**Research Article**

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# Homogenization of Smoluchowski-type equations with transmission boundary conditions

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**Abstract:** In this work, we prove a two-scale homogenization result for a set of diffusion-coagulation Smoluchowski-type equations with transmission boundary conditions. This system is meant to describe the aggregation and diffusion of pathological tau proteins in the cerebral tissue, a process associated with the onset and evolution of a large variety of tauopathies (such as Alzheimer's disease). We prove the existence, uniqueness, positivity and boundedness of solutions to the model equations derived at the microscale (that is the scale of single neurons). Then, we study the convergence of the homogenization process to the solution of a macro-model asymptotically consistent with the microscopic one.

**Keywords:** Smoluchowski equation; transmission boundary conditions; two-scale homogenization; tauopathies

**MSC 2020:** 35K55; 80M40; 92B05

## 1 Introduction

Let us consider a bounded open set $\Omega$ in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. This domain is decomposed into long cylindrical cavities, periodically distributed with period $\epsilon$, having generators parallel to the $z$-axis. More precisely, given a bounded domain $D$ in $\mathbb{R}^2$, let us set $\Omega := D \times [0, L]$ with $x \in D$ and $z \in [0, L]$. We denote by $\Gamma_L := \partial D \times [0, L]$ the lateral boundary and by $\Gamma_B := \overline{D} \times \{0, L\}$ the upper and lower sides of $\partial\Omega$. Let $Y = [0,1] \times [0,1]$ be the unit periodicity cell in $\mathbb{R}^2$ having the paving property, i.e. the disjoint union of translated copies of $Y$ can indeed cover the whole space. Let $X$ be an open subset of $Y$ with a smooth boundary $R = \partial X$, such that $\overline{X} \subset \text{Int}\, Y$, and $Z = Y \backslash X$. Then, we define $G_\epsilon$ to be the set of all translated images of $\epsilon\,\overline{X}$:

$$G_\epsilon := \cup \left\{ \epsilon(k + \overline{X}) \mid \epsilon(k + \overline{X}) \subset D, k \in \mathbb{Z}^2 \right\}$$

and $D_\epsilon = D \backslash G_\epsilon$, as well as

$$R_\epsilon := \cup \left\{ \partial(\epsilon(k + \overline{X})) \mid \epsilon(k + \overline{X}) \subset D, k \in \mathbb{Z}^2 \right\}.$$

Thus, the geometric structure of the domain in $\mathbb{R}^3$ is given by: $\Pi_\epsilon := G_\epsilon \times [0, L]$, $\Omega_\epsilon := D_\epsilon \times [0, L]$, and $\Gamma_\epsilon := R_\epsilon \times [0, L]$. We refer to Figure 1 to illustrate the geometry of the problem.

Throughout this paper, $\epsilon$ will denote the general term of a sequence of positive reals which converges to zero. Let us introduce two nonnegative vector-valued functions, $u^\epsilon : [0, T] \times \Pi_\epsilon \to \mathbb{R}^M$, $u^\epsilon(t, x, z) = \left(u_1^\epsilon, \dots, u_M^\epsilon\right)$

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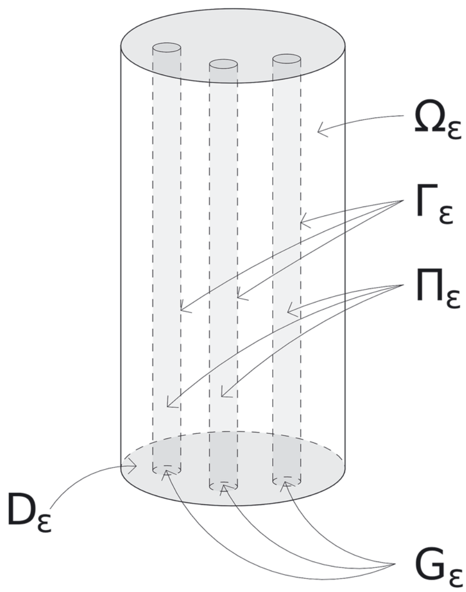


**Figure 1:** Geometry of the problem.

and $v^\epsilon\colon [0,T]\times\Omega_\epsilon \to \mathbb{R}^M$, $v^\epsilon(t,x,z) = \left(v_1^\epsilon,\dots,v_M^\epsilon\right)$, which solve the following system of discrete Smoluchowski-type equations [1] with transmission boundary conditions.

For $1 \le m \le M$ we have:

$$\begin{cases} \dfrac{\partial u_m^\epsilon}{\partial t} - \epsilon^2 D_m\,\Delta_x u_m^\epsilon - \tilde{D}_m\,\partial_z^2 u_m^\epsilon \quad = L_m(u^\epsilon) + f_m^\epsilon(t,x,z) & \text{in } [0,T]\times\Pi_\epsilon \\ \dfrac{\partial v_m^\epsilon}{\partial t} - d_m\,\Delta v_m^\epsilon = N_m(v^\epsilon) & \text{in } [0,T]\times\Omega_\epsilon \\ \epsilon\, D_m\,\nabla_x u_m^\epsilon\cdot\nu_\epsilon = -c_m(x,z)\big(u_m^\epsilon - v_m^\epsilon\big)_+ & \text{on } [0,T]\times\Gamma_\epsilon \\ d_m\,\nabla_x v_m^\epsilon\cdot\nu_\epsilon = \epsilon^2 D_m\,\nabla_x u_m^\epsilon\cdot\nu_\epsilon & \text{on } [0,T]\times\Gamma_\epsilon \\ \nabla_x v_m^\epsilon\cdot\nu_\epsilon = 0 & \text{on } [0,T]\times\Gamma_L \\ \partial_z u_m^\epsilon = \partial_z v_m^\epsilon = 0 & \text{on } [0,T]\times\Gamma_B \\ u_m^\epsilon(0,x,z) = U_m^\epsilon(x,z) & \text{in } \Pi_\epsilon \\ v_m^\epsilon(0,x,z) = 0 & \text{in } \Omega_\epsilon \end{cases} \tag{1.1}$$

where

$$L_1(u^\epsilon) = -\sum_{j=1}^{M} a_{1,j}u_1^\epsilon u_j^\epsilon \tag{1.2}$$

$$L_m(u^\epsilon) = \frac{1}{2}\sum_{j=1}^{m-1} a_{j,m-j}\,u_j^\epsilon\,u_{m-j}^\epsilon - \sum_{j=1}^{M} a_{m,j}u_m^\epsilon u_j^\epsilon \quad (1<m<M) \tag{1.3}$$

$$L_M(u^\epsilon) = \frac{1}{2}\sum_{\substack{j+k\ge M\\ k<M\\ j<M}} a_{j,k}\,u_j^\epsilon\,u_k^\epsilon \tag{1.4}$$

$$N_1(v^\epsilon) = -\sum_{j=1}^{M} b_{1,j}v_1^\epsilon v_j^\epsilon \tag{1.5}$$

$$N_m(v^\epsilon) = \frac{1}{2}\sum_{j=1}^{m-1} b_{j,m-j}\, v_j^\epsilon\, v_{m-j}^\epsilon - \sum_{j=1}^{M} b_{m,j} v_m^\epsilon v_j^\epsilon \quad (1 < m < M) \tag{1.6}$$

$$N_M(v^\epsilon) = \frac{1}{2}\sum_{\substack{j+k\geq M\\ k<M\\ j<M}} b_{j,k}\, v_j^\epsilon\, v_k^\epsilon \tag{1.7}$$

and

$$f_m^\epsilon(t,x,z) = \begin{cases} f^\epsilon(t,x,z) & \text{if } m=1 \\ 0 & \text{if } 1<m\leq M \end{cases}$$

$$U_m^\epsilon(x,z) = \begin{cases} U_1^\epsilon(x,z) > 0 & \text{if } m=1 \\ 0 & \text{if } 1<m\leq M \end{cases}$$

In the system of equations above, $\nu_\epsilon$ is the outer normal on $\Gamma_\epsilon$ with respect to $\Omega_\epsilon$; $f^\epsilon(t,x,z) \in C^1([0,T] \times \Pi_\epsilon)$, $c_i(x,z) \in L^\infty(\Gamma_\epsilon)(i = 1,\dots,M)$ and $U_1^\epsilon(x,z) \in L^\infty(\Pi_\epsilon)$ are given positive functions. In (1.1), the unknowns $u_m^\epsilon$ and $v_m^\epsilon$ $(1 \leq m < M)$ represent the concentration of $m$-clusters, that is, clusters consisting of $m$ identical elementary particles (monomers), while $u_M^\epsilon$ and $v_M^\epsilon$ take into account aggregations of more than $M-1$ monomers. We assume that the only reaction allowing clusters to coalesce to form larger clusters is a binary coagulation mechanism, while the approach of two clusters leading to aggregation results only from a diffusion process. In particular, we indicate with $D_i$, $\tilde{D}_i$, $d_i$ $(1 \leq i \leq M)$ the positive diffusion coefficients of $i$-clusters in different regions of the domain and along different directions. The kinetic coefficients $a_{i,j}$ and $b_{i,j}$ represent a reaction in which an $(i + j)$-cluster is formed from an $i$-cluster and a $j$-cluster. Therefore, they can be interpreted as 'coagulation rates' and are symmetric $a_{i,j} = a_{j,i} > 0$, $b_{i,j} = b_{j,i} > 0$ $(i, j = 1,\dots,M)$.

Our main statement shows that it is possible to homogenize the system of Equations (1.1) as $\epsilon \to 0$.

**Theorem 1.1.** *Let $v_m^\epsilon(t,x,z)$ and $u_m^\epsilon(t,x,z)$ $(1 \leq m \leq M)$ be a family of weak solutions to the system* (1.1) *(see Definition* 2.1*). The sequences $v_m^\epsilon$, $\nabla_x v_m^\epsilon$, $u_m^\epsilon$, $\epsilon\nabla_x u_m^\epsilon$ $(1 \leq m \leq M)$, two-scale converge to: $v_m(t,x,z)$, $\nabla_x v_m(t,x,z) + \nabla_y \tilde{v}_m(t,x,y,z)$, $u_m(t,x,y,z)$, $\nabla_y u_m(t,x,y,z)$, respectively. The limiting functions $v_m \in L^2(0,T;\ H^1(\Omega)) \bigcap H^1(0,T;\ L^2(\Omega))$, $\tilde{v}_m \in L^2([0,T]\times\Omega; H^1_\sharp(Y)/\mathbb{R})$, $u_m \in L^2([0,T]\times\Omega; H^1_\sharp(Y)) \bigcap H^1(0,T; L^2(\Omega\times Y))$ are the unique solutions of the following two-scale homogenized systems.*

*For $1 \leq m \leq M$ we have:*

$$\begin{aligned} &|Z|\,\frac{\partial v_m}{\partial t}(t,x,z) - div_x\big[d_m\, A\, \nabla_x v_m(t,x,z)\big] - d_m\, |Z|\, \partial_z^2 v_m(t,x,z) \\ &= \int_\Gamma c_m(x,z)\,(u_m(t,x,y,z) - v_m(t,x,z))_+\, \mathrm{d}\sigma(y) + |Z|\, N_m(v) \qquad \textit{in } [0,T]\times\Omega \end{aligned} \tag{1.8}$$

$$[A\, \nabla_x v_m(t,x,z)]\cdot \nu = 0 \qquad \textit{on } [0,T]\times\Gamma_L \tag{1.9}$$

$$\partial_z v_m(t,x,z) = 0 \qquad \textit{on } [0,T]\times\Gamma_B \tag{1.10}$$

$$v_m(t=0,x,z) = 0 \qquad \textit{in } \Omega \tag{1.11}$$

*where*

$$N_1(v) = -\sum_{j=1}^{M} b_{1,j} v_1(t,x,z)\, v_j(t,x,z)$$

$$N_m(v) = \frac{1}{2}\sum_{j=1}^{m-1} b_{j,m-j}\, v_j(t,x,z)\, v_{m-j}(t,x,z) - \sum_{j=1}^{M} b_{m,j} v_m(t,x,z)\, v_j(t,x,z) \quad (1<m<M)$$

$$N_M(v) = \frac{1}{2} \sum_{\substack{j+k \geq M \\ k<M \\ j<M}} b_{j,k}\, v_j(t,x,z)\, v_k(t,x,z)$$

*In* (1.8) *and* (1.9), *$A$ is a matrix with constant coefficients defined by*

$$A_{ij} = \int_Z (\nabla_y w_i + \hat{e}_i) \cdot (\nabla_y w_j + \hat{e}_j)\, \mathrm{d}y \tag{1.12}$$

*with $\hat{e}_i$ being the ith unit vector in $\mathbb{R}^3$, and $(w_i)_{1\leq i\leq 3}$ the family of solutions of the cell problem*

$$\begin{cases} -div_y[\nabla_y w_i + \hat{e}_i] = 0 & in\ Z \\ (\nabla_y w_i + \hat{e}_i)\cdot \nu = 0 & on\ \Gamma \\ y \to w_i(y) \quad Y-periodic \end{cases} \tag{1.13}$$

*appearing in the limiting function*

$$\tilde{v}_m(t,x,y,z) = \sum_{i=1}^{3} w_i(y) \frac{\partial v_m}{\partial x_i}(t,x,z) \quad (1 \leq m \leq M) \tag{1.14}$$

*Furthermore, for $1 \leq m \leq M$ we have:*

$$\begin{aligned} &\frac{\partial u_m}{\partial t}(t,x,y,z) - D_m \triangle_y u_m(t,x,y,z) - \tilde{D}_m\, \partial_z^2 u_m(t,x,y,z) \\ &= L_m(u) + f_m(t,x,y,z) \qquad t>0,\ (x,z)\in\Omega,\ y\in X \end{aligned} \tag{1.15}$$

$$\begin{aligned} &D_m \nabla_y u_m(t,x,y,z)\cdot \nu \\ &= -c_m(x,z)\,(u_m(t,x,y,z) - v_m(t,x,z))_+ \qquad t>0,\ (x,z)\in\Omega,\ y\in\Gamma \end{aligned} \tag{1.16}$$

$$\partial_z u_m(t,x,y,z) = 0 \qquad t>0,\ (x,z)\in \overline{D}\times\{0,L\},\ y\in X \tag{1.17}$$

$$u_m(t=0,x,y,z) = U_m(x,y,z) \qquad (x,z)\in\Omega,\ y\in X \tag{1.18}$$

*where*

$$L_1(u) = -\sum_{j=1}^{M} a_{1,j} u_1(t,x,y,z)\, u_j(t,x,y,z)$$

$$L_m(u) = \frac{1}{2}\sum_{j=1}^{m-1} a_{j,m-j}\, u_j(t,x,y,z)\, u_{m-j}(t,x,y,z) - \sum_{j=1}^{M} a_{m,j} u_m(t,x,y,z)\, u_j(t,x,y,z) \quad (1<m<M)$$

$$L_M(u) = \frac{1}{2} \sum_{\substack{j+k \geq M \\ k<M \\ j<M}} a_{j,k}\, u_j(t,x,y,z)\, u_k(t,x,y,z)$$

*and*

$$f_m(t,x,y,z) = \begin{cases} f(t,x,y,z) & if\ m=1 \\ 0 & if\ 1<m\leq M \end{cases}$$

$$U_m(x,y,z) = \begin{cases} U_1(x,y,z) > 0 & if\ m=1 \\ 0 & if\ 1<m\leq M \end{cases}$$

## 1.1 Motivation

The system of equations in (1.1), known under the generic name of *discrete Smoluchowski equations with diffusion*, is meant to model the aggregation and diffusion of the pathological tau protein in the brain, a process associated with the development of a large variety of cerebral diseases called tauopathies. Indeed, pathological accumulations of hyperphosphorylated tau protein aggregates, known as neurofibrillary tangles (NFTs), are detected in several neurodegenerative tauopathies, including Alzheimer's disease (AD) [2]–[5]. Tau is a highly soluble, natively unfolded protein which is predominantly located in the axons of neurons of the central nervous system. Here, its physiological function is to support assembly and stabilization of axonal microtubules. Under pathological conditions, tau can assume abnormal conformations, due to two transformations: hyperphosphorylation and misfolding. In particular, hyperphosphorylation has a negative impact on the biological function of tau proteins, since it inhibits the binding to microtubules, compromising their stabilization and axonal transport, and promotes self-aggregation. Thus, misfolded tau monomers constitute the building unit for the formation of oligomers, which in turn lead to highly structured and insoluble fibrils. For many years, cell autonomous mechanisms were believed to be responsible for the evolution of neurodegenerative diseases, implying that the same aggregation events occur independently in different brain cells. However, accumulating evidence now demonstrates that the progression of tau pathology reflects cell-to-cell propagation of the disease, achieved through the release of tau into the extracellular space and the uptake by surrounding healthy neurons [6], [7]. Extracellular tau then seeds physiological tau in the recipient cells propagating the pathological process through neural pathways made by bundles of axons (called tracts). This mechanism, often referred to as 'prion-like' propagation of tau pathology, has led to the idea that extracellular tau could be a novel therapeutic target to halt the spread of the disease [8]. For this reason, in the present work, we focus on a model which describes, in a simplified way, the tau diffusion along tracts starting from the microscopic release mechanism.

In this context, the set $\Pi_\epsilon$ represents a bundle of axons in the white matter (a tract), while the domain $\Omega_\epsilon$ indicates the extracellular region filled by cerebrospinal fluid. The variables $u_m^\epsilon \geq 0$ and $v_m^\epsilon \geq 0$ refer to the concentration of $m$-clusters of hyperphosphorylated tau spreading within the neuronal axons (represented by the long cylindrical cavities) and in extracellular space, respectively. Concerning the diffusion process, the choice made in our model to define two coefficients reflects the biological properties of the tau protein that diffuses differently in extracellular space and in the axon [2]. Moreover, within the neuronal axon, we further distinguish between two different diffusion coefficients, since tau propagates preferentially along the $z$-axis of the cylinder that represents the axon. The initial condition $U_1^\epsilon(x, z)$, given at $t = 0$ by the concentration of monomers diffusing within the axons, represents the amount of endogenous misfolded tau protein, while the source term $f^\epsilon(t, x, z)$, in the system (1.1) for $m = 1$, indicates the production of hyperphosphorylated tau monomers.

## 1.2 Background of this work

There is a large literature related to the use of the Smoluchowski equation in various physical contexts (e.g., Refs. [9]–[14]), but only a few works concerning its application in the biomedical field [15], [16]. Recently, the important role of the Smoluchowski equations in modelling at different scales the evolution of neurodegenerative diseases, such as AD, has been investigated in Refs. [15], [17]–[23]. In fact, the present work is part of a broader research effort carried on by various groups of researchers with diverse collaborations on mathematical models of the progression of AD, and represents an initial bridge between microscopic models of tau diffusion developed in biology and macroscopic mathematical models based on graph theory. To this end, the homogenization technique, introduced by the mathematicians in the seventies to carry out a sort of averaging process on the solutions of partial differential equations with rapidly varying coefficients or describing media with microstructures, has been applied [24]–[30].

It is nowadays generally accepted that tau protein, in synergetic combination with another protein, the so-called $\beta$-amyloid peptide, plays a key role in the development of AD (see Ref. [31]). We refer for instance to Ref. [15] for a discussion on macroscopic mathematical modeling of this interaction. Here, our interest is focused on the tau protein that diffuses through the neural pathway, whereas we ignore deliberately the action of the $\beta$-amyloid.

Unlike the approach proposed for instance in Refs. [15], [23], where the modeling of tau coagulation-diffusion processes has been carried out on a large scale, that is the scale of the neural network characterized by the connectivity of different regions through bundles of axons (tracts), in this paper only the mesoscopic dynamics within a portion of tract has been investigated. Starting from the derivation of model equations valid at the microscale, by using the so-called *two-scale homogenization* technique, we have proved that the solution two-scale converges to the solution of a macromodel asymptotically consistent with the original one. The notion of two-scale limit has been first introduced by Nguetseng [32] and Allaire [33] in the deterministic periodic setting, and later generalized to the stochastic framework by Zhikov and Piatnitsky [34]. Unlike other homogenization techniques (see Refs. [26], [27] for a review), the two-scale convergence method is self-contained in that, in a single step, one can derive the homogenized equations and prove the convergence of the sequence of solutions to the problem at hand.

To stress the novelty of the present paper, it is worth noting that two-scale homogenization techniques have been already used by the authors to pass from microscopic to macroscopic model of the diffusion of toxic proteins in the cerebral parenchyma affected by AD, but in completely different biological perspectives and geometries. Indeed, in Refs. [19], [20], the authors aimed to describe production, aggregation and diffusion of $\beta$-amyloid peptide in the cerebral tissue (macroscopic scale), a process associated with the development of AD, starting from the derivation of a model at the single neuron level (microscopic scale). Thus, the present paper differs from our previous works both for the biological meaning and, consequently, for the geometry of the problem and the boundary conditions introduced in order to take into account the peculiarities of tau protein propagation.

