T_k(u^\sigma) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, \infty) \\ u = u_{0,h} & \text{in } B_R \times \{0\}. \end{cases} \tag{5.2}$$

From standard results it follows that problem (5.2) has a solution $u_{h,k}^R$ in the sense of Definition 3.1. In addition, $u_{h,k}^R \in C([0, T]; L^q(B_R))$ for any $q > 1$.

(i) In view of Proposition 4.3 and Lemma 5.1, the solution $u_{h,k}^R$ to problem (5.2) satisfies estimate (4.3) for any $t \in (0, +\infty)$, uniformly w.r.t. R, k and h. By standard arguments we can pass to the limit as $R \to \infty$, $k \to \infty$ and $h \to \infty$ and we obtain a solution u to Eq (1.1) satisfying (2.3).

(ii) Due to Proposition 3.3, the solution $u_{h,k}^R$ to problem (5.2) satisfies estimate (3.11) for any $t \in (0, +\infty)$, uniformly w.r.t. R, k and h. Thus, the solution u fulfills (2.5).

(iii) We now furthermore suppose that $u_{0,h} \in L^q(M)$ and $u_{0,h} \longrightarrow u_0$ in $L^q(M)$. Due to Proposition 3.2, the solution $u_{h,k}^R$ to problem (5.2) satisfies estimate (3.4) for any $t \in (0, +\infty)$, uniformly w.r.t. R, k and h. Thus, the solution u also fulfills (2.7).

This completes the proof.

$\square$

6. Proof of Theorem 2.4

To prove Theorem 2.4 we need the following two results.

Lemma 6.1. *Assume* (1.2) *and, moreover, that* $\sigma > p - 1 + \frac{p}{N}$. *Assume that inequality* (1.3) *holds. Let* u *be a solution of problem* (3.2) *with* $u_0 \in L^\infty(B_R)$, $u_0 \ge 0$, *such that*

$$\|u_0\|_{L^{\sigma_0}(B_R)} \le \varepsilon_2,$$

for $\varepsilon_2 = \varepsilon_2(\sigma, p, N, C_{s,p}, \sigma_0) > 0$ sufficiently small and σ_0 as in (2.1). Let $S(t)$ and T be defined as in (4.5) and (5.1) respectively. Then

$$T = +\infty.$$

Proof. We suppose by contradiction that $T < +\infty$. Then, by (5.1) and (4.5), we can write:

$$1 = S(T) = \sup_{0 < t < T} t\|u(t)\|_{L^\infty(B_R)}^{\sigma-1}. \tag{6.1}$$

Due to Lemma 4.3 with the choice $r = q > \sigma_0$, (6.1) reduces to

$$
\begin{aligned}
1 = S(T) &\leq \sup_{0 < t < T} t \left\{ k\, t^{-\frac{N}{N(p-2)+pq}} \left(\sup_{\frac{t}{4} < \tau < t} \int_{B_R} u^q\, d\mu \right)^{\frac{p}{N(p-2)+pq}} \right\}^{(\sigma-1)} \\
&\leq \sup_{0 < t < T} k\, t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}} \left(\sup_{\frac{t}{4} < \tau < t} \|u(\tau)\|_{L^q(B_R)}^{\frac{qp(\sigma-1)}{N(p-2)+pq}} \right).
\end{aligned}
\tag{6.2}
$$

Define

$$I_1 := \sup_{\frac{t}{4} < \tau < t} \|u(\tau)\|_{L^q(B_R)}^{\frac{pq(\sigma-1)}{N(p-2)+pq}}. \tag{6.3}$$

In view of the choice $q > \sigma_0$, we can apply Proposition 3.3 with $q_0 = \sigma_0$ to (6.3), thus we get

$$
\begin{aligned}
I_1 &\leq \sup_{\frac{t}{4} < \tau < t} \left[C\, t^{-\gamma_q} \|u_0\|_{L^{q_0}(B_R)}^{\delta_q} \right]^{\frac{pq(\sigma-1)}{N(p-2)+pq}} \\
&\leq C\, t^{-\gamma_q \frac{pq(\sigma-1)}{N(p-2)+pq}} \|u_0\|_{L^{q_0}(B_R)}^{\delta_q \frac{pq(\sigma-1)}{N(p-2)+pq}},
\end{aligned}
\tag{6.4}
$$

where γ_q and δ_q are defined in (3.12). By substituting (6.4) into (6.2) we get

$$1 = S(T) \leq C\, k \sup_{0 < t < T} t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}-\gamma_q \frac{pq(\sigma-1)}{N(p-2)+pq}} \|u_0\|_{L^{q_0}(B_R)}^{\delta_q \frac{pq(\sigma-1)}{N(p-2)+pq}}.$$

