Low-Complexity Pearson-Based Detection for AWGN Channels with Offset

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***Abstract*—This work investigates the error performance of detection schemes based on the minimum Pearson distance in the context of additive white Gaussian noise channels with unknown and unbounded offset, constant throughout each channel use. We derive a lower bound on the word error rate under modified Pearson (MP) detection. Additionally, we introduce a new and low-complexity detection strategy, namely the Simplified Pearson (SP) detector. We analyze and compare the error performance of the SP detector with that of the MP detector.**

***Index Terms*—AWGN Channel with Offset, Low-Complexity Detection, Pearson Distance**

I. Introduction

Digital data stored in any memory device is affected by various impairments due to the physical nature of the considered support, whether it is optical, magnetic, electronic, or any other. For instance, these impairments can be due to several physical aspects of the device (e.g., temperature, humidity, charge magnitude, etc.) [1]. In general, any memory read is typically affected by a mismatch both in terms of mean and variance when compared to the nominal stored value.

In the context of memory systems, a suitable channel model for noisy memory reads is the additive white Gaussian noise (AWGN) channel model affected by unknown and random offset [2]. Based on each specific memory technology, the memory offset can be assumed to be constant throughout each channel use or varying within each read [3].

In this work, we focus on the first scenario. In the case of unknown and unbounded offset, constant throughout each channel use, the modified Pearson (MP) distance detector, presented in [1], achieves maximum likelihood (ML) decoding [4].

A. Contributions and Paper Outline

Both Pearson and MP detection can be computationally too complex for practical implementations. The main focus of this work is to investigate low-complexity alternative detectors suitable for more efficient implementations and with performance comparable to that of the MP detector.

Section II introduces the AWGN channel with offset and its features. In Sec. III, we briefly discuss the MP distance detection proposed in [1], closely related to the results presented in this work. In Sec. IV, we provide the error analysis of the MP detection. We show that, roughly speaking, the MP detector transforms an AWGN channel with offset into one without offset in terms of error performance. In Sec. V, we introduce and evaluate the performance of a low-complexity alternative solution to the MP detection, i.e., the Simplified Pearson (SP) detection. Finally, Sec. VI concludes the paper.

II. System Model

Let us model the memory reads as the output of a binary-input AWGN channel with random offset. The input-output relationship is given by

$$\mathbf{Y} = a(m(\mathbf{C}) + \mathbf{Z}) + b\mathbf{1} \tag{1}$$

$$= a(\mathbf{X} + \mathbf{Z}) + b\mathbf{1}, \tag{2}$$

where $\mathbf{Y} \in \mathbb{R}^K$ is a noisy read vector of length K, $a > 0$ is an unknown channel gain, and $\mathbf{C} \in \mathcal{C}$ is the K-dimensional binary sequence stored in the memory and chosen from the codebook $\mathcal{C}$. Moreover, $\mathbf{X} = m(\mathbf{C})$ is the modulated binary sequence, mapping $0 \mapsto 1$ and $1 \mapsto -1$. We denote the unknown offset by $b \in \mathbb{R}$, assumed to be constant within each memory read, and by $\mathbf{1}$ the all-1 vector. Finally, $\mathbf{Z} \in \mathbb{R}^K$ is a noise vector with independent and identically distributed (iid) entries $Z_i \sim \mathcal{N}(0, \sigma^2)$ for $i = 1, \dots, K$, and σ^2 is the variance of the additive noise components.

Throughout this work, we will consider the channel gain to be $a = 1$ without loss of generality, as it is always possible to account for any gain $a > 0$ by scaling the signal-to-noise ratio (SNR) accordingly.

III. Previous Works

Let us introduce the Pearson distance detection presented in [5] and the MP detection [1].

A. Pearson Distance Detection

Consider two binary sequences $\mathbf{x}$ and $\mathbf{y}$. The Pearson distance between the two sequences is defined as

$$\delta_{\mathrm{P}}(\mathbf{x}, \mathbf{y}) = 1 - \rho_{\mathbf{x},\mathbf{y}}, \tag{3}$$

where $\rho_{\mathbf{x},\mathbf{y}} = \left(\sum_{i=1}^{K}(x_i - \bar{\mathbf{x}})(y_i - \bar{\mathbf{y}})\right)/(\sigma_{\mathbf{x}}\sigma_{\mathbf{y}})$ is the Pearson correlation coefficient, with $\bar{\mathbf{x}} = \frac{1}{K}\sum_{i=1}^{K} x_i$, $\sigma_{\mathbf{x}}^2 = \sum_{i=1}^{K}(x_i - \bar{\mathbf{x}})^2$, and similarly for $\mathbf{y}$.