There is a large literature devoted to the study of transmission boundary conditions somehow similar to those of (1.1), especially in the framework of porous media [35]. The main results in this respect, which are relevant to our work, can be found in Refs. [36]–[38]. In particular, in Ref. [36], it is assumed that the porous medium is composed of periodically arranged cubic cells of size $\epsilon$, split up into a solid part (a ball surrounded by semi-permeable membranes) and a fluid part. In this setting, the diffusion and reactions of chemical species in the fluid and in the solid part are studied, while transmission boundary conditions are imposed on the interface. The same geometry is considered also in Ref. [37], where deposition effects under the influence of thermal gradients are analyzed. The model takes into account the motion of populations of colloidal particles dissolved in the water interacting together via Smoluchowski coagulation terms. The colloidal matter cannot penetrate the solid grain boundary, but it deposits there. This process is again described by transmission boundary conditions. A domain decomposed into long cylindrical cavities periodically distributed has been defined in Ref. [38] to model a porous medium consisting of a fluid part and solid bars. Chemical substances, dissolved in the fluid, are transported by diffusion and adsorbed on the surface of the bars (through transmission boundary conditions) where chemical reactions take place.

## 1.3 Outline of the paper

The paper is organized as follows. In Section 2 we prove the existence of weak solutions to the system of Smoluchowski-type Equations (1.1), while Sections 3 and 4 are devoted to the proof of their positivity, boundedness and uniqueness, respectively. *A priori* estimates on the derivatives of the solutions are also obtained and reported at the end of Section 3. In Section 5 we study the convergence of the homogenization process and prove our main result concerning the two-scale limit of the solutions to the set of Equations (1.1). Finally, in the Appendices we recall some basic Theorems related to functional analysis and on the two-scale convergence method.

## 2 Existence of solutions

Let us consider the following truncation of the nonlinear terms in the Smoluchowski-type Equations (1.1) for $1 \le i \le M$ [37]:

$$L_i^{\tilde{M}}(u^\epsilon) := L_i(\sigma_{\tilde{M}}(u_1^\epsilon), \sigma_{\tilde{M}}(u_2^\epsilon), \dots, \sigma_{\tilde{M}}(u_M^\epsilon)) \tag{2.1}$$

$$N_i^{\tilde{M}}(v^\epsilon) := N_i(\sigma_{\tilde{M}}(v_1^\epsilon), \sigma_{\tilde{M}}(v_2^\epsilon), \dots, \sigma_{\tilde{M}}(v_M^\epsilon)) \tag{2.2}$$

where

$$\sigma_{\tilde{M}}(s) := \begin{cases} 0, & s < 0 \\ s, & s \in [0, \tilde{M}] \\ \tilde{M}, & s > \tilde{M} \end{cases} \tag{2.3}$$

with $\tilde{M} > 0$ being a fixed threshold. If $\tilde{M}$ is large enough, the estimates derived later in this paper will give bounds that will remain below $\tilde{M}$. This means that the results obtained in the following hold also for the uncutted coagulation terms.

**Definition 2.1.** The functions $u_i^\epsilon \in H^1([0,T]; L^2(\Pi_\epsilon)) \cap L^\infty([0,T]; H^1(\Pi_\epsilon))$ and $v_i^\epsilon \in H^1([0,T]; L^2(\Omega_\epsilon)) \cap L^\infty([0,T]; H^1(\Omega_\epsilon))$ $(1 \le i \le M)$ are solutions to the problem (1.1) if the following relations hold, a.e. in $[0,T]$ and for a fixed value of $\epsilon > 0$.

If $1 \le i \le M$:

$$\begin{aligned} &\int_{\Pi_\epsilon} \partial_t u_i^\epsilon \, \psi_i \, \mathrm{d}x\, \mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_i \nabla_x u_i^\epsilon \cdot \nabla_x \psi_i \, \mathrm{d}x\, \mathrm{d}z + \int_{\Pi_\epsilon} \tilde{D}_i \, \partial_z u_i^\epsilon \cdot \partial_z \psi_i \, \mathrm{d}x\, \mathrm{d}z \\ &\quad + \epsilon \int_{\Gamma_\epsilon} c_i(x,z)\big(u_i^\epsilon - v_i^\epsilon\big)_+ \psi_i \, \mathrm{d}\sigma_\epsilon = \int_{\Pi_\epsilon} L_i^{\tilde{M}}(u^\epsilon)\, \psi_i \, \mathrm{d}x\, \mathrm{d}z + \int_{\Pi_\epsilon} f_i^\epsilon(t,x,z)\, \psi_i \, \mathrm{d}x\, \mathrm{d}z \end{aligned} \tag{2.4}$$

for all $\psi_i \in H^1(\Pi_\epsilon)$, and

$$\int_{\Omega_\epsilon} \partial_t v_i^\epsilon \, \phi_i \, \mathrm{d}x\, \mathrm{d}z + \int_{\Omega_\epsilon} d_i \nabla v_i^\epsilon \cdot \nabla \phi_i \, \mathrm{d}x\, \mathrm{d}z - \epsilon \int_{\Gamma_\epsilon} c_i(x,z)\big(u_i^\epsilon - v_i^\epsilon\big)_+ \phi_i \, \mathrm{d}\sigma_\epsilon = \int_{\Omega_\epsilon} N_i^{\tilde{M}}(v^\epsilon)\, \phi_i \, \mathrm{d}x\, \mathrm{d}z \tag{2.5}$$

for all $\phi_i \in H^1(\Omega_\epsilon)$, along with the initial conditions $u_i^\epsilon(0,x,z)$ and $v_i^\epsilon(0,x,z)$.

**Remark 2.1.** In Eqs. (2.4) and (2.5) the integrals over the boundary $\Gamma_\epsilon$ are well defined thanks to the regularity of the functions $u_i^\epsilon$ and $v_i^\epsilon$, as stated in Definition 2.1, and the existence of the interpolation-trace inequality given by Eq. (A.7) in Appendix A. With an abuse of notation, in Eqs. (2.4) and (2.5) we have indicated with $u_i^\epsilon$ and $v_i^\epsilon$ also the traces of these functions.

**Lemma 2.1.** *For a given small $\epsilon > 0$, the system* (1.1) *has a solution $u_i^\epsilon \in H^1([0,T]; L^2(\Pi_\epsilon)) \cap L^\infty([0,T]; H^1(\Pi_\epsilon))$ and $v_i^\epsilon \in H^1([0,T]; L^2(\Omega_\epsilon)) \cap L^\infty([0,T]; H^1(\Omega_\epsilon))$ $(1 \le i \le M)$ in the sense of the Definition* 2.1.

*Proof.* Let $\{\xi_j\}$ be an orthonormal basis of $H^1(\Pi_\epsilon)$ and $\{\eta_j\}$ of $H^1(\Omega_\epsilon)$. We denote by $u^\epsilon_{i,n}$ and $v^\epsilon_{i,n}$ $(1 \le i \le M)$ the Galerkin approximations of $u^\epsilon_i$ and $v^\epsilon_i$, respectively, that is

$$u^\epsilon_{i,n}(t,x,z) := \sum_{j=1}^{n} \alpha^n_{i,j}(t)\, \xi_j(x,z) \tag{2.6}$$

for all $t \in [0,T]$, $(x,z) \in \Pi_\epsilon$ and

$$v^\epsilon_{i,n}(t,x,z) := \sum_{j=1}^{n} \beta^n_{i,j}(t)\, \eta_j(x,z) \tag{2.7}$$

for all $t \in [0,T]$, $(x,z) \in \Omega_\epsilon$. Let now $i = 1$.

Since $\{\xi_j\}$ is an orthonormal basis of $H^1(\Pi_\epsilon)$, for each $n \in \mathbb{N}$, there exists

$$u^{\epsilon,0}_{1,n}(x,z) = \sum_{j=1}^{n} \alpha^{0,n}_{1,j}\, \xi_j(x,z) \tag{2.8}$$

so that $u^{\epsilon,0}_{1,n} \to U^\epsilon_1(x,z)$ in $H^1(\Pi_\epsilon)$ as $n \to \infty$.

Likewise, for each $n \in \mathbb{N}$, there exists

$$v^{\epsilon,0}_{1,n}(x,z) = \sum_{j=1}^{n} \beta^{0,n}_{1,j}\, \eta_j(x,z) \tag{2.9}$$

so that $v^{\epsilon,0}_{1,n} \to 0$ in $H^1(\Omega_\epsilon)$ as $n \to \infty$.

To derive the coefficients of the Galerkin approximations, we impose that the functions $u^\epsilon_{1,n}$ and $v^\epsilon_{1,n}$ satisfy Eqs. (2.4) and (2.5). Therefore, for all $\psi_1 \in span\{\xi_j\}_{j=1}^n$, Eq. (2.4) can be written as:

$$\begin{aligned}&\int_{\Pi_\epsilon} \partial_t u^\epsilon_{1,n}\, \psi_1 \,\mathrm{d}x\,\mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_1\, \nabla_x u^\epsilon_{1,n} \cdot \nabla_x \psi_1 \,\mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} \tilde{D}_1\, \partial_z u^\epsilon_{1,n} \cdot \partial_z \psi_1 \,\mathrm{d}x\,\mathrm{d}z \\ &\quad + \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left( u^\epsilon_{1,n} - v^\epsilon_{1,n} \right)_+ \psi_1 \,\mathrm{d}\sigma_\epsilon = \int_{\Pi_\epsilon} L_1^{\tilde{M}}\left(u^\epsilon_{i,n}\right) \psi_1 \,\mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\, \psi_1 \,\mathrm{d}x\,\mathrm{d}z\end{aligned} \tag{2.10}$$

By testing Eq. (2.10) with $\psi_1 = \xi_k$, we get

$$\begin{aligned}&\int_{\Pi_\epsilon} \partial_t \left[ \sum_{j=1}^{n} \alpha^n_{1,j}(t)\, \xi_j(x,z) \right] \xi_k \,\mathrm{d}x\,\mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_1 \left[ \sum_{j=1}^{n} \alpha^n_{1,j}(t)\, \nabla_x \xi_j(x,z) \right] \cdot \nabla_x \xi_k \,\mathrm{d}x\,\mathrm{d}z \\ &\quad + \int_{\Pi_\epsilon} \tilde{D}_1 \left[ \sum_{j=1}^{n} \alpha^n_{1,j}(t)\, \partial_z \xi_j(x,z) \right] \cdot \partial_z \xi_k \,\mathrm{d}x\,\mathrm{d}z + \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left[ \sum_{j=1}^{n} \alpha^n_{1,j}(t)\, \xi_j(x,z) - \sum_{j=1}^{n} \beta^n_{1,j}(t)\, \eta_j(x,z) \right]_+ \xi_k \,\mathrm{d}\sigma_\epsilon \\ &\quad = -\int_{\Pi_\epsilon} \xi_k \sum_{a=1}^{M} a_{1,a}\, \sigma_{\tilde{M}} \left[ \sum_{b=1}^{n} \alpha^n_{1,b}(t)\, \xi_b(x,z) \right] \sigma_{\tilde{M}} \left[ \sum_{c=1}^{n} \alpha^n_{a,c}(t)\, \xi_c(x,z) \right] \mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\, \xi_k \,\mathrm{d}x\,\mathrm{d}z.\end{aligned} \tag{2.11}$$

Hence, for $k \in \{1, \dots, n\}$:

$$\begin{aligned}\partial_t \alpha^n_{1,k}(t) + \sum_{j=1}^{n} A_{1jk}\, \alpha^n_{1,j}(t) = -F_{1,k}(\alpha^n_1(t), \beta^n_1(t)) - \int_{\Pi_\epsilon} \xi_k \sum_{a=1}^{M} a_{1,a}\, \sigma_{\tilde{M}} \left[ \sum_{b=1}^{n} \alpha^n_{1,b}(t)\, \xi_b(x,z) \right] \\ \times \sigma_{\tilde{M}} \left[ \sum_{c=1}^{n} \alpha^n_{a,c}(t)\, \xi_c(x,z) \right] \mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\, \xi_k \,\mathrm{d}x\,\mathrm{d}z\end{aligned} \tag{2.12}$$

where the coefficients $A_{1jk}$ are defined by

$$A_{1jk} := \epsilon^2 \int_{\Pi_\epsilon} D_1 \nabla_x \xi_j \cdot \nabla_x \xi_k \, dx\, dz + \int_{\Pi_\epsilon} \tilde{D}_1 \partial_z \xi_j \cdot \partial_z \xi_k \, dx\, dz \tag{2.13}$$

and

$$F_{1,k}(\alpha_1^n(t), \beta_1^n(t)) := \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left( u_{1,n}^\epsilon - v_{1,n}^\epsilon \right)_+ \xi_k \, d\sigma_\epsilon \tag{2.14}$$

We choose now $v_{1,n}^\epsilon$ satisfying Eq. (2.5) for all $\phi_1 \in \operatorname{span}\{\eta_j\}_{j=1}^n$, i.e.:

$$\int_{\Omega_\epsilon} \partial_t v_{1,n}^\epsilon \, \phi_1 \, dx\, dz + \int_{\Omega_\epsilon} d_1 \nabla v_{1,n}^\epsilon \cdot \nabla \phi_1 \, dx\, dz - \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left( u_{1,n}^\epsilon - v_{1,n}^\epsilon \right)_+ \phi_1 \, d\sigma_\epsilon = \int_{\Omega_\epsilon} N_1^{\tilde{M}} \left( v_{i,n}^\epsilon \right) \phi_1 \, dx\, dz \tag{2.15}$$

By testing Eq. (2.15) with $\phi_1 = \eta_k$, we get

$$\begin{aligned}
&\int_{\Omega_\epsilon} \partial_t \left[ \sum_{j=1}^n \beta_{1,j}^n(t)\, \eta_j(x,z) \right] \eta_k \, dx\, dz + \int_{\Omega_\epsilon} d_1 \left[ \sum_{j=1}^n \beta_{1,j}^n(t) \nabla \eta_j(x,z) \right] \cdot \nabla \eta_k \, dx\, dz \\
&- \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left[ \sum_{j=1}^n \alpha_{1,j}^n(t)\, \xi_j(x,z) - \sum_{j=1}^n \beta_{1,j}^n(t)\, \eta_j(x,z) \right]_+ \eta_k \, d\sigma_\epsilon \\
&= - \int_{\Omega_\epsilon} \eta_k \sum_{a=1}^M b_{1,a}\, \sigma_{\tilde{M}} \left[ \sum_{b=1}^n \beta_{1,b}^n(t)\, \eta_b(x,z) \right] \sigma_{\tilde{M}} \left[ \sum_{c=1}^n \beta_{a,c}^n(t)\, \eta_c(x,z) \right] dx\, dz
\end{aligned} \tag{2.16}$$

Hence, for $k \in \{1, \ldots, n\}$:

$$\begin{aligned}
\partial_t \beta_{1,k}^n(t) + \sum_{j=1}^n B_{1jk}\, \beta_{1,j}^n(t) &= G_{1,k}(\alpha_1^n(t), \beta_1^n(t)) \\
&- \int_{\Omega_\epsilon} \eta_k \sum_{a=1}^M b_{1,a}\, \sigma_{\tilde{M}} \left[ \sum_{b=1}^n \beta_{1,b}^n(t)\, \eta_b(x,z) \right] \sigma_{\tilde{M}} \left[ \sum_{c=1}^n \beta_{a,c}^n(t)\, \eta_c(x,z) \right] dx\, dz
\end{aligned} \tag{2.17}$$

where the coefficients $B_{1jk}$ are defined by

$$B_{1jk} := \int_{\Omega_\epsilon} d_1 \nabla \eta_j \cdot \nabla \eta_k \, dx\, dz \tag{2.18}$$

and

$$G_{1,k}(\alpha_1^n(t), \beta_1^n(t)) := \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left( u_{1,n}^\epsilon - v_{1,n}^\epsilon \right)_+ \eta_k \, d\sigma_\epsilon \tag{2.19}$$

Equations (2.12) and (2.17) represent a system of $2n$ ordinary differential equations for the coefficients $\alpha_1^n = \left(\alpha_{1,k}^n\right)_{k=1,\ldots,n}$, $\beta_1^n = \left(\beta_{1,k}^n\right)_{k=1,\ldots,n}$. By taking into account formulas (2.8) and (2.9), we assume as initial conditions:

$$\alpha_{1,k}^n(0) = \alpha_{1,k}^{0,n}; \quad \beta_{1,k}^n(0) = \beta_{1,k}^{0,n}. \tag{2.20}$$

Since the left-hand side of this system of ordinary differential equations is linear and the right-hand side has a sublinear growth (thanks to the Lipschitz property of the functions), one can conclude that the Cauchy problem (2.12), (2.17), and (2.20) has a unique solution extended to the whole interval $[0, T]$ (see the generalization of the Picard–Lindelöf theorem in Ref. [39], Theorem 5.1, p. 156). Moreover, $\alpha_1^n = \left(\alpha_{1,k}^n\right)_{k=1,\ldots,n}(t)$,

$\beta_1^n = \left(\beta_{1,k}^n\right)_{k=1,\dots,n}(t) \in H^1([0,T])$ for $t \in [0,T]$. We prove in the following the global Lipschitz property of $F_{1,k}$. The proof of the Lipschitz continuity of $G_{1,k}$ is similar.

Let $\left(u_{1,n}^\epsilon, v_{1,n}^\epsilon\right)$ and $\left(\tilde u_{1,n}^\epsilon, \tilde v_{1,n}^\epsilon\right)$ be of the form (2.6) and (2.7), with coefficients $(\alpha_1^n(t) = \left(\alpha_{1,k}^n\right)_{k=1,\dots,n}, \beta_1^n(t) = \left(\beta_{1,k}^n\right)_{k=1,\dots,n})$ and $(\tilde\alpha_1^n(t) = \left(\tilde\alpha_{1,k}^n\right)_{k=1,\dots,n}, \tilde\beta_1^n(t) = \left(\tilde\beta_{1,k}^n\right)_{k=1,\dots,n})$, respectively.

One has:

$$
\begin{aligned}
F_{1,k}(\alpha_1^n(t),&\beta_1^n(t)) - F_{1,k}(\tilde\alpha_1^n(t), \tilde\beta_1^n(t)) \\
&= \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left[\left(u_{1,n}^\epsilon - v_{1,n}^\epsilon\right)_+ - \left(\tilde u_{1,n}^\epsilon - \tilde v_{1,n}^\epsilon\right)_+\right] \xi_k(x,z)\, \mathrm{d}\sigma_\epsilon \\
&\le \epsilon\, C_1 \int_{\Gamma_\epsilon} c_1(x,z)\, |\left(u_{1,n}^\epsilon - \tilde u_{1,n}^\epsilon\right) - \left(v_{1,n}^\epsilon - \tilde v_{1,n}^\epsilon\right)|\, |\xi_k(x,z)|\, \mathrm{d}\sigma_\epsilon \\
&= \epsilon\, C_1 \int_{\Gamma_\epsilon} c_1(x,z) \Bigg| \sum_{j=1}^n (\alpha_{1,j}^n(t) - \tilde\alpha_{1,j}^n(t)) \xi_j(x,z) \\
&\qquad - \sum_{j=1}^n (\beta_{1,j}^n(t) - \tilde\beta_{1,j}^n(t)) \eta_j(x,z) \Bigg|\, |\xi_k(x,z)|\, \mathrm{d}\sigma_\epsilon \\
&\le \epsilon\, C_1 \sum_{j=1}^n |\alpha_{1,j}^n(t) - \tilde\alpha_{1,j}^n(t)| \int_{\Gamma_\epsilon} c_1(x,z)\, |\xi_j(x,z)|\, |\xi_k(x,z)|\, \mathrm{d}\sigma_\epsilon \\
&\qquad + \epsilon\, C_1 \sum_{j=1}^n |\beta_{1,j}^n(t) - \tilde\beta_{1,j}^n(t)| \int_{\Gamma_\epsilon} c_1(x,z)\, |\eta_j(x,z)|\, |\xi_k(x,z)|\, \mathrm{d}\sigma_\epsilon \\
&\le \epsilon\, C_1 \max\{c_{jk}\} \sum_{j=1}^n |\alpha_{1,j}^n(t) - \tilde\alpha_{1,j}^n(t)| \\
&\qquad + \epsilon\, C_1 \max\{d_{jk}\} \sum_{j=1}^n |\beta_{1,j}^n(t) - \tilde\beta_{1,j}^n(t)|
\end{aligned}
\tag{2.21}
$$

where the coefficients $(c_{jk})$ and $(d_{jk})$ are given by

$$
c_{jk} := \int_{\Gamma_\epsilon} c_1(x,z) |\xi_j(x,z)|\, |\xi_k(x,z)|\, \mathrm{d}\sigma_\epsilon \tag{2.22}
$$

$$
d_{jk} := \int_{\Gamma_\epsilon} c_1(x,z) |\eta_j(x,z)|\, |\xi_k(x,z)|\, \mathrm{d}\sigma_\epsilon \tag{2.23}
$$

for $j, k = 1, \dots, n$. Hence, we get

$$
|F_{1,k}\left(\alpha_1^n, \beta_1^n\right) - F_{1,k}\left(\tilde\alpha_1^n, \tilde\beta_1^n\right)| \le \epsilon\, C_2(n) \left[|\alpha_1^n - \tilde\alpha_1^n| + |\beta_1^n - \tilde\beta_1^n|\right]. \tag{2.24}
$$

The same conclusions can be drawn also when $1 < i \le M$ by applying exactly the arguments considered above.

