



# Evolution of Superoscillations in the Dirac Field

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## Abstract

Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. The study of the evolution of superoscillations as initial datum of field equations requires the notion of supershift, which generalizes the concept of superoscillations. The present paper has a dual purpose. The first one is to give an updated and self-contained explanation of the strategy to study the evolution of superoscillations by referring to the quantum-mechanical Schrödinger equation and its variations. The second purpose is to treat the Dirac equation in relativistic quantum theory. The treatment of the evolution of superoscillations for the Dirac equation can be deduced by recent results on the Klein–Gordon equation, but further additional considerations are in order, which are fully described in this paper.

**Keywords** Superoscillating functions · Convolution operators · Schrödinger equation · Dirac equation · Entire functions with growth conditions

**Mathematics Subject Classification** 32A15 · 32A10 · 47B38

## 1 Introduction to Superoscillations and the Supershift Property

Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. These functions (or sequences) appear in weak values in quantum mechanics (see [3, 12, 14, 26]) and in several other fields of science and technology, as it will be mentioned in the sequel. A natural problem, suggested by Aharonov, is to study the evolution of weak-values-superoscillations as initial datum of Schrödinger equation or as initial condition of other quantum field equations such as Klein–Gordon, see [10], or Dirac equation which is treated in this paper.

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In order to explain the topic and our main results we consider in this section some heuristic facts and properties of superoscillations and we introduce a diagram to visualize them. Furthermore, we discuss, from an intuitive point of view, the notion of supershift property of the solutions of differential equations which generalizes the notion of superoscillations. The supershift property is the crucial concept to investigate the evolution of superoscillatory initial datum by all field equations.

In the last decade a systematic study of superoscillating functions has been carried out also from the mathematical point of view. In fact, the rigorous treatment of the evolution of superoscillations, as initial datum of the Schrödinger equation, required some sophisticated mathematical tools, as it has been shown for example in [1, 2, 4–8, 16, 17, 27–29] and partially summarized in [9]. The prototypical function arising in Aharonov’s weak values is given by

$$F_n(x, a) = \left( \cos \left( \frac{x}{n} \right) + ia \sin \left( \frac{x}{n} \right) \right)^n \tag{1}$$

which, using Euler’s identities and Newton’s binomial, can be written as

$$F_n(x, a) = \sum_{j=0}^n C_j(n, a) e^{i(1-2j/n)x} \tag{2}$$

where  $a > 1$  and

$$C_j(n, a) = \binom{n}{j} \left( \frac{1+a}{2} \right)^{n-j} \left( \frac{1-a}{2} \right)^j. \tag{3}$$

If we fix  $x \in \mathbb{R}$  and we let  $n$  go to infinity, we obtain that

$$\lim_{n \rightarrow \infty} F_n(x, a) = e^{iax}. \tag{4}$$

The term superoscillations comes from the fact that the frequencies  $(1 - 2j/n)$  in (2) are in modulus bounded by 1 but the frequency  $a$ , in the limit function (4), can be arbitrarily large. The superoscillatory phenomenon appears because thanks to the form the coefficients  $C_j(n, a)$  the plane waves  $e^{i(1-2j/n)x}$  interfere in such a way that they produce the plane wave  $e^{iax}$ , where  $a$  can be very large with respect to one.

Superoscillations occur also when  $n$  is a fixed number, not only when  $n$  goes to infinity, simply the region of superoscillation is a small interval that contains the origin. One can clearly see that if  $x$  is very small compared to  $n$  it is  $\frac{x}{n} \sim 0$  so in this limit we have

$$\left( \cos \left( \frac{x}{n} \right) + ia \sin \left( \frac{x}{n} \right) \right)^n \sim \left( 1 + ia \frac{x}{n} \right)^n \sim e^{iax}.$$

We illustrate in the figure the superoscillations for fixed  $n$  and  $a$  plotting the logarithm of the real part of  $F_n$ . The logarithm is necessary to re-scale the function in order to better see the superoscillations (Fig. 1).

We note that there is no conflict with the theory of Fourier series because the sequence  $F_n(x, a)$  is not a Fourier sequence even though it looks like. In fact, the

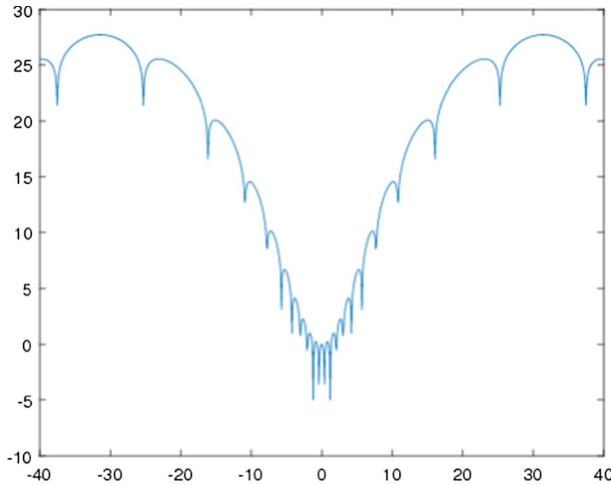


Fig. 1  $\log \operatorname{Re}(F_n(x, a)), a = 4, n = 20$

coefficients  $C_j(n, a)$  and the wave numbers  $k_j(n) := 1 - 2j/n$  depend both on  $j$  and on  $n$  while for Fourier series the numbers  $C_j(n)$  and  $k_j(n)$  depend on  $j$  and but not on  $n$ . Sequences such as  $F_n(x, a)$  are called generalized Fourier sequences and can produce superoscillations while Fourier series cannot.

