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Symmetries of slice monogenic functions

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Abstract. In this paper we consider the symmetry behavior of slice monogenic functions under Möbius transformations. We describe the group under which slice monogenic functions are taken into slice monogenic functions. We prove a transformation formula for composing slice monogenic functions with Möbius transformations and describe their conformal invariance. Finally, we explain two construction methods to obtain automorphic forms in the framework of this function class. We round off by presenting a precise algebraic characterization of the subset of slice monogenic linear fractional transformations within the set of general Möbius transformations.

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1. Introduction

The study of null solutions of the Dirac operator and of the Cauchy–Fueter operator is a very well established research field, see [5, 16, 19, 28, 29]. Such classes of functions are called monogenic functions. The more recent class of slice hyperholomorphic functions, see [12, 13, 24], has become a different theory which has several applications to operator theory, in particular to spectral theory for several operators as well as to quaternionic operators. See for instance [3, 13] and the original papers [8, 14].

In this paper we deal with slice monogenic functions, namely with slice hyperholomorphic functions with values in a Clifford algebra, which were introduced in [15] after the quaternionic case, see [25]. In this latter context, slice hyperholomorphic functions are called slice regular. For more details see the books [13, 24].

Both monogenic and slice monogenic function theories are natural generalizations of classical complex function theory but they are quite different from each other. Monogenic functions are meant as smooth functions in the kernel of the Dirac operator. Possible relations between the two theories were developed in the context of the Fueter–Sce mapping theorem, see [23, 38, 41], which allows to construct monogenic functions starting from holomorphic functions and more in general from slice monogenic functions, while its inverse map [9, 10] generates slice monogenic functions from axially monogenic functions.

Another link between slice monogenic functions and monogenic functions can be obtained using the Radon and dual Radon transforms, see [7]. The Radon transform for Clifford algebras valued functions maps monogenic functions to slice monogenic functions and, analogously, the dual Radon transform maps slice monogenic functions to monogenic functions. These important relations between the two function theories are relevant in the study of the Dirac operator.

An important property of the Dirac operator that plays a crucial role in several aspects of Noncommutative Geometry, in particular Riemannian geometry, is its invariance property under the full group of Möbius transformations (which is the Ahlfors–Vahlen group). This gives rise to monogenic spinor bundles over Riemannian conformally flat spin manifolds equipped with powerful function theoretic tools, see for instance [33]. The invariance of the Dirac operator under this group is reflected in a covariance property of the related Cauchy integral formula playing a fundamental role in Riemanian geometry, see also [6] which is dedicated more to the geometric aspects and spectral theory. In this paper we study the invariance properties of the class of slice monogenic functions under the group of Möbius transformations.

In particular, we precisely work out which kinds of Möbius transformations preserve axial symmetry. Moreover, we establish a quasi invariance property (up to an automorphic factor) for slice monogenic functions. The latter describes the conformal invariance of this class of functions. As an application, we prove a Borel–Pompeiu type formula which is conformally covariant if the integral is computed over an axially symmetric domain. Then we briefly look at Hardy spaces of slice monogenic functions. In contrast to the monogenic case we however meet substantial obstacles that do not allow us to establish isometry relations between Hardy spaces of slice monogenic functions that are connected by Möbius transformations. These obstacles are described.

After that, we turn to the construction of axially symmetric slice monogenic automorphic forms. Automorphic forms do not only play a crucial role in number theory, but also in spectral theory related to harmonic operators, see for instance [40]. Since about 30 years one is particularly interested in Maaß forms related arithmetic subgroups of the Ahlfors–Vahlen group which are eigenfunctions of the higher dimensional Laplace–Beltrami operator, see [21].

For the slice-monogenic setting we propose two different construction methods. The first way to obtain such function classes is offered by an Eisenstein–Poincaré series construction involving the slice monogenic automorphic weight factor used in the preceding section. In this construction we apply a slice monogenic generalization of the exponential function that guarantees the right convergence behavior. The slice monogenicity behavior of the whole series in turn is assured by the quasi-invariance property of slice monogenic functions that we proved in an earlier section.

Secondly, we can alternatively construct slice monogenic automorphic forms by applying Fueter's extension theorem to the classical holomorphic modular forms, for instance on the classical holomorphic Eisenstein series. Both construction methods are compared with each other.

Finally, we introduce the subset of slice monogenic linear fractional transformations; these transformations were discussed in the quaternionic case in [39] and they are called slice regular Möbius transformations. Our aim is to explain how exactly these slice monogenic linear fractional transformations can be characterized within the set of all Möbius transformations. As we shall discuss, in this framework the quaternionic case is special.

2. Preliminaries

In this section we introduce some preliminary results on Möbius transformations in $\mathbb{R}^{n+1}$ and the related analyticity concepts, within classes of Clifford algebra valued monogenic and slice monogenic functions.

2.1. Basics on Clifford algebras and notations. Throughout this paper let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of the Euclidean vector space $\mathbb{R}^n$. Further, we assume that the basis elements satisfy the relations $e_i e_j + e_j e_i = -2\delta_{ij}$, where δ_{ij} stands for the usual Kronecker symbol. The Clifford algebra $\mathbb{R}_n$ is the associative algebra generated by $\{e_1, e_2, \ldots, e_n\}$ over $\mathbb{R}$. A basis for the real Clifford algebra $\mathbb{R}_n$, considered as a vector space, is given by the element $e_0 = 1$, the canonical basis elements $e_1, e_2, \ldots, e_n$, as well as all their possible products

$$e_1e_2, \ldots, e_1e_n, \ldots, e_{n-1}e_n, \ldots e_1e_2 \cdots e_n.$$

In compact form, this is denoted by the set

$$\{e_A \mid A \subseteq \{1, \ldots, n\}\},$$

where $e_\emptyset = e_0 = 1$. Thus an arbitrary element of $\mathbb{R}_n$ has the form

$$a = \sum_{A \subseteq \{1,\ldots,n\}} a_A e_A$$

with real components a_A. Here we have set $e_A := e_{l_1} \cdots e_{l_r}$ where the integers $l_1, \ldots, l_r$ satisfy $1 \le l_1 < \cdots < l_r \le n$.

Next we introduce the Clifford conjugation by

$$\overline{a} := \sum_A a_A \overline{e_A},$$

where

$$\overline{e_A} = \overline{e_{l_r}} \cdots \overline{e_{l_1}}, \quad \overline{e_j} = -e_j, \ \ j = 1, \dots, n, \quad \overline{e_0} = e_0 = 1.$$

We also need the Clifford reversion defined by

$$\widetilde{a} := \sum_A a_A \widetilde{e_A},$$

where

$$\widetilde{e_A} = e_{l_r} \cdots e_{l_1}, \quad \widetilde{e_j} = e_j, \ \ j = 1, \dots, n, \quad \widetilde{e_0} = e_0 = 1.$$

We also have

$$\widetilde{a} = \sum_A (-1)^{|A|(|A|-1)/2} a_A e_A.$$

Furthermore, we consider the main involution defined by

$$e_A{}' = e_{l_1}{}' \cdots e_{l_r}{}', \quad e_j{}' = -e_j, \ \ j = 1, \dots, n, \quad e_0{}' = e_0 = 1.$$

One has the relation $\overline{a} = \widetilde{a'} = \widetilde{a}'$.

An important subspace of Clifford numbers is the set of so-called space of paravectors consisting of Clifford numbers of the form $a = a_0 + a_1 e_1 + \cdots + a_n e_n$ having a scalar and a vector part only. For simplicity this set will be identified with $\mathbb{R}^{n+1}$ via the map

$$(a_0, a_1, \dots, a_n) \mapsto a_0 + a_1 e_1 + \cdots + a_n e_n.$$

An element of the form $\underline{a} = a_1 e_1 + \cdots + a_n e_n$ is called 1-vector and the set of 1-vectors can be identified with $\mathbb{R}^n$ via the map

$$(a_1, \dots, a_n) \mapsto a_1 e_1 + \cdots + a_n e_n.$$

Note that sometimes we will write $\mathbb{R} \oplus \mathbb{R}^n$ instead of $\mathbb{R}^{n+1}$. A paravector x will be written in the form $x = x_0 + \underline{x}$. We also use the set

$$\mathbb{S}^{n-1} = \{a_1 e_1 + \cdots + a_n e_n : a_1^2 + \cdots + a_n^2 = 1\}$$

which can be identified with a sphere in $\mathbb{R}^n$ and whose elements have square -1.

The norm $\|a\|$ of paravector a is $\|a\| = \left(\sum_{i=0}^n a_i^2\right)^{1/2}$ namely the usual Euclidean norm. This norm can be extended to a pseudo-norm on the whole Clifford algebra by setting $\|a\| := \sqrt{\sum_A |a_A|^2}$.

Each non-zero paravector is invertible with inverse $a^{-1} = \frac{\overline{a}}{\|a\|^2}$.

2.2. Möbius transformations in $\mathbb{R}^{n+1}$. As is broadly well known, in dimension $n \geq 3$ the set of conformal maps coincides with that of Möbius transformations. Using Clifford algebras, Möbius transformations can be written very elegantly in terms of the action of 2×2 Clifford algebra valued matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ whose coefficients satisfy special conditions which will be listed below. For convenience, we introduce below the definition of the general Ahlfors–Vahlen group which will be denoted by $GAV(\mathbb{R} \oplus \mathbb{R}^n)$:

Definition 2.1. The group $GAV(\mathbb{R} \oplus \mathbb{R}^n)$ is the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

equipped with the product of matrices, whose coefficients $a, b, c, d \in \mathbb{R}_n$ satisfy the so-called Ahlfors–Vahlen conditions:

(i) a, b, c, d are products of paravectors from $\mathbb{R}^{n+1}$ (including 0);

(ii) $a\widetilde{d} - b\widetilde{c} \in \mathbb{R} \setminus \{0\}$;

(iii) $a\widetilde{b}, c\widetilde{d} \in \mathbb{R}^{n+1}$.