## 2.1 Uniform estimates

Let us now prove uniform estimates in $n$ for $u^\epsilon_{i,n}$ and $v^\epsilon_{i,n}$ $(1 \le i \le M)$. In the case $i = 1$, we take in Eq. (2.10) $\psi_1 = u^\epsilon_{1,n}(t,\cdot)$ as test function:

$$\frac{1}{2}\int_{\Pi_\epsilon} \partial_t\left(u^\epsilon_{1,n}\right)^2 \mathrm{d}x\,\mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_1 |\nabla_x u^\epsilon_{1,n}|^2 \mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} \tilde{D}_1 |\partial_z u^\epsilon_{1,n}|^2 \mathrm{d}x\,\mathrm{d}z + \epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ u^\epsilon_{1,n}\,\mathrm{d}\sigma_\epsilon$$
$$= -\int_{\Pi_\epsilon} \left[\sum_{j=1}^{M} a_{1,j}\,\sigma_{\tilde{M}}\left(u^\epsilon_{1,n}\right)\sigma_{\tilde{M}}\left(u^\epsilon_{j,n}\right)\right] u^\epsilon_{1,n}\,\mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\,u^\epsilon_{1,n}\,\mathrm{d}x\,\mathrm{d}z \tag{2.25}$$

Since the first term on the right-hand side is always negative due to the truncation of the coagulation terms (see Eq. (2.3)), one obtains:

$$\frac{1}{2}\partial_t \|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \epsilon^2 D_1 \|\nabla_x u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \tilde{D}_1 \|\partial_z u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)}$$
$$\le -\epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ u^\epsilon_{1,n}\,\mathrm{d}\sigma_\epsilon + \frac{1}{2}\|f^\epsilon(t,x,z)\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} \tag{2.26}$$

where we have applied the Hölder inequality to the last term on the right-hand side of Eq. (2.25). Since the function $f^\epsilon(t,x,z)$ is bounded in $L^2([0,T] \times \Pi_\epsilon)$, Eq. (2.26) reads:

$$\frac{1}{2}\partial_t \|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \epsilon^2 D_1 \|\nabla_x u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \tilde{D}_1 \|\partial_z u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)}$$
$$\le -\epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ u^\epsilon_{1,n}\,\mathrm{d}\sigma_\epsilon + C_f + \frac{1}{2}\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} \tag{2.27}$$

where $C_f$ is a positive constant. By testing now Eq. (2.15) with $\phi_1 = v^\epsilon_{1,n}(t,\cdot)$, we get

$$\frac{1}{2}\int_{\Omega_\epsilon} \partial_t\left(v^\epsilon_{1,n}\right)^2 \mathrm{d}x\,\mathrm{d}z + \int_{\Omega_\epsilon} d_1 |\nabla v^\epsilon_{1,n}|^2 \mathrm{d}x\,\mathrm{d}z - \epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ v^\epsilon_{1,n}\,\mathrm{d}\sigma_\epsilon$$
$$= -\int_{\Omega_\epsilon} \left[\sum_{j=1}^{M} b_{1,j}\,\sigma_{\tilde{M}}\left(v^\epsilon_{1,n}\right)\sigma_{\tilde{M}}\left(v^\epsilon_{j,n}\right)\right] v^\epsilon_{1,n}\,\mathrm{d}x\,\mathrm{d}z \tag{2.28}$$

Since again the term on the right-hand side is negative, we conclude:

$$\frac{1}{2}\partial_t \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + d_1 \|\nabla v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} \le \epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ v^\epsilon_{1,n}\,\mathrm{d}\sigma_\epsilon \tag{2.29}$$

Adding the inequalities (2.27) and (2.29), it follows that

$$\frac{1}{2}\partial_t \|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\partial_t \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + \epsilon^2 D_1 \|\nabla_x u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \tilde{D}_1 \|\partial_z u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)}$$
$$+ d_1 \|\nabla v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} \le C_f + \epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ \left(v^\epsilon_{1,n} - u^\epsilon_{1,n}\right)\mathrm{d}\sigma_\epsilon + \frac{1}{2}\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} \tag{2.30}$$

Let us estimate the second term on the right-hand side of (2.30):

$$
\begin{aligned}
I &= \epsilon \int_{\Gamma_\epsilon} c_1(x,z)\left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ \left(v^\epsilon_{1,n} - u^\epsilon_{1,n}\right) \mathrm{d}\sigma_\epsilon \\
&\leq \epsilon \int_{\Gamma_\epsilon} c_1(x,z)|u^\epsilon_{1,n} - v^\epsilon_{1,n}|^2 \, \mathrm{d}\sigma_\epsilon \\
&\leq \epsilon \|c_1(x,z)\|_{L^\infty(\Gamma_\epsilon)} \int_{\Gamma_\epsilon} \left(|u^\epsilon_{1,n}|^2 + |v^\epsilon_{1,n}|^2\right) \mathrm{d}\sigma_\epsilon \\
&\leq \epsilon \|c_1(x,z)\|_{L^\infty(\Gamma_\epsilon)} \left(\|u^\epsilon_{1,n}\|^2_{L^2(\Gamma_\epsilon)} + \|v^\epsilon_{1,n}\|^2_{L^2(\Gamma_\epsilon)}\right)
\end{aligned}
\tag{2.31}
$$

Applying the generalized interpolation-trace inequality (A.7) in Appendix A to each term inside the round brackets, one has:

$$
I \leq C_1\, \epsilon\, \eta \left(\|\nabla u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \|\nabla v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)}\right) + C_2\, \epsilon\, \eta^{-1}\left(\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)}\right) \tag{2.32}
$$

where $\eta$ is a small positive constant. If we take into account the estimate (2.32), the inequality (2.30) reads:

$$
\begin{aligned}
&\frac{1}{2}\partial_t \|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\partial_t \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + (\epsilon^2 D_1 - C_1\, \epsilon\, \eta)\|\nabla_x u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} \\
&\quad + (\tilde{D}_1 - C_1\, \epsilon\, \eta)\|\partial_z u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + (d_1 - C_1\, \epsilon\, \eta)\|\nabla v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} \leq C_f \\
&\quad + \left(C_2\, \epsilon\, \eta^{-1} + \frac{1}{2}\right)\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + C_2\, \epsilon\, \eta^{-1}\|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)}
\end{aligned}
\tag{2.33}
$$

If one chooses $\eta < \min\{\frac{\epsilon D_1}{C_1}, \frac{\tilde{D}_1}{\epsilon\, C_1}, \frac{d_1}{\epsilon\, C_1}\}$, the last three terms on the left-hand side are positive and Eq. (2.33) reduces to:

$$
\partial_t \|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \partial_t \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} \leq 2\, C_f + 2\, C_2\, \epsilon\, \eta^{-1}\left(\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)}\right) \tag{2.34}
$$

Integrating Eq. (2.34) over $[0,t]$ with $t \in [0,T]$, we get

$$
\begin{aligned}
\|u^\epsilon_{1,n}(t)\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}(t)\|^2_{L^2(\Omega_\epsilon)} &\leq \|u^{\epsilon,0}_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \|v^{\epsilon,0}_{1,n}\|^2_{L^2(\Omega_\epsilon)} \\
&\quad + 2\, C_f\, T + 2\, C_2\, \epsilon\, \eta^{-1} \int_0^t \mathrm{d}s \left(\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)}\right)
\end{aligned}
\tag{2.35}
$$

Since the sequences $\left(u^{\epsilon,0}_{1,n}\right)_{n\in\mathbb{N}}$ and $\left(v^{\epsilon,0}_{1,n}\right)_{n\in\mathbb{N}}$ converge in $H^1(\Pi_\epsilon)$ and $H^1(\Omega_\epsilon)$, respectively, they are bounded in $L^2$. Therefore, Eq. (2.35) reads

$$
\|u^\epsilon_{1,n}(t)\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}(t)\|^2_{L^2(\Omega_\epsilon)} \leq C + 2\, C_2\, \epsilon\, \eta^{-1} \int_0^t \mathrm{d}s \left(\|u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)}\right) \tag{2.36}
$$

Applying Gronwalls's inequality, we obtain

$$
\|u^\epsilon_{1,n}(t)\|^2_{L^2(\Pi_\epsilon)} + \|v^\epsilon_{1,n}(t)\|^2_{L^2(\Omega_\epsilon)} \leq C + 2\, C\, C_2\, \epsilon\, \eta^{-1} \int_0^t e^{2\, C_2\, \epsilon\, \eta^{-1}(t-s)}\, \mathrm{d}s \tag{2.37}
$$

Therefore, given $\epsilon \in [0,1]$, $\eta$ is fixed and for $t \in [0,T]$ we get:

$$\|u^{\epsilon}_{1,n}(t,\cdot)\|^2_{L^2(\Pi_\epsilon)} + \|v^{\epsilon}_{1,n}(t,\cdot)\|^2_{L^2(\Omega_\epsilon)} \le C_3 \tag{2.38}$$

where $C_3$ is a positive constant independent of $n$ and $\epsilon$.

By testing Eqs. (2.4) and (2.5), in the case $1 < i \le M$, with $\psi_i = u^{\epsilon}_{i,n}(t,\cdot)$ and $\phi_i = v^{\epsilon}_{i,n}(t,\cdot)$, respectively, one gets:

$$\begin{aligned}\frac{1}{2}\partial_t\|u^{\epsilon}_{i,n}\|^2_{L^2(\Pi_\epsilon)} &+ \epsilon^2 D_i\|\nabla_x u^{\epsilon}_{i,n}\|^2_{L^2(\Pi_\epsilon)} + \tilde{D}_i\|\partial_z u^{\epsilon}_{i,n}\|^2_{L^2(\Pi_\epsilon)}\\ &\le -\epsilon\int_{\Gamma_\epsilon} c_i(x,z)\left(u^{\epsilon}_{i,n} - v^{\epsilon}_{i,n}\right)_+ u^{\epsilon}_{i,n}\,\mathrm{d}\sigma_\epsilon + C + \frac{1}{4}\|u^{\epsilon}_{i,n}\|^2_{L^2(\Pi_\epsilon)}\end{aligned} \tag{2.39}$$

$$\frac{1}{2}\partial_t\|v^{\epsilon}_{i,n}\|^2_{L^2(\Omega_\epsilon)} + d_i\|\nabla v^{\epsilon}_{i,n}\|^2_{L^2(\Omega_\epsilon)} \le \epsilon\int_{\Gamma_\epsilon} c_i(x,z)\left(u^{\epsilon}_{i,n} - v^{\epsilon}_{i,n}\right)_+ v^{\epsilon}_{i,n}\,\mathrm{d}\sigma_\epsilon + \tilde{C} + \frac{1}{4}\|v^{\epsilon}_{i,n}\|^2_{L^2(\Omega_\epsilon)} \tag{2.40}$$

where $C$ and $\tilde{C}$ are two positive constants. Adding the inequalities (2.39) and (2.40), and exploiting the estimate (2.31) (which holds also when $1 < i \le M$) along with the interpolation-trace inequality (A.7), we obtain that also the functions $u^{\epsilon}_{i,n}$ and $v^{\epsilon}_{i,n}$ ($1 < i \le M$) satisfy Eq. (2.34). The rest of the proof carries over verbatim, leading to Eq. (2.38) also for the case $1 < i \le M$.

Thus, we can conclude that:

$\left\{u^{\epsilon}_{i,n}\right\}$ is bounded in $L^\infty([0,T];\ L^2(\Pi_\epsilon))$, and $\left\{v^{\epsilon}_{i,n}\right\}$ is bounded in $L^\infty([0,T];\ L^2(\Omega_\epsilon))$.

Let us now derive uniform estimates in $n$ for $\partial_t u^{\epsilon}_{i,n}$, $\nabla u^{\epsilon}_{i,n}$ and $\partial_t v^{\epsilon}_{i,n}$, $\nabla v^{\epsilon}_{i,n}$ ($1 \le i \le M$). In the case $i = 1$, we take, in Eq. (2.10), $\psi_1 = \partial_t u^{\epsilon}_{1,n}(t,\cdot)$ as test function:

$$\begin{aligned}&\int_{\Pi_\epsilon} |\partial_t u^{\epsilon}_{1,n}|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{\epsilon^2}{2}\int_{\Pi_\epsilon} D_1\,\partial_t\left(|\nabla_x u^{\epsilon}_{1,n}|^2\right)\mathrm{d}x\,\mathrm{d}z\\ &\quad + \frac{1}{2}\int_{\Pi_\epsilon} \tilde{D}_1\,\partial_t\left(|\partial_z u^{\epsilon}_{1,n}|^2\right)\mathrm{d}x\,\mathrm{d}z + \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\left(u^{\epsilon}_{1,n} - v^{\epsilon}_{1,n}\right)_+\left(\partial_t u^{\epsilon}_{1,n}\right)\mathrm{d}\sigma_\epsilon\\ &= -\int_{\Pi_\epsilon} w^{\epsilon}_{\tilde{M}}\left(\partial_t u^{\epsilon}_{1,n}\right)\mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} f^{\epsilon}(t,x,z)\left(\partial_t u^{\epsilon}_{1,n}\right)\mathrm{d}x\,\mathrm{d}z\end{aligned} \tag{2.41}$$

where $w^{\epsilon}_{\tilde{M}} =: \sum_{j=1}^{M} a_{1,j}\,\sigma_{\tilde{M}}\left(u^{\epsilon}_{1,n}\right)\sigma_{\tilde{M}}\left(u^{\epsilon}_{j,n}\right)$.

By taking into account the Hölder and Young inequalities, Eq. (2.41) becomes

$$\begin{aligned}&\int_{\Pi_\epsilon} |\partial_t u^{\epsilon}_{1,n}|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{\epsilon^2 D_1}{2}\int_{\Pi_\epsilon} \partial_t\left(|\nabla_x u^{\epsilon}_{1,n}|^2\right)\mathrm{d}x\,\mathrm{d}z + \frac{\tilde{D}_1}{2}\int_{\Pi_\epsilon} \partial_t\left(|\partial_z u^{\epsilon}_{1,n}|^2\right)\mathrm{d}x\,\mathrm{d}z\\ &\quad\le \frac{1}{2}\|w^{\epsilon}_{\tilde{M}}\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\|\partial_t u^{\epsilon}_{1,n}\|^2_{L^2(\Pi_\epsilon)} - \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\left(u^{\epsilon}_{1,n} - v^{\epsilon}_{1,n}\right)_+\left(\partial_t u^{\epsilon}_{1,n}\right)\mathrm{d}\sigma_\epsilon\\ &\quad + \eta^{-1}\|f^{\epsilon}(t,x,z)\|^2_{L^2(\Pi_\epsilon)} + \eta\|\partial_t u^{\epsilon}_{1,n}\|^2_{L^2(\Pi_\epsilon)}\end{aligned} \tag{2.42}$$

Exploiting the truncation of the coagulation terms and choosing $\eta = \frac{1}{4}$, we get:

$$\begin{aligned}&\frac{1}{4}\|\partial_t u^{\epsilon}_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{\epsilon^2}{2} D_1\,\partial_t\int_{\Pi_\epsilon} |\nabla_x u^{\epsilon}_{1,n}|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{\tilde{D}_1}{2}\,\partial_t\int_{\Pi_\epsilon} |\partial_z u^{\epsilon}_{1,n}|^2\,\mathrm{d}x\,\mathrm{d}z\\ &\quad\le C^1_{\tilde{M}} - \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\left(u^{\epsilon}_{1,n} - v^{\epsilon}_{1,n}\right)_+\left(\partial_t u^{\epsilon}_{1,n}\right)\mathrm{d}\sigma_\epsilon + 4\,\|f^{\epsilon}(t,x,z)\|^2_{L^2(\Pi_\epsilon)}\end{aligned} \tag{2.43}$$

where $C^1_{\tilde{M}}$ is a positive constant which depends on $\tilde{M}$.

Let us now test Eq. (2.15) with the function $\phi_1 = \partial_t v^\epsilon_{1,n}(t,\cdot)$:

$$\int_{\Omega_\epsilon} |\partial_t v^\epsilon_{1,n}|^2 \,dx\,dz + \frac{d_1}{2}\int_{\Omega_\epsilon} \partial_t\Big(|\nabla v^\epsilon_{1,n}|^2\Big)\,dx\,dz - \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)_+ \Big(\partial_t v^\epsilon_{1,n}\Big)\,d\sigma_\epsilon$$
$$= -\int_{\Omega_\epsilon}\left[\sum_{j=1}^{M} b_{1,j}\,\sigma_{\tilde{M}}\Big(v^\epsilon_{1,n}\Big)\,\sigma_{\tilde{M}}\Big(v^\epsilon_{j,n}\Big)\right]\Big(\partial_t v^\epsilon_{1,n}\Big)\,dx\,dz \tag{2.44}$$

By applying once again the Hölder and Young inequalities to the right-hand side as above, and exploiting Eq. (2.3), we end up with the following expression:

$$\frac{1}{2}\|\partial_t v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + \frac{d_1}{2}\,\partial_t\int_{\Omega_\epsilon} |\nabla v^\epsilon_{1,n}|^2\,dx\,dz \le C^2_{\tilde{M}} + \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)_+\Big(\partial_t v^\epsilon_{1,n}\Big)\,d\sigma_\epsilon \tag{2.45}$$

where $C^2_{\tilde{M}}$ is a positive constant which depends on $\tilde{M}$. Adding (2.43) and (2.45), and taking into account the $L^2$-boundedness of $f^\epsilon(t,x,z)$, one obtains:

$$\frac{1}{4}\|\partial_t u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\|\partial_t v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + \frac{\epsilon^2}{2}\,D_1\,\partial_t\int_{\Pi_\epsilon}|\nabla_x u^\epsilon_{1,n}|^2\,dx\,dz$$
$$+ \frac{\tilde{D}_1}{2}\partial_t\int_{\Pi_\epsilon}|\partial_z u^\epsilon_{1,n}|^2\,dx\,dz + \frac{d_1}{2}\,\partial_t\int_{\Omega_\epsilon}|\nabla v^\epsilon_{1,n}|^2\,dx\,dz$$
$$\le C + \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)_+\Big(\partial_t v^\epsilon_{1,n} - \partial_t u^\epsilon_{1,n}\Big)\,d\sigma_\epsilon \tag{2.46}$$

where $C$ is a positive constant. If we decompose now the function $\partial_t\Big(v^\epsilon_{1,n} - u^\epsilon_{1,n}\Big)$ on the right-hand side in its positive and negative parts, Eq. (2.46) can be rewritten as:

$$\frac{1}{4}\|\partial_t u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\|\partial_t v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + \frac{\epsilon^2}{2}\,D_1\,\partial_t\int_{\Pi_\epsilon}|\nabla_x u^\epsilon_{1,n}|^2\,dx\,dz$$
$$+ \frac{\tilde{D}_1}{2}\,\partial_t\int_{\Pi_\epsilon}|\partial_z u^\epsilon_{1,n}|^2\,dx\,dz + \frac{d_1}{2}\,\partial_t\int_{\Omega_\epsilon}|\nabla v^\epsilon_{1,n}|^2\,dx\,dz$$
$$\le C - \frac{\epsilon}{2}\int_{\Gamma_\epsilon} c_1(x,z)\partial_t\left[\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)^2_+\right]d\sigma_\epsilon \tag{2.47}$$

Integrating over $[0,t]$ with $t \in [0,T]$, we deduce:

$$\frac{1}{4}\int_0^t ds\,\|\partial_s u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{1}{2}\int_0^t ds\,\|\partial_s v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + \frac{\epsilon^2}{2}\,D_1\int_{\Pi_\epsilon}|\nabla_x u^\epsilon_{1,n}|^2\,dx\,dz - \frac{\epsilon^2}{2}\,D_1\int_{\Pi_\epsilon}|\nabla_x u^\epsilon_{1,n}(0)|^2\,dx\,dz$$
$$+ \frac{\tilde{D}_1}{2}\int_{\Pi_\epsilon}|\partial_z u^\epsilon_{1,n}|^2\,dx\,dz - \frac{\tilde{D}_1}{2}\int_{\Pi_\epsilon}|\partial_z u^\epsilon_{1,n}(0)|^2\,dx\,dz + \frac{d_1}{2}\int_{\Omega_\epsilon}|\nabla v^\epsilon_{1,n}|^2\,dx\,dz - \frac{d_1}{2}\int_{\Omega_\epsilon}|\nabla v^\epsilon_{1,n}(0)|^2\,dx\,dz \le C\,T$$
$$+ \frac{\epsilon}{2}\int_{\Gamma_\epsilon} c_1(x,z)\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)^2_+(0)\,d\sigma_\epsilon - \frac{\epsilon}{2}\int_{\Gamma_\epsilon} c_1(x,z)\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)^2_+\,d\sigma_\epsilon \tag{2.48}$$

Hence, taking into account that the last term on the right-hand side of Eq. (2.48) is negative, one has

$$\begin{aligned}&\int_0^t \mathrm{d}s\,\|\partial_s u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + 2\int_0^t \mathrm{d}s\,\|\partial_s v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + 2\,\epsilon^2 D_1\,\|\nabla_x u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)}\\&+ 2\tilde{D}_1\|\partial_z u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + 2\,d_1\|\nabla v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} \le 4\,C\,T\\&+ 2\,D_1\int_{\Pi_\epsilon}|\nabla_x u^\epsilon_{1,n}(0)|^2\,\mathrm{d}x\,\mathrm{d}z + 2\tilde{D}_1\int_{\Pi_\epsilon}|\partial_z u^\epsilon_{1,n}(0)|^2\,\mathrm{d}x\,\mathrm{d}z\\&+ 2\,d_1\int_{\Omega_\epsilon}|\nabla v^\epsilon_{1,n}(0)|^2\,\mathrm{d}x\,\mathrm{d}z + 2\int_{\Gamma_\epsilon} c_1(x,z)\Big(u^\epsilon_{1,n} - v^\epsilon_{1,n}\Big)^2_+(0)\,\mathrm{d}\sigma_\epsilon\end{aligned} \tag{2.49}$$

Since the sequences $u^\epsilon_{1,n}(0)$ and $v^\epsilon_{1,n}(0)$ are bounded, it follows:

$$\begin{aligned}&\int_0^T \mathrm{d}s\,\|\partial_s u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + 2\int_0^T \mathrm{d}s\,\|\partial_s v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} + 2\,\epsilon^2 D_1\,\|\nabla_x u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)}\\&+ 2\tilde{D}_1\,\|\partial_z u^\epsilon_{1,n}\|^2_{L^2(\Pi_\epsilon)} + 2\,d_1\,\|\nabla v^\epsilon_{1,n}\|^2_{L^2(\Omega_\epsilon)} \le \tilde{C} \qquad \text{for } t \in [0,T]\end{aligned} \tag{2.50}$$

where $\tilde{C}$ is a positive constant independent of $n$ and $\epsilon$.