In order to understand the meaning of the evolution of superoscillations as initial condition of a given quantum field equation we need the following interpretation of the generalized Fourier sequence  $F_n(x, a)$ . Consider  $x$  as a parameter and let us focus our attention on the points  $(1 - 2j/n)$ . For the sake of clarity we set

$$\varphi(\lambda, x) := e^{i\lambda x}$$

and with this position we write the sequence  $F_n(x, a)$  as

$$F_n(x, a) = \sum_{j=0}^n C_j(n, a) \varphi(1 - 2j/n, x).$$

Now consider  $x$  as a fixed parameter and recall we have set  $e^{iax} = \varphi(a, x)$ . Then the limit  $F_n(x, a) \rightarrow \varphi(a, x)$ , as  $n \rightarrow \infty$  means that we compute the function  $\lambda \mapsto \varphi(\lambda, x)$  in infinitely many points  $1 - 2j/n$  that belong to the interval  $[-1, 1]$  (band limited frequencies) and we determine the value of  $\varphi(\lambda, x)$  in the point  $\lambda = a \gg 1$ . When we consider the evolution of  $F_n(x, a)$  as initial value the free particle using Schrödinger equation

$$i\partial_t \psi(x, t) = -\partial_{xx}^2 \psi(x, t), \quad \psi(x, 0) = F_n(x, a)$$

we obtain the solution

$$\psi_n(x, t) = \sum_{k=0}^n C_k(n, a) e^{ix(1-2k/n)} e^{-it(1-2k/n)^2}.$$

We define the following function of  $\lambda$

$$\phi(\lambda, t, x) =: e^{ix\lambda} e^{-it\lambda^2},$$

where  $t$  and  $x$  are considered parameters, so that studying the evolution of superoscillations is tantamount to studying the limit of the solution

$$\psi_n(x, t) = \sum_{k=0}^n C_k(n, a) \phi(1 - 2k/n, t, x)$$

for  $n \rightarrow \infty$ . Since for the particular case of the free particle the solution of Schrödinger equation is written in terms of the exponential function we still speak of superoscillations, but in the general case of nontrivial potentials the functions  $\phi(\lambda, t, x)$  are not necessarily the exponential function.

In this case there arises the natural notion of supershift of the solution where we compute the function  $\lambda \mapsto \phi(\lambda, t, x)$  in infinitely many points  $1 - 2j/n$  that belong to the interval  $[-1, 1]$ , and we determine the value of  $\phi(\lambda, t, x)$  in the point  $\lambda = a \gg 1$ . For example for the quantum harmonic oscillator the function  $\phi(\lambda, t, x)$  can be computed by elementary functions and it is given by

$$\phi(\lambda, t, x) = (\cos t)^{-1/2} \exp(-(i/2)(x^2 + \lambda^2) \tan t + i\lambda x / \cos t).$$

As shown above, superoscillations are a particular case of supershift, which is the property to be proven when we study the evolution of superoscillations as initial datum of given field equation.

A different class of superoscillating functions with respect to the ones considered in this paper can be found in [23]. In antenna theory the phenomenon of superoscillations was discovered by Toraldo di Francia in [36] (see also the paper of Berry [20]). The literature on superoscillations is large, and without claiming completeness we mention the papers [19, 21, 22, 24, 25, 30, 31, 33, 35]. More information can be found in the introductory papers [11, 13, 15, 32].

Moreover, in the recent paper *Roadmap on superoscillations*, see [18], some of the most important achievements in the applications of superoscillations are well explained by the leading experts in this field. In fact, the main aim of the paper [18] is to give an exposition of various aspects of superoscillations that are accessible to non experts in this field. There are contributions in: optical superoscillatory focusing and imaging technologies, superoscillatory interference for super-resolution telescope, an antenna array approach to superoscillations, applications of superoscillations in ultrafast optics, superoscillations in magnetic holography and acoustics, superoscillations and information theory. The paper also contains many illustrations to clarify the experiments and it can be freely downloaded from the web.

This paper has two main purposes. The first one is to explain and motivate the general strategy to study the evolution of superoscillations in the context of quantum mechanics. The second one is to study superoscillations as initial datum of the Dirac

equation. The paper is organized as follows. In Sect. 2 we formulate and state our main results in precise mathematical term for the Dirac equation. In Sect. 3 we discuss the strategy to demonstrate our results, whose full proofs are then given in Sect. 4.

## 2 The Main Results for the Dirac Equation

In this section we formulate in precise mathematical terms the problem of the evolution of superoscillating initial datum for the Dirac equation. We summarize the main result on the supershift property for the Dirac equation, but we postpone the proofs to Sect. 4 of the paper. The sequence defined in (2) extends to the sequence of entire function  $F_n(z, a)$ , for  $z \in \mathbb{C}$ . The tools to study the evolution of superoscillations are convolution operators that naturally act on spaces of entire functions that contains the holomorphic extensions  $F_n(z, a)$ , so we recall some notions and results on entire functions. Let  $f$  be a non-constant entire function of a complex variable  $z$ . We define

$$M_f(r) = \max_{|z|=r} |f(z)|,$$

for  $r \geq 0$ . The non-negative real number  $\rho$  defined by

$$\rho = \limsup_{r \rightarrow \infty} \frac{\ln \ln M_f(r)}{\ln r}$$

is called the order of  $f$ . If  $\rho$  is finite then  $f$  is said to be of finite order and if  $\rho = \infty$  the function  $f$  is said to be of infinite order. In case  $f$  is of finite order we define the non-negative real number

$$\sigma = \limsup_{r \rightarrow \infty} \frac{\ln M_f(r)}{r^\rho},$$

which is refereed to as the type of  $f$ . More to the point, if  $\sigma \in (0, \infty)$  we say that  $f$  is of normal type, while we say that  $f$  is of minimal type if  $\sigma = 0$  and of maximal type if  $\sigma = \infty$ . The constant functions are said to be of minimal type of order zero. Let  $p \geq 1$ , we denote by  $\mathcal{A}_p$  the space of entire functions with either order lower than  $p$  or order equal to  $p$  and finite type. It consists of functions  $f$  for which there exist constants  $B, C > 0$  such that

$$|f(z)| \leq Ce^{B|z|^p}. \quad (5)$$

The extension of the functions  $F_n(x, a)$  defined in (2) are exponentially bounded and more general sequences that generalize the  $F_n(x, a)$  can be defined as follows:

$$Y_n(z) = \sum_{j=0}^n c_j(n) e^{ik_j(n)z}, \quad z \in \mathbb{C}, \quad (6)$$

with coefficients  $c_j(n) \in \mathbb{C}$  and  $k_j(n) \in \mathbb{R}$ . We say that  $Y_n$  is a superoscillating sequence (or function), if there exists some  $a \in \mathbb{R}$ , with  $|a| > 1$ , such that

- (i)  $\sup_{n \in \mathbb{N}_0, j \in \{0, \dots, n\}} |k_j(n)| \leq 1,$
- (ii)  $\lim_{n \rightarrow \infty} Y_n(z) = e^{iaz}$  on the compact sets of  $\mathbb{C}.$

Superoscillating functions as  $F_n$  do not only converge uniformly on the compact sets of  $\mathbb{C}$  but they converge in the space  $\mathcal{A}_1.$  More in general we recall that the sequence  $(f_n)_{n \in \mathbb{N}} \in \mathcal{A}_p$  converges to  $f_0 \in \mathcal{A}_p$  if there exists some  $B > 0$  such that

$$\lim_{n \rightarrow \infty} \sup_{z \in \mathbb{C}} |(f_n(z) - f_0(z))e^{-B|z|^p}| = 0. \tag{7}$$