An estimate of the stored sequence is evaluated as $\hat{\mathbf{c}}_{\mathrm{P}}(\mathbf{Y}) = \arg\min_{\mathbf{c}\in\mathcal{C}} \delta_{\mathrm{P}}(\mathbf{Y}, \mathbf{c})$. Notice that, to guarantee unambiguous detection, the Pearson detector requires the use of a codebook purged of the all-0 and all-1 sequences.

B. Modified Pearson Distance Detection

A modified version of the Pearson distance, improving the resistance against AWGN, is proposed in [2, Eq. (17)] and defined as

$$\delta_{\text{MP}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{K} (x_i - y_i + \bar{\mathbf{y}})^2. \tag{4}$$

Given the stored sequence $\mathbf{C}$ and upon reading its noisy version $\mathbf{Y}$, the MP detector outputs an estimate of $\mathbf{C}$ that is defined as follows

$$\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}) = \arg\min_{\mathbf{c}\in\mathcal{C}} \delta_{\text{MP}}(\mathbf{Y}, m(\mathbf{c})), \tag{5}$$

where the $\mathbf{c}$'s are tentative binary sequences extracted from the codebook $\mathcal{C}$. To ensure that the MP detection is unambiguous, the only requirement on the codebook $\mathcal{C}$ is to remove the all-1 sequence, see [2], therefore we have $\mathcal{C} = \mathbb{F}_2^K \setminus \{\mathbf{1}\}$. Throughout this work, we consider equiprobable codewords, i.e., $\mathbb{P}_{\mathbf{C}}(\mathbf{c}) = 1/(2^K - 1),\ \forall \mathbf{c} \in \mathcal{C}$. Notice that, the authors of [6] proved that (5) is the ML detection criterion for the scenario considered in this work, i.e., for $a = 1$ and for $b \in \mathbb{R}$.

An equivalent formulation of the MP detection, given by the cascade of Hamming weight estimation and Slepian detection [7], is proposed in [1]. The MP detector estimates the Hamming weight as follows

$$\hat{W}_{\text{MP}}(\mathbf{Y}) = \arg\min_{k\in[0,K-1]} \delta_k(\mathbf{Y}), \tag{6}$$

where $\delta_0(\mathbf{Y}) = 0$ and, for $w = 1, \ldots, K-1$, it holds

$$\delta_k(\mathbf{Y}) = \sum_{i=1}^{k} \left(Y_{i:K} - \bar{\mathbf{Y}} + \frac{K+1-2i}{K} \right) \tag{7}$$

$$= \delta_{k-1}(\mathbf{Y}) + Y_{k:K} - \bar{\mathbf{Y}} + \frac{K+1-2k}{K}, \tag{8}$$

where $Y_{i:K}$ is the ith order statistic in a set of K memory reads. Then, the Slepian detector assigns

$$\hat{c}_{\text{MP},i:K} = \begin{cases} 1 & i \le \hat{w}_{\text{MP}}, \\ 0 & i > \hat{w}_{\text{MP}}, \end{cases} \tag{9}$$

i.e., the first $\hat{w}_{\text{MP}}$ memory reads are detected as 1s, and the remaining reads are detected as 0s.

IV. Performance Analysis of the Modified Pearson Detector

Let us now evaluate the error performance of the AWGN channel with unbounded offset under MP detection.

A. Word Error Rate

Let us denote by E_{MP} and $\mathbb{P}_{\epsilon,\text{MP}}$, respectively, the number of symbol errors and the word error rate (WER), both under MP detection.

Theorem 1. *The WER of MP detection over an AWGN channel with random and unbounded offset is lower-bounded by*

$$\mathbb{P}_{\epsilon,\text{MP}} \ge \underline{\mathbb{P}}_{\epsilon,\text{MP}} = \frac{1 - (1 - Q\left(\frac{1}{\sigma}\right))^K - 2^{-K}}{1 - 2^{-K}} \tag{10}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$.

Proof. By definition, the MP detector cannot give the all-1 codeword as an answer, because the codebook is $\mathcal{C} = \mathbb{F}_2^K \setminus \{\mathbf{1}\}$.