Observe that

$$1 - \frac{N(\sigma-1)}{N(p-2)+pq} - \gamma_q \frac{pq(\sigma-1)}{N(p-2)+pq} = 0;$$

$$\delta_q \frac{pq(\sigma-1)}{N(p-2)+pq} = \sigma - p + 1 > 0;$$

hence

$$1 = S(T) < C\,\tilde{C}\, \varepsilon_2^{\sigma-p+1}.$$

Provided ε_2 is sufficiently small, a contradiction, i.e., $1 = S(T) < 1$. Thus $T = +\infty$. □

Proposition 6.2. *Assume (1.2) and, moreover, that $\sigma > p - 1 + \frac{p}{N}$. Let u be the solution to problem (3.2) with $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Let σ_0 be defined in (2.1) and $q > \sigma_0$. Assume that*

$$\|u_0\|_{L^{\sigma_0}(B_R)} < \varepsilon_2$$

with $\varepsilon_2 = \varepsilon_2(\sigma, p, N, C_{s,p}, \sigma_0) > 0$ sufficiently small. Then, for some $C = C(N, \sigma, p, q, \sigma_0) > 0$:

$$\|u(t)\|_{L^\infty(B_R)} \leq C\, t^{-\frac{1}{\sigma-1}} \|u_0\|_{L^{\sigma_0}(B_R)}^{1-\frac{p-2}{\sigma-1}} \quad \textit{for any } t \in (0, +\infty). \tag{6.5}$$

Proof. Due to Lemma 6.1,

$$S(t) \leq 1 \quad \text{for all} \ t \in (0, +\infty).$$

Therefore, by Lemma 4.3 and Proposition 3.3 with $q_0 = \sigma_0$, for all $t \in (0, +\infty)$

$$
\begin{aligned}
\|u(t)\|_{L^\infty(B_R)} &\leq \|u\|_{L^\infty(B_R \times (\frac{t}{2}, t))} \\
&\leq k \, t^{-\frac{N}{N(p-2)+pq}} \left[\sup_{\frac{t}{4} < \tau < t} \|u(\tau)\|_{L^q(B_R)}^q \right]^{\frac{p}{N(p-2)+pq}} \\
&\leq C \, t^{-\frac{N}{N(p-2)+pq} - \gamma_q \frac{pq}{N(p-2)+pq}} \|u_0\|_{L^{\sigma_0}(B_R)}^{\delta_q \frac{pq}{N(p-2)+pq}},
\end{aligned}
\tag{6.6}
$$

where $C = C(\sigma, p, N, q, \sigma_0) > 0$, γ_q and δ_q as in (3.12) with $q_0 = \sigma_0$. Observe that

$$-\frac{N}{N(p-2)+pq} - \gamma_q \frac{pq}{N(p-2)+pq} = -\frac{1}{\sigma-1}, \tag{6.7}$$

and

$$\delta_q \frac{pq}{N(p-2)+pq} = \frac{\sigma-p+1}{\sigma-1}. \tag{6.8}$$

By combining (6.6) with (6.7) and (6.8) we get the thesis. $\square$

Proof of Theorem 2.4. We use the same argument discussed in the proof of Theorem 2.2. In fact, let $\{u_{0,l}\}_{l \geq 0}$ be a sequence of functions such that

$$(a) \ u_{0,l} \in L^\infty(M) \cap C_c^\infty(M) \ \text{ for all } l \geq 0,$$
$$(b) \ u_{0,l} \geq 0 \ \text{ for all } l \geq 0,$$
$$(c) \ u_{0,l_1} \leq u_{0,l_2} \ \text{ for any } l_1 < l_2,$$
$$(d) \ u_{0,l} \longrightarrow u_0 \ \text{ in } L^{\sigma_0}(M) \quad \text{as } l \rightarrow +\infty,$$

where σ_0 has been defined in (2.1). Observe that, due to assumptions (c) and (d), $u_{0,l}$ satisfies (2.10). For any $R > 0$, $k > 0$, $l > 0$, we consider problem (5.2) with the sequence $u_{0,h}$ replaced by the sequence $u_{0,l}$. From standard results it follows that problem (5.2) has a solution $u_{l,k}^R$ in the sense of Definition 3.1; moreover, $u_{l,k}^R \in C([0,T]; L^q(B_R))$ for any $q > 1$.

Due to Proposition 6.2, Proposition 3.3 and Lemma 3.2, the solution $u_{l,k}^R$ to problem (5.2) satisfies estimates (6.5), (3.11) and (3.4) for $t \in (0, +\infty)$, uniformly w.r.t. R, k and l. Thus, by standard arguments we can pass to the limit as $R \rightarrow \infty$, $k \rightarrow \infty$ and $l \rightarrow \infty$ and we obtain a solution u to Eq (1.1) satisfying (2.11), (2.5) and (2.7). $\square$