Remark 2.2. As explained in [2, p. 220] the conditions (ii) and (iii) also imply that $d\widetilde{b}, c\widetilde{a} \in \mathbb{R}^{n+1}$.

Remark 2.3. The Ahlfors–Vahlen conditions arise naturally from the group axiomatic requirement that every element in the group must also have an inverse element in the group and that this inverse is uniquely defined. The uniqueness of the inverse also implies that the left and right inverse coincides with each other. From the property

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \widetilde{d} & -\widetilde{b} \\ -\widetilde{c} & \widetilde{a} \end{pmatrix} = \begin{pmatrix} \widetilde{d} & -\widetilde{b} \\ -\widetilde{c} & \widetilde{a} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \det(M) & 0 \\ 0 & \det(M) \end{pmatrix}$$

where $\det(M) := a\widetilde{d} - b\widetilde{c}$, it follows that the elements on the anti-diagonal must all vanish. This fact implies (iii). For the invertibility we need that the expression $\det(M)$, called the pseudo-determinant of the Ahlfors–Vahlen matrix, does not vanish. In fact $\det(M) \in \mathbb{R}\setminus\{0\}$, otherwise we would not get the Dieudonné property that $\det(AB) = \det(A)\det(B)$. In fact this is a general requirement also claimed in [18, 31] and elsewhere.

Following for example [20], Möbius transformations are defined as action of this matrix group on $\mathbb{R}^{n+1}$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x\right) \mapsto M\langle x\rangle = (ax+b)(cx+d)^{-1} \in \mathbb{R}^{n+1}.$$

In the case where a, b, c, d are products of vectors from $\mathbb{R}^n$ the associated group $GAV(\mathbb{R}^n)$ acts transitively on right half-space $x_0 > 0$, or, respectively, the group $GAV(\mathbb{R} \oplus \mathbb{R}^{n-1})$ acts transitively on upper half-space $x_n > 0$. The Ahlfors–Vahlen conditions ensure that the Möbius transformations are maps from $\mathbb{R}^{n+1}$ to itself or from upper-half space to upper half-space, respectively.

Again, following [20] the whole group $GAV(\mathbb{R} \oplus \mathbb{R}^n)$ can be generated by four different types of matrices:

(i) Translations:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

where $b \in \mathbb{R}^{n+1}$, induce Möbius transformations of the form $M\langle x\rangle = x + b$.

(ii) Inversion (reflection at the unit sphere):

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

representing the Möbius transformation $M\langle x\rangle = -x^{-1}$.

(iii) Modified rotations:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

where $a \in \mathbb{S}^{n-1}$, induce $M\langle x\rangle = axa$.

(iv) Dilations:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$, induce $M\langle x\rangle = \lambda^2 x$.

Note that condition (iii) can be also stated for $a \in \mathbb{R}^{n+1}$. Alternatively one can consider in (iv) matrices of the form $\left(\begin{smallmatrix} \lambda^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\lambda \in \mathbb{R}\backslash\{0\}$.

Remark 2.4. The group $GAV(\mathbb{R}^{n+1})$ is exactly the group that leaves the Dirac operator invariant. In other words, the Dirac (or generalized Cauchy Riemann operator) is invariant under the full Ahlfors–Vahlen group coinciding with the conformal group.

Remark 2.5. Geometrically, a rotation in $\mathbb{R}^{n+1}$ has the form $x \mapsto ax\widetilde{a}^{-1}$ where a may be a product of paravectors of unit length from $\mathbb{R}^{n+1}$ so, in the simplest case, a is a single paravector. In the particular case where a is a 1-vector in the unit sphere $\mathbb{S}^{n-1}$, the transformation simplifies to $x \mapsto -axa$. However, as we directly see in that simplified case, the determinant expression in the sense of [21] of such a rotation, i.e., $a\widetilde{d} - b\widetilde{c}$, is negative. In fact, in the case of vectors in the unit sphere it precisely equals -1. Since we will focus on subgroups consisting of matrices having positive determinant, equal to 1, we prefer to use the generator in the form listed above, because $x \mapsto ax\widetilde{a}$ is then associated to a matrix of determinant equal to $+1$. This transformation is not really a rotation in the classical sense. Therefore, we refer to it as *modified rotation*.

Remark 2.6. The orientation preserving transformations are represented by the subgroup

$$SAV(\mathbb{R}\oplus\mathbb{R}^n) := \{M \in GAV(\mathbb{R}\oplus\mathbb{R}^n) \mid \det(M) = 1\},$$

where $\det(M) := a\tilde{d} - b\tilde{c}$ is the determinant of a Clifford algebra valued matrix. More precisely, the group $SAV(\mathbb{R}\oplus\mathbb{R}^n)$ is a normal subgroup in $GAV(\mathbb{R}\oplus\mathbb{R}^n)$ of index 2 generated only by the basic types of matrices of type (i), (ii) and (iii) above.

2.3. Two classes of hypercomplex functions. In this subsection we briefly recall two different basic concepts that generalize holomorphic function theory to higher dimensional real vector spaces. Concretely speaking, we look at Clifford algebra valued monogenic functions and at Clifford algebra valued slice monogenic functions; the latter function class is the focus of this paper. We briefly explain the connections between these two function classes as well as some of their important properties concerning this paper. In particular, we recall Fueter's theorem that provides us with a key link between them. We start by introducing the set of monogenic functions:

Monogenic functions. Let $U \subseteq \mathbb{R}\oplus\mathbb{R}^n$ be an open set. Then a real differentiable function $f: U \to \mathbb{R}_n$ that satisfies $Df = 0$ (respectively $fD = 0$), where

$$D := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}e_1 + \cdots + \frac{\partial}{\partial x_n}e_n$$

is the generalized Cauchy–Riemann operator (or Dirac operator), is called left monogenic (respectively right monogenic), cf. for instance [5]. Due to the non-commutativity of $\mathbb{R}_n$ for $n > 1$, both classes of functions do not coincide with each other. However, f is left monogenic if and only if $\tilde{f}$ is right monogenic. The generalized Cauchy–Riemann operator factorizes the Euclidean Laplacian

$$\Delta = \sum_{j=0}^{n} \frac{\partial^2}{\partial x_j^2},$$

viz $\overline{D}D = \Delta$. Every real component of a monogenic function hence is harmonic.

An important property of the D-operator is its quasi-invariance under Möbius transformations acting on the Euclidean space $\mathbb{R}\oplus\mathbb{R}^n$. The following result can be found e.g. in [36]:

Theorem 2.7. *Let $M \in GAV(\mathbb{R}\oplus\mathbb{R}^n)$ and let f be a left monogenic function in the variable $y = M\langle x\rangle = (ax+b)(cx+d)^{-1}$. Then the function*

$$g(x) := \frac{\overline{cx+d}}{\|cx+d\|^{n+1}} f(M\langle x\rangle)$$

is left monogenic in the variable x for any $M \in GAV(\mathbb{R}\oplus\mathbb{R}^n)$.

Slice monogenic functions. This class of functions is widely studied nowadays, see e.g. the book [13] and the references therein for more details. Originally they have been introduced as follows. Let $U \subseteq \mathbb{R} \oplus \mathbb{R}^n$ be a domain and let $f: U \to \mathbb{R}_n$ be a real differentiable function. Note that every nonreal paravector x can be written in the form $u + \omega v$ where $\omega \in \mathbb{S}^{n-1}$. Fix $\omega \in \mathbb{S}^{n-1}$ and let f_ω be the restriction of f to the complex plane $\mathbb{C}_\omega = \{u + \omega v, \mid u, v \in \mathbb{R}\}$. Assume that for every $\omega \in \mathbb{S}^{n-1}$

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + \omega \frac{\partial}{\partial v}\right) f_\omega(u + \omega v) = 0, \tag{1}$$

for $u + \omega v \in U$. We could consider this one as the definition of slice monogenic function, however it will be somewhat too general. In fact, in order to have a nice function theory one need to put restrictions on the open sets U that may be considered.

In fact, the natural sets U where slice monogenic functions are defined are *axially symmetric*, namely if $u + \omega_0 v \in U$ for a given $\omega_0 \in \mathbb{S}^{n-1}$ then $u + \omega v \in U$ for all $\omega \in \mathbb{S}^{n-1}$. Moreover, we will say that a domain U is a *slice domain*, if $U \cap \mathbb{C}_\omega$ is connected for all $\omega \in \mathbb{S}^{n-1}$. On axially symmetric slice domains a function is slice monogenic in the above sense if and only if it is of the form

$$f(x) = f(u + \omega v) = \alpha(u, v) + \omega\beta(u, v),$$

see [15].

Thus it makes sense to consider the following modified and more specific definition, see [26]:

Definition 2.8. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain, let $D \subseteq \mathbb{R}^2$ be such that $u + \omega v \in U$ whenever $(u, v) \in D$ and let $f: U \to \mathbb{R}_n$. The function f is slice monogenic if there exist two differentiable functions $\alpha, \beta: D \subseteq \mathbb{R}^2 \to \mathbb{R}_n$ satisfying $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = -\beta(u, -v)$ and the Cauchy–Riemann system

$$\begin{cases} \partial_u\alpha - \partial_v\beta = 0, \\ \partial_u\beta + \partial_v\alpha = 0, \end{cases} \tag{2}$$

such that

$$f(u + \omega v) = \alpha(u, v) + \omega\beta(u, v). \tag{3}$$

The class of slice monogenic functions defined on U will be denoted by $\mathcal{SM}(U)$.