We now come to the definition of supershift property. Let  $\mathcal{I} \subseteq \mathbb{R}$  be an interval with  $[-1, 1] \subset \mathcal{I}, \Omega \subseteq \mathbb{R}^d,$  for  $d \in \mathbb{N},$  be a domain and let  $\varphi : \mathcal{I} \times [0, T] \times \Omega \rightarrow \mathbb{R}, T > 0$  be a continuous function on  $\mathcal{I}.$  We set

$$\varphi_\lambda(t, x) := \varphi(\lambda, t, x),$$

and consider a sequence of points  $(\lambda_{k,n})$  such that

$$(\lambda_{k,n}) \in [-1, 1] \text{ for } k = 0, \dots, n \text{ and } n = 0, \dots, +\infty,$$

and a sequence of complex numbers  $(c_k(n))$  for  $k = 0, \dots, n$  and  $n = 0, \dots, +\infty.$  We define the functions

$$\psi_n(t, x) = \sum_{k=0}^n c_k(n) \varphi_{\lambda_{k,n}}(t, x). \tag{8}$$

If we have that  $\lim_{n \rightarrow \infty} \psi_n(t, x) = \varphi_a(t, x)$  for some  $a \in \mathcal{I}$  and with  $|a| > 1,$  we say that the function  $\lambda \rightarrow \varphi_\lambda(t, x),$  for  $t$  and  $x$  fixed, admits a supershift in  $\lambda.$

Finally we state our results for the Dirac equation. Let  $\gamma^\mu$  be the Gamma matrices and let  $\varphi_0^+(x)$  and  $\varphi_0^-(x)$  represent the wave functions for the positive and negative energy level. Now consider, for  $a, b \in \mathbb{R}$  and  $m > 0,$  the Cauchy problem

$$(i\gamma^\mu \partial_\mu - m)\varphi(t, x) = 0, \quad t \neq 0, \quad x \in \mathbb{R}^3, \quad \text{and} \quad \varphi(0, x) = \left[ \varphi_{0,a}^+(x) \quad \varphi_{0,b}^-(x) \right]^T, \quad x \in \mathbb{R}^3 \tag{9}$$

where

$$\varphi_{0,a}^+(x) = \begin{pmatrix} e^{iax_1} \\ 0 \end{pmatrix}, \quad \varphi_{0,b}^-(x) = \begin{pmatrix} e^{ibx_1} \\ 0 \end{pmatrix}, \tag{10}$$

and denote by  $\varphi_{1,a}(t, x), \varphi_{2,b}(t, x), \varphi_{3,b}(t, x)$  and  $\varphi_{4,b}(t, x)$  the components of the solution  $\varphi(t, x).$  Suppose that  $\psi_{0,n}^+(x)$  and  $\psi_{0,n}^-(x)$  are the superoscillating initial data of the Cauchy problem

$$(i\gamma^\mu \partial_\mu - m)\psi(t, x) = 0, \quad t \neq 0, \quad x \in \mathbb{R}^3, \quad \text{and} \quad \psi_n(0, x) = \left[ \psi_{0,n}^+(x) \quad \psi_{0,n}^-(x) \right]^T, \quad x \in \mathbb{R}^3 \tag{11}$$

where

$$\psi_{0,n}^+(x) = \begin{pmatrix} F_n(x_1, a) \\ 0 \end{pmatrix}, \quad \psi_{0,n}^-(x) = \begin{pmatrix} F_n(x_1, b) \\ 0 \end{pmatrix}. \quad (12)$$

Then the solution of (11) can be written in terms of the solution of the Cauchy problem (9) as

$$\begin{aligned} \psi_{j,n}(t, x) &= \sum_{k=0}^n C_k(n, a) \varphi_{j,1-2k/n}(t, x), \quad j = 1, 4, \\ \psi_{j,n}(t, x) &= \sum_{k=0}^n C_k(n, b) \varphi_{j,1-2k/n}(t, x), \quad j = 2, 3, \end{aligned} \quad (13)$$

and the supershift property holds, that is

$$\lim_{n \rightarrow \infty} \psi_{j,n}(t, x) = \varphi_{j,a}(t, x), \quad \text{for } j = 1, 4 \quad \lim_{n \rightarrow \infty} \psi_{j,n}(t, x) = \varphi_{j,b}(t, x), \quad \text{for } j = 2, 3.$$

The precise mathematical proof of the above result is not trivial and all the details are given in Sect. 4.

### 3 Supershift and Techniques Based on Infinite Order Differential Operators

In this section we recall some facts leading to the definition of supershift and we summarize the main mathematical tools used to study superoscillations. From the prototypical example described in the previous section one can extend the definition of superoscillations to the case of several variables. To this aim, we give the definition of generalized Fourier sequence in several variables, which is a variation of the definition introduced in [8].

**Definition 3.1** (*Generalized Fourier sequence in several variables*) For  $m \in \mathbb{N}$  we assume that  $(x_1, \dots, x_m) \in \mathbb{R}^m$ . Let  $(\lambda_{k,j}(n))$ ,  $k = 0, \dots, n$  for  $n = 0, \dots, \infty$ , be real-valued sequences for  $j = 1, \dots, m$ . We call *generalized Fourier sequence in several variables* a sequence of the form

$$f_n(x_1, \dots, x_m) = \sum_{k=0}^n c_k(n) \prod_{j=1}^m e^{ix_j \lambda_{k,j}(n)}, \quad (14)$$

where  $(c_k(n))$ ,  $k = 0, \dots, n$ ,  $n = 0, \dots, +\infty$  is a real-valued sequence.

**Definition 3.2** (*Superoscillating sequence*) A generalized Fourier sequence in several variables  $f_n(x_1, \dots, x_m)$ , is said to be a *superoscillating sequence* if, for

$$\sup_{k=0, \dots, n, n \in \mathbb{N}_0} |\lambda_{k,j}(n)| \leq 1, \quad \text{for } j = 1, \dots, m$$

we have

$$\lim_{n \rightarrow \infty} f_n(x_1, \dots, x_m) = \prod_{j=1}^m e^{ix_j g_j}$$

and there exists a compact subset of  $\mathbb{R}^m$ , which will be called a *superoscillation set*, on which  $f_n$  converges uniformly to  $\prod_{j=1}^m e^{ix_j g_j}$ , where  $|g_j| > 1$  for  $j = 1, \dots, m$ .