7. Proof of Theorem 2.7

Lemma 7.1. *Assume* (1.2), *$p > 2$, and $q > \max\{\sigma_0, 1\}$. Let u be a solution to problem* (3.2) *with $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$, such that*

$$\|u_0\|_{L^q(B_R)} \leq \delta_1, \tag{7.1}$$

for $\delta_1 > 0$ sufficiently small. Let $S(t)$ be as in (4.5), *then*

$$T := \sup\{t > 0 : S(t) \leq 1\} > 1. \tag{7.2}$$

Proof. By (4.5) and (7.2) one has

$$1 = S(T) = \sup_{0<t<T} t\|u(t)\|_{L^\infty(B_R)}^{\sigma-1}. \tag{7.3}$$

By Lemma (4.3) applied with $r = q > \max\left\{\frac{N}{p}(\sigma - p + 1), 1\right\}$, (7.3) gives

$$
\begin{aligned}
1 = S(T) &\le \sup_{0<t<T} t\left\{k\, t^{-\frac{N}{N(p-2)+pq}}\left(\sup_{\frac{t}{4}<\tau<t} \int_{B_R} u^q\, d\mu\right)^{\frac{p}{N(p-2)+pq}}\right\}^{(\sigma-1)} \\
&\le \sup_{0<t<T} k\, t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}}\left(\sup_{\frac{t}{4}<\tau<t} \|u(\tau)\|_{L^q(B_R)}^{\frac{q\,p(\sigma-1)}{N(p-2)+pq}}\right).
\end{aligned}
\tag{7.4}
$$

By applying Proposition 3.6 to (7.4) and due to (7.1), we get

$$
\begin{aligned}
1 = S(T) &\le \sup_{0<t<T} k\, t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}}\|u_0\|_{L^q(B_R)}^{\frac{q\,p(\sigma-1)}{N(p-2)+pq}} \\
&\le k\, T^{1-\frac{N(\sigma-1)}{N(p-2)+pq}}\, \delta_1^{\frac{q\,p(\sigma-1)}{N(p-2)+pq}}.
\end{aligned}
$$

The thesis follows for $\delta_1 > 0$ small enough.

$\square$

Lemma 7.2. *Assume* (1.2)*, $p > 2$ and $s > \max\{\sigma_0, 1\}$. Let u be a solution to problem* (3.2) *with $u_0 \in L^\infty(B_R)$, $u_0 \ge 0$, such that*

$$\|u_0\|_{L^s(B_R)} \le \delta_1, \quad \|u_0\|_{L^{\sigma\frac{N}{p}}(B_R)} \le \delta_1, \tag{7.5}$$

for $\delta_1 > 0$ sufficiently small. Let $S(t)$ be as in (4.5)*, then*

$$T := \sup\{t \ge 0 : S(t) \le 1\} = +\infty. \tag{7.6}$$

Proof. We suppose by contradiction that

$$T < +\infty.$$

Then, by (7.6), the definition of $S(t)$ in (4.5) and by Lemma 7.1 we can write,

$$
\begin{aligned}
1 = S(T) &= \sup_{0<t<T} t\|u(t)\|_{L^\infty(B_R)}^{\sigma-1} \\
&\le \sup_{0<t<1} t\|u(t)\|_{L^\infty(B_R)}^{\sigma-1} + \sup_{1<t<T} t\|u(t)\|_{L^\infty(B_R)}^{\sigma-1} \\
&=: J_1 + J_2.
\end{aligned}
\tag{7.7}
$$

Now, by Lemma 4.3, applied with $r = s$, and Lemma 3.5 with $q = s$, we can write

$$
\begin{aligned}
J_1 &\le \sup_{0<t<1} t\left\{k\, t^{-\frac{N}{N(p-2)+ps}}\left(\sup_{\frac{t}{4}<\tau<t} \int_{B_R} u^s\, d\mu\right)^{\frac{p}{N(p-2)+ps}}\right\}^{(\sigma-1)} \\
&\le \sup_{0<t<1} k\, t^{1-\frac{N(\sigma-1)}{N(p-2)+ps}}\|u_0\|_{L^s(B_R)}^{\frac{ps(\sigma-1)}{N(p-2)+ps}}.
\end{aligned}
\tag{7.8}
$$

On the other hand, for any $q > s$, by Lemma 4.3, applied with $r = q$, and Proposition 3.6 with $q_0 = s$, we get

$$
\begin{aligned}
J_2 &\leq \sup_{1<t<T} t \left\{ k\, t^{-\frac{N}{N(p-2)+pq}} \left(\sup_{\frac{t}{4}<\tau<t} \int_{B_R} u^q \, d\mu \right)^{\frac{p}{N(p-2)+pq}} \right\}^{(\sigma-1)} \\
&\leq \sup_{1<t<T} k\, t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}} \sup_{\frac{t}{4}<\tau<t} \|u(\tau)\|_{L^q(B_R)}^{\frac{pq(\sigma-1)}{N(p-2)+pq}} \\
&\leq \sup_{1<t<T} k\, t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}} \sup_{\frac{t}{4}<\tau<t} \left(Ct^{-\frac{s}{p-2}\left(\frac{1}{s}-\frac{1}{q}\right)} \|u_0\|_{L^s(B_R)}^{\frac{s}{q}} \right)^{\frac{pq(\sigma-1)}{N(p-2)+pq}} \\
&\leq \sup_{1<t<T} \frac{C\,k}{4} t^{1-\frac{N(\sigma-1)}{N(p-2)+pq}-\frac{spq(\sigma-1)}{(p-2)[N(p-2)+pq]}\left(\frac{1}{s}-\frac{1}{q}\right)} \|u_0\|_{L^s(B_R)}^{\frac{ps(\sigma-1)}{N(p-2)+pq}} .
\end{aligned}
\tag{7.9}
$$