More generally, let U be an axially symmetric open set. Furthermore, let $f: U \to \mathbb{R}_n$ be a function of the form

$$f(u + \omega v) = \alpha(u, v) + \omega\beta(u, v)$$

with $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = -\beta(u, -v)$. The function f is called a *slice function*. We say that a slice function belongs to the class $\mathcal{C}^k$ on U if α, β belong to the class $\mathcal{C}^k$ on D.

Slice functions and, in particular, slice monogenic functions satisfy the following:

Theorem 2.9 (Representation formula). *Let U be an axially symmetric set in $x, z \in U$, let f be a slice function and $x \in [z]$, i.e. $x = u + \omega v$, $z = u + iv$, $\omega, i \in \mathbb{S}^{n-1}$. Then*

$$f(x) = \frac{1 - \omega i}{2} f(z) + \frac{1 + \omega i}{2} f(\bar{z}).$$

The pointwise multiplication of two slice monogenic functions does not give a slice monogenic function in general. However, it is possible to define a suitable product, called $*$-product, which is an inner operation in the set of slice monogenic functions, see [13, 26]. It is defined as follows: given an axially symmetric set $U \subseteq \mathbb{R}^{n+1}$ and $f, g \in \mathcal{SM}(U)$ with

$$f(x) = f(u+\omega v) = \alpha(u, v)+\omega\beta(u, v), \quad g(x) = g(u+\omega v) = \gamma(u, v)+\omega\delta(u, v),$$

we define

$$\begin{aligned}(f * g)(x) &= (f * g)(u + \omega v)\\ &= (\alpha(u, v)\gamma(u, v) - \beta(u, v)\delta(u, v))\\ &\qquad + \omega(\beta(u, v)\gamma(u, v) + \alpha(u, v)\delta(u, v)).\end{aligned}$$

This multiplication coincides with the standard notion of multiplication of two polynomials or of two converging power series in a non-commutative ring. Specifically, if $f(x) = \sum_{k\geq 0} x^k a_k$ and $g(x) = \sum_{k\geq 0} x^k b_k$ then

$$(f * g)(x) = \sum_{n\geq 0} x^n \bigg(\sum_{k=0}^{n} a_k b_{n-k} \bigg).$$

It is also possible to define an inverse with respect to the $*$-product. For further information on slice monogenic functions we refer the reader to [13]. Here we only mention another result which will be useful in the sequel. To state it we need to introduce the Cauchy kernel:

$$S^{-1}(s, x) = -\big(x^2 - 2\mathrm{Re}(s)x + \|s\|^2\big)^{-1}(x - \bar{s}). \tag{4}$$

where $x \notin [s]$, i.e. if $s = u + \omega v$ then $x \neq u + \omega' v$ for all $\omega' \in \mathbb{S}^{n-1}$.

Theorem 2.10 (Borel–Pompeiu Formula). *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric open bounded set such that $\partial(U \cap \mathbb{C}_\omega)$ is a union of a finite number of continuously differentiable Jordan curves for every $\omega \in S^{n-1}$. Let $f : \overline{U} \subseteq \mathbb{R}^{n+1}$ be a slice function of class $\mathcal{C}^1$ and set $ds_\omega = -\omega ds$. For every $x \in U$, and $\omega \in S^{n-1}$, we have*

$$f(x) = \frac{1}{2\pi}\bigg(\int_{\partial(U\cap\mathbb{C}_\omega)} S^{-1}(s, x) ds_\omega f(s) + \int_{U\cap\mathbb{C}_\omega} S^{-1}(s, x)\overline{\partial}_\omega f(s) ds_\omega \wedge d\bar{s}\bigg). \tag{5}$$

Remark 2.11. As we discussed in the introduction, the class of slice monogenic functions and the class of monogenic functions can be related. Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$ and let f be slice monogenic in U. By the Fueter–Sce mapping theorem, the function $\Delta^{n-1/2} f$ is monogenic, see [9], more precisely axially monogenic. Given an axially monogenic function $\breve{f}$, it makes sense to ask whether it is possible to construct a so-called Fueter primitive, that is a slice monogenic function f such that $\Delta^{n-1/2} f = \breve{f}$. The answer is positive and the construction of the Fueter primitive is given in [10]. This result can be further generalized to monogenic functions, see [11].

2.4. Möbius transformations preserving axial symmetry. In this section we turn to introduce a subgroup of Möbius transformations that leaves invariant the axial symmetry property of a set. The proper analogy to the general Ahlfors–Vahlen group in this particular context will be the set stabilizer of the real line. The latter is generated by the inversion, dilations, translations in the x_0-direction only, and by modified rotations. Specifically, we introduce the following:

Definition 2.12. The group $GRAV(\mathbb{R} \oplus \mathbb{R}^n)$ is defined by

$$GRAV(\mathbb{R} \oplus \mathbb{R}^n) := \left\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle, \tag{6}$$

where $b \in \mathbb{R}$, $a \in \mathbb{S}^{n-1}$ and $\lambda \in \mathbb{R}\setminus\{0\}$.

Proposition 2.13. *The elements in the group $GRAV(\mathbb{R} \oplus \mathbb{R}^n)$ take axially symmetric sets into axially symmetric sets.*

Proof. Let us consider an axially symmetric open set U and the generators of the group $GRAV(\mathbb{R} \oplus \mathbb{R}^n)$. An inversion M takes $u_0 + \omega v_0 \in U$ into

$$-\frac{u_0 - \omega v_0}{\|u_0 - \omega v_0\|^2} \in M(U),$$

where $M(U)$ denotes the transformed set of U via M. When ω varies in $\mathbb{S}^{n-1}$, we obtain that $u_0 + \omega v_0 \in U$ if and only if the corresponding element belongs to $M(U)$. Dilations clearly take an axially symmetric set into another axially symmetric set. Translations T in the u_0-direction only take $u_0 + \omega v_0 \in U$ into

$$(u_0 + b) + \omega v_0 \in T(U), \quad b \in \mathbb{R},$$

and again the axial symmetry is preserved. Finally, let us consider $u + \omega v \in U$ and $a \in \mathbb{S}^{n-1}$. We have

$$a(u + \omega v)a = aua + a\omega av = -u + a\omega av.$$

Using the formula $a\omega a = 2(a, \overline{\omega})a - |a|^2\overline{\omega}$, we obtain

$$a\omega a = -2(a, \omega)a + \omega,$$

where $(\cdot\,,\cdot)$ denotes the (Euclidean) scalar product. We deduce that $a\omega a$ is a 1-vector and since $|a\omega a| = 1$, $a\omega a = \omega' \in \mathbb{S}^{n-1}$. Thus

$$a(u + \omega v)a = -u + \omega' v.$$

Let $-u + \omega' v \in M(U)$ and let $-u + \omega' v = a(u + \omega v)a$. For all $\omega'' \in \mathbb{S}^{n-1}$ the element $-u + \omega'' v \in M(U)$ since $-u + \omega'' v = a(u + \widetilde{\omega} v)a$ with $\widetilde{\omega} = a\omega'' a$. The proof is completed. □

Remark 2.14. Notice that the other transformations, for example rotations not preserving the real axis, are clearly not preserving the axial symmetry of a set.

Remark 2.15. It is worthwhile to mention that the group of all Möbius transformations is isomorphic to the projective quotient group $GRAV/\{\lambda I\}$ where I is the identity matrix and where $\lambda \in \mathbb{R}\backslash\{0\}$. This results from the identity

$$(az + b)(cz + d)^{-1} = (\lambda az + \lambda b)(\lambda cz + \lambda d)^{-1},$$

which holds for all $\lambda \in \mathbb{R}\backslash\{0\}$; compare with [2, p. 221, line 13].

The natural analogue of the special Ahlfors–Vahlen group in this context is then played by the group

$$SRAV(\mathbb{R} \oplus \mathbb{R}^n) := \{M \in GRAV(\mathbb{R} \oplus \mathbb{R}^n) \mid \det(M) = 1\}$$

which is generated only by the first three types of matrices listed in (6). Dilations are not needed.

Remark 2.16. In algebraic terms this subgroup is the set stabilizer of the general Ahlfors–Vahlen group fixing the real line.

Remark 2.17. The counterpart of the complex upper half-plane in the slice monogenic setting is the set

$$H := \bigcup_{\omega \in \mathbb{S}^{n-1}} \mathbb{C}^+_\omega.$$

Here, by $\mathbb{C}^+_\omega$ we mean the complex upper half-plane associated to the imaginary unit ω and where the real line is excluded. By construction, the groups $GRAV$ and $SRAV$ leave H invariant.

2.5. Arithmetic subgroups leaving H invariant. In this section we introduce the proper discrete analogue of the group $SRAV(\mathbb{R}\oplus\mathbb{R}^n)$ playing the role of the $SL(2,\mathbb{Z})$ in the classical framework of $SL(2,\mathbb{R})$. As we can consider translations in the direction of the real line only, the only translation group that we can consider in the axially symmetric context is the lattice $\mathbb{Z}$. The associated Möbius transformations are generated by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, inducing the generating transformation $T\langle x\rangle = x+1$. Then we consider the inversion matrix $J := \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. The elements T and J are the generators for $SL(2,\mathbb{Z})$ leaving of the axial symmetry invariant. However, we now add to the generators of $SL(2,\mathbb{Z})$ all the discrete rotations that leave the real line invariant. To this end let us consider a two-sided ideal $\mathcal{I} \subset \mathbb{R}_n$ with a rational basis in a rational Clifford algebra generated by elements $a \in \mathbb{S}^{n-1}$ with the properties that $a\widetilde{a} \in \mathbb{Z}$ and that $axa \in \mathbb{Z}\oplus\mathbb{Z}^n$ for all $x \in \mathbb{Z}\oplus\mathbb{Z}^n$. All discrete rotation matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

with a satisfying the mentioned admissibility conditions can be added as further generators. More precisely, we give the following:

Definition 2.18. The axial symmetric modular group Γ_{RAV} is defined by

$$\Gamma_{RAV} := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\rangle,$$

where $a \in \mathbb{S}^{n-1}$ is such that $a\widetilde{a} \in \mathbb{Z}$ and $axa \in \mathbb{Z}\oplus\mathbb{Z}^n$ for all $x \in \mathbb{Z}\oplus\mathbb{Z}^n$.