A very important particular case of generalized Fourier sequence in several variables is the following

$$\mathcal{Y}_n(x_1, \dots, x_m) = \sum_{k=0}^n C_k(n, a) e^{ix_1(1-2k/n)^{q_1}} \dots e^{ix_m(1-2k/n)^{q_m}}$$

where  $C_k(n, a)$  are given by (3) and  $q_j \in \mathbb{N}$ , for  $j = 1, \dots, m$ .

From an historical point of view the main problem, which was first suggested by Aharonov, was to study the evolution of superoscillations by Schrödinger equation for the force free field with rigorous mathematical tools. This example was the starting point to understand the most natural functional setting in which one can study the evolution of superoscillations. We have already mentioned the solution of the Schrödinger equation with superoscillatory initial datum for the free particle

$$i\partial_t \psi(x, t) = -\partial_{xx}^2 \psi(x, t), \quad \psi(x, 0) = F_n(x, a) \tag{15}$$

is given by

$$\psi_n(x, t) = \sum_{k=0}^n C_k(n, a) e^{ix(1-2k/n)} e^{-it(1-2k/n)^2} \tag{16}$$

The crucial problem is to show that during the evolution, as  $n$  goes to infinity, the superoscillatory behavior of the limit function persists. It turns out that the limit

$$\lim_{n \rightarrow \infty} \psi_n(x, t) = e^{iax - ia^2 t},$$

is uniform on compact sets in  $\mathbb{R}^2$ . This means that the limit function  $e^{iax - ia^2 t}$  has a superoscillatory behavior. The main tools used to compute rigorously such a limit are convolution operators acting on spaces of entire functions. For such operators a crucial fact is to prove a continuity theorem. Specifically, for the free particle the convolution operator associated with this problem is

$$U(t, \partial_x) := \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \partial_x^{2m}$$

so that the evolution problem can be written as

$$\psi_n(x, t) = U(t, \partial_x) F_n(x, a)$$

for every  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$ . Clearly, the function in (16) is a generalized Fourier sequence in two variables, and for  $a > 1$  it is a superoscillatory function.



Inspired by the example of the free particle, if in Eq. (15) one replaces the second derivative with respect to the space variable  $x$  by the partial derivative  $\partial_x^p \psi(x, t)$  of order  $p \in \mathbb{N}$ , it is possible to prove that we still have a superoscillating function. In this case we have ( $p$  is even)

$$\psi_n(x, t) = \sum_{k=0}^n C_k(n, a) e^{ix(1-2k/n)} e^{-i(1-2k/n)^p t}$$

which converges, for  $n \rightarrow \infty$ , and uniformly on compact sets, to  $\psi(x, t) = e^{iax - ia^p t}$ . More in general, there is a very large class of superoscillatory functions that can be generated starting from convolution equations of the type

$$i\partial_t \psi(x, t) = \mathcal{P}(\partial_x) \psi(x, t), \quad \psi(x, 0) = F_n(x, a) \quad (17)$$

where the second derivative with respect to time in (15) has been replaced by a suitable convolution operator

$$\mathcal{P}(\partial_x) := \sum_{\ell=0}^{\infty} a_{\ell} \partial_x^{\ell}$$

where the complex coefficients  $a_{\ell}$  satisfy give growth conditions.

For the above class of differential equations, or more in general for convolution equations, the solutions remain in the class of generalized Fourier sequences, but when we take into account the Schrödinger equation with a nonconstant potential it turns out that the solution is not a generalized Fourier series in several variable. For instance, when we consider a linear potential, which means that the evolution takes place in a constant electric field, the solution of the Cauchy problem

$$i\partial_t \psi(t, x) = -\frac{1}{2} \partial_x^2 \psi(t, x) - x\psi(t, x), \quad \psi(0, x) = \sum_{k=0}^n C_k(n, a) e^{ix(1-2k/n)}, \quad (18)$$

is given by

$$\psi_n(t, x) = \sum_{k=0}^n C_k(n, a) e^{-it^3/6} e^{-i(1-2k/n)t((1-2k/n)+t)/2} e^{i((1-2k/n)+t)x}, \quad (19)$$

which is not of the form (14). So, by computing the limit for  $n \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} \psi_n(t, x) = e^{-it^3/6} e^{-iat(a+t)/2} e^{i(a+t)x}.$$

Hence, the class of superoscillations proves too narrow. In order to overcome this limitation, the notion of supershift has been introduced.

**Definition 3.3** (*Supershift*) Let  $\mathcal{I} \subseteq \mathbb{R}$  be an interval with  $[-1, 1] \subset \mathcal{I}$ ,  $\Omega \subseteq \mathbb{R}^d$ , for  $d \in \mathbb{N}$ , be a domain and let  $\varphi : \mathcal{I} \times [0, T] \times \Omega \rightarrow \mathbb{R}$ ,  $T > 0$  be a continuous function on  $\mathcal{I}$ . We set

$$\varphi_\lambda(t, x) := \varphi(\lambda, t, x),$$

and we consider a sequence of points  $(\lambda_{k,n})$  such that

$$(\lambda_{k,n}) \in [-1, 1] \text{ for } k = 0, \dots, n \text{ and } n = 0, \dots, +\infty.$$

We define the functions

$$\psi_n(t, x) = \sum_{k=0}^n c_k(n) \varphi_{\lambda_{k,n}}(t, x), \tag{20}$$

where  $(c_k(n))$  is a sequence of complex numbers for  $k = 0, \dots, n$  and  $n = 0, \dots, +\infty$ . If  $\lim_{n \rightarrow \infty} \psi_n(t, x) = \varphi_a(t, x)$  for some  $a \in \mathcal{I}$  with  $|a| > 1$ , we say that the function  $\lambda \rightarrow \varphi_\lambda(t, x)$ , for  $t$  and  $x$  fixed, admits a supershift in  $\lambda$ .

Let us stress that the term supershift comes from the fact that  $\mathcal{I}$  can be an arbitrarily large (it can also be  $\mathbb{R}$ ) and so also  $a$  can be arbitrarily far away from the interval  $[-1, 1]$  where the function  $\varphi_\lambda$  is computed, see (20). In the case of the sequence  $(F_n)$  in (2), we have  $\lambda_{k,n} = 1 - 2k/n$ ,  $\varphi_{\lambda_{k,n}}(t, x) = e^{i(1-2k/n)x}$  and  $c_k(n)$  are the coefficients  $C_k(n, a)$  defined in (3).