Now, let us consider input codewords drawn equally probable from $\mathcal{C}' = \mathbb{F}_2^K$, i.e., with $\mathbb{P}_{\mathbf{C}'}(\mathbf{c}) = 2^{-K}$ for all $\mathbf{c} \in \mathcal{C}'$. Then, we can write

$$\mathbb{P}_{\epsilon,\text{MP}}$$

$$= \mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}) \ne \mathbf{C}) \tag{11}$$

$$= \mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}') \ne \mathbf{C}' | \mathbf{C}' \ne \mathbf{1}) \tag{12}$$

$$= \frac{\mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}') \ne \mathbf{C}',\, \mathbf{C}' \ne \mathbf{1})}{1 - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})} \tag{13}$$

$$= \frac{\mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}') \ne \mathbf{C}') - \mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}') \ne \mathbf{1} | \mathbf{C}' = \mathbf{1}) \mathbb{P}_{\mathbf{C}'}(\mathbf{1})}{1 - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})} \tag{14}$$

$$= \frac{\mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}') \ne \mathbf{C}') - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})}{1 - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})} \tag{15}$$

$$\ge \frac{\mathbb{P}(\hat{\mathbf{c}}_{\text{ML}}(\mathbf{Y}') \ne \mathbf{C}') - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})}{1 - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})} \tag{16}$$

$$\ge \frac{\mathbb{P}(\hat{\mathbf{c}}_{\text{ML}}(\mathbf{Y}' - b\mathbf{1}) \ne \mathbf{C}') - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})}{1 - \mathbb{P}_{\mathbf{C}'}(\mathbf{1})} \tag{17}$$

$$= \frac{1 - (1 - Q\left(\frac{1}{\sigma}\right))^K - 2^{-K}}{1 - 2^{-K}}, \tag{18}$$

where $\mathbf{Y}' = m(\mathbf{C}') + b + \mathbf{Z}$. Notice that (15) holds because the MP detector never *chooses* the codeword $\mathbf{1}$ and, therefore, $\mathbb{P}(\hat{\mathbf{c}}_{\text{MP}}(\mathbf{Y}') \ne \mathbf{1} | \mathbf{C}' = \mathbf{1}) = 1$. The lower bound in (16) arises from the definition of an ML detector, $\hat{\mathbf{c}}_{\text{ML}}$. Step (17) follows from providing side information b to the receiver. Finally, (18) holds because $\hat{\mathbf{c}}_{\text{ML}}$ function as the Euclidean detector for an AWGN channel without offset, and the noise variance is σ^2. □

An upper bound[1] on $\mathbb{P}_{\epsilon,\text{MP}}$ is derived in [2] and given by

$$\mathbb{P}_{\epsilon,\text{MP}} < \overline{\mathbb{P}}_{\epsilon,\text{MP}} = K \cdot Q\left(\frac{1}{\sigma} \sqrt{1 - \frac{1}{K}} \right). \tag{19}$$

In Fig. 1, we compare the upper and lower bounds on the WER to a numerical evaluation of $\mathbb{P}_{\epsilon,\text{MP}}$, for $K = 16, 64, 128$. For any K, the lower bound $\underline{\mathbb{P}}_{\epsilon,\text{MP}}$ is tighter at low SNR. On the other hand, for small values of K, the upper bound is tighter at high SNR. As K increases, the lower bound closely matches the numerical evaluation of $\mathbb{P}_{\epsilon,\text{MP}}$ at any SNR.

Remark 1. For large values of K, we have

$$\underline{\mathbb{P}}_{\epsilon,\text{MP}} \approx \mathbb{P}_{\epsilon,\text{AWGN}(\sigma^2)} = 1 - \left(1 - Q\left(\frac{1}{\sigma} \right) \right)^K, \tag{20}$$

where $\mathbb{P}_{\epsilon,\text{AWGN}(\sigma^2)}$ is the WER under Euclidean detection for an AWGN channel without offset and noise variance σ^2. In Fig. 1, we show that for large K we have $\underline{\mathbb{P}}_{\epsilon,\text{MP}} \approx \mathbb{P}_{\epsilon,\text{MP}}$. Therefore, as K increases, the error performance of the AWGN channel with unbounded offset and under MP detection tends to the error performance of an AWGN channel without offset under Euclidean detection.

[1] In [2, Eq. (28)], the correct scaling term for the argument of the Q function is $1/\sigma$ instead of $1/(2\sigma)$.

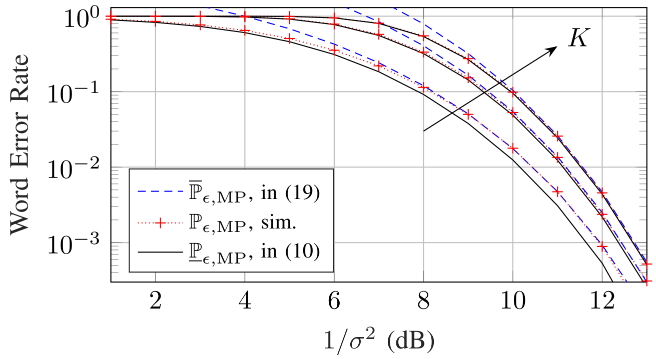

Figure 1. Bounds and Monte Carlo evaluation of the WER under MP detection vs SNR, for $K = 16, 64, 128$.