The group Γ_{RAV} provides us with the axial symmetric analogue of the $SL(2,\mathbb{Z})$ (and it coincide with $SL(2,\mathbb{Z})$ in the complex setting i.e. when $n = 1$). As a discrete subgroup of $SRAV$ it acts discontinuously on the generalized upper half-space H introduced in the previous subsection.

For integers $N \geq 1$ we also introduce the congruence subgroups of level N by:

$$\Gamma_{RAV}[N] = \left\{ M = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma_{RAV} \mid a-1, d-1, b, c \equiv 0 \ (\mathrm{mod}\ N) \right\}.$$

For further applications we point out that up from $N \geq 3$ the matrix $-I$, where I is the identity matrix, is not included in this group.

3. Invariance of slice monogenic functions

In this section we prove the invariance of slice monogenicity with respect to transformations in $GRAV(\mathbb{R}\oplus\mathbb{R}^n)$.

Theorem 3.1. *Let $M \in GRAV(\mathbb{R}\oplus\mathbb{R}^n)$, and let f be a function that is slice monogenic over an axially symmetric open set $U \subseteq \mathbb{R}^{n+1}$. Then the function*

$$F(x) := \frac{\overline{cx+d}}{\|cx+d\|^2} f\big(M\langle x\rangle\big)$$

is slice monogenic for all $x \in M^{-1}(U)$.

Proof. We split the proof into three steps in which we prove that the statement is true for the generators of $GRAV(\mathbb{R} \oplus \mathbb{R}^n)$. We check this property for translations along the real line, the inversion and rotations fixing the real line.

Step 1. Let us first consider translations along the real line $M\langle x\rangle = x + b$, $b \in \mathbb{R}$, and $f \in \mathcal{SM}(U)$.

Then, $a = d = 1$, $c = 0$ and the function $F(x) = f(x + b)$ is slice monogenic since it is the composition of a slice monogenic function with an intrinsic slice monogenic function (namely a slice monogenic function with real coefficients), see [13].

Step 2. Let us consider the case of the inversion $M\langle x\rangle = -x^{-1}$, Then $a = d = 0$, $b = 1$, $c = -1$, and the function $F(x) = -x^{-1} f(-x^{-1})$ is slice monogenic since it is the composition of the slice monogenic function f with an intrinsic slice monogenic function.

Step 3. In the case of a modified rotation, we consider $M\langle x\rangle = ax\widetilde{a} = axa$, where $a \in \mathbb{S}^{n-1}$, $d = \widetilde{a}^{-1} = a^{-1}$, $b = c = 0$. We have $F(x) = \overline{a}^{-1} f(axa) = af(axa)$. We have to show that F is slice monogenic on U. To this end, we write $x = u + \omega v$, $\omega \in \mathbb{S}^{n-1}$ and we compute

$$(\partial_u + \omega\partial_v)F(u + \omega v) = (\partial_u + \omega\partial_v)(af(a(u + \omega v)a)).$$

Since $f(u + \omega v) = \alpha(u, v) + \omega\beta(u, v)$, we have

$$\begin{aligned}
&(\partial_u + \omega\partial_v)(af(a(u + \omega v)a)) = (\partial_u + \omega\partial_v)(af(-u + a\omega a v))\\
&\quad = (\partial_u + \omega\partial_v)(a\alpha(-u, v) - \omega a\beta(-u, v))\\
&\quad = (\partial_u + \omega\partial_v)(a\alpha(-u, -v) + \omega a\beta(-u, -v))\\
&\quad = -a\partial_u\alpha(-u, -v) - \omega a\partial_u\beta(-u, -v) - \omega a\partial_v\alpha(-u, -v) + a\partial_v\beta(-u, -v)\\
&\quad = -a(\partial_u\alpha(-u, -v) - \partial_v\beta(-u, -v)) - \omega a(\partial_u\beta(-u, -v) + \partial_v\alpha(-u, -v)) = 0,
\end{aligned}$$

since the pair α, β satisfies the Cauchy–Riemann equations.

Step 4. Let us finally consider dilations $M\langle x\rangle = \lambda^2 x$, $\lambda \in \mathbb{R}$, so that $F(x) = \lambda f(\lambda^2 x)$. The function F is obviously slice monogenic since if we write $x = u + \omega v$ we trivially have

$$(\partial_u + \omega\partial_v)F(\partial_u + \omega\partial_v) = (\partial_u + \omega\partial_v)(\lambda f(\lambda^2(u + \omega v))) = 0.$$

We conclude that the statement is true for all generators, and so for any element of the group. To see this, consider two generators M and N. Then the maps

$$F: z \to J(M, z) f\big(M\langle z\rangle\big) \quad \text{and} \quad g: z \to J(N, z) F\big(N\langle z\rangle\big)$$

are slice regular for any f, F slice regular. In view of the cocyle relation for the automorphic factor, see [32] and references therein, we have

$$g(z) = J(N, z) J\big(M, N\langle z\rangle\big) f\big(M\langle N\langle z\rangle\rangle\big) = J((MN), z) f\big((MN)\langle z\rangle\big)$$

is slice regular, for any generator M and N. Since any element of the group can be generated as products of the generators listed above (for which we have shown the slice regularity), it follows that the function $z \to J(M,z)f(M\langle z\rangle)$ is slice regular for any arbitrary $M \in GRAV$. This concludes the proof. □

Remark 3.2. It is worthwhile to note that Step 3 in the proof of the previous result works even if we consider a rotation $x \mapsto = ax\tilde{a}^{-1} = ax\overline{a}$, performed using $a \in \mathbb{R}^{n+1}$, $\|a\| = 1$. In fact we have that $ax\overline{a} = a(u+\omega v)\overline{a}) = u + a\omega\overline{a}\, v$ and

$$F(u+\omega v) = \overline{a} f(u + a\omega\overline{a} v).$$

Furthermore, since α, β satisfy the Cauchy–Riemann equations, we have

$$\begin{aligned}(\partial_u + \omega\partial_v)(\overline{a} f(a(u+\omega v)a)) &= (\partial_u + \omega\partial_v)(\overline{a} f(u + a\omega\overline{a} v)) \\ &= (\partial_u + \omega\partial_v)(\overline{a}\alpha(u,v) + \omega\overline{a}\beta(u,v)) \\ &= \overline{a}\partial_u\alpha(u,v) + \omega\overline{a}\partial_u\beta(u,v) + \omega\overline{a}\partial_v\alpha(u,v) - \overline{a}\partial_v\beta(u,v) \\ &= \overline{a}(\partial_u\alpha(u,v) - \partial_v\beta(u,v)) + \omega\overline{a}(\partial_u\beta(u,v) + \partial_v\alpha(u,v)) = 0.\end{aligned}$$

However note that we have excluded these transformations from our discussion, see Remark 2.5.

Remark 3.3. The invariance obtained in Theorem 3.1 is, formally, of the same type as the one of hypermonogenic functions considered for instance in [34] and elsewhere.

4. Axial conformal invariance of slice monogenic function

We begin this section by studying the conformal invariance of the Cauchy kernel (4). We consider the Möbius transformation $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ and we set

$$J(M,x) := \frac{\overline{cx+d}}{\|cx+d\|^2} = (cx+d)^{-1}.$$

We have:

Lemma 4.1. *Let $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ be the Möbius transformation over $\mathbb{R}^{n+1} \cup \{\infty\}$ associated to a matrix belonging to $GRAV(\mathbb{R}^{n+1})$. Then*

$$S^{-1}\big(M\langle y\rangle, M\langle x\rangle\big) = J(M,x)^{-1} S^{-1}(y,x)\widetilde{J(M,y)^{-1}}. \tag{7}$$

Proof. We show that the formula holds for the generators of the group.

Step 1. Let us consider a translation along the real axis $M\langle x\rangle = x+b$. In this case, $J(M,x)=1$. We have

$$\begin{aligned} S^{-1}&\big(M(\langle y\rangle), M(\langle x\rangle)\big)\\ &= -\big((x+b)^2 - 2(\mathrm{Re}(y+b))(x+b) + \|y+b\|^2\big)^{-1}(x-\overline{y})\\ &= -\big(x^2 - 2\mathrm{Re}(y)x + \|y\|^2\big)^{-1}(x-\overline{y}) = S^{-1}(y,x) \end{aligned}$$

and the statement holds.

Step 2. Let $M\langle x\rangle = -x^{-1}$ so that $J(M,x) = -x^{-1}$. We have

$$\begin{aligned} S^{-1}(-y^{-1},-x^{-1}) &= \big((-x^{-1})^2 - 2\mathrm{Re}(-y^{-1})(-x^{-1}) + \|y^{-1}\|^2\big)^{-1}(-x^{-1}+\overline{y}^{-1})\\ &= \big(x^{-2} - 2\mathrm{Re}(y^{-1})(x^{-1}) + \|y\|^{-2}\big)^{-1}(-x^{-1}+\overline{y}^{-1})\\ &= \big[x^{-2}\big(\|y\|^2 - 2\mathrm{Re}(y^{-1})\|y\|^2x + x^2\big)\|y\|^{-2}\big]^{-1}x^{-1}(-\overline{y}+x)\overline{y}^{-1}\\ &= \|y\|^2\big(\|y\|^2 - 2\mathrm{Re}(y^{-1})\|y\|^2x + x^2\big)^{-1}x(-\overline{y}+x)\overline{y}^{-1}\\ &= x\big(\|y\|^2 - 2\mathrm{Re}(y^{-1})\|y\|^2x + x^2\big)^{-1}(-\overline{y}+x)\overline{y}^{-1}\|y\|^2\\ &= J(M,x)^{-1}S^{-1}(y,x)\widetilde{J(M,y)^{-1}}, \end{aligned}$$

in view of $\widetilde{J(M,y)^{-1}} = \widetilde{-y} = -y$.