By substituting (7.8) and (7.9) into (7.7) we get

$$
1 = S(T) \leq \sup_{0<t<1} k\, t^a \|u_0\|_{L^s(B_R)}^{\frac{ps(\sigma-1)}{N(p-2)+ps}} + \sup_{1<t<T} \frac{C\,k}{4} t^b \|u_0\|_{L^s(B_R)}^{\frac{ps(\sigma-1)}{N(p-2)+pq}} ,
\tag{7.10}
$$

where we have set

$$
a = 1 - \frac{N(\sigma-1)}{N(p-2)+ps}, \quad \text{and} \quad b = 1 - \frac{N(\sigma-1)}{N(p-2)+pq} - \frac{spq(\sigma-1)}{(p-2)[N(p-2)+pq]}\left(\frac{1}{s}-\frac{1}{q}\right).
$$

Now, observe that, since $s > \max\left\{\frac{N}{p}(\sigma-p+1), 1\right\}$ and $q > s$,

$$
a > 0; \quad \text{and} \quad b < 0.
$$

Hence, (7.10), due to assumption (7.5), reads

$$
1 = S(T) < k\, \delta_1^{\frac{ps(\sigma-1)}{N(p-2)+ps}} + \frac{C\,k}{4} \delta_1^{\frac{ps(\sigma-1)}{N(p-2)+pq}} .
$$

Provided that δ_1 is sufficiently small, thus yielding $1 = S(T) < 1$, a contradiction. Thus $T = +\infty$. □

Proposition 7.3. *Assume* (1.2), *$p > 2$ and $s > \max\{\sigma_0, 1\}$. Let u be a solution to problem* (3.2) *with $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$, such that*

$$
\|u_0\|_{L^s(B_R)} \leq \varepsilon_1, \quad \|u_0\|_{L^{\sigma \frac{N}{p}}(B_R)} \leq \varepsilon_1,
$$

with $\varepsilon_1 = \varepsilon_1(\sigma, p, N, C_{s,p}, C_p, s)$ sufficiently small. Then, for any $t \in (0, +\infty)$, for some $\Gamma = \Gamma(\sigma, p, N, q, s, C_{s,p}, C_p) > 0$

$$
\|u(t)\|_{L^\infty(B_R)} \leq \Gamma\, t^{-\frac{1}{p-2}\left(1-\frac{ps}{N(p-2)+pq}\right)} \|u_0\|_{L^s(B_R)}^{\frac{ps}{N(p-2)+pq}} .
\tag{7.11}
$$

Proof. Due to Lemma 7.2,

$$
S(t) \leq 1 \quad \text{for all } t \in (0, +\infty].
$$

Therefore, by Lemma 4.3 and Proposition 3.6 applied with $q_0 = s$, for any $q > s$, we get, for all $t \in (0, +\infty)$

$$\|u(t)\|_{L^\infty(B_R)} \le \|u\|_{L^\infty\left(B_R \times \left(\frac{t}{2}, t\right)\right)}$$

$$\le k\, t^{-\frac{N}{N(p-2)+pq}} \left[\sup_{\frac{t}{4} < \tau < t} \|u(\tau)\|^q_{L^q(B_R)}\right]^{\frac{p}{N(p-2)+pq}}$$

$$\le \Gamma\, t^{-\frac{N}{N(p-2)+pq} - \frac{s}{p-2}\left(\frac{1}{s} - \frac{1}{q}\right)\frac{pq}{N(p-2)+pq}} \|u_0\|^{\frac{s}{q}\frac{pq}{N(p-2)+pq}}_{L^s(B_R)}.$$

Observing that

$$-\frac{N}{N(p-2)+pq} - \frac{s}{p-2}\left(\frac{1}{s} - \frac{1}{q}\right)\frac{pq}{N(p-2)+pq} = -\frac{1}{p-2}\left(1 - \frac{ps}{N(p-2)+pq}\right),$$

we get the thesis. □

Proof of Theorem 2.7. We proceed as in the proof of the previous Theorems. Let $\{u_{0,h}\}_{h\ge 0}$ be a sequence of functions such that

(a) $u_{0,h} \in L^\infty(M) \cap C^\infty_c(M)$ for all $h \ge 0$,

(b) $u_{0,h} \ge 0$ for all $h \ge 0$,

(c) $u_{0,h_1} \le u_{0,h_2}$ for any $h_1 < h_2$, (7.12)

(d) $u_{0,h} \longrightarrow u_0$ in $L^s(M)$ as $h \to +\infty$.