In the case $1 < i \le M$, by testing Eqs. (2.4) and (2.5) with $\psi_i = \partial_t u^\epsilon_{i,n}(t,\cdot)$ and $\phi_i = \partial_t v^\epsilon_{i,n}(t,\cdot)$, respectively, one obtains:

$$\begin{aligned}&\frac{1}{4}\|\partial_t u^\epsilon_{i,n}\|^2_{L^2(\Pi_\epsilon)} + \frac{\epsilon^2}{2}\,D_i\,\partial_t\int_{\Pi_\epsilon}|\nabla_x u^\epsilon_{i,n}|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{\tilde{D}_i}{2}\,\partial_t\int_{\Pi_\epsilon}|\partial_z u^\epsilon_{i,n}|^2\,\mathrm{d}x\,\mathrm{d}z\\&\le C - \epsilon\int_{\Gamma_\epsilon} c_i(x,z)\Big(u^\epsilon_{i,n} - v^\epsilon_{i,n}\Big)_+\Big(\partial_t u^\epsilon_{i,n}\Big)\,\mathrm{d}\sigma_\epsilon\end{aligned} \tag{2.51}$$

$$\frac{1}{4}\|\partial_t v^\epsilon_{i,n}\|^2_{L^2(\Omega_\epsilon)} + \frac{d_i}{2}\,\partial_t\int_{\Omega_\epsilon}|\nabla v^\epsilon_{i,n}|^2\,\mathrm{d}x\,\mathrm{d}z \le \tilde{C} + \epsilon\int_{\Gamma_\epsilon} c_i(x,z)\Big(u^\epsilon_{i,n} - v^\epsilon_{i,n}\Big)_+\Big(\partial_t v^\epsilon_{i,n}\Big)\,\mathrm{d}\sigma_\epsilon \tag{2.52}$$

due to the boundedness of the coagulation terms given by (2.3). In Eqs. (2.51) and (2.52), $C$ and $\tilde{C}$ are two positive constants which depend on $\tilde{M}$. Adding the two inequalities (2.51) and (2.52), and applying exactly the same arguments considered for $i = 1$, we obtain that also the functions $u^\epsilon_{i,n}$ and $v^\epsilon_{i,n}$ $(1 < i \le M)$ satisfy Eq. (2.50).

Thus, combining the estimates (2.38) and (2.50), one concludes that:

$\Big\{u^\epsilon_{i,n}\Big\}$ is bounded in $H^1([0,T];\ L^2(\Pi_\epsilon)) \cap L^\infty([0,T];\ H^1(\Pi_\epsilon))$, and $\Big\{v^\epsilon_{i,n}\Big\}$ is bounded in $H^1([0,T];\ L^2(\Omega_\epsilon)) \cap L^\infty([0,T];\ H^1(\Omega_\epsilon))$. Hence, we can state the following proposition.

**Proposition 2.1.** *Since $\Big(u^\epsilon_{i,n}\Big)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T];\ H^1(\Pi_\epsilon))$, by the Banach–Alaoglu theorem we may assume that, up to a subsequence, $u^\epsilon_{i,n} \rightharpoonup u^\epsilon_i$ weakly* in $L^\infty([0,T];\ H^1(\Pi_\epsilon))$, i.e. for all $\psi \in L^1([0,T];\ H^1(\Pi_\epsilon))$*

$$\begin{aligned}&\int_0^T\int_{\Pi_\epsilon}\Big(\psi(t,x,z)u^\epsilon_{i,n}(t,x,z) + \nabla\psi(t,x,z)\cdot\nabla u^\epsilon_{i,n}(t,x,z)\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z\\&\to \int_0^T\int_{\Pi_\epsilon}\big(\psi(t,x,z)u^\epsilon_i(t,x,z) + \nabla\psi(t,x,z)\cdot\nabla u^\epsilon_i(t,x,z)\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z\end{aligned} \tag{2.53}$$

*as* $n \to \infty$. *Since* $L^2([0,T];\ H^1(\Pi_\epsilon)) \subset L^1([0,T];\ H^1(\Pi_\epsilon))$, *formula* (2.53) *holds also for* $\psi \in L^2([0,T];\ H^1(\Pi_\epsilon))$.

*In addition,* $\left(u^\epsilon_{i,n}\right)_{n\in\mathbb{N}}$ *is bounded in* $H^1([0,T];\ L^2(\Pi_\epsilon))$, *thus we can also assume that* $u^\epsilon_{i,n} \rightharpoonup u^\epsilon_i$ *weakly in* $H^1([0,T];\ L^2(\Pi_\epsilon))$ *as* $n \to \infty$.

*Finally, since in particular*

$$\left(u^\epsilon_{i,n}\right)_{n\in\mathbb{N}} \text{ is bounded in } \quad H^1([0,T];L^2(\Pi_\epsilon)) \cap L^\infty([0,T];H^1(\Pi_\epsilon)),$$

*by the Aubin–Lions–Simon theorem (*[40], *Theorem II.5.16, p. 102) (see Appendix* B*) we can infer that* $u^\epsilon_{i,n} \to u^\epsilon_i$ *strongly in* $C^0([0,T];\ L^2(\Pi_\epsilon))$ *as* $n \to \infty$.

An analogous proposition can be proved also for the sequence $\left(v^\epsilon_{i,n}\right)_{n\in\mathbb{N}}$.

Now, integrating Eq. (2.10) with respect to time and using as a test function $\psi_1 = \phi(t)\zeta_1(x,z)$, with $\phi \in \mathcal{D}([0,T])$ and $\zeta_1 \in H^1(\Pi_\epsilon)$, we get

$$\begin{aligned}
&\int_0^T \int_{\Pi_\epsilon} \partial_t u^\epsilon_{1,n}\, \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z + \epsilon^2 \int_0^T \int_{\Pi_\epsilon} D_1\, \nabla_x u^\epsilon_{1,n} \cdot \nabla_x \zeta_1\, \phi(t)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z \\
&\quad + \int_0^T \int_{\Pi_\epsilon} \tilde{D}_1\, \partial_z u^\epsilon_{1,n} \cdot \partial_z \zeta_1\, \phi(t)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z + \epsilon \int_0^T \int_{\Gamma_\epsilon} c_1(x,z) \left(u^\epsilon_{1,n} - v^\epsilon_{1,n}\right)_+ \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}\sigma_\epsilon \\
&\quad = \int_0^T \int_{\Pi_\epsilon} L^{\tilde{M}}_1\left(u^\epsilon_{i,n}\right) \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z + \int_0^T \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\, \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z
\end{aligned} \tag{2.54}$$

where we denote by $\mathcal{D}([0,T])$ the set of indefinitely differentiable functions whose support is a compact set.

Exploiting the convergence results stated in Proposition 2.1 we can pass to the limit as $n \to \infty$ to obtain:

$$\begin{aligned}
&\int_0^T \int_{\Pi_\epsilon} \partial_t u^\epsilon_1\, \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z + \epsilon^2 \int_0^T \int_{\Pi_\epsilon} D_1\, \nabla_x u^\epsilon_1 \cdot \nabla_x \zeta_1\, \phi(t)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z \\
&\quad + \int_0^T \int_{\Pi_\epsilon} \tilde{D}_1\, \partial_z u^\epsilon_1 \cdot \partial_z \zeta_1\, \phi(t)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z + \epsilon \int_0^T \int_{\Gamma_\epsilon} c_1(x,z) \left(u^\epsilon_1 - v^\epsilon_1\right)_+ \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}\sigma_\epsilon \\
&\quad = \int_0^T \int_{\Pi_\epsilon} L^{\tilde{M}}_1\left(u^\epsilon_i\right) \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z + \int_0^T \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\, \phi(t)\, \zeta_1(x,z)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}z
\end{aligned} \tag{2.55}$$

where we have taken into account that the term $L^{\tilde{M}}_1\left(u^\epsilon_i\right)$ is Lipschitz continuous. Eq. (2.55), which holds for arbitrary $\phi(t) \in \mathcal{D}([0,T])$ and $\zeta_1(x,z) \in H^1(\Pi_\epsilon)$, is exactly the variational Equation (2.4).

By using the same arguments handled above for $u^\epsilon_1$, we can prove that the functions $v^\epsilon_1$ and $u^\epsilon_i, v^\epsilon_i$ $(1 < i \le M)$ satisfy Eqs. (2.4) and (2.5).

It remains to show that the initial conditions hold. By the Aubin–Lions–Simon theorem it follows that $u^\epsilon_{1,n} \to u^\epsilon_1$ strongly in $C^0([0,T];\ L^2(\Pi_\epsilon))$ as $n \to \infty$. Since $u^\epsilon_{1,n}(t=0) = u^{\epsilon,0}_{1,n}$ and $u^{\epsilon,0}_{1,n} \to U^\epsilon_1(x,z)$ strongly in $H^1(\Pi_\epsilon)$ as $n \to \infty$, we conclude that $u^\epsilon_1(0,x,z) = U^\epsilon_1(x,z)$. Following the same line of arguments, we also obtain $v^\epsilon_1(0,x,z) = 0$ along with $u^\epsilon_i(0,x,z) = v^\epsilon_i(0,x,z) = 0$ $(1 < i \le M)$. □

## 3 Positivity and boundedness of solutions

**Lemma 3.1.** *For a given small $\epsilon > 0$, let $u_i^\epsilon(t,x,z)$ and $v_i^\epsilon(t,x,z)$ $(1 \le i \le M)$ be solutions of the system* (1.1) *in the sense of the Definition* 2.1. *Then $0 \le u_i^\epsilon(t,x,z) < M_i$ a.e. in $(0,T) \times \Pi_\epsilon$ and $0 \le v_i^\epsilon(t,x,z) < \overline{M}_i$ a.e. in $(0,T) \times \Omega_\epsilon$, where $M_i > 0$ and $\overline{M}_i > 0$ are constants independent of $\tilde{M}$, $u_i^\epsilon$, $v_i^\epsilon$ and $\epsilon$.*

*Proof.* In the case $i = 1$, let us test Eq. (2.4) with the function $\psi_1 = -u_1^{\epsilon,-}(t,\cdot)$:

$$\begin{aligned}\int_{\Pi_\epsilon} \partial_t u_1^\epsilon \left(-u_1^{\epsilon,-}\right) \mathrm{d}x\,\mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_1 \nabla_x u_1^\epsilon \cdot \nabla_x \left(-u_1^{\epsilon,-}\right) \mathrm{d}x\,\mathrm{d}z \\ + \int_{\Pi_\epsilon} \tilde{D}_1\, \partial_z u_1^\epsilon \cdot \partial_z \left(-u_1^{\epsilon,-}\right) \mathrm{d}x\,\mathrm{d}z + \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \big(u_1^\epsilon - v_1^\epsilon\big)_+ \left(-u_1^{\epsilon,-}\right) \mathrm{d}\sigma_\epsilon \\ = \int_{\Pi_\epsilon} L_1^{\tilde{M}}(u^\epsilon) \left(-u_1^{\epsilon,-}\right) \mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} f^\epsilon(t,x,z) \left(-u_1^{\epsilon,-}\right) \mathrm{d}x\,\mathrm{d}z \end{aligned} \tag{3.1}$$

Decomposing the function $u_1^\epsilon$ in its positive and negative parts, we get:

$$\begin{aligned}\int_{\Pi_\epsilon} \frac{\partial u_1^{\epsilon,-}}{\partial t} u_1^{\epsilon,-} \,\mathrm{d}x\,\mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_1 \,|\nabla_x u_1^{\epsilon,-}|^2 \,\mathrm{d}x\,\mathrm{d}z \\ + \int_{\Pi_\epsilon} \tilde{D}_1 \,|\partial_z u_1^{\epsilon,-}|^2 \,\mathrm{d}x\,\mathrm{d}z = \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \big(u_1^\epsilon - v_1^\epsilon\big)_+ u_1^{\epsilon,-} \,\mathrm{d}\sigma_\epsilon \\ + \int_{\Pi_\epsilon} \left[\sum_{j=1}^{M} a_{1,j}\, \sigma_{\tilde{M}}\big(u_1^\epsilon\big)\, \sigma_{\tilde{M}}\Big(u_j^\epsilon\Big)\right] u_1^{\epsilon,-} \,\mathrm{d}x\,\mathrm{d}z - \int_{\Pi_\epsilon} f^\epsilon(t,x,z)\, u_1^{\epsilon,-} \,\mathrm{d}x\,\mathrm{d}z \end{aligned} \tag{3.2}$$

Since the last but one term on the right-hand side is always zero and the last one is negative, one obtains:

$$\frac{1}{2}\partial_t \|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)} + \epsilon^2 D_1 \,\|\nabla_x u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)} + \tilde{D}_1 \,\|\partial_z u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)} \le \epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left(u_1^\epsilon - v_1^\epsilon\right)_+ u_1^{\epsilon,-} \,\mathrm{d}\sigma_\epsilon \tag{3.3}$$

Let us now take $\phi_1 = -v_1^{\epsilon,-}(t,\cdot)$ as test function in Eq. (2.5) and decompose the function $v_1^\epsilon$ in its positive and negative parts:

$$\begin{aligned}\int_{\Omega_\epsilon} \frac{\partial v_1^{\epsilon,-}}{\partial t} v_1^{\epsilon,-} \,\mathrm{d}x\,\mathrm{d}z + \int_{\Omega_\epsilon} d_1 \,|\nabla v_1^{\epsilon,-}|^2 \,\mathrm{d}x\,\mathrm{d}z = -\epsilon \int_{\Gamma_\epsilon} c_1(x,z) \big(u_1^\epsilon - v_1^\epsilon\big)_+ v_1^{\epsilon,-} \,\mathrm{d}\sigma_\epsilon \\ + \int_{\Omega_\epsilon} \left[\sum_{j=1}^{M} b_{1,j}\, \sigma_{\tilde{M}}\big(v_1^\epsilon\big)\, \sigma_{\tilde{M}}\Big(v_j^\epsilon\Big)\right] v_1^{\epsilon,-} \,\mathrm{d}x\,\mathrm{d}z \end{aligned} \tag{3.4}$$

Since the last term on the right-hand side is always zero, Eq. (3.4) yields:

$$\frac{1}{2}\partial_t \|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)} + d_1 \,\|\nabla v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)} = -\epsilon \int_{\Gamma_\epsilon} c_1(x,z) \left(u_1^\epsilon - v_1^\epsilon\right)_+ v_1^{\epsilon,-} \,\mathrm{d}\sigma_\epsilon \tag{3.5}$$

Adding Eqs. (3.3) and (3.5) we obtain:

$$\begin{aligned}&\frac{1}{2}\partial_t\|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\frac{1}{2}\partial_t\|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}+\epsilon^2 D_1\,\|\nabla_x u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\tilde{D}_1\,\|\partial_z u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+d_1\,\|\nabla v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\\&\le \epsilon\int\limits_{\Gamma_\epsilon} c_1(x,z)\left(u_1^\epsilon-v_1^\epsilon\right)_+\left(u_1^{\epsilon,-}-v_1^{\epsilon,-}\right)\mathrm{d}\sigma_\epsilon\end{aligned}\tag{3.6}$$

Let us now estimate the term on the right-hand side of Eq. (3.6):

$$\begin{aligned}&\epsilon\int\limits_{\Gamma_\epsilon} c_1(x,z)\left(u_1^\epsilon-v_1^\epsilon\right)_+\left(u_1^{\epsilon,-}-v_1^{\epsilon,-}\right)\mathrm{d}\sigma_\epsilon\\&\quad\le \epsilon\int\limits_{\Gamma_\epsilon} c_1(x,z)\,\mathcal{H}\big(u_1^\epsilon-v_1^\epsilon\big)\big(u_1^\epsilon-v_1^\epsilon\big)\,u_1^{\epsilon,-}\,\mathrm{d}\sigma_\epsilon\\&\quad\le \epsilon\int\limits_{\Gamma_\epsilon} c_1(x,z)\,\mathcal{H}\big(u_1^\epsilon-v_1^\epsilon\big)\big(u_1^{\epsilon,-}\,v_1^{\epsilon,-}\big)\,\mathrm{d}\sigma_\epsilon\le\epsilon\|c_1(x,z)\|_{L^\infty(\Gamma_\epsilon)}\int\limits_{\Gamma_\epsilon} u_1^{\epsilon,-}\,v_1^{\epsilon,-}\,\mathrm{d}\sigma_\epsilon\end{aligned}\tag{3.7}$$

where we denote by $\mathcal{H}(\cdot)$ the Heaviside function:

$$\mathcal{H}(s):=\begin{cases}0, & s<0\\ 1, & s\ge 0.\end{cases}$$

Exploiting the Hölder and Young inequalities along with the generalized interpolation-trace inequality (A.7), Eq. (3.7) becomes

$$\begin{aligned}&\epsilon\int\limits_{\Gamma_\epsilon} c_1(x,z)\left(u_1^\epsilon-v_1^\epsilon\right)_+\left(u_1^{\epsilon,-}-v_1^{\epsilon,-}\right)\mathrm{d}\sigma_\epsilon\\&\quad\le\frac{\epsilon}{2}\|c_1(x,z)\|_{L^\infty(\Gamma_\epsilon)}\Big[\|u_1^{\epsilon,-}\|^2_{L^2(\Gamma_\epsilon)}+\|v_1^{\epsilon,-}\|^2_{L^2(\Gamma_\epsilon)}\Big]\\&\quad\le C_1\,\epsilon\,\eta\Big[\|\nabla u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\|\nabla v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\Big]+C_2\,\epsilon\,\eta^{-1}\Big[\|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\Big]\end{aligned}\tag{3.8}$$

where $\eta$ is a small constant. Finally from Eqs. (3.6) and (3.8) it follows that

$$\begin{aligned}&\frac{1}{2}\partial_t\|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\frac{1}{2}\partial_t\|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}+\big(\epsilon^2 D_1-C_1\,\epsilon\,\eta\big)\,\|\nabla_x u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}\\&\quad+(\tilde{D}_1-C_1\,\epsilon\,\eta)\,\|\partial_z u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+(d_1-C_1\,\epsilon\,\eta)\,\|\nabla v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\\&\le C_2\,\epsilon\,\eta^{-1}\Big[\|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\Big]\end{aligned}\tag{3.9}$$

If one chooses $\eta<\min\{\frac{\epsilon\,D_1}{C_1},\frac{\tilde{D}_1}{\epsilon\,C_1},\frac{d_1}{\epsilon\,C_1}\}$, Eq. (3.9) reduces to:

$$\frac{1}{2}\partial_t\|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\frac{1}{2}\partial_t\|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\le C_2\,\epsilon\,\eta^{-1}\Big[\|u_1^{\epsilon,-}\|^2_{L^2(\Pi_\epsilon)}+\|v_1^{\epsilon,-}\|^2_{L^2(\Omega_\epsilon)}\Big]\tag{3.10}$$

Setting the initial conditions: $u_1^{\epsilon,-}(0)\equiv 0$ and $v_1^{\epsilon,-}(0)\equiv 0$, Gronwall's lemma gives:

$$\|u_1^{\epsilon,-}(t,\cdot)\|^2_{L^2(\Pi_\epsilon)}+\|v_1^{\epsilon,-}(t,\cdot)\|^2_{L^2(\Omega_\epsilon)}\le 0\tag{3.11}$$

that is, $u_1^\epsilon\ge 0$ a.e. in $\Pi_\epsilon$ and $v_1^\epsilon\ge 0$ a.e. in $\Omega_\epsilon$, for all $t\in[0,T]$.