### 3.1 Schrödinger-Like Equations

By using the notion of supershift, it is now possible to formulate in precise mathematical terms the problem of the evolution of superoscillations for a quantum mechanical system described by the Hamiltonian operator  $\mathcal{H}$ . We consider the evolution problem

$$i\partial_t \psi(t, x) = \mathcal{H}(t, x) \psi(t, x), \quad \psi(0, x) = F_n(x, a) \tag{21}$$

and we denote by  $\psi_n(t, x)$  the solution of the Cauchy problem (21), by  $\varphi_a(t, x)$  we indicate the solution of the Schrödinger equation in (21) where we replace  $F_n$  by the initial datum  $e^{iax}$ . Given the solutions  $\psi_n(t, x)$  and  $\varphi_a(t, x)$ , owing to the linearity of Schrödinger equation, the function  $\psi_n(t, x)$  can be written as

$$\psi_n(t, x) = \sum_{k=0}^n C_j(n, a) \varphi_{1-2k/n}(t, x).$$

The persistence of superoscillations means that the supershift property holds. Hence, we have to show that

$$\lim_{n \rightarrow \infty} \psi_n(t, x) = \varphi_a(t, x)$$

or, equivalently, that the function  $\lambda \rightarrow \varphi_\lambda(t, x)$  admits a supershift in the sense Definition 3.3.

When the Green function of the time-dependent Schrödinger equation is known, there is a standard procedure to study the evolution problem (21) and to determine

under which conditions the supershift property holds. The first step consists in writing the solution of the Cauchy problem (21) in term of the function  $\varphi_a(t, x)$  which is the solution of the problem

$$i\partial_t\varphi(t, x) = \mathcal{H}\varphi(t, x), \quad \varphi(0, x) = e^{iax}. \tag{22}$$

The Green function  $K$  of Schrödinger equation with Hamiltonian  $\mathcal{H}$  gives

$$\varphi_a(t, x) = \int_{\Omega} e^{ia y} K(x, y, t) dy, \tag{23}$$

where, in most typical cases,  $\Omega$  amounts to the real line  $\mathbb{R}$  or the half line  $\mathbb{R}^+$ . Accordingly, the solution of problem (21) is given by

$$\psi_n(t, x) = \sum_{k=0}^n C_k(n, a) \int_{\Omega} e^{i(1-2k/n)y} K(x, y, t) dy.$$

Then, in order to study the supershift property, some crucial considerations are in order:

(I) The first important step is to establish in what sense the integral (23) converges. This fact can be assured by regularizing the integral or by considering the integral defined as a Fresnel integral. Here complex analysis plays an important role in the definition of the integral.

(II) Once that point (I) is enforced, we have to assure that the function

$$a \mapsto \int_{\Omega} e^{ia y} K(x, y, t) dy$$

is analytic and then we need suitable estimates, which allows us to exchange the series with the integral in such a way that we obtain

$$\varphi_a(t, x) = \sum_{m=0}^{\infty} \frac{a^m}{m!} \int_{\Omega} (iy)^m K(x, y, t) dy. \tag{24}$$

This is quite delicate fact to handle because the integral is computed, in general, on an unbounded domain  $\Omega$ .

(III) Finally, suppose formula (24) holds, we set

$$\varphi_a(t, x) = \sum_{m=0}^{\infty} \lambda_m(x, t) a^m, \tag{25}$$

where

$$\lambda_m(x, t) := \int_{\Omega} \frac{(iy)^m}{m!} K(x, y, t) dy,$$

and by introducing the auxiliary complex variable  $\xi$  we represent the function  $\varphi_a(t, x)$  as

$$\varphi_a(t, x) = \mathcal{U}(x, t, \partial_\xi) e^{ia\xi} |_{\xi=0}$$

where  $\mathcal{U}(x, t, \partial_\xi)$  is the convolution operator defined by

$$\mathcal{U}(x, t, \partial_\xi) := \sum_{m=0}^\infty \frac{\lambda_m(x, t)}{i^m} \partial_\xi^m$$

acting on  $e^{ia\xi}$  and  $|_{\xi=0}$  denotes the restriction to  $\xi = 0$ .

The above steps make sense when we can prove that the operator  $\mathcal{U}(x, t, \partial_\xi)$  acts continuously on a space of entire functions that contains the exponential function. For example, if we show that

$$\mathcal{U}(x, t, \partial_\xi) : \mathcal{A}_1 \rightarrow \mathcal{A}_1$$

acts continuously, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_n(t, x) |_{\xi=0} &= \lim_{n \rightarrow \infty} \mathcal{U}(x, t, \partial_\xi) F_n(\xi, a) |_{\xi=0} \\ &= \mathcal{U}(x, t, \partial_\xi) \lim_{n \rightarrow \infty} F_n(\xi, a) |_{\xi=0} \\ &= \mathcal{U}(x, t, \partial_\xi) e^{ia\xi} |_{\xi=0} \\ &= \varphi_a(t, x) \end{aligned}$$

which in turn gives the sought-after supershift property. This follows because the sequence  $F_n(\xi, a)$  converges to  $e^{ia\xi}$  in  $\mathcal{A}_1$  and in particular it converges uniformly on the compact set in  $\mathbb{C}$ .

The above procedure is very general and it works also for different types of evolution equations recently it has been applied to the Klein–Gordon equation (see [10]). In the next section, by taking advantage of some preliminary results presented in [10], we study the supershift property for the Dirac field.

### 4 Proofs of the Main Results for the Dirac Equation

In this section we recall a well known representation formula for the solution of Klein–Gordon equation in  $\mathbb{R}^3$  expressed by smooth functions like the Bessel function  $J_1$ . This formula is a consequence of the knowledge of the Green function and it expresses the solution of the Cauchy problem

$$(\partial_t^2 - \Delta + m^2)\psi(t, x) = 0, \quad \psi(0, x) = f(x), \quad \psi_t(0, x) = g(x). \tag{26}$$

Precisely, we consider  $y = (y_1, y_2, y_3) \in \mathbb{R}^3$  with  $y$  written in spherical coordinates, i.e.,  $y_1 = \rho \sin \phi \cos \theta$ ,  $y_2 = \rho \sin \phi \sin \theta$ ,  $y_3 = \rho \cos \phi$ . We will set  $d^3y = r^2 dr d\Omega_y$ , where  $d\Omega_y = \sin \phi d\theta d\phi$ , for  $\rho \in [0, \infty)$ ,  $\phi \in [0, \pi]$ ,  $\theta \in [0, 2\pi]$ .