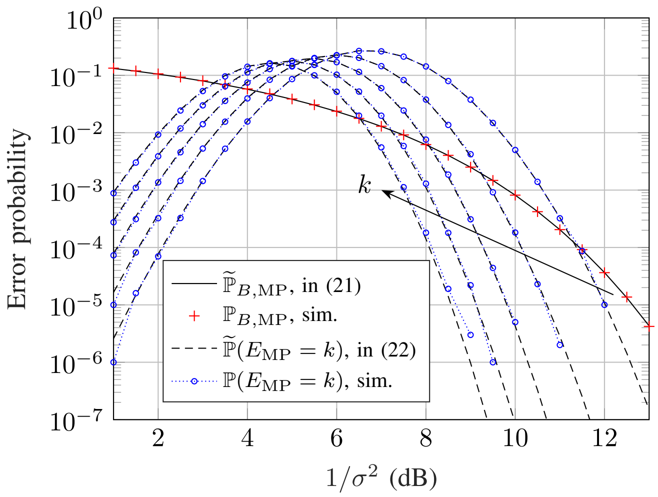

Figure 2. Error probability curves under MP detection vs SNR, for $K = 128$ and $k = 2, \ldots, 6$.

B. Bit Error Rate

In view of Remark 1, we posit that for sufficiently large K one can approximate the bit error rate (BER) of the MP detector as follows

$$\mathbb{P}_{B,\mathrm{MP}} \approx \widetilde{\mathbb{P}}_{B,\mathrm{MP}} = \frac{\mathbb{E}\left[\mathcal{B}\left(K, Q\left(\frac{1}{\sigma}\right)\right)\right]}{K} = Q\left(\frac{1}{\sigma}\right), \qquad (21)$$

where $\mathcal{B}(n,p)$ is the binomial distribution with n trials and success probability p per trial. Furthermore, the probability of having k bit errors can be approximated by

$$\widetilde{\mathbb{P}}(E_{\mathrm{MP}} = k) = \binom{K}{k} (\widetilde{\mathbb{P}}_{B,\mathrm{MP}})^k (1 - \widetilde{\mathbb{P}}_{B,\mathrm{MP}})^{K-k}. \qquad (22)$$

Let us assess the accuracy of the proposed approximations via numerical simulations.

In Fig. 2, we show BER and error probability curves for $K = 128$. The black line is the analytical approximated BER, while the red + markers show the Monte Carlo evaluated BER. The black dashed lines and the blue dotted lines with circle markers, show, respectively, the analytical approximation $\widetilde{\mathbb{P}}(E_{\mathrm{MP}} = k)$ and the numerical evaluation of $\mathbb{P}(E_{\mathrm{MP}} = k)$, both for $k = 2, \ldots, 6$. The close match between the analytical and simulated curves shows us the high level of accuracy given by the approximation in (21) and (22).

V. Simplified Pearson

The MP detector requires the computation of sample average $\bar{\mathbf{Y}}$ over all memory reads and the computation of K metrics, see (8). Note that, by (8), the values $\delta_k(\mathbf{Y})$ can be computed recursively. Moreover, at high SNR, the function $k \mapsto \delta_k(\mathbf{Y})$ is sharp, i.e., the difference $\delta_k(\mathbf{Y}) - \delta_{k-1}(\mathbf{Y})$ exhibits a sudden change of sign at some k.

We define the Simplified Pearson (SP) detector as the approximate Pearson detector implemented by estimating the codeword's Hamming weight $\hat{W}_{\mathrm{SP}}(\mathbf{Y})$ as the smallest k such that $\delta_k(\mathbf{Y}) - \delta_{k-1}(\mathbf{Y}) > 0$, i.e.,

$$\hat{W}_{\mathrm{SP}}(\mathbf{Y}) = \min\left\{k : \left(Y_k - \bar{\mathbf{Y}} + \frac{K+1-2k}{K}\right) > 0\right\} - 1. \qquad (23)$$

The mentioned features make SP detection particularly appealing for any kind of memory technology based on ramp sensing [8]. Indeed, when paired with ramp sensing, the SP detector becomes particularly efficient and can be implemented simply by counting the trigger events satisfying (23).