In the case $1 < i \le M$, by testing Eqs. (2.4) and (2.5) with $\psi_i = -u_i^{\epsilon,-}(t,\cdot)$ and $\phi_i = -v_i^{\epsilon,-}(t,\cdot)$, respectively, and decomposing the functions $u_i^\epsilon$ and $v_i^\epsilon$ in the positive and negative parts, one gets:

$$\begin{aligned}
&\int_{\Pi_\epsilon} \frac{\partial u_i^{\epsilon,-}}{\partial t}\, u_i^{\epsilon,-}\,\mathrm{d}x\,\mathrm{d}z + \epsilon^2 \int_{\Pi_\epsilon} D_i\,|\nabla_x u_i^{\epsilon,-}|^2\,\mathrm{d}x\,\mathrm{d}z + \int_{\Pi_\epsilon} \tilde{D}_i\,|\partial_z u_i^{\epsilon,-}|^2\,\mathrm{d}x\,\mathrm{d}z \\
&\quad = \epsilon \int_{\Gamma_\epsilon} c_i(x,z)\big(u_i^\epsilon - v_i^\epsilon\big)_+\, u_i^{\epsilon,-}\,\mathrm{d}\sigma_\epsilon - \frac{1}{2}\int_{\Pi_\epsilon} \left[\sum_{j=1}^{i-1} a_{j,i-j}\,\sigma_{\tilde{M}}\Big(u_j^\epsilon\Big)\,\sigma_{\tilde{M}}\Big(u_{i-j}^\epsilon\Big)\right] u_i^{\epsilon,-}\,\mathrm{d}x\,\mathrm{d}z \\
&\quad\quad + \int_{\Pi_\epsilon} \left[\sum_{j=1}^{M} a_{i,j}\,\sigma_{\tilde{M}}\big(u_i^\epsilon\big)\,\sigma_{\tilde{M}}\Big(u_j^\epsilon\Big)\right] u_i^{\epsilon,-}\,\mathrm{d}x\,\mathrm{d}z
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\int_{\Omega_\epsilon} \frac{\partial v_i^{\epsilon,-}}{\partial t}\, v_i^{\epsilon,-}\,\mathrm{d}x\,\mathrm{d}z + \int_{\Omega_\epsilon} d_i\,|\nabla v_i^{\epsilon,-}|^2\,\mathrm{d}x\,\mathrm{d}z \\
&\quad = -\epsilon \int_{\Gamma_\epsilon} c_i(x,z)\big(u_i^\epsilon - v_i^\epsilon\big)_+\, v_i^{\epsilon,-}\,\mathrm{d}\sigma_\epsilon - \frac{1}{2}\int_{\Omega_\epsilon} \left[\sum_{j=1}^{i-1} b_{j,i-j}\,\sigma_{\tilde{M}}\Big(v_j^\epsilon\Big)\,\sigma_{\tilde{M}}\Big(v_{i-j}^\epsilon\Big)\right] v_i^{\epsilon,-}\,\mathrm{d}x\,\mathrm{d}z \\
&\quad\quad + \int_{\Omega_\epsilon} \left[\sum_{j=1}^{M} b_{i,j}\,\sigma_{\tilde{M}}\big(v_i^\epsilon\big)\,\sigma_{\tilde{M}}\Big(v_j^\epsilon\Big)\right] v_i^{\epsilon,-}\,\mathrm{d}x\,\mathrm{d}z
\end{aligned} \tag{3.13}$$

Since in both Eqs. (3.12) and (3.13) the last but one term on the right-hand side is negative and the last one always zero, we obtain that also the functions $u_i^{\epsilon,-}$ and $v_i^{\epsilon,-}$ $(1 < i \le M)$ satisfy Eqs. (3.3) and (3.5), respectively. Therefore, applying exactly the same arguments considered for $i = 1$, we conclude that $u_i^\epsilon \ge 0$ a.e. in $\Pi_\epsilon$ and $v_i^\epsilon \ge 0$ a.e. in $\Omega_\epsilon$, for all $t \in [0,T]$.

Let us now prove the boundedness of solutions. In the case $i = 1$, we test Eq. (2.4) with $\psi_1 = p\big(u_1^\epsilon\big)^{p-1}(p \ge 2)$:[1]

$$\begin{aligned}
&p\int_{\Pi_\epsilon} \partial_t u_1^\epsilon\,\big(u_1^\epsilon\big)^{p-1}\,\mathrm{d}x\,\mathrm{d}z + \epsilon^2\, p\int_{\Pi_\epsilon} D_1\,\nabla_x u_1^\epsilon\cdot\nabla_x\big(u_1^\epsilon\big)^{p-1}\,\mathrm{d}x\,\mathrm{d}z \\
&\quad + p\int_{\Pi_\epsilon} \tilde{D}_1\,\partial_z u_1^\epsilon\cdot\partial_z\big(u_1^\epsilon\big)^{p-1}\,\mathrm{d}x\,\mathrm{d}z + \epsilon\, p\int_{\Gamma_\epsilon} c_1(x,z)\big(u_1^\epsilon - v_1^\epsilon\big)_+\,\big(u_1^\epsilon\big)^{p-1}\mathrm{d}\sigma_\epsilon \\
&\quad = -p\int_{\Pi_\epsilon}\left[\sum_{j=1}^{M} a_{1,j}\,\sigma_{\tilde{M}}\big(u_1^\epsilon\big)\,\sigma_{\tilde{M}}\Big(u_j^\epsilon\Big)\right]\big(u_1^\epsilon\big)^{p-1}\,\mathrm{d}x\,\mathrm{d}z + p\int_{\Pi_\epsilon} f^\epsilon(t,x,z)\,\big(u_1^\epsilon\big)^{p-1}\,\mathrm{d}x\,\mathrm{d}z
\end{aligned} \tag{3.14}$$

Since the last term on the left-hand side is positive and the first term on the right-hand side is negative, Eq. (3.14) implies:

$$\begin{aligned}
&\int_{\Pi_\epsilon} \frac{\partial}{\partial t}\big(u_1^\epsilon\big)^p\,\mathrm{d}x\,\mathrm{d}z + \epsilon^2\, p\,(p-1)\int_{\Pi_\epsilon} D_1\,\big(u_1^\epsilon\big)^{p-2}\,|\nabla_x u_1^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z \\
&\quad + p\,(p-1)\int_{\Pi_\epsilon} \tilde{D}_1\,\big(u_1^\epsilon\big)^{p-2}\,|\partial_z u_1^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z \le p\int_{\Pi_\epsilon} f^\epsilon(t,x,z)\,\big(u_1^\epsilon\big)^{p-1}\,\mathrm{d}x\,\mathrm{d}z
\end{aligned} \tag{3.15}$$

---

**1** The choice of the test function $\psi_1$ is not fully correct, since $\psi_1$ does not belong to $H^1(\Pi_\epsilon)$. Nevertheless, by a standard argument, this difficulty can be bypassed (here and later on) replacing the power $\big(u_1^\epsilon\big)^{p-1}$ with an approximation which is linear for $u_1^\epsilon \ge N$ ($N$ large) and taking eventually the limit as $N \to \infty$. We refer, for instance, to the proof of Theorem 8.15 in Ref. [44].

Taking into account that the last two terms on the left-hand side are positive and integrating over $[0, t]$ with $t \in [0, T]$, we obtain:

$$\int_0^t \int_{\Pi_\epsilon} \frac{\partial}{\partial s} \left(u_1^\epsilon\right)^p \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z \le p \int_0^t \int_{\Pi_\epsilon} f^\epsilon(s,x,z) \left(u_1^\epsilon\right)^{p-1} \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z \tag{3.16}$$

Hence,

$$\int_{\Pi_\epsilon} \left(u_1^\epsilon\right)^p \mathrm{d}x\,\mathrm{d}z \le \int_{\Pi_\epsilon} (u_1^\epsilon(0))^p \,\mathrm{d}x\,\mathrm{d}z + p\,\|f^\epsilon\|_{L^\infty([0,T]\times\Pi_\epsilon)} \int_0^t \int_{\Pi_\epsilon} \left(u_1^\epsilon\right)^{p-1} \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z \tag{3.17}$$

In order to estimate the last term on the right-hand side of Eq. (3.17), it is now convenient to use Young's inequality in the following form [41]:

$$a\,b \le \eta\, a^{p'} + \eta^{1-p}\, b^p, \quad \forall a \ge 0,\ b \ge 0, \quad p' = \frac{p}{p-1} \tag{3.18}$$

with: $a = \left(u_1^\epsilon\right)^{p-1}$, $b = p\,\|f^\epsilon\|_{L^\infty([0,T]\times\Pi_\epsilon)}$ and for an arbitrary $\eta > 0$. Therefore, we get:

$$\begin{aligned} & p\,\|f^\epsilon\|_{L^\infty([0,T]\times\Pi_\epsilon)} \int_0^t \int_{\Pi_\epsilon} \left(u_1^\epsilon\right)^{p-1} \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z \\ & \le \int_0^t \mathrm{d}s \int_{\Pi_\epsilon} p^p\,\|f^\epsilon\|^p_{L^\infty([0,T]\times\Pi_\epsilon)}\,\eta^{1-p}\,\mathrm{d}x\mathrm{d}z + \int_0^t \mathrm{d}s \int_{\Pi_\epsilon} \eta \left(u_1^\epsilon\right)^p \mathrm{d}x\,\mathrm{d}z \\ & \le p^{p-1}\,\|f^\epsilon\|^p_{L^\infty([0,T]\times\Pi_\epsilon)}\,\eta^{1-p}\,|\Pi_\epsilon|\,T + \eta \int_0^t \mathrm{d}s \int_{\Pi_\epsilon} \left(u_1^\epsilon\right)^p \mathrm{d}x\,\mathrm{d}z \end{aligned} \tag{3.19}$$

The last line above has been obtained by making the following approximation, which holds since, to finalize the proof, we will take the limit $p \to \infty$: $p^p \simeq (p-1)^{p-1} = \frac{(p-1)^{p-1}}{p^{p-1}}\, p^{p-1} \le p^{p-1}$, due to: $\left(\frac{p}{p-1}\right)^{1-p} \le 1$.

Choosing $\eta = p$ and using the inequality (3.19) in (3.17), we conclude that:

$$\|u_1^\epsilon\|^p_{L^p(\Pi_\epsilon)} \le \|U_1^\epsilon\|^p_{L^p(\Pi_\epsilon)} + \|f^\epsilon\|^p_{L^\infty([0,T]\times\Pi_\epsilon)}|\Pi_\epsilon|\,T + p\int_0^t \mathrm{d}s\,\|u_1^\epsilon(s)\|^p_{L^p(\Pi_\epsilon)} \tag{3.20}$$

Applying Gronwall's inequality it follows that:

$$\|u_1^\epsilon\|^p_{L^p(\Pi_\epsilon)} \le \left[\|U_1^\epsilon\|^p_{L^p(\Pi_\epsilon)} + \|f^\epsilon\|^p_{L^\infty([0,T]\times\Pi_\epsilon)}\,|\Pi_\epsilon|\,T\right] e^{pt} \tag{3.21}$$

And finally

$$\begin{aligned} \sup_{t\in[0,T]} \lim_{p\to\infty} \left[\int_{\Pi_\epsilon} \left(u_1^\epsilon\right)^p \mathrm{d}x\,\mathrm{d}z\right]^{1/p} & \le \left[\|U_1^\epsilon\|_{L^\infty(\Pi_\epsilon)} + \|f^\epsilon\|_{L^\infty([0,T]\times\Pi_\epsilon)}\right] e^T \\ & \le M_1 \end{aligned} \tag{3.22}$$

where $M_1$ is a positive constant due to the boundedness of the initial condition $U_1^\epsilon(x,z)$ and of the source term $f^\epsilon(t,x,z)$.

We test now Eq. (2.5) with $\phi_1 = (v_1^\epsilon(t,\cdot) - \overline{M}_1)^+$, where $\overline{M}_1$ is a positive constant:

$$\int_{\Omega_\epsilon} \partial_t\left(v_1^\epsilon - \overline{M}_1\right)\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}x\,\mathrm{d}z + \int_{\Omega_\epsilon} d_1 \nabla\left(v_1^\epsilon - \overline{M}_1\right)\cdot\nabla\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}x\,\mathrm{d}z$$

$$- \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\big(u_1^\epsilon - v_1^\epsilon\big)_+\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon = \int_{\Omega_\epsilon} N_1^{\tilde{M}}(v^\epsilon)\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}x\,\mathrm{d}z \tag{3.23}$$

Decomposing the function $\left(v_1^\epsilon - \overline{M}_1\right)$ in its positive and negative parts, one obtains:

$$\frac{1}{2}\int_{\Omega_\epsilon} \partial_t\left|\left(v_1^\epsilon - \overline{M}_1\right)^+\right|^2 \mathrm{d}x\,\mathrm{d}z + \int_{\Omega_\epsilon} d_1\left|\nabla\left(v_1^\epsilon - \overline{M}_1\right)^+\right|^2 \mathrm{d}x\,\mathrm{d}z \tag{3.24}$$

$$= \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\big(u_1^\epsilon - v_1^\epsilon\big)_+\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon - \int_{\Omega_\epsilon}\left[\sum_{j=1}^{M} b_{1,j}\,\sigma_{\tilde{M}}\left(v_1^\epsilon\right)\sigma_{\tilde{M}}\left(v_j^\epsilon\right)\right]\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}x\,\mathrm{d}z$$

Since the last term on the right-hand side is negative, we conclude:

$$\frac{1}{2}\partial_t\left\|\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega_\epsilon)} + d_1\left\|\nabla\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega_\epsilon)} \le \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\big(u_1^\epsilon - v_1^\epsilon\big)_+\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon \tag{3.25}$$

Let us now estimate the term on the right-hand side of (3.25):

$$\begin{aligned}
&\epsilon\int_{\Gamma_\epsilon} c_1(x,z)\big(u_1^\epsilon - v_1^\epsilon\big)_+\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon\\
&\quad\le \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\mathcal{H}\big(u_1^\epsilon - v_1^\epsilon\big)\big(u_1^\epsilon - M_1\big)\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon\\
&\qquad - \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\mathcal{H}\big(u_1^\epsilon - v_1^\epsilon\big)\big(v_1^\epsilon - M_1\big)\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon\\
&\quad\le \epsilon\int_{\Gamma_\epsilon} c_1(x,z)\mathcal{H}\big(u_1^\epsilon - v_1^\epsilon\big)\big(u_1^\epsilon - M_1\big)^+\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon
\end{aligned} \tag{3.26}$$

if one chooses $M_1 \le \overline{M}_1$. By applying the Hölder and Young inequalities, along with (A.7), one gets:

$$\begin{aligned}
&\epsilon\int_{\Gamma_\epsilon} c_1(x,z)\big(u_1^\epsilon - v_1^\epsilon\big)_+\left(v_1^\epsilon - \overline{M}_1\right)^+ \mathrm{d}\sigma_\epsilon\\
&\quad\le \frac{\epsilon}{2}\|c_1(x,z)\|_{L^\infty(\Gamma_\epsilon)}\left[\left\|\big(u_1^\epsilon - M_1\big)^+\right\|^2_{L^2(\Gamma_\epsilon)} + \left\|\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Gamma_\epsilon)}\right]\\
&\quad\le C_1\,\epsilon\,\eta\left\|\nabla\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega\epsilon)} + C_2\,\epsilon\,\eta^{-1}\left\|\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega\epsilon)}
\end{aligned} \tag{3.27}$$

where $\eta$ is a small constant and $u_1^\epsilon$ satisfies the inequality (3.22). Therefore, from Eq. (3.25):

$$\frac{1}{2}\partial_t\left\|\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega_\epsilon)} + (d_1 - C_1\epsilon\eta)\left\|\nabla\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega_\epsilon)} \le C_2\,\epsilon\,\eta^{-1}\left\|\left(v_1^\epsilon - \overline{M}_1\right)^+\right\|^2_{L^2(\Omega_\epsilon)} \tag{3.28}$$

If we choose $\eta < \frac{d_1}{\epsilon\, C_1}$, Eq. (3.28) reduces to:

$$\frac{1}{2}\partial_t \| \left( v_1^\epsilon - \overline{M}_1 \right)^+ \|^2_{L^2(\Omega_\epsilon)} \le C_2\, \epsilon\, \eta^{-1} \, \| \left( v_1^\epsilon - \overline{M}_1 \right)^+ \|^2_{L^2(\Omega_\epsilon)} \tag{3.29}$$

and the Gronwall lemma gives:

$$\| (v_1^\epsilon(t,\cdot) - \overline{M}_1)^+ \|^2_{L^2(\Omega_\epsilon)} \le \| \left( v_1^\epsilon - \overline{M}_1 \right)^+ (0) \|^2_{L^2(\Omega_\epsilon)} \, \exp\left( 2\, C_2\, \epsilon\, \eta^{-1}\, T \right) \tag{3.30}$$

for all $t \in [0, T]$. Since $v_1^\epsilon(0) = 0$, then:

$$\| \left( v_1^\epsilon - \overline{M}_1 \right)^+ (0) \|^2_{L^2(\Omega_\epsilon)} = 0$$

and from Eq. (3.30) it follows:

$$v_1^\epsilon \le \overline{M}_1. \tag{3.31}$$

In the case $1 < i \le M$ we proceed by induction and test Eq. (2.4) with $\psi_i = p \left( u_i^\epsilon \right)^{p-1} (p \ge 2)$:

$$\begin{aligned} & p \int_{\Pi_\epsilon} \partial_t u_i^\epsilon \, \left( u_i^\epsilon \right)^{p-1} \mathrm{d}x \, \mathrm{d}z + \epsilon^2 \, p \int_{\Pi_\epsilon} D_i \, \nabla_x u_i^\epsilon \cdot \nabla_x \left( u_i^\epsilon \right)^{p-1} \mathrm{d}x \, \mathrm{d}z \\ & \quad + p \int_{\Pi_\epsilon} \tilde{D}_i \, \partial_z u_i^\epsilon \cdot \partial_z \left( u_i^\epsilon \right)^{p-1} \mathrm{d}x \, \mathrm{d}z + \epsilon \, p \int_{\Gamma_\epsilon} c_i(x,z) \left( u_i^\epsilon - v_i^\epsilon \right)_+ \left( u_i^\epsilon \right)^{p-1} \mathrm{d}\sigma_\epsilon \\ & \quad = - p \int_{\Pi_\epsilon} \left[ \sum_{j=1}^{M} a_{i,j} \, \sigma_{\tilde{M}} \left( u_i^\epsilon \right) \sigma_{\tilde{M}} \left( u_j^\epsilon \right) \right] \left( u_i^\epsilon \right)^{p-1} \mathrm{d}x \, \mathrm{d}z + \frac{p}{2} \int_{\Pi_\epsilon} \left[ \sum_{j=1}^{i-1} a_{j,i-j} \, \sigma_{\tilde{M}} \left( u_j^\epsilon \right) \sigma_{\tilde{M}} \left( u_{i-j}^\epsilon \right) \right] \left( u_i^\epsilon \right)^{p-1} \mathrm{d}x \, \mathrm{d}z \end{aligned} \tag{3.32}$$

Since the last term on the left-hand side is positive and the first term on the right-hand side is negative, Eq. (3.32) reduces to:

$$\begin{aligned} & \int_{\Pi_\epsilon} \frac{\partial}{\partial t} \left( u_i^\epsilon \right)^p \mathrm{d}x \, \mathrm{d}z + \epsilon^2 \, p \, (p-1) \int_{\Pi_\epsilon} D_i \, \left( u_i^\epsilon \right)^{p-2} |\nabla_x u_i^\epsilon|^2 \, \mathrm{d}x \, \mathrm{d}z + p \, (p-1) \int_{\Pi_\epsilon} \tilde{D}_i \, \left( u_i^\epsilon \right)^{p-2} |\partial_z u_i^\epsilon|^2 \, \mathrm{d}x \, \mathrm{d}z \\ & \quad \le \frac{p}{2} \int_{\Pi_\epsilon} \left[ \sum_{j=1}^{i-1} a_{j,i-j} \, \sigma_{\tilde{M}} \left( u_j^\epsilon \right) \sigma_{\tilde{M}} \left( u_{i-j}^\epsilon \right) \right] \left( u_i^\epsilon \right)^{p-1} \mathrm{d}x \, \mathrm{d}z \end{aligned} \tag{3.33}$$