**Theorem 4.1** *Let  $f \in C^3(\mathbb{R}^3)$  and  $g \in C^2(\mathbb{R}^3)$ . Then the solution of the Cauchy problem (26) is given by*

$$\begin{aligned} \psi(t, x) = & \frac{t}{4\pi} \int_{\partial B_t(0)} g(x+y)d\Omega_y - \frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} g(x+y)d^3y \\ & + \partial_t \left( \frac{t}{4\pi} \int_{\partial B_t(0)} f(x+y)d\Omega_y - \frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} f(x+y)d^3y \right) \end{aligned} \tag{27}$$

where  $J_1$  is the Bessel function, the spherical coordinates are given by  $d^3y = r^2 dr d\Omega_y$ , and  $B_t(0)$  is the ball with radius  $t$  centered at the origin. Moreover,  $\psi$  belongs to in  $C^2(\mathbb{R}^4)$ .

**Proof** See Theorem 3.3.1 in [34]. □

Here we introduce some basic facts to study the evolution of superoscillations in the Dirac field. We recall the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the Gamma matrices, which can be written in terms of the Pauli matrices, are given by

$$\gamma^0 = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{pmatrix}, \quad \gamma^\ell = \begin{pmatrix} 0 & \sigma^\ell \\ -\sigma^\ell & 0 \end{pmatrix}, \quad \ell = 1, 2, 3.$$

The components of the solution of the Dirac equation satisfy the Klein–Gordon equation. We now focus on the Dirac equation with the associated Cauchy problem

$$(i\gamma^\mu \partial_\mu - m)\psi(t, x) = 0, \quad t \neq 0 \quad x \in \mathbb{R}^3, \quad \psi(0, x) = \psi_0(x). \tag{28}$$

The solution  $\psi(t, x)$  has components  $\psi_j$ , for  $j = 1, 2, 3, 4$ , and we write

$$\psi(t, x) = [\psi_1(t, x) \quad \psi_2(t, x) \quad \psi_3(t, x) \quad \psi_4(t, x)]^T,$$

where  $T$  indicates the transposed vector, the components  $\psi_j$  are explicitly given by

$$\begin{aligned} \psi_j(t, x) = & \frac{t}{4\pi} \int_{\partial B_t(0)} g_j(x+y)d\Omega_y - \frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} g_j(x+y)d^3y \\ & + \partial_t \left( \frac{t}{4\pi} \int_{\partial B_t(0)} f_j(x+y)d\Omega_y - \frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} f_j(x+y)d^3y \right) \end{aligned} \tag{29}$$

for  $j = 1, 2, 3, 4$ , with the initial data that are given by

$$f_j(x) := (\psi_0(x))_j, \quad g_j(x) := -i(\alpha^k \partial_k \psi_0)_j(x) + m(\beta \psi_0)_j(x), \quad j = 1, 2, 3, 4$$

where we have used the matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^\ell = \begin{pmatrix} 0 & \sigma^\ell \\ \sigma^\ell & 0 \end{pmatrix}, \quad \ell = 1, 2, 3$$

and  $I$  is the  $2 \times 2$  identity matrix. In order to study superoscillations, we then need to consider the Cauchy problem associated with the plane waves. So, we start with the following result.

**Theorem 4.2** *Let  $\varphi_0^+(x)$  and  $\varphi_0^-(x)$  represent the wave functions for the positive and negative energy level. Consider the Cauchy problem*

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)\varphi(t, x) &= 0, \quad t \neq 0, \quad x \in \mathbb{R}^3, \quad \text{and} \quad \varphi(0, x) = \left[ \varphi_{0,a}^+(x) \quad \varphi_{0,b}^-(x) \right]^T, \quad x \in \mathbb{R}^3 \\ \varphi_{0,a}^+(x) &= \begin{pmatrix} e^{iax_1} \\ 0 \end{pmatrix}, \quad \varphi_{0,b}^-(x) = \begin{pmatrix} e^{ibx_1} \\ 0 \end{pmatrix}, \end{aligned} \tag{30}$$

where  $a$  and  $b$  are real numbers and  $m > 0$ . Define the functions:

$$\begin{aligned} U_1(b, t) &:= \frac{t}{4\pi} \int_{\partial B_t(0)} e^{iby_1} d\Omega_y, \\ U_2(b, t) &:= -\frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{iby_1} d^3y. \end{aligned} \tag{31}$$

Then the components of the solution of the Dirac equation are given by

$$\begin{aligned} \varphi_{1,a}(t, x) &= mU_1(a, t)e^{iax_1} + mU_2(a, t)e^{iax_1} + \partial_t U_1(a, t)e^{iax_1} + \partial_t U_2(a, t)e^{iax_1}, \\ \varphi_{2,b}(t, x) &= bU_1(b, t)e^{ibx_1} + bU_2(b, t)e^{ibx_1}, \\ \varphi_{3,b}(t, x) &= mU_1(b, t)e^{ibx_1} + mU_2(b, t)e^{ibx_1} + \partial_t U_1(b, t)e^{ibx_1} + \partial_t U_2(b, t)e^{ibx_1}, \\ \varphi_{4,a}(t, x) &= bU_1(a, t)e^{iax_1} + aU_2(a, t)e^{iax_1}. \end{aligned} \tag{32}$$

**Proof** Let us write explicitly the function  $g$ :

$$g(x) := -i(\alpha^k \partial_k \psi_0(x)) + m(\beta \psi_0(x))$$

so we have that

$$\begin{aligned} g(x) &= -i \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_1} \psi_0^+(x) \\ \partial_{x_1} \psi_0^-(x) \end{pmatrix} - i \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_2} \psi_0^+(x) \\ \partial_{x_2} \psi_0^-(x) \end{pmatrix} \\ &\quad - i \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_3} \psi_0^+(x) \\ \partial_{x_3} \psi_0^-(x) \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \psi_0^+(x) \\ \psi_0^-(x) \end{pmatrix} \end{aligned}$$

and, taking into account with the initial datum (30), we obtain

$$g(x) = -i \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_1} \psi_0^+(x) \\ \partial_{x_1} \psi_0^-(x) \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \psi_0^+(x) \\ \psi_0^-(x) \end{pmatrix}$$

so

$$g_+(x) = -i\sigma^1 \partial_{x_1} \psi_0^-(x) + m\psi_0^+(x) = b\sigma^1 \psi_0^-(x) + m\psi_0^+(x)$$

and

$$g_-(x) = -i\sigma^1 \partial_{x_1} \psi_0^+(x) - m\psi_0^-(x) = a\sigma^1 \psi_0^+(x) - m\psi_0^-(x)$$

and the components of  $g_{\pm}$  are explicitly given by

$$g_+(x) = \begin{pmatrix} me^{iax_1} \\ be^{ibx_1} \end{pmatrix}, \quad g_-(x) = \begin{pmatrix} me^{ibx_1} \\ ae^{iax_1} \end{pmatrix}.$$