A. Error Analysis on the Hamming Weight Estimation

Notice that $\bar{\mathbf{X}} = (K - 2w)/K$, where $w = \mathrm{w}(\mathbf{C})$ is the Hamming weight of the binary word $\mathbf{C}$. Therefore, we have

$$\bar{\mathbf{Y}} = \frac{K - 2w}{K} + b + \bar{\mathbf{Z}}, \qquad (24)$$

where $\bar{\mathbf{Z}} = \frac{1}{K}\sum_{k=1}^{K} Z_k$, with $Z_k \sim \mathcal{N}(0, \sigma^2)$, $\forall k$. Notice also that

$$Y_k - \bar{\mathbf{Y}} + 1 + \frac{1-2k}{K} = Y_k - b - \bar{\mathbf{Z}} + \frac{1 + 2(w-k)}{K}, \qquad (25)$$

and therefore (23) can be rewritten as

$$\hat{W}_{\mathrm{SP}}(\mathbf{Y}) = \min\left\{k : \left(Y_i - b - \bar{\mathbf{Z}} + \frac{1 + 2(w-k)}{K}\right) > 0\right\} - 1. \qquad (26)$$

Let $\epsilon_w = \hat{W}_{\mathrm{SP}}(\mathbf{Y}) - w$ be the estimation error of the Hamming weight w. We have $\epsilon_w = \hat{k} - 1 - w$ with $\hat{k}$ satisfying the condition in (26). The complementary cumulative distribution function (cdf) of ϵ_w is approximately

$$\mathbb{P}(\epsilon_w > \epsilon) \approx \mathbb{P}\left(Y_{w+\epsilon+1} - b - \bar{\mathbf{Z}} \leq \frac{1+2\epsilon}{K}\right), \qquad (27)$$

with $\epsilon \in \mathbb{Z} \cap [-K, K]$ and under the assumption that, if the condition in (26) is not satisfied for a given $\tilde{k}$, then it is not either for $\forall k < \tilde{k}$. In (27), $Y_{w+\epsilon+1}$ is the $(w+\epsilon+1)$th sample in the complete ordered statistic of the read samples.

Lemma 1. *Consider two classes of independent memory reads* $\mathbf{U}^{(1)}$ *and* $\mathbf{U}^{(2)}$*, such that*

$$U_k^{(i)} \sim \mathcal{N}(\mu_i, \sigma^2), \qquad k = 1, \ldots, n_i, \qquad (28)$$

for $i = 1, 2$*. Denote by* $\mathbf{U}$ *the ordered statistic of the mixed population given by* $\mathbf{U}^{(1)}$ *and* $\mathbf{U}^{(2)}$*. Then, the cdf of the* j*th ordered statistic* U_j *is*

$$F_{U_j}(u) = \sum_{k=j}^{n_1+n_2} \{\mathcal{B}(n_1, \Phi_1(u)) * \mathcal{B}(n_2, \Phi_2(u))\}(k), \quad (29)$$

where Φ_i is the $\mathcal{N}(\mu_i, \sigma^2)$ cdf, and $*$ denotes convolution.

Proof. Let V_i be the number of samples that fall below a threshold u for the ith class. Then, $V_i \sim \mathcal{B}(n_i, \Phi_i)$, where V_1 and V_2 are independent and

$$\Phi_i(u) = \mathbb{P}(U_k^{(i)} \leq u) = \Phi\left(\frac{u - \mu_i}{\sigma}\right), \quad i = 1, 2. \quad (30)$$

Then, the cdf of U_j is given by

$$\mathbb{P}(U_j \leq u) = \sum_{k=j}^{n_1+n_2} \mathbb{P}(U_k \leq u, U_{k+1} > u) \quad (31)$$

$$= \sum_{k=j}^{n_1+n_2} \mathbb{P}(V_1 + V_2 = k) \quad (32)$$

$$= \sum_{k=j}^{n_1+n_2} \{\mathcal{B}(n_1, \Phi_1(u)) * \mathcal{B}(n_2, \Phi_2(u))\}(k). \quad (33)$$

□

Let us consider the two classes of reads

$$\mathbf{Y}^{(-1)} = \{Y_i : \ X_i = -1, \ i = 1, \dots, K\}, \quad (34a)$$

$$\mathbf{Y}^{(+1)} = \{Y_i : \ X_i = +1, \ i = 1, \dots, K\}. \quad (34b)$$

Moreover, let us denote by $\mathbf{U}$ the ordered statistic of the mixed population $\mathbf{Y}^{(-1)}$ and $\mathbf{Y}^{(+1)}$. Under the simplifying assumption that $Y_{(w+\epsilon+1)}$ and $\bar{\mathbf{Z}}$ are independent and thanks to Lemma 1, we have

$$\mathbb{P}(\epsilon_w > \epsilon) \approx \mathbb{P}\left(Y_{w+\epsilon+1} \leq b + \bar{\mathbf{Z}} + \frac{1+2\epsilon}{K}\right) \quad (35)$$

$$= \mathbb{E}\left[F_{U_{w+\epsilon+1}}\left(b + \bar{\mathbf{Z}} + \frac{1+2\epsilon}{K}\right)\right], \quad (36)$$

with expectation over $\bar{\mathbf{Z}} \sim \mathcal{N}(0, \sigma^2/K^2)$, and for the probability mass function of ϵ_w we have

$$P_{\epsilon_w}(\epsilon) = \mathbb{P}(\epsilon_w > \epsilon - 1) - \mathbb{P}(\epsilon_w > \epsilon). \quad (37)$$