From standard results it follows that problem (5.2) has a solution $u^R_{h,k}$ in the sense of Definition 3.1 with $u_{0,h}$ as in (7.12); moreover, $u^R_{h,k} \in C([0,\infty); L^q(B_R))$ for any $q > 1$. Due to Proposition 7.3, 3.6 and Lemmata 3.5 and 7.2, the solution $u^R_{h,k}$ to problem (5.2) satisfies estimates (3.31), (3.42) and (7.11) for any $t \in (0, +\infty)$, uniformly w.r.t. R, k and h. Thus, by standard arguments, we can pass to the limit as $R \to +\infty$, $k \to +\infty$ and $h \to +\infty$ and we obtain a solution u to problem (1.1), which fulfills (2.12), (2.13) and (2.14).

□

8. Porous medium equation with reaction

We now consider the following nonlinear reaction-diffusion problem:

$$\begin{cases} u_t = \Delta u^m + u^\sigma & \text{in } M \times (0, T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{8.1}$$

where M is an $N-$dimensional complete noncompact Riemannian manifold of infinite volume, Δ being the Laplace-Beltrami operator on M and $T \in (0, \infty]$. We shall assume throughout this section that

$$N \ge 3, \qquad m > 1, \qquad \sigma > m,$$

so that we are concerned with the case of *degenerate diffusions* of porous medium type (see [37]), and that the initial datum u_0 is nonnegative. Let $L^q(M)$ be the space of those measurable functions f such

that $|f|^q$ is integrable w.r.t. the Riemannian measure μ. We shall always assume that M supports the Sobolev inequality, namely that:

$$\text{(Sobolev inequality)} \quad \|v\|_{L^{2^*}(M)} \leq \frac{1}{C_s}\|\nabla v\|_{L^2(M)} \quad \text{for any} \;\; v \in C_c^\infty(M), \tag{8.2}$$

where C_s is a positive constant and $2^* := \frac{2N}{N-2}$. In one of our main results, we shall also suppose that M supports the Poincaré inequality, namely that:

$$\text{(Poincaré inequality)} \quad \|v\|_{L^2(M)} \leq \frac{1}{C_p}\|\nabla v\|_{L^2(M)} \quad \text{for any} \;\; v \in C_c^\infty(M), \tag{8.3}$$

for some $C_p > 0$.

Solutions to (8.1) will be meant in the very weak, or distributional, sense, according to the following definition.

Definition 8.1. *Let M be a complete noncompact Riemannian manifold of infinite volume, of dimension $N \geq 3$. Let $m > 1$, $\sigma > m$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$. We say that the function u is a solution to problem* (8.1) *in the time interval $[0, T)$ if*

$$u \in L^\sigma_{loc}(M \times (0, T))$$

and for any $\varphi \in C_c^\infty(M \times [0, T])$ such that $\varphi(x, T) = 0$ for any $x \in M$, u satisfies the equality:

$$-\int_0^T \int_M u\,\varphi_t\,d\mu\,dt = \int_0^T \int_M u^m\,\Delta\varphi\,d\mu\,dt + \int_0^T \int_M u^\sigma\,\varphi\,d\mu\,dt$$
$$+ \int_M u_0(x)\,\varphi(x, 0)\,d\mu.$$

First we consider the case that $\sigma > m + \frac{2}{N}$ and the Sobolev inequality holds on M. In order to state our results we define

$$\sigma_1 := (\sigma - m)\frac{N}{2}. \tag{8.4}$$

Observe that $\sigma_1 > 1$ whenever $\sigma > m + \frac{2}{N}$. We comment that the next results improve and in part correct some of the results of [17]. The proofs are omitted since they are identical to the previous ones.

Theorem 8.2. *Let M be a complete, noncompact, Riemannian manifold of infinite volume and of dimension $N \geq 3$, such that the Sobolev inequality* (8.2) *holds. Let $m > 1$, $\sigma > m + \frac{2}{N}$, $s > \sigma_1$ and $u_0 \in L^s(M) \cap L^1(M)$, $u_0 \geq 0$.*