Step 3. Let us consider the modified rotation $M\langle x\rangle = axa$, $a\in\mathbb{S}^{n-1}$ so that $J(M,x)=a$. Since $\mathrm{Re}(aya) = -\mathrm{Re}(y)$, we have:

$$\begin{aligned} S^{-1}(aya,axa) &= -\big(-ax^2a - 2\mathrm{Re}(aya)(axa) + \|aya\|^2\big)^{-1}(axa-\overline{aya})\\ &= -\big(-ax^2a + 2\mathrm{Re}(aya)x - a^2\|y\|^2\big)^{-1}a(x-\overline{y})a\\ &= \big[a\big(x^2 - 2\mathrm{Re}(y)x + \|y\|^2\big)a\big]^{-1}a(x-\overline{y})a\\ &= a^{-1}\big(x^2 - 2\mathrm{Re}(y)x + \|y\|^2\big)^{-1}(x-\overline{y})a\\ &= a^{-1}S^{-1}(y,x)a^{-1}\\ &= J(M,x)^{-1}S^{-1}(y,x)\widetilde{J(M,y)}^{-1}. \end{aligned}$$

Step 4. In the case of a dilation $M\langle x\rangle = \lambda^2x$, $\lambda\in\mathbb{R}$, $J(M,x)=\lambda$, we have

$$\begin{aligned} S^{-1}(\lambda^2y,\lambda^2x) &= -\big(\lambda^4x^2 - 2\mathrm{Re}(\lambda^2y)(\lambda^2x) + \|\lambda^2y\|^2\big)^{-1}(\lambda^2x - \overline{\lambda^2y})\\ &= -\lambda^{-4}\big(x^2 - 2\mathrm{Re}(y)(x) + \|y\|^2\big)^{-1}\lambda^2(x-\overline{y})\\ &= -\lambda^{-4}\big(x^2 - 2\mathrm{Re}(y)(x) + \|y\|^2\big)^{-1}\lambda^2(x-\overline{y})\\ &= \lambda^{-1}S^{-1}(y,x)\lambda^{-1} = J(M,x)^{-1}S^{-1}(y,x)\widetilde{J(M,y)}^{-1}. \qquad\square \end{aligned}$$

As a consequence of Lemma 4.1 and of the Cauchy integral formula for slice monogenic functions, see [13] but also (5), we can now deduce the axial conformal invariance for the set of slice monogenic functions. We will present two types of results, starting with the following:

Theorem 4.2. *Suppose that $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ is a Möbius transformation associated with a matrix from $SRAV(\mathbb{R}^{n+1})$. Let K be a compact in $\mathbb{R}^{n+1}$ and suppose that $g\colon K \to \mathbb{R}_n$ is a $\mathcal{C}^1(K)$ slice function. Then, for any $\omega \in S^{n-1}$ the following integral invariance formula holds*

$$J(M,x) \int_{K\cap\mathbb{C}_\omega} S^{-1}(z, M\langle x\rangle) g(z)\, dA(z) = \int_{M^{-1}(K\cap\mathbb{C}_\omega)} S^{-1}(y,x) g(y)\, j_4(M,y)\, dA(y), \tag{8}$$

where $z := M\langle y\rangle = (ay+b)(cy+d)^{-1}$, $dA(z) = dz_\omega \wedge d\overline{z}$ and

$$j_4(M,x) := \frac{\widetilde{cx+d}}{\|cx+d\|^4}.$$

Proof. According to Lemma 4.1 we know that

$$S^{-1}\big(M\langle y\rangle, M\langle x\rangle\big) = J(M,x)^{-1} S^{-1}(y,x) \widetilde{J(M,y)^{-1}}.$$

The area differential $dA(z)$ has the following transformation behavior under Möbius transformations (see [1]):

$$dA(z) = dA\big(M\langle y\rangle\big) = dA\big((ay+b)(cy+d)^{-1}\big) = \frac{dA(y)}{\|cy+d\|^4}.$$

Combining this with the left hand-side of formula (8) we obtain that

$$\begin{aligned} J(M,x) &\int_{K\cap\mathbb{C}_\omega} S^{-1}\big(z, M\langle x\rangle\big) g(z)\, dA(z) \\ &= J(M,x) \int_{M^{-1}(K\cap\mathbb{C}_\omega)} J(M,x)^{-1} S^{-1}(y,x) \widetilde{J(M,y)^{-1}} g(y)\, \frac{dA(y)}{\|cy+d\|^4} \\ &= \int_{M^{-1}(K\cap\mathbb{C}_\omega)} S^{-1}(y,x) \widetilde{j_4(M,y)} g(y)\, dA(y), \end{aligned}$$

since

$$\frac{\widetilde{J(M,y)^{-1}}}{\|cy+d\|^4} = \left(\frac{\widetilde{cy+d}}{\|cy+d\|^2}\right)^{-1} \frac{1}{\|cy+d\|^4}$$
$$= \frac{(\widetilde{cy+d})}{\|cy+d\|^4}$$
$$= j_4(M,y),$$

and the stated relation follows. □

More in general we have:

Theorem 4.3. *Suppose that* $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ *is a Möbius transformation associated with a matrix in* $SRAV(\mathbb{R}^{n+1})$. *Let* K *be a compact set in* $\mathbb{R}^{n+1}$ *and suppose that* $g\colon K \to \mathbb{R}_n$ *is a* $\mathcal{C}^1(K)$ *slice function. Then, for any* $\omega \in \mathbb{S}^{n-1}$ *the following integral invariance formula holds*

$$\begin{aligned}\frac{1}{2\pi} J(M,x_0)\Big(\int_{\partial(K\cap\mathbb{C}_\omega)} & S^{-1}(y,y_0)\, dy_\omega\, f(y) \\ &+ \int_{K\cap\mathbb{C}_\omega} S^{-1}(y,y_0)\overline{\partial}_\omega f(y)\, dy_\omega \wedge d\,\overline{y}\Big) \\ = \frac{1}{2\pi}\Big(\int_{M^{1}(\partial(K\cap\mathbb{C}_\omega))} & S^{-1}(x,x_0)\overline{\widetilde{J(M,x)}}\, dx_\omega\, f(x) \\ &+ \int_{M^{-1}(K\cap\mathbb{C}_\omega)} S^{-1}(x,x_0)\, j_4(M,x)\,\overline{\partial}_\omega f(x)\, dx_\omega \wedge d\,\overline{x}\Big),\end{aligned} \tag{9}$$

where $y = M\langle x\rangle$, $y_0 = M\langle x_0\rangle$.

Proof. The invariance formula follows from the Borel–Pompeiu formula (5), from (7) and (8). □

Remark 4.4. It is important to note that the integrals at the left hand side of (9) are computed on a curve or on a compact set in a complex plane $\mathbb{C}_\omega$. The integrals on the right hand side are still computed on a curve or on a two-dimensional set but they are not anymore contained in a complex plane but in the surface corresponding to $M^{-1}(\mathbb{C}_\omega)$.

Remark 4.5. As is customary in the literature, we have considered Möbius transformations associated to matrices M with determinant $\det(M) = 1$. In the general case, the Jacobian of the transformation has a factor depending on the determinant of M.

Even though most of the integral formulas for slice monogenic functions hold on slices, namely on complex planes $\mathbb{C}_\omega$ for some $\omega \in \mathbb{S}^{n-1}$, the Borel–Pompeiu formula can be proved using integration in higher dimensions. We quote below the result, see e.g. [27, Corollary 2.7]:

Proposition 4.6. *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric open bounded set with $\mathcal{C}^1$ boundary. Let $f : \bar{U} \subseteq \mathbb{R}^{n+1}$ be a slice function of class $\mathcal{C}^1$. For every $x \in U$ we have*

$$f(x) = \frac{1}{2\pi} \int_{\partial U} C(s,x) n(s) f(s)\, d\sigma(s) - \frac{1}{\pi} \int_U C(s,x) \partial_{\bar{s}} f(s)\, dV(s), \tag{10}$$

where n denotes the outer normal vector field, $d\sigma$ and dV denote the volume form on ∂U and U, respectively, and

$$C(s,x) = \frac{\Gamma(n/2)}{\pi^{n/2}} \frac{S^{-1}(s,x)}{(s_1^2 + \cdots + s_n^2)^{(n-1)/2}}.$$

In order to study the conformal invariance of the integral formula (10), we need to prove an analogue of Lemma 4.1 for the kernel $C(s,x)$. To this end we need to introduce the group

$$\Gamma^{\infty}_{RAV}(\mathbb{R} \oplus \mathbb{R}^n) := \left\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\rangle,$$

$b \in \mathbb{R}, a \in \mathbb{S}^{n-1}$.

Lemma 4.7. *Let $M\langle x \rangle$ be a Möbius transformation over $\mathbb{R}^{n+1} \cup \{\infty\}$ associated to a matrix belonging to $\Gamma^{\infty}_{RAV}(\mathbb{R}^{n+1})$. Then*

$$C\big(M(\langle y \rangle), M(\langle x \rangle)\big) = J(M,x)^{-1} C(y,x) \widetilde{J(M,y)^{-1}}. \tag{11}$$

Proof. To study how the kernel $C(y,x)$ transforms under Möbius transformations in $\Gamma^{\infty}_{RAV}(\mathbb{R} \oplus \mathbb{R}^n)$ is equivalent to study how the kernel

$$\frac{S^{-1}(y,x)}{(y_1^2 + \cdots + y_n^2)^{(n-1)/2}}$$

transforms. It is immediate that $y_1^2 + \cdots + y_n^2$ does not change neither under translations fixing the real axis nor and under modified rotations. So, the statement follows from Lemma 4.1. □

The following statement provides us with a generalization of Theorem 6 from Ryan's paper [37] to the slice monogenic framework. For convenience, to a Möbius transformation $M\langle x \rangle = (ax+b)(cx+d)^{-1}$ we associate the following generalized weight factor of weight α:

$$j_\alpha(M,x) := \frac{\widetilde{cx+d}}{\|cx+d\|^\alpha}.$$

The next result contains another invariance formula.