Finally, for any given Hamming weight distribution $\mathbb{P}(W = w)$, the probability of wrong Hamming weight estimate is

$$\mathbb{P}(\hat{W}_{\text{SP}} - w = \epsilon) = \sum_{w=0}^{K-1} P_{\epsilon_w}(\epsilon)\mathbb{P}(W = w). \quad (38)$$

Let us assume that the zero crossing of the difference function $k \mapsto D_k(\mathbf{Y}) = \delta_k(\mathbf{Y}) - \delta_{k-1}(\mathbf{Y})$ happens once, where

$$D_k(\mathbf{Y}) = Y_{k:K} - \bar{\mathbf{Y}} + \frac{K+1-2k}{K}, \qquad k = 1, \dots, K. \quad (39)$$

We want to characterize the distribution of $\hat{W}_{\text{SP}}$ when the Hamming weight of the written codeword is w, i.e.

$$F_{\hat{W}_{\text{SP}};w}(k) \triangleq \mathbb{P}(\hat{W}_{\text{SP}} < k) \approx \mathbb{P}(D_k \geq 0). \quad (40)$$

Let us assume again that if, for a given $\tilde{k}$, the condition in (26) is not satisfied, then it is not either for $\forall k < \tilde{k}$. Given the just mentioned assumption, (40) follows because $D_k \geq 0$ implies that $\hat{W}_{\text{SP}}$ has surely been set to a value less than k. Notice that also the converse implication is true, i.e., if $\hat{W}_{\text{SP}} = t < k$, then $D_{t+1} \geq 0$, hence $D_k \geq 0$. Therefore, the two events $\hat{W}_{\text{SP}} < k$ and $D_k \geq 0$ are equivalent under the mentioned assumption on k and (26).

By (24), we have

$$F_{\hat{W}_{\text{SP}};w}(k) \approx \mathbb{P}(D_k \geq 0) \quad (41)$$

$$= \mathbb{P}\left(Y_{k:K} - b - \bar{\mathbf{Z}} + \frac{1 - 2(k-w)}{K} \geq 0\right) \quad (42)$$

$$= 1 - \mathbb{P}\left(Y_{k:K} \leq b + \bar{\mathbf{Z}} - \frac{1 - 2(k-w)}{K}\right) \quad (43)$$

$$\approx 1 - \mathbb{E}\left[F_{U_k}\left(b + \bar{\mathbf{Z}} - \frac{1 - 2(k-w)}{K}\right)\right], \quad (44)$$

where the last step follows from (36). Finally, we can write

$$\mathbb{P}(\hat{W}_{\text{SP}} = k) = \sum_{w=0}^{K-1} \mathbb{P}(W = w)\mathbb{P}(\hat{W}_{\text{SP}} = k | W = w), \quad (45)$$

where $\mathbb{P}(\hat{W}_{\text{SP}} = k | W = w) = F_{\hat{W}_{\text{SP}};w}(k+1) - F_{\hat{W}_{\text{SP}};w}(k)$.

B. Bit Error Rate

Let us now evaluate the bit error statistics. We denote by E_{SP} the number of bit errors at the output of the SP detector.

Notice that the event $\{\hat{W}_{\text{SP}} = k\}$ implies the event $\{D_k < 0 < D_{k+1}\}$ and if we assume a single zero crossing of the metrics $\{D_k\}$, then $\mathbb{P}(\hat{W}_{\text{SP}} = k) = \mathbb{P}(D_k < 0 < D_{k+1})$. The event $\{D_k < 0 < D_{k+1}\}$ can be rewritten as follows

$$Y_{k:K} < b + \bar{\mathbf{Z}} + \frac{2(k-W)-1}{K} < Y_{(k+1):K} - \frac{2}{K}. \quad (46)$$

Notice that (46) defines an upper threshold for $Y_{k:K}$, and a lower threshold for $Y_{(k+1):K}$:

$$Y_{k:K} < t_{-1} \triangleq b + \bar{\mathbf{Z}} + \frac{2(k-W)-1}{K}, \quad (47)$$

$$Y_{(k+1):K} > t_{+1} \triangleq b + \bar{\mathbf{Z}} + \frac{2(k-W)+1}{K}. \quad (48)$$