With the above computations we obtain the set of conditions:

$$g_1(x) = me^{iax_1}, \quad g_2(x) = be^{ibx_1}, \quad g_3(x) = me^{ibx_1}, \quad g_4(x) = ae^{iax_1}$$

and also

$$f_1(x) = e^{iax_1}, \quad f_2(x) = 0, \quad f_3(x) = e^{ibx_1}, \quad f_4(x) = 0.$$

Based on Theorem 4.1, the components of the Dirac equation can be written as

$$\begin{aligned} \varphi_{1,a}(t, x) = & \frac{mt}{4\pi} \int_{\partial B_t(0)} e^{ia(x_1+y_1)} d\Omega_y - \frac{m^2}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{ia(x_1+y_1)} d^3y \\ & + \partial_t \left( \frac{t}{4\pi} \int_{\partial B_t(0)} e^{ia(x_1+y_1)} d\Omega_y - \frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{ia(x_1+y_1)} d^3y \right), \end{aligned} \quad (33)$$

$$\varphi_{2,b}(t, x) = \frac{bt}{4\pi} \int_{\partial B_t(0)} e^{ib(x_1+y_1)} d\Omega_y - \frac{bm}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{ib(x_1+y_1)} d^3y, \quad (34)$$

$$\begin{aligned} \varphi_{3,b}(t, x) = & \frac{mt}{4\pi} \int_{\partial B_t(0)} e^{ib(x_1+y_1)} d\Omega_y - \frac{m^2}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{ib(x_1+y_1)} d^3y \\ & + \partial_t \left( \frac{t}{4\pi} \int_{\partial B_t(0)} e^{ib(x_1+y_1)} d\Omega_y - \frac{m}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{ib(x_1+y_1)} d^3y \right), \end{aligned} \quad (35)$$

$$\varphi_{4,a}(t, x) = \frac{at}{4\pi} \int_{\partial B_t(0)} e^{ia(x_1+y_1)} d\Omega_y - \frac{ma}{4\pi} \int_{B_t(0)} \frac{J_1(m\sqrt{t^2 - |y|^2})}{\sqrt{t^2 - |y|^2}} e^{ia(x_1+y_1)} d^3y, \quad (36)$$

from which we reach the desired conclusion by using the functions  $U_1$  and  $U_2$  defined in (31).  $\square$

Let us note that it is enough to study the functions  $(\varphi_{1,a}, \varphi_{4,a})$  since the functions  $(\varphi_{2,b}, \varphi_{3,b})$  are obtained by commuting  $a$  and  $b$ . As it was proven in [10], the functions  $U_1$  and  $U_2$  can be written in terms of convolution operators acting continuously on the space of entire functions  $\mathcal{A}_1$ . For  $t \in [0, T]$ , with  $T > 0$ , we define the following convolution operators

$$\mathcal{U}_j(t, \partial_{x_1}) := \sum_{\ell=0}^{\infty} B_{j,\ell}(t) \partial_{x_1}^{\ell} : \mathcal{A}_1 \rightarrow \mathcal{A}_1, \quad j = 1, 2, 3, 4, 5 \tag{37}$$

where the coefficients  $B_{j,\ell}(t)$  are given by

$$B_{1,\ell}(t) := \frac{1}{4\pi} \frac{t^{\ell}}{\ell!} K(\ell + 1, \ell), \tag{38}$$

$$B_{2,\ell}(t) := -\frac{m}{4\pi} \frac{1}{\ell!} K(\ell + 1, \ell) \int_0^t \rho^{2+\ell} \frac{J_1(m\sqrt{t^2 - \rho^2})}{\sqrt{t^2 - \rho^2}} d\rho, \tag{39}$$

$$B_{3,\ell}(t) := \frac{1}{4\pi} \frac{t^{\ell+1}}{\ell!} K(\ell + 2, \ell + 1), \tag{40}$$

$$B_{4,\ell}(t) := -\frac{m^2 t}{4\pi} \frac{1}{\ell!} K(\ell + 1, \ell) \int_0^t \frac{J_2(m\sqrt{t^2 - \rho^2})}{t^2 - \rho^2} \rho^{\ell+2} d\rho, \tag{41}$$

$$B_{5,\ell}(t) := \frac{m^2}{8\pi} \frac{t^{\ell+2}}{\ell!} K(\ell + 1, \ell), \tag{42}$$

with the function  $K(p, q)$  being defined by

$$K(p, q) := \int_0^{\pi} \sin^p \phi d\phi \int_0^{2\pi} \cos^q \theta d\theta, \quad p, q \in \mathbb{N}_0. \tag{43}$$

Moreover, we also need the following lemma.

**Lemma 4.3** *For  $t \in [0, T]$ , with  $T > 0$ , we define the following convolution operators*

$$\tilde{\mathcal{U}}_j(t, \partial_{x_1}) := \sum_{\ell=0}^{\infty} \tilde{B}_{j,\ell}(t) \partial_{x_1}^{\ell+1} : \mathcal{A}_1 \rightarrow \mathcal{A}_1, \quad j = 1, 2 \tag{44}$$

where the coefficients  $\tilde{B}_{j,\ell}(t)$  are given by

$$\tilde{B}_{1,\ell}(t) := \frac{1}{4\pi i} \frac{t^{\ell+1}}{\ell!} K(\ell + 1, \ell), \tag{45}$$



$$\tilde{B}_{2,\ell}(t) := \frac{1}{i} B_{2,\ell}(t). \quad (46)$$

Then we have

$$bU_1(b, t)e^{ibx_1} = \tilde{U}_1(t, \partial_{x_1})e^{ibx_1} \quad (47)$$

and

$$bU_2(b, t)e^{ibx_1} = \tilde{U}_2(t, \partial_{x_1})e^{ibx_1} \quad (48)$$

and the operators  $\tilde{U}_j(t, \partial_{x_1})$ , for  $j = 1, 2$ , act continuously on  $\mathcal{A}_1$ .