Theorem 4.8. *Suppose that* $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ *is a Möbius transformation associated with a matrix in* $\Gamma^{\infty}_{RAV}(\mathbb{R}^{n+1})$. *Let* K *be a compact axially symmetric set in* $\mathbb{R}^{n+1}$ *and suppose that* $g\colon K \to \mathbb{R}_n$ *is a slice function of class* $\mathcal{C}^1$. *Then we have the following volume integral invariance formula*

$$J(M,x)\int\limits_K C(z,M\langle x\rangle)g(z)\,dV(z) = \int\limits_{M^{-1}(K)} C(y,x)j_{2n+2}(M,y)g(y)\,dV(y), \tag{12}$$

where $z := M\langle y\rangle = (ay+b)(cy+d)^{-1}$.

Proof. According to the previously proved in Lemma 4.1 we know that

$$C\big(M\langle y\rangle, M\langle x\rangle\big) = J(M,x)^{-1}C(y,x)\widetilde{J(M,y)^{-1}}.$$

The volume differential has the following transformation behavior under Möbius transformations:

$$dV(z) = dV\big(M\langle y\rangle\big) = dV\big((ay+b)(cy+d)^{-1}\big) = \frac{dV(y)}{\|cy+d\|^{2n+2}}.$$

Combining this with the left hand side of formula (12) we obtain that

$$\begin{aligned} J(M,x)&\int_K C(z,M\langle x\rangle)g(z)\,dV(z)\\ &= J(M,x)\int_{M^{-1}(K)} J(M,x)^{-1}C(y,x)\widetilde{J(M,y)^{-1}}g(y)\,\frac{dV(y)}{\|cy+d\|^{2n+2}}\\ &= \int_{M^{-1}(K)} C(y,x)j_{2n+2}(M,y)g(y)\,dV(y), \end{aligned}$$

since

$$\begin{aligned} \frac{\widetilde{J(M,y)^{-1}}}{\|cy+d\|^{2n+2}} &= (\widetilde{cy+d})\frac{1}{\|cy+d\|^{2n+2}}\\ &= j_{2n+2}(M,y), \end{aligned}$$

and so the stated equality follows. □

Also the surface measure $d\sigma$ transforms nicely under a Möbius transformation in $\mathbb{R}^{n+1}$, namely in the way

$$d\sigma(z) = d\sigma\big(M\langle y\rangle\big) = \frac{d\sigma(y)}{\|cy+d\|^{2n}},$$

quite analogously to the volume differential. It just differs in the exponent of the factor. Thus we deduce the following result.

Proposition 4.9. *Let $K \subset \mathbb{R}^{n+1}$ be a bounded domain with $\mathcal{C}^1$ boundary ∂K with a piecewise continuous exterior normal field $n(z)$ and suppose that $g\colon \partial K \to \mathbb{R}_n$ is a $\mathcal{C}^1(\partial K)$ slice function. Let $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ be a Möbius transformation over $\mathbb{R}^{n+1} \cup \{\infty\}$ associated to a matrix belonging to $\Gamma^{\infty}_{RAV}(\mathbb{R}^{n+1})$. Then we have:*

$$J(M,x)\int_{\partial K} C\big(z, M\langle x\rangle\big)n(z)g(z)\,d\sigma(z) = \int_{M^{-1}(\partial K)} C(y,x)\,j_{2n}(y)n(y)g(y)\,d\sigma(y),$$

where $z = M\langle y\rangle$.

Proof. We follow the same chain of arguments as in the proof of the previous theorem. The only difference is

$$\frac{\widetilde{J(M,y)^{-1}}}{\|cy+d\|^{2n}} = (\widetilde{cy+d})\frac{1}{\|cy+d\|^{2n}} = j_{2n}(M,y),$$

and so the statement follows. □

Next we present an axial conformal invariance of the Borel–Pompeiu formula which can be obtained combining the previous results.

Proposition 4.10. *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric open bounded set. Let $f\colon \overline{U} \subseteq \mathbb{R}^{n+1}$ be a slice function of class $\mathcal{C}^1$. Let $M\langle x\rangle = (ax+b)(cx+d)^{-1}$ be a Möbius transformation over $\mathbb{R}^{n+1} \cup \{\infty\}$ associated to a matrix belonging to $\Gamma^{\infty}_{RAV}(\mathbb{R}^{n+1})$. We have*

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\partial K} C\big(z, M\langle x\rangle\big)n(z)g(z)d\sigma(z) - \frac{1}{\pi}\int_K C\big(z, M\langle x\rangle\big)g(z)\,dV(z) \\ &= J(M,x)^{-1}\Bigg[\frac{1}{2\pi}\int_{M^{-1}(\partial K)} C(y,x)\,j_{2n}(M,y)\,n(y)\,g(y)\,d\sigma(y) \\ &\qquad - \frac{1}{\pi}\int_{M^{-1}(\partial K)} C(y,x)\,j_{2n+2}(M,y)\,\partial_{\bar{y}}g(y)\,dV(y)\Bigg]. \end{aligned}$$

Proof. It follows from Theorem 4.8 and Proposition 4.9. □

Remark 4.11. One can consider Hardy spaces of slice monogenic functions. If $K \subset \mathbb{R}^{n+1}$ is a domain with a C^1-boundary, then the associated Hardy space of slice monogenic functions is the closure of the set of functions that are slice monogenic inside of K with a continuous extension to ∂K and that are additionally square-integrable over K. This can be done completely analogously to the context of Clifford algebra valued monogenic functions presented in [5].

However, a quite natural question arising in this framework is whether there are isometries between Hardy spaces of functions that can be transformed into each other by applying Möbius transformations. In the monogenic setting, there exists such an isometry. See for instance [18]. In the slice monogenic framework we meet some obstacles.

If $K \subset \mathbb{R}^{n+1}$ is a compact subset in a complex plane, i.e. of codimension $(n-1)$, then the integral transformation with a Möbius transformation actually involves a right conformal weight factor. Concretely, we have

$$d\sigma(M\langle x\rangle) = \frac{1}{\|cx+d\|^4}\, d\sigma(x) = \frac{(cx+d)\overline{(cx+d)}}{\|cx+d\|^4}\, dA(x).$$

However, in the general case it may happen that a domain K lying in a two-dimensional complex plane will not be mapped in general to a domain $f(K)$ lying in another complex plane.

If we alternatively consider the integration over the boundary of a domain in $\mathbb{R}^{n+1}$ of codimension zero, then we meet the problem that the integral measure does not transform with the slice monogenic conformal weight factor, but instead in the following way (see [18])

$$d\sigma\big(M\langle x\rangle\big) = \frac{1}{\|cx+d\|^{2n}}\, d\sigma(x) = \frac{(cx+d)\overline{(cx+d)}}{\|cx+d\|^{2n+2}}\, d\sigma(x)$$

so that will we not get an isometry between the associated Hardy spaces, except for the case $n = 1$ which is the classical complex case.

Domains of other codimensions do not involve the right exponent of the weight factor that is needed for an isometry, either.

5. Application to automorphic forms

In this section, we outline two different ways to construct non-trivial slice monogenic automorphic forms on congruence subgroups of the axial symmetric arithmetic subgroups Γ_{RAV}. The study of automorphic forms on discrete arithmetic groups represents a central research topic in complex and harmonic analysis (cf. [22,40]) and provides an analytic toolkit to study many problems in number theory and algebra on the one hand and in quantum field theories, quantum gravity and gauge field theories on the other hand, see for instance [30,35]. In the framework of monogenic, hypermonogenic and holomorphic Cliffordian functions this topic has been studied in [32] and [17]. Since the setting of slice monogenic functions lies on the interface between classical complex holomorphic function theory and these mentioned classes of Clifford analysis, it is interesting to explore this field also in the slice monogenic framework. A classical result is that all automorphic forms can be described in terms of Eisenstein and Poincaré series. Therefore we are motivated to propose similar constructions of Eisenstein and Poincaré series in the slice monogenic setting. We propose two construction approaches:

5.1. Direct construction of slice monogenic automorphic forms. In this subsection we present a direct construction method of automorphic forms in the slice monogenic setting within the context of the axial symmetric arithmetic subgroups. First of all we introduce the following:

Definition 5.1. Let L be an arithmetic subgroup of Γ_{RAV}. A slice monogenic function $f\colon H \to \mathbb{R}_n$ is called a slice monogenic automorphic form on L if it satisfies

$$f(x) = J(cx+d) f\big(M\langle x\rangle\big)$$

for all $x \in H$ and all $M \in L$.

In the case where L is just a discrete translation group we are simply dealing with slice monogenic periodic functions. A simple example is given by the slice monogenic exponential function, introduced below. These functions serve as useful building blocks to apply the following Poincaré series construction in order to obtain non-trivial examples of slice monogenic automorphic forms on larger arithmetic subgroups.

Remark 5.2. By $L[N]$ we denote the congruence subgroup of level N of L, while $L^{\infty}[N]$ consists of all matrices belonging to $L[N]$ where the $(2,1)$ entry is zero, i.e. all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ that belong to $L[N]$.