Let us consider the classes (-1) and $(+1)$ defined in (34) and define the intervals $T_{-1} = (-\infty, t_{-1})$, $T_0 = [t_{-1}, t_{+1}]$, and $T_{+1} = (t_{+1}, \infty)$. For each class, the probability of having n_1 elements in T_{-1}, n_2 elements in T_0, and n_3 elements in T_{+1} can be modeled by the Trinomial distribution

$$\mathcal{T}\begin{pmatrix} n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{pmatrix} = \frac{(n_1+n_2+n_3)!}{n_1!n_2!n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}, \quad (49)$$

where p_1, p_2, and p_3 are the probabilities, for any given Y_i, of falling in one of the intervals T_{-1}, T_0, and T_{+1}, respectively. The two trinomials, for classes (-1) and $(+1)$, are independent. We are interested in the event $n_2 = 0$ for both classes. Bit errors occur when elements of class (-1) fall in T_{+1} and elements of class $(+1)$ in T_{-1}. We can observe k elements in the *wrong* intervals in $k+1$ ways, i.e., i elements of class (-1) in T_{+1}, and $k-i$ elements of class $(+1)$ in T_{-1}

for $i = 0, \ldots, k$. Therefore, the probability of having k bit errors given Hamming weight w, i.e., $\mathbb{P}(E_{\text{SP}} = k \mid W = w)$, can be approximated by

$$\widetilde{\mathbb{P}}(E_{\text{SP}} = k \mid W = w) = \sum_{i=0}^{k} \mathcal{T}\begin{pmatrix} w-i & 0 & i \\ p_{A1} & 1-p_{A1}-p_{A2} & p_{A2} \end{pmatrix} \cdot \mathcal{T}\begin{pmatrix} k-i & 0 & K-w-(k-i) \\ p_{B1} & 1-p_{B1}-p_{B2} & p_{B2} \end{pmatrix}, \quad (50)$$

where the first trinomial refers to the class (-1) and is characterized by

$$p_{A1} = \Phi\left(\frac{K-1+2(k-2i)}{K\sigma_n}\right), \quad (51)$$

$$p_{A2} = 1-\Phi\left(\frac{K+1+2(k-2i)}{K\sigma_n}\right), \quad (52)$$

while the second trinomial refers to the class $(+1)$ with

$$p_{B1} = \Phi\left(\frac{-K-1+2(k-2i)}{K\sigma_n}\right), \quad (53)$$

$$p_{B2} = 1-\Phi\left(\frac{-K+1+2(k-2i)}{K\sigma_n}\right), \quad (54)$$

and $\sigma_n = \sigma\sqrt{\frac{K-1}{K}}$. Therefore, notice that

$$\widetilde{\mathbb{P}}(E_{\text{SP}} = k) = \mathbb{E}\left[\widetilde{\mathbb{P}}(E_{\text{SP}} = k \mid W)\right], \quad (55)$$

and we can approximate the BER, under SP detection, as

$$\widetilde{\mathbb{P}}_{B,\text{SP}} = \frac{\sum_{k=1}^{K} k \cdot \widetilde{\mathbb{P}}(E_{\text{SP}} = k)}{K}. \quad (56)$$

Remark 2. When $\lim_{K\to\infty} k/K = 0$, we have

$$\begin{aligned} &\lim_{K\to\infty} \widetilde{\mathbb{P}}(E_{\text{SP}} = k \mid W = w) \\ &= \{\mathcal{B}(w, 1-\Phi(1/\sigma)) * \mathcal{B}(K-w, \Phi(-1/\sigma))\}(k) && (57) \\ &= \{\mathcal{B}(w, 1-\Phi(1/\sigma)) * \mathcal{B}(K-w, 1-\Phi(1/\sigma))\}(k) && (58) \\ &= \mathcal{B}(K, 1-\Phi(1/\sigma))(k) && (59) \\ &= \mathbb{P}(E_{\text{AWGN}(\sigma^2)} = k), && (60) \end{aligned}$$

where $E_{\text{AWGN}(\sigma^2)}$ is the number of symbol errors at the output of Euclidean detection for an AWGN channel without offset and with noise variance σ^2, i.e.,

$$E_{\text{AWGN}(\sigma^2)} \sim \mathcal{B}\left(K, Q\left(\frac{1}{\sigma}\right)\right). \quad (61)$$

Equivalently, T_0 disappears, we have two identical binomials for the errors of both classes, and the total number of errors can be written as a single binomial. Therefore, as K increases, the error performance of the SP detector tends to that of the MP detector.