Taking into account that the last two terms on the left-hand side are positive and integrating over $[0, t]$ with $t \in [0, T]$, we obtain:

$$\int_0^t \int_{\Pi_\epsilon} \frac{\partial}{\partial s} \left( u_i^\epsilon \right)^p \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}z \le \frac{p}{2} \int_0^t \int_{\Pi_\epsilon} \left[ \sum_{j=1}^{i-1} a_{j,i-j} \, \sigma_{\tilde{M}} \left( u_j^\epsilon \right) \sigma_{\tilde{M}} \left( u_{i-j}^\epsilon \right) \right] \left( u_i^\epsilon \right)^{p-1} \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}z \tag{3.34}$$

Exploiting the boundedness of $u_j^\epsilon (1 \le j \le i-1)$ in $L^\infty(0, T;\ L^\infty(\Pi_\epsilon))$ and setting the initial conditions, one gets:

$$\int_{\Pi_\epsilon} \left( u_i^\epsilon \right)^p \mathrm{d}x \, \mathrm{d}z \le \frac{p}{2} \int_0^t \int_{\Pi_\epsilon} \left[ \sum_{j=1}^{i-1} a_{j,i-j} \, K_j \, K_{i-j} \right] \left( u_i^\epsilon \right)^{p-1} \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}z \tag{3.35}$$

where $K_j (1 \le j \le i-1)$ are positive constants. In order to estimate the term on the right-hand side of Eq. (3.35), we use the Young inequality (3.18) with: $a = \left(u_i^\epsilon\right)^{p-1}$ and $b = p\left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right]$. We find:

$$
\begin{aligned}
& p \int_0^t \int_{\Pi_\epsilon} \left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right] \left(u_i^\epsilon\right)^{p-1} \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}z \\
& \le \int_0^t \mathrm{d}s \int_{\Pi_\epsilon} p^p \left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right]^p \eta^{1-p}\, \mathrm{d}x \mathrm{d}z + \int_0^t \mathrm{d}s \int_{\Pi_\epsilon} \eta \left(u_i^\epsilon\right)^p \mathrm{d}x\, \mathrm{d}z \\
& \le p^{p-1} \left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right]^p \eta^{1-p}\, |\Pi_\epsilon|\, T + \eta \int_0^t \mathrm{d}s \int_{\Pi_\epsilon} \left(u_i^\epsilon\right)^p \mathrm{d}x\, \mathrm{d}z
\end{aligned}
\tag{3.36}
$$

Choosing $\eta = p$ and using the inequality (3.36) in (3.35), we obtain:

$$
\|u_i^\epsilon\|^p_{L^p(\Pi_\epsilon)} \le \left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right]^p |\Pi_\epsilon|\, T + p \int_0^t \mathrm{d}s\, \|u_i^\epsilon(s)\|^p_{L^p(\Pi_\epsilon)}
\tag{3.37}
$$

The Gronwall lemma applied to (3.37) leads to the estimate:

$$
\|u_i^\epsilon\|^p_{L^p(\Pi_\epsilon)} \le \left[\sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\right]^p |\Pi_\epsilon|\, T\, e^{pt}
\tag{3.38}
$$

Hence,

$$
\sup_{t\in[0,T]} \lim_{p\to\infty} \left[\int_{\Pi_\epsilon} \left(u_i^\epsilon\right)^p \mathrm{d}x\, \mathrm{d}z\right]^{1/p} \le \sum_{j=1}^{i-1} a_{j,i-j} K_j K_{i-j}\, e^T \le M_i
\tag{3.39}
$$

where $M_i$ is a positive constant.

We test now Eq. (2.5) with $\phi_i = p\left[(v_i^\epsilon(t,\cdot) - \overline{m}_i)^+\right]^{p-1}$ $(p \ge 2)$, where $\overline{m}_i$ is a positive constant such that $\overline{m}_i > M_i$:

$$
\begin{aligned}
& p \int_{\Omega_\epsilon} \partial_t\left(v_i^\epsilon - \overline{m}_i\right) \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^{p-1} \mathrm{d}x\, \mathrm{d}z + p \int_{\Omega_\epsilon} d_i \nabla\left(v_i^\epsilon - \overline{m}_i\right) \cdot \nabla\left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^{p-1} \mathrm{d}x\, \mathrm{d}z \\
& \quad - \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z)\left(u_i^\epsilon - v_i^\epsilon\right)_+ \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^{p-1} \mathrm{d}\sigma_\epsilon \\
& = -p \int_{\Omega_\epsilon} \left[\sum_{j=1}^{M} b_{i,j}\, \sigma_{\tilde{M}}\left(v_i^\epsilon\right) \sigma_{\tilde{M}}\left(v_j^\epsilon\right)\right] \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^{p-1} \mathrm{d}x\, \mathrm{d}z \\
& \quad + \frac{p}{2} \int_{\Omega_\epsilon} \left[\sum_{j=1}^{i-1} b_{j,i-j}\, \sigma_{\tilde{M}}\left(v_j^\epsilon\right) \sigma_{\tilde{M}}\left(v_{i-j}^\epsilon\right)\right] \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^{p-1} \mathrm{d}x\, \mathrm{d}z
\end{aligned}
\tag{3.40}
$$

Decomposing the function $\left(v_i^\epsilon - \overline{m}_i\right)$ in its positive and negative parts and taking into account the boundedness of $v_j^\epsilon$ $(1 \le j \le i-1)$ in $L^\infty(0,T;\ L^\infty(\Omega_\epsilon))$ due to the inductive hypothesis, one obtains:

$$\int_{\Omega_\epsilon} \partial_t \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^p \,\mathrm{d}x\,\mathrm{d}z + p\,(p-1) \int_{\Omega_\epsilon} d_i \,|\nabla \big(v_i^\epsilon - \overline{m}_i\big)^+|^2 \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-2} \,\mathrm{d}x\,\mathrm{d}z$$

$$\leq \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z) \big(u_i^\epsilon - v_i^\epsilon\big)_+ \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}\sigma_\epsilon + \frac{p}{2} \int_{\Omega_\epsilon} \left[ \sum_{j=1}^{i-1} b_{j,i-j}\, K_j\, K_{i-j} \right] \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}x\,\mathrm{d}z$$

$$- p \int_{\Omega_\epsilon} \left[ \sum_{j=1}^{M} b_{i,j}\, \sigma_{\tilde{M}}\big(v_i^\epsilon\big)\, \sigma_{\tilde{M}}\Big(v_j^\epsilon\Big) \right] \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}x\,\mathrm{d}z \tag{3.41}$$

where $K_j (1 \leq j \leq i-1)$ are positive constants. Since the last term on the left-hand side is positive and the one on the right-hand side is negative, Eq. (3.41) reduces to:

$$\int_{\Omega_\epsilon} \partial_t \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^p \,\mathrm{d}x\,\mathrm{d}z \leq \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z) \big(u_i^\epsilon - v_i^\epsilon\big)_+ \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}\sigma_\epsilon$$

$$+ \frac{p}{2} \int_{\Omega_\epsilon} \left[ \sum_{j=1}^{i-1} b_{j,i-j}\, K_j\, K_{i-j} \right] \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}x\,\mathrm{d}z \tag{3.42}$$

Let us now estimate the first term, $I_1$, on the right-hand side of Eq. (3.42):

$$I_1 = \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z) \mathcal{H}\big(u_i^\epsilon - v_i^\epsilon\big) \big(u_i^\epsilon - v_i^\epsilon\big) \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}\sigma_\epsilon$$

$$\leq \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z) \mathcal{H}\big(u_i^\epsilon - v_i^\epsilon\big) \big(u_i^\epsilon - \overline{m}_i\big) \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}\sigma_\epsilon$$

$$- \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z) \mathcal{H}\big(u_i^\epsilon - v_i^\epsilon\big) \big(v_i^\epsilon - \overline{m}_i\big) \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}\sigma_\epsilon$$

$$\leq \epsilon\, p \int_{\Gamma_\epsilon} c_i(x,z) \big(u_i^\epsilon - \overline{m}_i\big)^+ \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}\sigma_\epsilon \tag{3.43}$$

where the last step in inequality (3.43) has been obtained decomposing the functions $\big(u_i^\epsilon - \overline{m}_i\big)$ and $\big(v_i^\epsilon - \overline{m}_i\big)$ in positive and negative parts. By applying the Hölder inequality and exploiting the generalized interpolation-trace inequality (A.7), one has:

$$I_1 \leq \epsilon\, p\, \|c_i(x,z)\|_{L^\infty(\Gamma_\epsilon)}\, \|\big(u_i^\epsilon - \overline{m}_i\big)^+\|_{L^2(\Gamma_\epsilon)}\, \|\Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1}\|_{L^2(\Gamma_\epsilon)}$$

$$\leq \epsilon\, p\, \|c_i(x,z)\|_{L^\infty(\Gamma_\epsilon)}\, \|\Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1}\|_{L^2(\Gamma_\epsilon)} \Big[ 2\,C\,\eta \|\nabla \big(u_i^\epsilon - \overline{m}_i\big)^+\|^2_{L^2(\Pi_\epsilon)}$$

$$+ 8\,C\,\eta^{-1} \|\big(u_i^\epsilon - \overline{m}_i\big)^+\|^2_{L^2(\Pi_\epsilon)} \Big]^{1/2} \tag{3.44}$$

The right-hand side of this inequality vanishes since $u_i^\epsilon \in L^\infty(0,T; L^\infty(\Pi_\epsilon))$, which implies: $\big(u_i^\epsilon - \overline{m}_i\big)^+ = 0$, because of the choice of $\overline{m}_i > M_i$. Therefore, on the one hand, by definition, $I_1 \geq 0$, while on the other side $I_1 \leq 0$, which leads to $I_1 = 0$.

Therefore, Eq. (3.42) reduces to:

$$\int_{\Omega_\epsilon} \partial_t \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^p \,\mathrm{d}x\,\mathrm{d}z \leq p \int_{\Omega_\epsilon} \left[ \sum_{j=1}^{i-1} b_{j,i-j}\, K_j\, K_{i-j} \right] \Big[ \big(v_i^\epsilon - \overline{m}_i\big)^+ \Big]^{p-1} \mathrm{d}x\,\mathrm{d}z \tag{3.45}$$

Integrating over $[0,t]$ with $t \in [0,T]$, and estimating the term on the right-hand side by using Young's inequality in the form (3.18), with: $a = \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^{p-1}$ and $b = p\left[\sum_{j=1}^{i-1} b_{j,i-j} K_j K_{i-j}\right]$, we get

$$\int_0^t \int_{\Omega_\epsilon} \frac{\partial}{\partial s}\left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^p \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}z$$

$$\leq p^{p-1}\left[\sum_{j=1}^{i-1} b_{j,i-j} K_j K_{i-j}\right]^p \eta^{1-p}\,|\Omega_\epsilon|\,T + \eta \int_0^t \int_{\Omega_\epsilon} \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^p \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}z$$

Choosing $\eta = p$ and setting the initial conditions, it follows that:

$$\|\left(v_i^\epsilon - \overline{m}_i\right)^+\|^p_{L^p(\Omega_\epsilon)} \leq \left[\sum_{j=1}^{i-1} b_{j,i-j} K_j K_{i-j}\right]^p |\Omega_\epsilon|\,T + p\int_0^t \mathrm{d}s\, \|\left(v_i^\epsilon - \overline{m}_i\right)^+\|^p_{L^p(\Omega_\epsilon)} \tag{3.46}$$

Finally, the Gronwall Lemma leads to the estimate

$$\|\left(v_i^\epsilon - \overline{m}_i\right)^+\|^p_{L^p(\Omega_\epsilon)} \leq \left[\sum_{j=1}^{i-1} b_{j,i-j} K_j K_{i-j}\right]^p |\Omega_\epsilon|\,T\,e^{p\,t} \tag{3.47}$$

Hence,

$$\sup_{t\in[0,T]} \lim_{p\to\infty} \left[\int_{\Omega_\epsilon} \left[\left(v_i^\epsilon - \overline{m}_i\right)^+\right]^p \mathrm{d}x\, \mathrm{d}z\right]^{1/p} \leq \left[\sum_{j=1}^{i-1} b_{j,i-j} K_j K_{i-j}\right] e^T \tag{3.48}$$

Therefore, since the positive part of the function $\left(v_i^\epsilon - \overline{m}_i\right)$ is bounded, it follows that: $v_i^\epsilon(t,x,z) \leq \overline{M}_i$, where $\overline{M}_i > \overline{m}_i$ is a positive constant. □

**Lemma 3.2.** *For a given small $\epsilon > 0$, let $u_i^\epsilon(t,x,z)$ and $v_i^\epsilon(t,x,z)$ $(1 \leq i \leq M)$ be solutions of the system* (1.1) *in the sense of the Definition* 2.1. *Then, the following estimates hold:*

$$\epsilon\|\nabla_x u_i^\epsilon\|_{L^\infty(0,T;L^2(\Pi_\epsilon))} \leq C_i^x \tag{3.49}$$

$$\|\partial_z u_i^\epsilon\|_{L^\infty(0,T;L^2(\Pi_\epsilon))} \leq C_i^z \tag{3.50}$$

$$\|\partial_t u_i^\epsilon\|_{L^2(0,T;L^2(\Pi_\epsilon))} \leq C_i^t \tag{3.51}$$

$$\|\nabla v_i^\epsilon\|_{L^\infty(0,T;L^2(\Omega_\epsilon))} \leq D_i \tag{3.52}$$

$$\|\partial_t v_i^\epsilon\|_{L^2(0,T;L^2(\Omega_\epsilon))} \leq D_i^t \tag{3.53}$$

*where $1 \leq i \leq M$ and $C_i^x, C_i^z, C_i^t, D_i, D_i^t$ are positive constants independent of $\tilde{M}$ and $\epsilon$.*

*Proof.* In the case $i = 1$, let $u_{1,n}^\epsilon$ and $v_{1,n}^\epsilon$ be the approximate solutions defined in the proof of Lemma 2.1. Then the inequality (2.50) holds with a constant $\tilde{C} \geq 0$ independent of $n$ and $\epsilon$. Using the convergence results reported in Proposition 2.1 and the lower-semicontinuity of the norm from Theorem B.2 [26] (see Appendix B), we get from Eq. (2.50):

$$\|\partial_t u_1^\epsilon\|^2_{L^2(0,T;L^2(\Pi_\epsilon))} + 2\,\|\partial_t v_1^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} + 2\,\epsilon^2 D_1\,\|\nabla_x u_1^\epsilon\|^2_{L^\infty(0,T;L^2(\Pi_\epsilon))}$$

$$+ 2\,\tilde{D}_1\,\|\partial_z u_1^\epsilon\|^2_{L^\infty(0,T;L^2(\Pi_\epsilon))} + 2\,d_1\,\|\nabla v_1^\epsilon\|^2_{L^\infty(0,T;L^2(\Omega_\epsilon))}$$

$$\leq \liminf_{n\to\infty} \|\partial_t u^\epsilon_{1,n}\|^2_{L^2(0,T;L^2(\Pi_\epsilon))} + \liminf_{n\to\infty} 2\,\|\partial_t v^\epsilon_{1,n}\|^2_{L^2(0,T;L^2(\Omega_\epsilon))}$$
$$+ \liminf_{n\to\infty} 2\,\epsilon^2 D_1\,\|\nabla_x u^\epsilon_{1,n}\|^2_{L^\infty(0,T;L^2(\Pi_\epsilon))} + \liminf_{n\to\infty} 2\,\tilde{D}_1\,\|\partial_z u^\epsilon_{1,n}\|^2_{L^\infty(0,T;L^2(\Pi_\epsilon))}$$
$$+ \liminf_{n\to\infty} 2\,d_1\,\|\nabla v^\epsilon_{1,n}\|^2_{L^\infty(0,T;L^2(\Omega_\epsilon))} \leq \tilde{C} \tag{3.54}$$

For the case $1 < i \leq M$, the proof carries over verbatim, since also the functions $u^\epsilon_{i,n}$ and $v^\epsilon_{i,n}$ $(1 < i \leq M)$ satisfy the inequality (2.50). □

# 4 Uniqueness of solutions

**Theorem 4.1.** *Let* $u^\epsilon_i, \tilde{u}^\epsilon_i \in H^1([0,T]; L^2(\Pi_\epsilon))$ *and* $v^\epsilon_i, \tilde{v}^\epsilon_i \in H^1([0,T]; L^2(\Omega_\epsilon))$ $(1 \leq i \leq M)$ *be solutions to the problem* (1.1) *in the sense of Definition* 2.1.

*Then,* $u^\epsilon_i \equiv \tilde{u}^\epsilon_i$ *and* $v^\epsilon_i \equiv \tilde{v}^\epsilon_i$ $(1 \leq i \leq M)$.

*Proof.* From now on, we drop the index $\epsilon$ and we set

$$U_i := u_i - \tilde{u}_i \qquad \text{and} \qquad V_i := v_i - \tilde{v}_i. \tag{4.1}$$

Then, in Eqs. (2.1) and (2.2), we choose

$$\tilde{M} > \max\{M_i, \overline{M}_i,\ i = 1, \dots, M\},$$

so that

$$L^{\tilde{M}}_i(u) =: L_i(u) = L_i(u_1, u_2, \dots, u_M)$$

and

$$N^{\tilde{M}}_i(v) =: N_i(v) = N_i(v_1, v_2, \dots, v_M).$$

Let us now write Eq. (2.4) for $(u_i, v_i)$ and $(\tilde{u}_i, \tilde{v}_i)$ $(i = 1, \dots, M)$. Subtracting the resulting equations, we get:

$$\int_{\Pi_\epsilon} \partial_t U_i\,\psi_i \,dx\,dz + \epsilon^2 \int_{\Pi_\epsilon} D_i \nabla_x U_i \cdot \nabla_x \psi_i \,dx\,dz + \int_{\Pi_\epsilon} \tilde{D}_i\,\partial_z U_i \cdot \partial_z \psi_i \,dx\,dz$$
$$= -\epsilon \int_{\Gamma_\epsilon} c_i(x,z)[(u_i - v_i)_+ - (\tilde{u}_i - \tilde{v}_i)_+]\,\psi_i \,d\sigma_\epsilon + \int_{\Pi_\epsilon} (L_i(u) - L_i(\tilde{u}))\,\psi_i \,dx\,dz \tag{4.2}$$

for all $\psi_i \in H^1(\Pi_\epsilon)$.

Since the positive part function is 1-Lipschitz, the following estimate holds:

$$|(u_i - v_i)_+ - (\tilde{u}_i - \tilde{v}_i)_+| \leq |u_i - \tilde{u}_i| + |v_i - \tilde{v}_i|.$$

Thus

$$\left|\int_{\Gamma_\epsilon} c_i(x,z)[(u_i - v_i)_+ - (\tilde{u}_i - \tilde{v}_i)_+]\,\psi_i \,d\sigma_\epsilon\right| \leq C \int_{\Gamma_\epsilon} |(u_i - v_i)_+ - (\tilde{u}_i - \tilde{v}_i)_+|\,|\psi_i| \,d\sigma_\epsilon$$
$$\leq C \int_{\Gamma_\epsilon} (|u_i - \tilde{u}_i| + |v_i - \tilde{v}_i|)\,|\psi_i| \,d\sigma_\epsilon = C \int_{\Gamma_\epsilon} (|U_i| + |V_i|)\,|\psi_i| \,d\sigma_\epsilon \tag{4.3}$$

where $C$ is a positive constant. Moreover, we have

$$|L_i(u) - L_i(\tilde{u})| \leq \sum_{j=1}^{M} a_{i,j} |u_i u_j - \tilde{u}_i \tilde{u}_j| \leq C(\tilde{M}) \sum_{j=1}^{M} |U_j| \tag{4.4}$$

since $L_i$ is a Lipschitz continuous function, and $C(\tilde{M})$ is a positive constant.

Thus, Eq. (4.2) becomes

$$\begin{aligned}
&\int_{\Pi_\epsilon} \partial_t U_i \, \psi_i \, dx \, dz + \epsilon^2 \int_{\Pi_\epsilon} D_i \nabla_x U_i \cdot \nabla_x \psi_i \, dx \, dz + \int_{\Pi_\epsilon} \tilde{D}_i \, \partial_z U_i \cdot \partial_z \psi_i \, dx \, dz \\
&\quad \leq C \int_{\Gamma_\epsilon} (|U_i| + |V_i|) \, |\psi_i| \, d\sigma_\epsilon + C(\tilde{M}) \sum_{j=1}^{M} \int_{\Pi_\epsilon} |U_j| \, |\psi_i| \, dx \, dz
\end{aligned} \tag{4.5}$$

for all $\psi_i \in H^1(\Pi_\epsilon)$.