Recalling that we can write a paravector x is the form $x := x_0 + \underline{x}$ where $\underline{x} = x_1 e_1 + \cdots + x_n e_n$, we have the following result:

Theorem 5.3. *Let $F\colon H \to \mathbb{R}_n$ be*

$$F(x) := \exp\Big(\frac{x\underline{x}}{\|\underline{x}\|}\Big).$$

For $N \geq 3$ the slice monogenic Poincaré series

$$P(x) = \sum_{M : L^{\infty}[N]\backslash L} J(M,x) F\big(M\langle x\rangle\big)$$

converges normally on each compact subset of H and satisfies

$$P(x) = J(M,x) P\big(M\langle x\rangle\big)$$

for each $x \in H$ and all $M \in L[N]$.

Proof. First of all we observe that $F(x)$ is slice monogenic according to (1).

In fact let us take the restriction to the complex plane $\mathbb{C}_\omega$ by rewriting

$$x = x_0 + \underline{x} = x_0 + \frac{\underline{x}}{\|\underline{x}\|}\|\underline{x}\| = u + \omega v$$

with $u = x_0$, $v = \|\underline{x}\|$, $\omega = \frac{\underline{x}}{\|\underline{x}\|}$. Then we have

$$F(u + \omega v) = \exp((u + \omega v)\omega)$$

which is trivially in the kernel of $\partial_u + \omega\partial_v$ so F is slice monogenic according to the definition in (1). We recall, see Remark 2.17, that $H = \cup_{\omega\in\mathbb{S}^{n-1}}\mathbb{C}^+_\omega$. Moreover

$$\begin{aligned} F(x + 2k\pi) &= F((u + 2k\pi) + \omega v) \\ &= \exp(-v + (u + 2k\pi)\omega) = \exp(-v + u\omega) = F(x) \end{aligned}$$

for any $k \in \mathbb{Z}$ and F is 2π-periodic. Note that $e^{(u+\omega v)\omega} = e^{-v+u\omega v}$ so that $\|e^{(u+\omega v)\omega}\| = e^{-v}$. A Möbius transformation takes $\mathbb{C}^+_\omega$ to $\mathbb{C}^+_{\omega'}$: this fact can be easily verified on the generators. It is immediate for the translations and the dilations. The inversion takes $u + \omega v$ to $(u^2 + v^2)^{-1}(-u + \omega v)$. Finally, a modified rotation takes $u + \omega v$ to $-u + a\omega a v = -u + \omega' v$ so the assertion follows.

The convergence can now be established along the classical convergence proof for Clifford Poincaré series presented in [32] relying on the additional argument that we have asymptotically

$$\left\| \frac{\overline{cx + d}}{\|cx + d\|^2} \exp\left(\frac{M\langle x\rangle \cdot \underline{M\langle x\rangle}}{\|\underline{M\langle x\rangle}\|}\right)\right\| \sim \frac{C}{r}\exp(-r)$$

when $r \to \infty$ and the constant C depends on the dimension n of $\mathbb{R}_n$. Due to the dominance of the decreasing exponential factor, the convergence is guaranteed by the classical Poincaré series convergence argument. Slice monogenicity then follows from the facts that the series is normally convergent and that each summand is slice monogenic itself, as a consequence of Theorem 3.1. □

Remark 5.4. The reason for claiming $N \geq 3$ is to guarantee that the negative identity matrix is not included in the series in order to avoid the vanishing behavior of the series.

5.2. Construction of automorphic forms from complex holomorphic automorphic forms. Another possibility to construct slice monogenic automorphic forms consists of applying the intermediate step in the Fueter's construction to classical holomorphic modular forms defined in one complex variable on upper half-plane. Notice that the classical modular forms exhibit an invariance property of the form

$$f(z) = (cz + d)^{-2k} f\big((az + b)/(cx + d)\big) \quad k \geq 2$$

under Möbius transformation in $SL(2,\mathbb{Z})$, so their slice monogenic extension will not be an automorphic form under all transformations of the full group Γ_{RAV} but just of a subgroup that is isomorphic to the $SL(2,\mathbb{Z})$, because Fueter's theorem only inherits the invariance under the $SL(2,\mathbb{Z})$. This is clearly reflected when applying Fueter's theorem to the most classical examples of holomorphic Eisenstein series having the form

$$E_{2k}(z) := \sum_{(c,d)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} (cz+d)^{-2k}.$$

Its slice monogenic counterpart has the form

$$\mathcal{E}_{2k}(x) := \sum_{(c,d)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} (cx+d)^{-2k} \tag{13}$$

which written out explicitly, using the Representation Formula (see Theorem 2.9), is

$$\begin{aligned}\mathcal{E}_{2k}(x) &= \sum_{(c,d)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \frac{1+\omega i}{2}(c\overline{z}+d)^{-2k} + \frac{1-\omega i}{2}(cz+d)^{-2k} \\ &= \frac{1+\omega i}{2}E_{2k}(\overline{z}) + \frac{1-\omega i}{2}E_{2k}(z),\end{aligned}$$

where $z\in\mathbb{C}_i$, $x\in\mathbb{R}^{n+1}$, $x=u+\omega v$, and where E_{2k} is the standard holomorphic Eisenstein series of positive integer weight $2k$ in the complex plane $\mathbb{C}_i$.

We note that for general c,d we should have used $(cz+d)^{-2k*}$ in the series expansion (13), however $c,d\in\mathbb{Z}$ so $(cz+d)^{-2k*}=(cz+d)^{-2k}$.

From this representation we can easily derive that $\mathcal{E}_{2k}(z)=\mathcal{E}_{2k}(z+b)$ for all $b\in\mathbb{R}$. Analogously, it is also easily seen that

$$\mathcal{E}_{2k}(z) = z^{-2k}\mathcal{E}_{2k}(-z^{-1})$$

by the analogous properties of E_{2k}. Indeed, these series only exhibits invariance on the transformations $z\mapsto z+b$ and $z\mapsto -z^{-1}$ which correspond to those of $SL(2,\mathbb{Z})$ which actually only form a subgroup of the Γ_{RAV}.

6. Slice monogenic fractional linear functions

In the previous sections we have studied the invariance properties of slice monogenic functions under Möbius transformations. Obviously, these transformations are not all slice monogenic. They are so when the coefficients are real (which is a sufficient condition). However, if we take the $*$-product instead of the pointwise product, we can construct some linear fractional transformations which are slice monogenic. They are defined as:

$$\mathcal{M}\langle x\rangle = (xc+d)^{-*} * (xa+b),$$

and, a priori, a, b, c, d are suitable elements in the Clifford algebra $\mathbb{R}_n$. A similar class of functions has been introduced in [39] for slice regular functions of a quaternionic variable.

Remark 6.1. One should note that, despite the formal analogy with the classical complex case where the multiplication and the inverse are substituted by the $*$-multiplication and $*$-inverse respectively, the above expression equals

$$\begin{aligned}\mathcal{M}\langle x\rangle &= \big(x^2|c|^2 + x(c\bar{d} + d\bar{c}) + |d|^2\big)^{-1}\big((x\bar{c} + \bar{d}) * (xa + b)\big)\\ &= \big(x^2|c|^2 + x(c\bar{d} + d\bar{c}) + |d|^2\big)^{-1}\big(x^2(\bar{c}a) + x(\bar{c}b + \bar{d}a) + \bar{d}b\big),\ x\in\mathbb{R}^{n+1}.\end{aligned}$$

Also the slice monogenic fractional linear functions can be described in terms of the action of the group $GAV(\mathbb{R}\oplus\mathbb{R}^n)$ where the action is defined as

$$GAV(\mathbb{R}\oplus\mathbb{R}^n)\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1},\quad (A,x)\mapsto\mathcal{M}_A\langle x\rangle := (xc+d)^{-*} * (xa+b),$$

with

$$A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

Proposition 6.2. *Let $A, B \in GAV(\mathbb{R}\oplus\mathbb{R}^n)$, then $\mathcal{M}_{[A^T B^T]^T} = \mathcal{M}_B\circ\mathcal{M}_A$, where ^{T} stands for the transpose of a matrix.*

Proof. We write

$$A^T = \begin{pmatrix} a & c\\ b & d\end{pmatrix},\quad B = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},\quad B^T = \begin{pmatrix} a_1 & c_1\\ b_1 & d_1\end{pmatrix}.$$

Using computation similar to those in [39], that we repeat here for the reader's convenience, we have

$$\begin{aligned}\mathcal{M}_B\big(\mathcal{M}_A\langle x\rangle\big) &= \mathcal{M}_B\big(\mathcal{M}_A\langle x\rangle\big) = \mathcal{M}_B\big((xc+d)^{-*} * (xa+b)\big)\\ &= \big((xc+d)^{-*} * (xa+b)c_1 + d_1\big)^{-*} * \big((xc+d)^{-*} * (xa+b)a_1 + b_1\big)\\ &= \big[(xc+d)^{-*} * ((xa+b)c_1 + (xc+d)d_1\big]^{-*}\\ &\qquad * \big[(xc+d)^{-*} * ((xa+b)a_1 + (xc+d)b_1\big]\\ &= \big(x(ac_1+cd_1) + bc_1 + dd_1\big)^{-*} * \big(x(aa_1+cb_1) + ba_1 + db_1\big)\\ &= \mathcal{M}_{[A^T B^T]^T}\langle x\rangle.\end{aligned}$$

□

It is a natural question to ask which is the invariance with respect to this new class of transformations. The first task is to assign the conditions under which $\mathcal{M}$ takes $\mathbb{R}^{n+1}$ to itself.

Unlike what happens with the Möbius transformations studied in the previous sections, the set of slice monogenic fractional linear functions is not isomorphic to $GAV(\mathbb{R}\oplus\mathbb{R}^n)/\{\mu I\}$ $(\mu\in\mathbb{R}\backslash\{0\})$. In fact we have:

Proposition 6.3. *The set of slice monogenic fractional linear functions is in one-to-one correspondence with the set of right cosets of*

$$\mathcal{N} := \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\rangle, \quad \alpha \in \mathbb{R}^{n+1} \setminus \{0\}$$

in $GAV(\mathbb{R} \oplus \mathbb{R}^n)$.