**Proof** We observe that from the definition of the function  $U_1$  we have

$$bU_1(b, t) = \frac{1}{4\pi i} \sum_{\ell=0}^{\infty} \frac{i^{\ell+1} b^{\ell+1} t^{\ell+1}}{\ell!} K(\ell + 1, \ell).$$

and hence we obtain (47), similarly we get (48). The continuity of the operators  $\tilde{U}_j(t, \partial_{x_1})$  for  $j = 1, 2$  follows with similar computations of the Lemmas 4.2 and 4.3 in [10].  $\square$

**Lemma 4.4** The components of the solution of the Dirac equation given in (32) can be represented in terms of convolution operators as

$$\begin{aligned} \varphi_{1,a}(t, x) &= (mt + 1)\mathcal{U}_1(t, \partial_{x_1}) + m\mathcal{U}_2(t, \partial_{x_1}) + \mathcal{U}_3(t, \partial_{x_1}) + \mathcal{U}_4(t, \partial_{x_1}) + \mathcal{U}_5(t, \partial_{x_1})e^{iax_1}, \\ \varphi_{2,b}(t, x) &= (\tilde{\mathcal{U}}_1(t, \partial_{x_1}) + \tilde{\mathcal{U}}_2(t, \partial_{x_1}))e^{ibx_1}, \\ \varphi_{3,b}(t, x) &= (mt + 1)\mathcal{U}_1(t, \partial_{x_1}) + m\mathcal{U}_2(t, \partial_{x_1}) + \mathcal{U}_3(t, \partial_{x_1}) + \mathcal{U}_4(t, \partial_{x_1}) + \mathcal{U}_5(t, \partial_{x_1})e^{ibx_1}, \\ \varphi_{4,a}(t, x) &= (\tilde{\mathcal{U}}_1(t, \partial_{x_1}) + \tilde{\mathcal{U}}_2(t, \partial_{x_1}))e^{iax_1}. \end{aligned} \quad (49)$$

**Proof** The functions  $\varphi_{1,a}(t, x)$  and  $\varphi_{3,b}$  in (49) are a direct consequence of (32) and of the relations

$$\begin{aligned} U_1(a, t)e^{iax_1} &= t\mathcal{U}_1(t, \partial_{x_1})e^{iax_1} \\ U_2(a, t)e^{iax_1} &= \mathcal{U}_2(t, \partial_{x_1})e^{iax_1} \\ \partial_t U_1(a, t)e^{iax_1} &= \mathcal{U}_1(t, \partial_{x_1})e^{iax_1} + \mathcal{U}_3((t, \partial_{x_1})e^{iax_1}, \\ \partial_t U_2(a, t)e^{iax_1} &= \mathcal{U}_4(t, \partial_{x_1})e^{iax_1} + \mathcal{U}_5(t, \partial_{x_1})e^{iax_1}. \end{aligned}$$

where the operators  $\mathcal{U}_j(t, \partial_{x_1})$ , for  $j = 1, 2, 3, 4, 5$ , are defined in (37) with coefficients (45), (46), (40), (41) and (42). These operators are continuous on the space of entire functions  $\mathcal{A}_1$  as proved in Lemmas 4.2, 4.3, 4.5 and 4.4 in [10]. The representations of  $\varphi_{2,b}(t, x)$  and  $\varphi_{4,a}(t, x)$  follows from Lemma 4.3.  $\square$

We are now in the position to state the main result of this section, that is the super-shift property for the Dirac equation. It is encoded in the following theorem:

**Theorem 4.5** *Consider the Cauchy problem*

$$(i\gamma^\mu \partial_\mu - m)\psi(t, x) = 0, \quad t \neq 0, \quad x \in \mathbb{R}^3, \quad \text{and} \quad \psi_n(0, x) = \left[ \psi_{0,n}^+(x) \quad \psi_{0,n}^-(x) \right]^T, \quad x \in \mathbb{R}^3$$

where  $\psi_{0,n}^+(x)$  and  $\psi_{0,n}^-(x)$  are the superoscillating initial data:

$$\psi_{0,n}^+(x) = \begin{pmatrix} F_n(x_1, a) \\ 0 \end{pmatrix}, \quad \psi_{0,n}^-(x) = \begin{pmatrix} F_n(x_1, b) \\ 0 \end{pmatrix} \tag{50}$$

for  $a$  and  $b$  real numbers and let  $m > 0$ . Then the solution can be written as

$$\begin{aligned} \psi_{1,n}(t, x) &= \sum_{k=0}^n C_k(n, a) \varphi_{1,1-2k/n}(t, x), \\ \psi_{2,n}(t, x) &= \sum_{k=0}^n C_k(n, b) \varphi_{2,1-2k/n}(t, x), \\ \psi_{3,n}(t, x) &= \sum_{k=0}^n C_k(n, b) \varphi_{3,1-2k/n}(t, x), \\ \psi_{4,n}(t, x) &= \sum_{k=0}^n C_k(n, a) \varphi_{4,1-2k/n}(t, x), \end{aligned} \tag{51}$$

and the supershift property holds, that is we have

$$\lim_{n \rightarrow \infty} \psi_{j,n}(t, x) = \varphi_{j,a}(t, x), \quad \text{for } j = 1, 4 \quad \lim_{n \rightarrow \infty} \psi_{j,n}(t, x) = \varphi_{j,b}(t, x), \quad \text{for } j = 2, 3.$$

**Proof** The proof is a direct consequence of the continuity of the operators

$$\mathcal{U}_j(t, \partial_{x_1}) := \sum_{\ell=0}^{\infty} B_{j,\ell}(t) \partial_{x_1}^\ell : \mathcal{A}_1 \rightarrow \mathcal{A}_1, \quad j = 1, 2, 3, 4, 5 \tag{52}$$

defined in (37) and from the continuity of the operators

$$\tilde{\mathcal{U}}_j(t, \partial_{x_1}) := \sum_{\ell=0}^{\infty} \tilde{B}_{j,\ell}(t) \partial_{x_1}^{\ell+1} : \mathcal{A}_1 \rightarrow \mathcal{A}_1, \quad j = 1, 2 \tag{53}$$

defined in (44). □

## 5 Conclusions

Superoscillations are an important notion in quantum mechanics, as they appear in Aharonov's theory of weak values. The study of superoscillations requires sophisticated mathematical tools in order describe their evolution in time according to Schrödinger equation or other relativistic field equations. The mathematical methods to investigate the evolution of superoscillations have improved in the last ten years, and in this paper we have offered a summary of these methods by using Schrödinger equation as a model. In particular, we explicitly explained why we need to generalize the notion superoscillation with the notion of supershift. That put us in a position to develop our main technical result, namely the treatment of the evolution of superoscillation via the Dirac equation.

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