Now let us take:

$$\psi_i = 2U_i.$$

Then, Eq. (4.5) can be rewritten as follows:

$$\begin{aligned}
&\int_{\Pi_\epsilon} \partial_t U_i^2 \, dx \, dz + 2\epsilon^2 \int_{\Pi_\epsilon} D_i |\nabla_x U_i|^2 \, dx \, dz + 2 \int_{\Pi_\epsilon} \tilde{D}_i \, |\partial_z U_i|^2 \, dx \, dz \\
&\quad \leq 2C \int_{\Gamma_\epsilon} (|U_i| + |V_i|) \, |U_i| \, d\sigma_\epsilon + 2C(\tilde{M}) \sum_{j=1}^{M} \int_{\Pi_\epsilon} |U_j| \, |U_i| \, dx \, dz \\
&\quad \leq C_1 \int_{\Gamma_\epsilon} \left( |U_i|^2 + |V_i|^2 \right) d\sigma_\epsilon + C_1(\tilde{M}) \sum_{j=1}^{M} \int_{\Pi_\epsilon} |U_j|^2 \, dx \, dz
\end{aligned} \tag{4.6}$$

where $C_1$ and $C_1(\tilde{M})$ are positive constants. Integrating over $[0, t]$ and taking into account that $U_i(0) = 0$, we get

$$\begin{aligned}
&\int_{\Pi_\epsilon} U_i^2 \, dx \, dz + 2\epsilon^2 \int_0^t \int_{\Pi_\epsilon} D_i |\nabla_x U_i|^2 \, dx \, dz \, ds + 2 \int_0^t \int_{\Pi_\epsilon} \tilde{D}_i \, |\partial_z U_i|^2 \, dx \, dz \, ds \\
&\leq C_1 \int_0^t \int_{\Gamma_\epsilon} \left( |U_i|^2 + |V_i|^2 \right) d\sigma_\epsilon \, ds + C_1(\tilde{M}) \sum_{j=1}^{M} \int_0^t \int_{\Pi_\epsilon} |U_j|^2 \, dx \, dz \, ds.
\end{aligned} \tag{4.7}$$

Let us now write Eq. (2.5) for $(u_i, v_i)$ and $(\tilde{u}_i, \tilde{v}_i)$ $(i = 1, \ldots, M)$. Subtracting the resulting equations, choosing $\phi_i = 2V_i$ and using the same arguments reported above for $U_i$, we obtain:

$$\begin{aligned}
&\int_{\Omega_\epsilon} V_i^2 \, dx \, dz + 2 \int_0^t \int_{\Omega_\epsilon} d_i |\nabla V_i|^2 \, dx \, dz \, ds \\
&\quad \leq C_1 \int_0^t \int_{\Gamma_\epsilon} \left( |U_i|^2 + |V_i|^2 \right) d\sigma_\epsilon \, ds + C_1(\tilde{M}) \sum_{j=1}^{M} \int_0^t \int_{\Omega_\epsilon} |V_j|^2 \, dx \, dz \, ds.
\end{aligned} \tag{4.8}$$

Adding Eqs. (4.7) and (4.8), and applying the interpolation trace inequality (A.7) (in Appendix A) to the boundary term on $\Gamma_\epsilon$, one has:

$$\|U_i\|^2_{L^2(\Pi_\epsilon)} + 2\,\epsilon^2 D_i \int_0^t \|\nabla_x U_i\|^2_{L^2(\Pi_\epsilon)}\,\mathrm{d}s + 2\,\tilde{D}_i \int_0^t \|\partial_z U_i\|^2_{L^2(\Pi_\epsilon)}\,\mathrm{d}s + \|V_i\|^2_{L^2(\Omega_\epsilon)} + 2\,d_i \int_0^t \|\nabla V_i\|^2_{L^2(\Omega_\epsilon)}\,\mathrm{d}s$$

$$\leq C_1 \eta \int_0^t \left(\|\nabla U_i\|^2_{L^2(\Pi_\epsilon)} + \|\nabla V_i\|^2_{L^2(\Omega_\epsilon)}\right)\mathrm{d}s + C_2\,\eta^{-1} \int_0^t \left(\|U_i\|^2_{L^2(\Pi_\epsilon)} + \|V_i\|^2_{L^2(\Omega_\epsilon)}\right)\mathrm{d}s$$

$$+\, C_1(\tilde{M}) \sum_{j=1}^{M} \int_0^t \int_{\Pi_\epsilon} |U_j|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}s + C_1(\tilde{M}) \sum_{j=1}^{M} \int_0^t \int_{\Omega_\epsilon} |V_j|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}s \tag{4.9}$$

where $\eta$ is a small positive constant. If we choose $\eta < \min\left\{\frac{2\epsilon^2 D_i}{C_1}, \frac{2\tilde{D}_i}{C_1}, \frac{2d_i}{C_1}\right\}$, Eq. (4.9) becomes

$$\|U_i\|^2_{L^2(\Pi_\epsilon)} + \|V_i\|^2_{L^2(\Omega_\epsilon)} \leq C_2\,\eta^{-1} \int_0^t \left(\|U_i\|^2_{L^2(\Pi_\epsilon)} + \|V_i\|^2_{L^2(\Omega_\epsilon)}\right)\mathrm{d}s$$

$$+\, C_1(\tilde{M}) \sum_{j=1}^{M} \int_0^t \int_{\Pi_\epsilon} |U_j|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}s + C_1(\tilde{M}) \sum_{j=1}^{M} \int_0^t \int_{\Omega_\epsilon} |V_j|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}s \tag{4.10}$$

Summing up for $i = 1, \dots, M$ and putting $U^2 := \sum_{i=1}^M U_i^2$ and $V^2 := \sum_{i=1}^M V_i^2$, we get eventually

$$\|U\|^2_{L^2(\Pi_\epsilon)} + \|V\|^2_{L^2(\Omega_\epsilon)} \leq C_3 \int_0^t \left(\|U\|^2_{L^2(\Pi_\epsilon)} + \|V\|^2_{L^2(\Omega_\epsilon)}\right)\mathrm{d}s \tag{4.11}$$

where $C_3$ is a positive constant and, by Gronwall's lemma, it follows that: $U \equiv V \equiv 0$. □

# 5 Homogenization

The behavior of the solutions $u_i^\epsilon, v_i^\epsilon$ $(1 \leq i \leq M)$ of the set of Eq. (1.1) as $\epsilon \to 0$ will now be studied. In order to pass to the limit, it is necessary to obtain equations and estimates in $\Omega$.

**Lemma 5.1.** *Let us consider the sets defined in Section* 1.

*1. There exists a linear continuous extension operator*

$$\tilde{P}\colon H^1(X) \to H^1(Y) \tag{5.1}$$

*such that*

$$\tilde{P}u = u \qquad \text{in } X \tag{5.2}$$

*and*

$$\|\tilde{P}\,u\|_{L^2(Y)} \leq C\|u\|_{L^2(X)} \tag{5.3}$$

$$\|\nabla(\tilde{P}\,u)\|_{L^2(Y)} \leq C\|\nabla u\|_{L^2(X)} \tag{5.4}$$

*where C is a positive constant.*

*2. There exists a family of linear continuous extension operators*

$$\tilde{P}_\epsilon : H^1(G_\epsilon) \to H^1(D) \tag{5.5}$$

*such that*

$$\tilde{P}_\epsilon u^\epsilon = u^\epsilon \qquad \text{in } G_\epsilon \tag{5.6}$$

*and*

$$\|\tilde{P}_\epsilon u^\epsilon\|_{L^2(D)} \le C\|u^\epsilon\|_{L^2(G_\epsilon)} \tag{5.7}$$

$$\|\nabla\left(\tilde{P}_\epsilon u^\epsilon\right)\|_{L^2(D)} \le C\|\nabla u^\epsilon\|_{L^2(G_\epsilon)} \tag{5.8}$$

*where the constant $C > 0$ does not depend on $\epsilon$.*

*Proof.* 1. First we extend $u$ into a neighbourhood $X_0$ of $R = \partial X$ with smooth boundary, such that $\operatorname{clos}(X) \subset X_0$ and $\operatorname{clos}(X_0) \subset \operatorname{int}(Y)$. Since we have assumed that $R$ is sufficiently smooth, we can construct a diffeomorphism as follows:

$$\Phi : R \times ]-\delta, \delta[ \to X_0 \tag{5.9}$$

$$\Phi(y, \lambda) = x \tag{5.10}$$

Exploiting this coordinate transformation, the function $u$ can be extended by reflection:

$$u^*(x) = u^*(\Phi(y,\lambda)) = \begin{cases} u(\Phi(y,\lambda)) & \lambda \ge 0 \\ u(\Phi(y,-\lambda)) & \lambda < 0 \end{cases} \tag{5.11}$$

Let us now consider the following smooth function

$$\Psi : Y \to [0,1] \tag{5.12}$$

such that $\operatorname{supp}\Psi \subseteq Z$ and $\Psi = 1$ in $Z \setminus X_0$. Then, we define

$$\tilde{u}(x) := (1-\Psi)(u^*(x) - m) + m \tag{5.13}$$

where

$$m := \frac{1}{|X|}\int_X u(y)\,\mathrm{d}y \tag{5.14}$$

Let us prove that $\tilde{u}(x)$ is an extension of $u(x)$, which satisfies (5.3) and (5.4). One gets:

$$\begin{aligned} \|\tilde{u}\|^2_{L^2(Z)} &= \int_Z |(1-\Psi)(u^* - m) + m|^2\,\mathrm{d}x \\ &= \int_{Z\cap X_0} |(1-\Psi)u^* + \Psi m|^2\,\mathrm{d}x + \int_{Z\setminus X_0} m^2\,\mathrm{d}x \end{aligned} \tag{5.15}$$

Taking into account the following estimates

$$\int_{Z\cap X_0}\left[\frac{1}{|X|}\int_X u(y)\,\mathrm{d}y\right]^2 \mathrm{d}x \le \int_{Z\cap X_0}\frac{1}{|X|}\left(\int_X u^2\,\mathrm{d}y\right)\mathrm{d}x \le \frac{|Z|}{|X|}\|u\|^2_{L^2(X)} \tag{5.16}$$

$$\int_{Z\setminus X_0}\left[\frac{1}{|X|}\int_X u(y)\,\mathrm{d}y\right]^2 \mathrm{d}x \le \int_{Z\setminus X_0}\frac{1}{|X|}\left(\int_X u^2\,\mathrm{d}y\right)\mathrm{d}x \le \frac{|Z|}{|X|}\|u\|^2_{L^2(X)} \tag{5.17}$$

and the Minkowski inequality, Eq. (5.15) can be rewritten as

$$\|\tilde{u}\|^2_{L^2(Z)} \le 2 \int_{Z\cap X_0} (1-\Psi)^2\, |u^*|^2 \,\mathrm{d}x + 2 \int_{Z\cap X_0} \Psi^2\, m^2 \,\mathrm{d}x + \frac{|Z|}{|X|}\|u\|^2_{L^2(X)} \le C(Z, X, \Psi)\|u\|^2_{L^2(X)} \tag{5.18}$$

For the derivative we obtain:

$$\begin{aligned}\|\nabla\tilde{u}\|^2_{L^2(Z)} &= \int_Z |\nabla[(1-\Psi)(u^* - m)]|^2 \,\mathrm{d}x \\ &= \int_Z |(u^* - m)\nabla(1-\Psi) + (1-\Psi)\nabla u^*|^2 \,\mathrm{d}x \end{aligned}\tag{5.19}$$

By using the Minkowski and Poincaré inequalities, Eq. (5.19) becomes:

$$\begin{aligned}\|\nabla\tilde{u}\|^2_{L^2(Z)} &\le 2 \int_{Z\cap X_0} |(u^* - m)\nabla(1-\Psi)|^2 \,\mathrm{d}x + 2\int_{Z\cap X_0} |(1-\Psi)\nabla u^*|^2 \,\mathrm{d}x \\ &\le c \int_{Z\cap X_0} |(u^* - m)|^2 \,\mathrm{d}x + c^* \int_{Z\cap X_0} |\nabla u^*|^2 \,\mathrm{d}x \le C_1(\Psi)\|\nabla u\|^2_{L^2(X)}\end{aligned}\tag{5.20}$$

2. The construction of $\tilde{P}_\epsilon$ is obvious by summation over the individual cells. □

**Lemma 5.2.** *Let us consider the sets defined in Section* 1.

*1. There exists a linear continuous extension operator*

$$P\colon H^1(Z) \to H^1(Y) \tag{5.21}$$

*such that*

$$Pv = v \qquad \text{in } Z \tag{5.22}$$

*and*

$$\|P\,v\|_{L^2(Y)} \le C\|v\|_{L^2(Z)} \tag{5.23}$$

$$\|\nabla(P\,v)\|_{L^2(Y)} \le C\|\nabla v\|_{L^2(Z)} \tag{5.24}$$

*where C is a positive constant.*

*2. There exists a family of linear continuous extension operators*

$$P_\epsilon\colon H^1(D_\epsilon) \to H^1(D) \tag{5.25}$$

*such that*

$$P_\epsilon v^\epsilon = v^\epsilon \qquad \text{in } D_\epsilon \tag{5.26}$$

*and*

$$\|P_\epsilon v^\epsilon\|_{L^2(D)} \le C\|v^\epsilon\|_{L^2(D_\epsilon)} \tag{5.27}$$

$$\|\nabla\left(P_\epsilon v^\epsilon\right)\|_{L^2(D)} \le C\|\nabla v^\epsilon\|_{L^2(D_\epsilon)} \tag{5.28}$$

*where the constant $C > 0$ does not depend on $\epsilon$.*

*Proof.* This Lemma can be proved by applying the same arguments considered in the proof of Lemma 5.1 and in Ref. [42] (p. 25).

**Remark 5.1.** Analogous extension theorems hold also for $u_i^\epsilon(t,x,z)$, defined in $[0,T]\times\Pi_\epsilon = [0,T]\times G_\epsilon\times[0,L]$, and $v_i^\epsilon(t,x,z)$, defined in $[0,T]\times\Omega_\epsilon=[0,T]\times D_\epsilon\times[0,L]$ $(1\le i\le M)$, since $z$ and $t$ can be considered as parameters, because of the geometry of the domain.

We briefly explain the argument for the function $u_i^\epsilon$; similarly one can argue for $v_i^\epsilon$. Since $[0,T]\times\Pi_\epsilon$ is a Lipschitz domain by density we can assume that $u_i^\epsilon\in C^\infty([0,T]\times\bar{G}_\epsilon\times[0,L])$, and we can define its extension to $[0,T]\times\Omega=[0,T]\times D\times[0,L]$ by $\tilde{P}_{\epsilon,0}u_i^\epsilon$, as follows.

If $(t,x,z)\in[0,T]\times\Omega$, we set

$$\big(\tilde{P}_{\epsilon,0}u_i^\epsilon\big)(t,x,z) = (t,\big(\tilde{P}_\epsilon u_i^\epsilon\big)(x),z).$$

Then

$$\big(\nabla_x\tilde{P}_{\epsilon,0}u_i^\epsilon\big)(t,x,z) = (t,\big(\nabla_x\tilde{P}_\epsilon u_i^\epsilon\big)(x),z),$$

so that, by Tonelli theorem

$$\begin{aligned}
&\int_0^T\int_0^L\int_D|\nabla_x\tilde{P}_{\epsilon,0}u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_0^T\int_0^L\int_D|\tilde{P}_{\epsilon,0}u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\\
&=\int_0^T\int_0^L\int_D|\nabla_x\tilde{P}_\epsilon u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+\int_0^T\int_0^L\int_D|\tilde{P}_\epsilon u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\\
&\le C\int_0^T\int_0^L\int_{G_\epsilon}|\nabla_x u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+C\int_0^T\int_0^L\int_{G_\epsilon}|u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\\
&=C\int_0^T\int_{\Pi_\epsilon}|\nabla_x u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t+C\int_0^T\int_{\Pi_\epsilon}|u_i^\epsilon|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\\
&\le C\int_0^T\|u_i^\epsilon\|^2_{H^1(\Pi_\epsilon)}\,\mathrm{d}t
\end{aligned}$$

From now on, we identify $v_i^\epsilon$ with its extension $P_\epsilon v_i^\epsilon$ according to Lemma 5.2 and $u_i^\epsilon$ with its extension $\tilde{P}_\epsilon u_i^\epsilon$ according to Lemma 5.1. In order to study the limiting behavior of the set of Eq. (1.1), we use the notion of two-scale convergence. Some definitions and results on two-scale convergence, introduced in Refs. [32], [33], are reported in Appendix C.

**Proposition 5.1.** *Let $v_i^\epsilon$ $(1\le i\le M)$ be the extension of the solutions to the system* (1.1). *Then, up to a subsequence:*

$$v_i^\epsilon(t,x,z)\rightharpoonup v_i(t,x,z)\quad\textit{weakly in}\quad L^2(0,T;H^1(\Omega)) \tag{5.29}$$

$$\partial_t v_i^\epsilon(t,x,z)\rightharpoonup \partial_t v_i(t,x,z)\quad\textit{weakly in}\quad L^2([0,T]\times\Omega) \tag{5.30}$$

$$v_i^\epsilon(t,x,z)\to v_i(t,x,z)\quad\textit{strongly in}\quad C^0([0,T];L^2(\Omega)) \tag{5.31}$$

*Proof.* The convergence results (5.29)–(5.31) follow immediately from the a priori estimates given in Lemma 3.2. □

**Remark 5.2.** Since $v_i^\epsilon(t,x,z)$ converges weakly to $v_i(t,x,z)$ $(1 \le i \le M)$ in $L^2(0,T;\ H^1(\Omega)) \cap H^1(0,T;\ L^2(\Omega))$, Theorem C.5 in Appendix C implies the two-scale convergence to the same $v_i(t,x,z)$, and there exists a function $\tilde{v}_i \in L^2([0,T] \times \Omega; H^1_\sharp(Y)/\mathbb{R})$ such that up to a subsequence $\nabla_x v_i^\epsilon(t,x,z)$ two-scale converges to $\nabla_x v_i(t,x,z) + \nabla_y \tilde{v}_i(t,x,y,z)$ (where $y \in Y$ is the microscopic variable).

Moreover, the interpolation-trace inequality (A.7) in Appendix A and Theorem C.7 in Appendix C allow one to infer the two-scale convergence of $v_i^\epsilon$ on the boundary $\Gamma_\epsilon$.

Next we prove the convergence of $u_i^\epsilon$ $(1 \le i \le M)$.

**Proposition 5.2.** *Let $u_i^\epsilon(t,x,z)$ $(1 \le i \le M)$ be the extension of the solutions to the system* (1.1). *Then, up to a subsequence:*

$$\begin{aligned}
&u_i^\epsilon(t,x,z) \to u_i(t,x,y,z)\\
&\partial_t u_i^\epsilon(t,x,z) \to \partial_t u_i(t,x,y,z)\\
&\partial_z u_i^\epsilon(t,x,z) \to \partial_z u_i(t,x,y,z)\\
&\epsilon \nabla_x u_i^\epsilon(t,x,z) \to \nabla_y u_i(t,x,y,z)
\end{aligned} \tag{5.32}$$

*in the two-scale sense with $u_i(t,x,y,z) \in L^2([0,T] \times \Omega; H^1_\sharp(Y)) \cap H^1(0,T; L^2(\Omega \times Y))$ (where $y \in Y$ is the microscopic variable).*

*Proof.* The convergence results (5.32) follow immediately from the a priori estimates given in Lemma 3.2 and Theorem C.6 in Appendix C. □

*Proof of the main Theorem 1.1.* In the case $m = 1$, let us rewrite Eq. (2.5) in the form:

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t v_1^\epsilon\, \chi\left(\frac{x}{\epsilon}\right) \phi_1 \,dt\,dx\,dz + d_1 \int_0^T \int_\Omega \nabla_x v_1^\epsilon\, \chi\left(\frac{x}{\epsilon}\right) \nabla_x \phi_1 \,dt\,dx\,dz\\
&\qquad + d_1 \int_0^T \int_\Omega \partial_z v_1^\epsilon\, \chi\left(\frac{x}{\epsilon}\right) \partial_z \phi_1 \,dt\,dx\,dz - \epsilon \int_0^T \int_{\Gamma_\epsilon} c_1(x,z) \left(u_1^\epsilon - v_1^\epsilon\right)_+ \phi_1 \,dt\,d\sigma_\epsilon\\
&\quad = -\int_0^T \int_\Omega \left[\sum_{j=1}^M b_{1,j}\, v_1^\epsilon\, v_j^\epsilon\right] \chi\left(\frac{x}{\epsilon}\right) \phi_1 \,dt\,dx\,dz
\end{aligned} \tag{5.33}$$

where $\chi(\frac{x}{\epsilon})$ is the characteristic function of $\Omega_\epsilon$. Inserting in Eq. (5.33) the following test function:

$$\phi_1 := \phi_1^0(t,x,z) + \epsilon \tilde{\phi}_1\left(t,x,\frac{x}{\epsilon},z\right)$$

where $\phi_1^0 \in C^1([0,T] \times \overline{\Omega})$ and $\tilde{\phi}_1 \in C^1([0,T] \times \overline{\Omega}; C^\infty_\#(Y))$, we obtain:

$$\int_0^T \int_\Omega \partial_t v_1^\epsilon(t,x,z)\,\chi\Big(\frac{x}{\epsilon}\Big)\Big[\phi_1^0(t,x,z)+\epsilon\,\tilde\phi_1\Big(t,x,\frac{x}{\epsilon},z\Big)\Big]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z$$
$$+ d_1\int_0^T\int_\Omega \nabla_x v_1^\epsilon\,\chi\Big(\frac{x}{\epsilon}\Big)\Big[\nabla_x\phi_1^0+\epsilon\nabla_x\tilde\phi_1\Big(t,x,\frac{x}{\epsilon},z\Big)+\nabla_y\tilde\phi_1\Big(t,x,\frac{x}{\epsilon},z\Big)\Big]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z$$
$$+ d_1\int_0^T\int_\Omega \partial_z v_1^\epsilon(t,x,z)\,\chi\Big(\frac{x}{\epsilon}\Big)\Big[\partial_z\phi_1^0(t,x,z)+\epsilon\partial_z\tilde\phi_1\Big(t,x,\frac{x}{\epsilon},z\Big)\Big]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z$$
$$-\epsilon\int_0^T\int_{\Gamma_\epsilon} c_1(x,z)\,(u_1^\epsilon(t,x,z)-v_1^\epsilon(t,x,z))_+\Big[\phi_1^0+\epsilon\tilde\phi_1\Big(t,x,\frac{x}{\epsilon},z\Big)\Big]\,\mathrm{d}t\,\mathrm{d}\sigma_\epsilon$$
$$=-\int_0^T\int_\Omega\Big[\sum_{j=1}^M b_{1,j}\,v_1^\epsilon(t,x,z)\,v_j^\epsilon(t,x,z)\Big]\,\chi\Big(\frac{x}{\epsilon}\Big)\Big[\phi_1^0+\epsilon\tilde\phi_1\Big(t,x,\frac{x}{\epsilon},z\Big)\Big]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z \tag{5.34}$$