Proof. In this framework, we consider the following generators of $GAV(\mathbb{R} \oplus \mathbb{R}^n)$:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $b \in \mathbb{R}^{n+1}$, $a \in \mathbb{S}^{n-1}$ and where $\lambda \in \mathbb{R}\setminus\{0\}$. With this choice we have:

$$\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, x \right) = (x \cdot 0 + 1)^{-*} * (x + b) \quad = x + b$$
$$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, x \right) = (-x \cdot 1 + 0)^{-*}(x \cdot 0 + 1) = -x^{-1}$$
$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, x \right) = (x \cdot 0 + 1)^{-*}(\lambda^2 x + 0) \quad = \lambda^2 x.$$

The subset $\mathcal{N}$ is evidently a subgroup of $GAV(\mathbb{R} \oplus \mathbb{R}^n)$, moreover

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, x \right) = (x\alpha)^{-*} * (x\alpha) = x,$$

which shows that all transformation matrices of the form αI, $\alpha \in \mathbb{R}^{n+1}\setminus\{0\}$, gives the identity. Let us now consider the right cosets $\mathcal{N} A$ where $A \in GAV(\mathbb{R} \oplus \mathbb{R}^n)$. To any slice monogenic linear fractional transformation can be associated to a right coset, precisely to $\mathcal{M}\langle x \rangle = (xc + d)^{-*} * (xa + b)$ we can associate $\mathcal{N} A$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In fact, any element in $\mathcal{N} A$ is of the form αA, $\alpha \in \mathbb{R}^{n+1}\setminus\{0\}$, and

$$\begin{aligned} \mathcal{M}_{\alpha A}\langle x \rangle &= (x\alpha c + \alpha d)^{-*} * (x\alpha a + \alpha b) \\ &= (\alpha * (xc + d))^{-*} * \alpha * (xa + b) \\ &= (xc + d)^{-*} * \alpha^{-1} * \alpha * (xa + b) = \mathcal{M}_{\alpha A}\langle x \rangle. \end{aligned}$$

And, similarly, to any right coset $\mathcal{N} A$ we associate the unique transformation $\mathcal{M}_A$. □

An immediate consequence is:

Corollary 6.4. *The set of slice monogenic fractional linear functions is in one-to-one correspondence with a subset of all Möbius transformations.*

Remark 6.5. We also note that the subgroup $\mathcal{N}$ defined in Proposition 6.3 is not normal. This is immediately seen by taking for example a translation T and computing $T^{-1}\alpha I T$ to see that it does not belong to $\mathcal{N}$. Thus the set of right cosets does not inherit the group structure of $GAV(\mathbb{R} \oplus \mathbb{R}^n)$.

Remark 6.6. If we consider the generator $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, then we have

$$\mathcal{M}\langle x \rangle = xa^2 = -x.$$

Finally, we have the following result:

Proposition 6.7. *Slice monogenic Möbius transformation preserving the axial symmetry of an open set are associated with matrices in*

$$GRAV(\mathbb{R} \oplus \mathbb{R}^n) = \left\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle,$$

where $b \in \mathbb{R}$, $a \in \mathbb{S}^{n-1}$, $\lambda \in \mathbb{R} \setminus \{0\}$. The set of slice monogenic fractional linear functions is given by the set of right cosets of $\mathcal{N}$ in $GRAV(\mathbb{R} \oplus \mathbb{R}^n)$.

Proof. The assertion follows from the fact that translations preserving axial symmetry are those with $b \in \mathbb{R}$ and from the proof of Proposition 6.3. □

As a consequence of Proposition 6.3 we have the following analog of Theorem 3.1 and Lemma 4.1:

Proposition 6.8. *Let $A \in GRAV(\mathbb{R} \oplus \mathbb{R}^n)$, and let f be a function that is slice monogenic over an axially symmetric open set $U \subseteq \mathbb{R}^{n+1}$. Then the function*

$$F(x) := f\left(\mathcal{M}_A\langle x \rangle\right)$$

is slice monogenic for all $x \in \mathcal{M}_A^{-1}(U)$.

Proof. The proof is based on the fact that the statement is true for the generators of $GRAV(\mathbb{R} \oplus \mathbb{R}^n)$. Let A be a generator; in this case, $\mathcal{M}_A$ turns out to be an intrinsic slice monogenic function and thus the composition $f(\mathcal{M}_A\langle x \rangle)$ is slice monogenic. □

The behavior of the Cauchy kernel $S^{-1}(y, x)$, is slice monogenic on the left in x and on the right in y, composed with slice monogenic Möbius transformations is described in the following result. To this end, we set

$$J(\widetilde{\mathcal{M}}, y) = (\widetilde{a}y + \widetilde{b}) *_r (\widetilde{c}y + \widetilde{d})^{-*_r}.$$

Proposition 6.9. *The following formula holds:*

$$S^{-1}\big(\widetilde{\mathcal{M}}\langle y\rangle, \mathcal{M}\langle x\rangle\big) = (a\widetilde{d} - c\widetilde{b})^{-1} S^{-1}(y,x) * (J(\widetilde{\mathcal{M}}, y))^{-1} * J(\mathcal{M}, x)^{-*},$$

where

$$J(\mathcal{M}, x) = (xc + d)^{-*}, \quad J(\widetilde{\mathcal{M}}, y) = (\widetilde{c}y + \widetilde{d})^{-*_r}$$

and the $\star$-product is computed in the variable x.

Proof. It is convenient to write $S^{-1}(y,x)$ in the form $(y-x)^{-*}$ so that

$$\begin{aligned}\big(\widetilde{\mathcal{M}}\langle y\rangle, \mathcal{M}\langle x\rangle\big)^{-*} &= \big[(xc+d)^{-*} * \big[(xc+d)(\widetilde{a}y+\widetilde{b}) - (xa+b)(\widetilde{c}y+\widetilde{d})\big] \\ &\qquad *_r (\widetilde{c}y+\widetilde{d})^{-*_r}\big]^{-*} \\ &= \big[(xc+d)^{-*} * [(d\widetilde{a} - b\widetilde{c})y - x(a\widetilde{d} - c\widetilde{b})] *_r (\widetilde{c}y+\widetilde{d})^{-*_r}\big]^{-*} \\ &= (a\widetilde{d} - c\widetilde{b})^{-1}\big[(xc+d)^{-*} * (y-x) *_r (\widetilde{c}y+\widetilde{d})^{-*_r}\big]^{-*} \\ &= (a\widetilde{d} - c\widetilde{b})^{-1}\big[(y-x) *_r (\widetilde{c}y+\widetilde{d})^{-*_r}\big]^{-*} * (xc+d),\end{aligned}$$

where the $*$-products are computed in the variable x and the $*_r$-products are computed in y. We used the fact that $c\widetilde{a} \in \mathbb{R}^{n+1}$ and thus it equals $a\widetilde{c}$ (and similarly for $d\widetilde{b}$) and the fact that the pseudodeterminant $a\widetilde{d} - c\widetilde{b}$ is real and so

$$\widetilde{(a\widetilde{d} - c\widetilde{b})} = a\widetilde{d} - c\widetilde{b}.$$

We then have

$$\begin{aligned}\big[(y-x) *_r (\widetilde{c}y+\widetilde{d})^{-*_r}\big]^{-*} &= \big[(y-x) *_r (\widetilde{c}y+\widetilde{d})^{-*_r}\big]^{-*} \\ &= \big[(\widetilde{c}y+\widetilde{d})^{-*_r} * (y-x)\big]^{-*} \\ &= (y-x)^{-*} * \big((\widetilde{c}y+\widetilde{d})^{-*_r}\big)^{-1}.\end{aligned}$$

The statement follows. □

Remark 6.10. Using the generators of $GAV(\mathbb{R} \oplus \mathbb{R}^n)$ one has that for translations and dilations

$$S^{-1}\big(\widetilde{\mathcal{M}}\langle y\rangle, \mathcal{M}\langle x\rangle\big) = S^{-1}(y,x).$$

For the inversion

$$S^{-1}\big(\widetilde{\mathcal{M}}\langle y\rangle, \mathcal{M}\langle x\rangle\big) = S^{-1}(y,x) * y * x = xS^{-1}(y,x)y.$$

Finally, for the transformations associated with matrices of the form aI, $a \in \mathbb{S}^{n-1}$ we have

$$S^{-1}\big(\widetilde{\mathcal{M}}\langle y\rangle, \mathcal{M}\langle x\rangle\big) = -S^{-1}(y,x) * (a^{-1})^{-1} * a = -S^{-1}(y,x)a^2 = S^{-1}(y,x).$$

However, the analogue of the invariance integral formula (8) does not look so simple in this case.

Remark 6.11. The group considered in [39] is $GL(2, \mathbb{H})$ and so the only condition on the matrix elements is that the Dieudonné determinant of a 2×2 quaternionic matrix does not vanish. This fact translates into the condition $\|b - ac^{-1}d\|\,\|b\| \neq 0$ when $c \neq 0$, and $\|ad\| \neq 0$ when $c = 0$. The remaining Ahlfors–Vahlen conditions are automatically fulfilled because the quaternions are closed under multiplication. Thus, any invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ defines

$$\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, x\right) = (x \cdot c + d)^{-*} * (xa + b) = (qc + d)^{-*} * (qa + b),$$

thus, in the quaternionic case, the group of transformations is larger. In fact a slice monogenic transformation is either affine or there exist $a, b, p \in \mathbb{H}$ such that $\mathcal{M}\langle x \rangle = (x - p)^{-*} * (xa + b)$, $x \in \mathbb{H}$, see [39, Proposition 5.3].

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