(i) Assume that

$$\|u_0\|_{L^s(M)} < \varepsilon_0, \quad \|u_0\|_{L^1(M)} < \varepsilon_0,$$

with $\varepsilon_0 = \varepsilon_0(\sigma, m, N, C_s) > 0$ sufficiently small. Then problem (8.1) *admits a solution for any $T > 0$, in the sense of Definition 8.1. Moreover, for any $\tau > 0$, one has $u \in L^\infty(M \times (\tau, +\infty))$ and there exists a constant $\Gamma > 0$ such that, one has*

$$\|u(t)\|_{L^\infty(M)} \leq \Gamma\,t^{-\alpha}\,\|u_0\|_{L^1(M)}^{\frac{2}{N(m-1)+2}} \quad \text{for all } t > 0,$$

where

$$\alpha := \frac{N}{N(m-1) + 2}\,.$$

(ii) Let $\sigma_1 \le q < \infty$ and

$$\|u_0\|_{L^{\sigma_1}(M)} < \hat{\varepsilon}_0$$

for $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(\sigma, m, N, C_s, q) > 0$ small enough. Then there exists a constant $C = C(m, \sigma, N, \varepsilon_0, C_s, q) > 0$ such that

$$\|u(t)\|_{L^q(M)} \le C\, t^{-\gamma_q} \|u_0\|_{L^{\sigma_1}(M)}^{\delta_q} \quad \text{for all } t > 0,$$

where

$$\gamma_q = \frac{1}{\sigma - 1}\left[1 - \frac{N(\sigma - m)}{2q}\right], \quad \delta_q = \frac{\sigma - m}{\sigma - 1}\left[1 + \frac{N(m-1)}{2q}\right].$$

(iii) Finally, for any $1 < q < \infty$, if $u_0 \in L^q(M) \cap L^{\sigma_1}(M)$ and

$$\|u_0\|_{L^{\sigma_1}(M)} < \varepsilon$$

with $\varepsilon = \varepsilon(\sigma, m, N, r, C_s, q) > 0$ sufficiently small, then

$$\|u(t)\|_{L^q(M)} \le \|u_0\|_{L^q(M)} \quad \text{for all } t > 0.$$

Theorem 8.3. *Let M be a complete, noncompact manifold of infinite volume and of dimension $N \ge 3$, such that the Sobolev inequality (8.2) holds. Let $m > 1$, $\sigma > m + \frac{2}{N}$ and $u_0 \in L^{\sigma_1}(M)$, $u_0 \ge 0$ where σ_1 has been defined in (8.4). Assume that*

$$\|u_0\|_{L^{\sigma_1}(M)} < \varepsilon_0$$

with $\varepsilon_0 = \varepsilon_0(\sigma, m, N, r, C_s) > 0$ sufficiently small. Then problem (8.1) admits a solution for any $T > 0$, in the sense of Definition 8.1. Moreover, for any $\tau > 0$, one has $u \in L^\infty(M \times (\tau, +\infty))$ and there exists a constant $\Gamma > 0$ such that, one has

$$\|u(t)\|_{L^\infty(M)} \le \Gamma\, t^{-\frac{1}{\sigma-1}} \|u_0\|_{L^{\sigma_1}(M)}^{\frac{\sigma-m}{\sigma-1}} \quad \text{for all } t > 0.$$

Moreover, the statements in (ii) and (iii) of Theorem 8.2 hold.

In the next theorem, we address the case that $\sigma > m$, supposing that both the inequalities (8.2) and (8.3) hold on M.

Theorem 8.4. *Let M be a complete, noncompact manifold of infinite volume and of dimension $N \ge 3$, such that the Sobolev inequality (8.2) and the Poincaré inequality (8.3) hold. Let*

$$m > 1, \quad \sigma > m,$$

and $u_0 \in L^s(M) \cap L^{\sigma\frac{N}{2}}(M)$ where $s > \max\{1, \sigma_1\}$, $u_0 \ge 0$. Assume that

$$\|u_0\|_{L^s(M)} < \varepsilon_1, \quad \|u_0\|_{L^{\sigma\frac{N}{2}}(M)} < \varepsilon_1,$$

holds with $\varepsilon_1 = \varepsilon_1(m, \sigma, N, r, C_p, C_s) > 0$ sufficiently small. Then problem (8.1) admits a solution for any $T > 0$, in the sense of Definition 8.1. Moreover for any $\tau > 0$ and for any $q > s$ one has $u \in L^\infty(M \times (\tau, +\infty))$ and for all $t > 0$ one has

$$\|u(t)\|_{L^\infty(B_R)} \le \Gamma\, t^{-\beta_{q,s}} \|u_0\|_{L^s(B_R)}^{\frac{2s}{N(m-1)+2q}},$$

where

$$\beta_{q,s} := \frac{1}{m-1}\left(1 - \frac{2s}{N(m-1)+2q}\right) > 0.$$

Moreover, let $s \le q < \infty$ and

$$\|u_0\|_{L^s(M)} < \hat{\varepsilon}_1,$$

for some $\hat{\varepsilon}_1 = \hat{\varepsilon}_1(\sigma, m, N, r, C_p, C_s, q, s) > 0$ sufficiently small. Then there exists a constant $C = C(\sigma, m, N, \varepsilon_1, C_s, C_p, q, s) > 0$ such that

$$\|u(t)\|_{L^q(M)} \le C t^{-\gamma_q} \|u_0\|_{L^s(M)}^{\delta_q} \quad \text{for all } t > 0,$$

where

$$\gamma_q := \frac{s}{m-1}\left[\frac{1}{s} - \frac{1}{q}\right], \qquad \delta_q := \frac{s}{q}.$$