Passing to the two-scale limit we get

$$\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}x\,\mathrm{d}z\int_Z \partial_t v_1(t,x,z)\,\phi_1^0(t,x,z)\,\mathrm{d}y$$
$$+ d_1\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}x\,\mathrm{d}z\int_Z [\nabla_x v_1+\nabla_y\tilde v_1(t,x,y,z)]\,\big[\nabla_x\phi_1^0+\nabla_y\tilde\phi_1(t,x,y,z)\big]\,\mathrm{d}y$$
$$+ d_1\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}x\,\mathrm{d}z\int_Z \partial_z v_1(t,x,z)\,\partial_z\phi_1^0(t,x,z)\,\mathrm{d}y$$
$$-\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}x\,\mathrm{d}z\int_\Gamma c_1(x,z)\,(u_1(t,x,y,z)-v_1(t,x,z))_+\,\phi_1^0(t,x,z)\,\mathrm{d}\sigma(y)$$
$$=-\int_0^T \mathrm{d}t\int_\Omega \mathrm{d}x\,\mathrm{d}z\int_Z\Big[\sum_{j=1}^M b_{1,j}\,v_1(t,x,z)\,v_j(t,x,z)\Big]\,\phi_1^0(t,x,z)\,\mathrm{d}y \tag{5.35}$$

An integration by parts shows that Eq. (5.35) is a variational formulation associated with the following homogenized system:

$$-div_y[d_1(\nabla_x v_1(t,x,z)+\nabla_y\tilde v_1(t,x,y,z))]=0,\qquad t>0,\ \ (x,z)\in\Omega,\ \ y\in Z \tag{5.36}$$

$$[\nabla_x v_1(t,x,z)+\nabla_y\tilde v_1(t,x,y,z)]\cdot\nu=0,\qquad t>0,\ \ (x,z)\in\Omega,\ \ y\in\Gamma \tag{5.37}$$

$$|Z|\frac{\partial v_1}{\partial t}(t,x,z) - div_x\left[d_1\int_Z \mathrm{d}y\,(\nabla_x v_1(t,x,z) + \nabla_y \tilde{v}_1(t,x,y,z))\right]$$

$$- d_1\,|Z|\,\partial_z^2 v_1(t,x,z) + |Z|\sum_{j=1}^{M} b_{1,j}\,v_1(t,x,z)\,v_j(t,x,z)$$

$$= \int_\Gamma c_1(x,z)\,(u_1(t,x,y,z) - v_1(t,x,z))_+\,\mathrm{d}\sigma(y) \qquad \text{in } [0,T]\times\Omega \tag{5.38}$$

$$\left[\int_Z (\nabla_x v_1(t,x,z) + \nabla_y \tilde{v}_1(t,x,y,z))\,\mathrm{d}y\right]\cdot \nu = 0 \qquad \text{on } [0,T]\times\Gamma_L \tag{5.39}$$

$$\partial_z v_1(t,x,z) = 0 \qquad \text{on } [0,T]\times\Gamma_B \tag{5.40}$$

By continuity we have that

$$v_1(t=0,x,z) = 0 \qquad \text{in } \Omega. \tag{5.41}$$

Since we have assumed that the diffusion coefficient is constant and we have proved that the limiting function $v_1(t,x,z)$ does not depend on the microscopic variable $y$, Eqs. (5.36) and (5.37) reduce to:

$$\triangle_y \tilde{v}_1(t,x,y,z) = 0, \qquad t>0,\quad (x,z)\in\Omega,\quad y\in Z \tag{5.42}$$

$$\nabla_y \tilde{v}_1(t,x,y,z)\cdot\nu = -\nabla_x v_1(t,x,z)\cdot\nu, \qquad t>0,\quad (x,z)\in\Omega,\quad y\in\Gamma \tag{5.43}$$

Then, $\tilde{v}_1(t,x,y,z)$ satisfying Eqs. (5.42) and (5.43) can be written as

$$\tilde{v}_1(t,x,y,z) = \sum_{i=1}^{3} w_i(y)\,\frac{\partial v_1}{\partial x_i}(t,x,z) \tag{5.44}$$

where $(w_i)_{1\le i\le 3}$ is the family of solutions of the cell problem

$$\begin{cases} -\mathrm{div}_y[\nabla_y w_i + \hat{e}_i] = 0 & \text{in } Z \\ (\nabla_y w_i + \hat{e}_i)\cdot\nu = 0 & \text{on } \Gamma \\ y \to w_i(y) \quad Y-\text{periodic} & \end{cases} \tag{5.45}$$

with $\hat{e}_i$ being the $i$th unit vector in $\mathbb{R}^3$.

Inserting the relation (5.44) in Eqs. (5.38) and (5.39), we get

$$\begin{aligned} &|Z|\,\frac{\partial v_1}{\partial t}(t,x,z) - \mathrm{div}_x\big[d_1\,A\,\nabla_x v_1(t,x,z)\big] - d_1\,|Z|\,\partial_z^2 v_1(t,x,z) + |Z|\sum_{j=1}^{M} b_{1,j}\,v_1(t,x,z)\,v_j(t,x,z) \\ &= \int_\Gamma c_1(x,z)\,(u_1(t,x,y,z) - v_1(t,x,z))_+\,\mathrm{d}\sigma(y) \qquad \text{in } [0,T]\times\Omega \end{aligned} \tag{5.46}$$

$$[A\,\nabla_x v_1(t,x,z)]\cdot\nu = 0 \qquad \text{on } [0,T]\times\Gamma_L \tag{5.47}$$

where $A$ is a matrix with constant coefficients defined by

$$A_{ij} = \int_Z (\nabla_y w_i + \hat{e}_i)\cdot(\nabla_y w_j + \hat{e}_j)\,\mathrm{d}y.$$

Let us now rewrite Eq. (2.4) as follows:

$$\int_0^T \int_\Omega \partial_t u_1^\epsilon \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \psi_1 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z + \epsilon^2 D_1 \int_0^T \int_\Omega \nabla_x u_1^\epsilon \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \nabla_x \psi_1 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z$$
$$+ \tilde{D}_1 \int_0^T \int_\Omega \partial_z u_1^\epsilon \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \partial_z \psi_1 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z + \epsilon \int_0^T \int_{\Gamma_\epsilon} c_1(x,z) \left(u_1^\epsilon - v_1^\epsilon\right)_+ \psi_1 \, \mathrm{d}t \, \mathrm{d}\sigma_\epsilon$$
$$= -\int_0^T \int_\Omega \left[\sum_{j=1}^M a_{1,j} \, u_1^\epsilon \, u_j^\epsilon\right] \tilde{\chi}\left(\frac{x}{\epsilon}\right) \psi_1 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z + \int_0^T \int_\Omega f^\epsilon \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \psi_1 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z \tag{5.48}$$

where $\tilde{\chi}(\frac{x}{\epsilon})$ is the characteristic function of $\Pi_\epsilon$. If we choose the test function as:

$$\psi_1 := \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right)$$

where $\tilde{\psi}_1 \in C^1([0,T] \times \overline{\Omega}; C_\#^\infty(Y))$, Eq. (5.48) reads:

$$\int_0^T \int_\Omega \partial_t u_1^\epsilon(t,x,z) \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right) \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z$$
$$+ \epsilon^2 D_1 \int_0^T \int_\Omega \nabla_x u_1^\epsilon \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \left[\nabla_x \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right) + \frac{1}{\epsilon} \nabla_y \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right)\right] \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z$$
$$+ \tilde{D}_1 \int_0^T \int_\Omega \partial_z u_1^\epsilon(t,x,z) \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \partial_z \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right) \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z$$
$$+ \epsilon \int_0^T \int_{\Gamma_\epsilon} c_1(x,z) \, (u_1^\epsilon(t,x,z) - v_1^\epsilon(t,x,z))_+ \, \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right) \, \mathrm{d}t \, \mathrm{d}\sigma_\epsilon(x,z)$$
$$= -\int_0^T \int_\Omega \left[\sum_{j=1}^M a_{1,j} \, u_1^\epsilon(t,x,z) \, u_j^\epsilon(t,x,z)\right] \tilde{\chi}\left(\frac{x}{\epsilon}\right) \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right) \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z$$
$$+ \int_0^T \int_\Omega f^\epsilon(t,x,z) \, \tilde{\chi}\left(\frac{x}{\epsilon}\right) \tilde{\psi}_1\left(t, x, \frac{x}{\epsilon}, z\right) \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}z \tag{5.49}$$

Passing to the two-scale limit we obtain:

$$\begin{aligned}
&\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}x\,\mathrm{d}z \int_X \partial_t u_1(t,x,y,z)\,\tilde{\psi}_1(t,x,y,z)\,\mathrm{d}y \\
&\quad + D_1 \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}x\,\mathrm{d}z \int_X \nabla_y u_1(t,x,y,z)\,\nabla_y \tilde{\psi}_1(t,x,y,z)\,\mathrm{d}y \\
&\quad + \tilde{D}_1 \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}x\,\mathrm{d}z \int_X \partial_z u_1(t,x,y,z)\,\partial_z \tilde{\psi}_1(t,x,y,z)\,\mathrm{d}y \\
&\quad + \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}x\,\mathrm{d}z \int_\Gamma c_1(x,z)\,(u_1(t,x,y,z) - v_1(t,x,z))_+\,\tilde{\psi}_1(t,x,y,z)\,\mathrm{d}\sigma(y) \\
&= -\int_0^T \mathrm{d}t \int_\Omega \mathrm{d}x\,\mathrm{d}z \int_X \left[\sum_{j=1}^{M} a_{1,j}\,u_1(t,x,y,z)\,u_j(t,x,y,z)\right] \tilde{\psi}_1(t,x,y,z)\,\mathrm{d}y \\
&\quad + \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}x\,\mathrm{d}z \int_X f(t,x,y,z)\,\tilde{\psi}_1(t,x,y,z)\,\mathrm{d}y
\end{aligned} \tag{5.50}$$

An integration by parts shows that Eq. (5.50) is a variational formulation associated with the following homogenized system:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}(t,x,y,z) - D_1 \triangle_y u_1(t,x,y,z) - \tilde{D}_1\,\partial_z^2 u_1(t,x,y,z) \\
&\quad = -\sum_{j=1}^{M} a_{1,j}\,u_1(t,x,y,z)\,u_j(t,x,y,z) + f(t,x,y,z), \quad t>0,\ (x,z)\in\Omega,\ y\in X
\end{aligned} \tag{5.51}$$

$$D_1 \nabla_y u_1(t,x,y,z)\cdot \nu = -c_1(x,z)\,(u_1(t,x,y,z) - v_1(t,x,z))_+, \quad t>0,\ (x,z)\in\Omega,\quad y\in\Gamma \tag{5.52}$$

$$\partial_z u_1(t,x,y,z) = 0, \qquad t>0,\quad (x,z)\in \overline{D}\times\{0,L\},\quad y\in X \tag{5.53}$$

To conclude, by continuity, we have that

$$u_1(t=0,x,y,z) = U_1(x,y,z) \qquad (x,z)\in\Omega,\quad y\in X \tag{5.54}$$

The proof in the case $1 < m \le M$ is achieved by applying exactly the same arguments considered when $m = 1$.

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# Appendix A

In the following, we generalize the interpolation-trace inequality given in Ref. [43]:

$$\|u\|^2_{L^2(\Gamma)} \le C\|\nabla u\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)} \tag{A.1}$$

which is valid for any function $u(x) \in H^1(\Omega)$ with:

$$\int_\Omega u(x)\,\mathrm{d}x = 0 \tag{A.2}$$

where $\Gamma$ is an $(n-1)$-dimensional boundary of an $n$-dimensional domain $\Omega$. Indeed, in Ref. [43] (Eqs. (2.25) and (2.27), Chap. 2) the authors report an estimate similar to the one we will derive, but we will write down a short proof in order to show explicitly the dependencies of all constants from the geometry of our problem.

Let us consider now functions $u(x) \in H^1(\Omega)$ which do not satisfy (A.2). In this case, one can define

$$u_0 := \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x \tag{A.3}$$

and apply (A.1) to $(u-u_0)$. Hence,

$$\begin{aligned}\int_\Gamma |u(x)|^2\,\mathrm{d}\sigma(x) &\le 2\Big[\|u-u_0\|^2_{L^2(\Gamma)} + \|u_0\|^2_{L^2(\Gamma)}\Big]\\ &\le 2\,C\,\|\nabla(u-u_0)\|_{L^2(\Omega)}\|u-u_0\|_{L^2(\Omega)} + \frac{2|\Gamma|}{|\Omega|^2}\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2\\ &\le 2\,C\Big[\eta\|\nabla u\|^2_{L^2(\Omega)} + \eta^{-1}\|u-u_0\|^2_{L^2(\Omega)}\Big] + \frac{2|\Gamma|}{|\Omega|^2}\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2\end{aligned} \tag{A.4}$$

where the Young inequality has been used with $\eta$ being a small constant. By exploiting the Minkowski and Hölder inequalities, respectively, to estimate the terms:

$$\|u-u_0\|^2_{L^2(\Omega)} \le 2\Big[\|u\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}\Big] \tag{A.5}$$

$$\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2 \le |\Omega|\,\|u\|^2_{L^2(\Omega)} \tag{A.6}$$

Equation (A.4) reads

$$\|u\|^2_{L^2(\Gamma)} \le 2\,C\,\eta\|\nabla u\|^2_{L^2(\Omega)} + \left[8\,C\,\eta^{-1} + \frac{2|\Gamma|}{|\Omega|}\right]\|u\|^2_{L^2(\Omega)} \tag{A.7}$$

# Appendix B

**Theorem B.1.** *(Aubin–Lions–Simon)*

*Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of $B_1$ in $B_2$ is continuous and that the embedding of $B_0$ in $B_1$ is compact. Let $p, r$ such that $1 \le p, r \le +\infty$. For $T > 0$, we define*

$$E_{p,r} = \{v \in L^p([0,T], B_0), \partial_t v \in L^r([0,T], B_2)\}.$$

*(i) If $p < +\infty$, $E_{p,r}$ is compactly embedded in $L^p([0,T], B_1)$.*
*(ii) If $p = +\infty$ and if $r > 1$, $E_{p,r}$ is compactly embedded in $C^0([0,T], B_1)$.*

**Theorem B.2.** *(Lower-semicontinuity of the norm)*
*Let E and F be Banach spaces and $F'$ be the dual space of F.*
*(i) Let $\{x_n\}$ be a sequence weakly convergent to x in E. Then, the norm on E is lower semi-continuous with respect to the weak convergence, i.e.*

$$\|x\|_E \le \lim\inf_{n\to\infty} \|x_n\|_E \tag{B.1}$$

*(ii) Let $\{x_n\}$ be a sequence* weakly* *convergent to x in $F'$. Then, the norm on $F'$ is lower semi-continuous with respect to the* weak* *convergence, i.e.*

$$\|x\|_{F'} \le \lim\inf_{n\to\infty} \|x_n\|_{F'} \tag{B.2}$$

# Appendix C

**Definition C.1.** A sequence of functions $v^\epsilon$ in $L^2([0,T] \times \Omega)$ two-scale converges to $v_0 \in L^2([0,T] \times \Omega \times Y)$ if

$$\lim_{\epsilon\to 0} \int_0^T \int_\Omega v^\epsilon(t,x)\,\phi\Big(t,x,\frac{x}{\epsilon}\Big)\,\mathrm{d}t\,\mathrm{d}x = \int_0^T \int_\Omega \int_Y v_0(t,x,y)\,\phi(t,x,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \tag{C.1}$$

for all $\phi \in C^1([0,T] \times \overline{\Omega}; C^\infty_\#(Y))$.

**Theorem C.3.** *(Compactness theorem) If $v^\epsilon$ is a bounded sequence in $L^2([0,T] \times \Omega)$, then there exists a function $v_0(t,x,y)$ in $L^2([0,T] \times \Omega \times Y)$ such that, up to a subsequence, $v^\epsilon$ two-scale converges to $v_0$.*

**Theorem C.4.** *Let $v^\epsilon$ be a sequence of functions in $L^2([0,T] \times \Omega)$ which two-scale converges to a limit $v_0 \in L^2([0,T] \times \Omega \times Y)$. Suppose, furthermore, that*

$$\lim_{\epsilon\to 0} \int_0^T \int_\Omega |v^\epsilon(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x = \int_0^T \int_\Omega \int_Y |v_0(t,x,y)|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \tag{C.2}$$

*Then, for any sequence $w^\epsilon$ in $L^2([0,T] \times \Omega)$ that two-scale converges to a limit $w_0 \in L^2([0,T] \times \Omega \times Y)$, we have*

$$\lim_{\epsilon\to 0} \int_0^T \int_\Omega v^\epsilon(t,x)\,w^\epsilon(t,x)\,\phi\Big(t,x,\frac{x}{\epsilon}\Big)\,\mathrm{d}t\,\mathrm{d}x = \int_0^T \int_\Omega \int_Y v_0(t,x,y)\,w_0(t,x,y)\,\phi(t,x,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \tag{C.3}$$

*for all $\phi \in C^1([0,T] \times \overline{\Omega}; C^\infty_\#(Y))$.*

In the following, we identify $H^1(\Omega) = W^{1,2}(\Omega)$, where the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ v | v \in L^p(\Omega), \frac{\partial v}{\partial x_i} \in L^p(\Omega), i = 1,2,3 \right\}$$

and we denote by $H^1_\#(Y)$ the closure of $C^\infty_\#(Y)$ for the $H^1$-norm.

**Theorem C.5.** *Let $v^\epsilon$ be a bounded sequence in $L^2([0,T]; H^1(\Omega))$ that converges weakly to a limit $v(t,x)$ in $L^2([0,T]; H^1(\Omega))$. Then, $v^\epsilon$ two-scale converges to $v(t,x)$, and there exists a function $v_1(t,x,y)$ in $L^2([0,T] \times \Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, $\nabla v^\epsilon$ two-scale converges to $\nabla_x v(t,x) + \nabla_y v_1(t,x,y)$.*

**Theorem C.6.** *Let $v^\epsilon$ and $\epsilon\nabla v^\epsilon$ be two bounded sequences in $L^2([0,T]\times\Omega)$. Then, there exists a function $v_1(t,x,y)$ in $L^2([0,T]\times\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence, $v^\epsilon$ and $\epsilon\nabla v^\epsilon$ two-scale converge to $v_1(t,x,y)$ and $\nabla_y v_1(t,x,y)$, respectively.*

**Theorem C.7.** *Let $v^\epsilon$ be a sequence in $L^2([0,T]\times\Gamma_\epsilon)$ such that*

$$\epsilon\int_0^T\int_{\Gamma_\epsilon}|v^\epsilon(t,x)|^2\,\mathrm{d}t\,\mathrm{d}\sigma_\epsilon(x)\le C \tag{C.4}$$

*where $C$ is a positive constant, independent of $\epsilon$. There exist a subsequence (still denoted by $\epsilon$) and a two-scale limit $v_0(t,x,y)\in L^2([0,T]\times\Omega;\ L^2(\Gamma))$ such that $v^\epsilon(t,x)$ two-scale converges to $v_0(t,x,y)$ in the sense that*

$$\lim_{\epsilon\to0}\epsilon\int_0^T\int_{\Gamma_\epsilon}v^\epsilon(t,x)\,\phi\Big(t,x,\frac{x}{\epsilon}\Big)\,\mathrm{d}t\,\mathrm{d}\sigma_\epsilon(x)=\int_0^T\int_\Omega\int_\Gamma v_0(t,x,y)\,\phi(t,x,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\sigma(y) \tag{C.5}$$

*for any function $\phi\in C^1([0,T]\times\overline{\Omega}; C^\infty_\#(Y))$.*

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