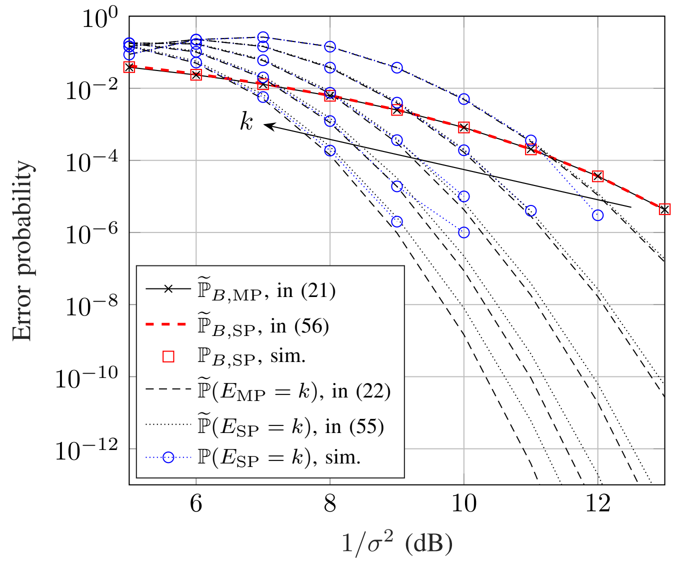

Figure 3. Error probability curves under SP detection vs SNR, for $K = 128$ and for $k = 2, \ldots, 6$.

Numerical Results: Let us now provide numerical results on the performance of the SP detector. In Fig. 3, the black solid line with crosses is the approximated BER performance for the MP detector. The red squares markers and the red dashed line are, respectively, the simulated and the analytical approximation of the BER for the SP detector. The black dashed lines, the black dotted lines, and the blue dotted lines with circle markers are, respectively, the probabilities of k errors given by the MP and SP analytical approximation and the SP simulation. All the mentioned error probabilities are evaluated for $k = 2, \ldots, 6$. We have already shown that the analytical approximations of $\mathbb{P}_{B,\text{MP}}$ and $\mathbb{P}(E_{\text{MP}} = k)$ closely match the MP error probabilities obtained via numerical simulation, see Fig. 2. Therefore, to improve the readability of Fig. 3, for the MP detector we show only $\widetilde{\mathbb{P}}_{B,\text{MP}}$ and $\widetilde{\mathbb{P}}(E_{\text{MP}} = k)$ and neglect the corresponding simulated curves.

Notice that the performance of the SP detector closely matches that of the MP detector. Moreover, Fig. 3 shows that the analytical approximation of the error curves for the SP detector agrees with the Monte Carlo simulation. Therefore, if compared to the MP detector, the SP detector is a lower complexity solution with a fairly negligible loss in terms of performance.

VI. Conclusion

Given an additive white Gaussian noise (AWGN) channel with unknown and unbounded offset, we derived a tight lower bound on the word error rate under modified Pearson (MP) detection. We also showed that the MP detector provides error performance that closely matches that of an Euclidean detector for an AWGN channel without offset. Moreover, we proposed a new detection scheme based on a simplification of the MP detection, namely the Simplified Pearson (SP) detection. We showed that the SP detector, although being characterized by a lower computational complexity, retains error performance close to that of the MP detector.

References

[1] K. A. S. Immink, K. Cai, and J. H. Weber, “Dynamic threshold detection based on pearson distance detection,” *IEEE Transactions on Communications*, vol. 66, no. 7, pp. 2958–2965, 2018.

[2] K. A. S. Immink and J. H. Weber, “Minimum Pearson distance detection for multilevel channels with gain and/or offset mismatch,” *IEEE Transactions on Information Theory*, vol. 60, no. 10, pp. 5966–5974, 2014.

[3] K. A. S. Immink and J. H. Weber, “A new detection method for noisy channels with time-varying offset,” *IEEE Transactions on Information Theory*, vol. 68, no. 5, pp. 3108–3114, 2022.

[4] J. H. Weber and K. A. S. Immink, “Maximum likelihood decoding for Gaussian noise channels with gain or offset mismatch,” *IEEE Communications Letters*, vol. 22, no. 6, pp. 1128–1131, 2018.

[5] J. H. Weber, K. A. S. Immink, and S. R. Blackburn, “Pearson codes,” *IEEE Transactions on Information Theory*, vol. 62, no. 1, pp. 131–135, 2015.

[6] R. Bu and J. H. Weber, “Maximum likelihood decoding for multi-level cell memories with scaling and offset mismatch,” in *ICC 2019-2019 IEEE International Conference on Communications (ICC)*. IEEE, 2019, pp. 1–6.

[7] D. Slepian, “Permutation modulation,” *Proceedings of the IEEE*, vol. 53, no. 3, pp. 228–236, 1965.

[8] M. Sforzin, G. Mirichigni, A. Orlando, and P. Amato, “Self-referenced read methodology for ems,” in *2018 IEEE International Memory Workshop (IMW)*. IEEE, 2018, pp. 1–4.