Finally, for any $1 < q < \infty$, if $u_0 \in L^q(M) \cap L^s(M) \cap L^{\sigma\frac{N}{2}}(M)$ and

$$\|u_0\|_{L^s(M)} < \varepsilon,$$

for some $\varepsilon = \varepsilon(\sigma, m, N, C_p, C_s, q) > 0$ sufficiently small, then

$$\|u(t)\|_{L^q(M)} \le \|u_0\|_{L^q(M)} \quad \text{for all } t > 0.$$

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Conflict of interest

The authors declare no conflict of interest.

References

1. E. Acerbi, G. Mingione, Gradient estimates for a class of parabolic systems, *Duke Math. J.*, **136** (2007), 285–320. https://doi.org/10.1215/S0012-7094-07-13623-8

2. N. D. Alikakos, L^p bounds of solutions of reaction-diffusion equations, *Commun. Part. Diff. Eq.*, **4** (1979), 827–868. https://doi.org/10.1080/03605307908820113

3. C. Bandle, M. A. Pozio, A. Tesei, The Fujita exponent for the Cauchy problem in the hyperbolic space, *J. Differ. Equations*, **251** (2011), 2143–2163. https://doi.org/10.1016/j.jde.2011.06.001

4. P. Bénilan, *Opérateurs accrétifs et semi-groupes dans les espaces L^p ($1 \le p \le +\infty$)*, Japan-France seminar, Japan Society for the Advancement of Science, 1978.

5. P. Bénilan, M. G. Crandall, Completely accretive operators, semigroup theory and evolution equations (Delft, 1989), In: *Lecture Notes in Pure and Applied Mathematics, Volume 135*, Dekker, 1991, 41–75.

6. V. Bögelein, F. Duzaar, G. Mingione, *The regularity of general parabolic systems with degenerate diffusion*, Memoirs of the American Mathematical Society, 2013. https://doi.org/10.1090/S0065-9266-2012-00664-2

7. X. Chen, M. Fila, J. S. Guo, Boundedness of global solutions of a supercritical parabolic equation, *Nonlinear Anal.*, **68** (2008), 621–628. https://doi.org/10.1016/j.na.2006.11.023

8. T. Coulhon, D. Hauer, *Regularisation effects of nonlinear semigroups*, SMAI - Mathématiques et Applications, Springer, to appear.

9. K. Deng, H. A. Levine, The role of critical exponents in blow-up theorems: the sequel, *J. Math. Anal. Appl.*, **243** (2000), 85–126. https://doi.org/10.1006/jmaa.1999.6663

10. H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. I*, **13** (1966), 109–124.

11. Y. Fujishima, K. Ishige, Blow-up set for type I blowing up solutions for a semilinear heat equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31** (2014), 231–247. https://doi.org/10.1016/j.anihpc.2013.03.001

12. V. A. Galaktionov, The conditions for there to be no global solutions of a class of quasilinear parabolic equations, *USSR Computational Mathematics and Mathematical Physics*, **22** (1982), 73–90. https://doi.org/10.1016/0041-5553(82)90037-4

13. V. A. Galaktionov, Blow-up for quasilinear heat equations with critical Fujita's exponents, *Proc. Roy. Soc. Edinb. A*, **124** (1994), 517–525. https://doi.org/10.1017/S0308210500028766

14. V. A. Galaktionov, H. A. Levine, A general approach to critical Fujita exponents in nonlinear parabolic problems, *Nonlinear Anal.*, **34** (1998), 1005–1027. https://doi.org/10.1016/S0362-546X(97)00716-5

15. A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.*, **36** (1999), 135–249. https://doi.org/10.1090/s0273-0979-99-00776-4

16. A. Grigor'yan, *Heat kernel and analysis on manifolds*, Providence, RI: American Mathematical Society, 2009.

17. G. Grillo, G. Meglioli, F. Punzo, Global existence of solutions and smoothing effects for classes of reaction-diffusion equations on manifolds, *J. Evol. Equ.*, **21** (2021), 2339–2375. https://doi.org/10.1007/s00028-021-00685-3

18. G. Grillo, G. Meglioli, F. Punzo, Smoothing effects and infinite time blowup for reaction-diffusion equations: an approach via Sobolev and Poincaré inequalities, *J. Math. Pure. Appl.*, **151** (2021), 99–131. https://doi.org/10.1016/j.matpur.2021.04.011

19. G. Grillo, G. Meglioli, F. Punzo, Blow-up versus global existence of solutions for reaction-diffusion equations on classes of Riemannian manifolds, *Annali di Matematica Pura e Applicata*, in press. https://doi.org/10.1007/s10231-022-01279-7

20. Q. Gu, Y. Sun, J. Xiao, F. Xu, Global positive solution to a semi-linear parabolic equation with potential on Riemannian manifold, *Calc. Var.*, **59** (2020), 170. https://doi.org/10.1007/s00526-020-01837-y

21. D. Hauer, Regularizing effect of homogeneous evolution equations with perturbation, *Nonlinear Anal.*, **206** (2021), 112245. https://doi.org/10.1016/j.na.2021.112245

22. K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic differential equations, *Proc. Japan Acad.*, **49** (1973), 503–505. https://doi.org/10.3792/pja/1195519254

23. K. Kobayashi, T. Sirao, H. Tanaka, On the growing up problem for semilinear heat equations, *J. Math. Soc. Japan*, **29** (1977), 407–424. https://doi.org/10.2969/jmsj/02930407

24. T. Kuusi, G. Mingione, Gradient regularity for nonlinear parabolic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **12** (2013), 755–822. https://doi.org/10.2422/2036-2145.201103_006

25. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva, *Linear and quasilinear equations of parabolic type*, Providence, RI: American Mathematical Society, 1968.

26. H. A. Levine, The role of critical exponents in blow-up theorems, *SIAM Rev.*, **32** (1990), 262–288. https://doi.org/10.1137/1032046

27. A. V. Martynenko, A. F. Tedeev, On the behavior of solutions of the Cauchy problem for a degenerate parabolic equation with nonhomogeneous density and a source, *Comput. Math. Math. Phys.*, **48** (2008), 1145–1160. https://doi.org/10.1134/S0965542508070087

28. P. Mastrolia, D. D. Monticelli, F. Punzo, Nonexistence of solutions to parabolic differential inequalities with a potential on Riemannian manifolds, *Math. Ann.*, **367** (2017), 929–963. https://doi.org/10.1007/s00208-016-1393-2

29. G. Meglioli, D. D. Monticelli, F. Punzo, Nonexistence of solutions to quasilinear parabolic equations with a potential in bounded domains, *Calc. Var.*, **61** (2022), 23. https://doi.org/10.1007/s00526-021-02132-0

30. E. Mitidieri, S. I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Proc. Steklov Inst. Math.*, **234** (2001), 1–362.

31. E. Mitidieri, S. I. Pohozaev, Towards a unified approach to nonexistence of solutions for a class of differential inequalities, *Milan J. Math.*, **72** (2004), 129–162. https://doi.org/10.1007/s00032-004-0032-7

32. S. I. Pohozaev, A. Tesei, Nonexistence of local solutions to semilinear partial differential inequalities, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **21** (2004), 487–502. https://doi.org/10.1016/j.anihpc.2003.06.002

33. F. Punzo, A. Tesei, On a semilinear parabolic equation with inverse-square potential, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.*, **21** (2010), 359–396. https://doi.org/10.4171/RLM/578

34. F. Punzo, Blow-up of solutions to semilinear parabolic equations on Riemannian manifolds with negative sectional curvature, *J. Math. Anal. Appl.*, **387** (2012), 815–827. https://doi.org/10.1016/j.jmaa.2011.09.043

35. P. Quittner, The decay of global solutions of a semilinear heat equation, *Discrete Contin. Dyn. Syst.*, **21** (2008), 307–318. https://doi.org/10.3934/dcds.2008.21.307

36. P. Souplet, Morrey spaces and classification of global solutions for a supercritical semilinear heat equation in $\mathbb{R}^N$, *J. Funct. Anal.*, **272** (2017), 2005–2037. https://doi.org/10.1016/j.jfa.2016.09.002

37. J. L. Vázquez, *The porous medium equation: mathematical theory*, Oxford: Oxford University Press, 2007. https://doi.org/10.1093/acprof:oso/9780198569039.001.0001

38. L. Véron, Effets régularisants de semi-groupes non linéaires dans des espaces de Banach, *Ann. Fac. Sci. Toulouse Math. (5)*, **1** (1979), 171–200.

39. Z. Wang, J. Yin, A note on semilinear heat equation in hyperbolic space, *J. Differ. Equations*, **256** (2014), 1151–1156. https://doi.org/10.1016/j.jde.2013.10.011

40. Z. Wang, J. Yin, Asymptotic behaviour of the lifespan of solutions for a semilinear heat equation in hyperbolic space, *Proc. Roy. Soc. Edinb. A*, **146** (2016), 1091–1114. https://doi.org/10.1017/S0308210515000785

41. F. B. Weissler, L^p-energy and blow-up for a semilinear heat equation, *Proc. Sympos. Pure Math.*, **45** (1986), 545–551.

42. E. Yanagida, Behavior of global solutions of the Fujita equation, *Sugaku Expositions*, **26** (2013), 129–147.

43. Q. S. Zhang, Blow-up results for nonlinear parabolic equations on manifolds, *Duke Math. J.*, **97** (1999), 515–539. https://doi.org/10.1215/S0012-7094-99-09719-3